STUDIA MATHEMATICA 221 (2) (2014)

On the relationships between Fourier–Stieltjes coefficients and spectra of measures

by

Przemysław Ohrysko and Michał Wojciechowski (Warszawa)

Abstract. We construct examples of uncountable compact subsets of complex numbers with the property that any Borel measure on the circle group with Fourier coefficients taking values in this set has a natural spectrum. For measures with Fourier coefficients tending to 0 we construct an open set with this property. We also give an example of a singular measure whose spectrum is contained in our set.

1. Introduction. Let $M(\mathbb{T})$ denote the convolution algebra of Borel measures on the unit circle group. For details of notation and basic definitions see [Ka]. The closure of the set of Fourier coefficients of $\mu \in M(\mathbb{T})$ is obviously a subset of the spectrum $\sigma(\mu)$ of μ. However, as was observed by Wiener and Pitt (see [WP]) in general it is a proper subset. There are several different proofs of this phenomenon (cf. [S], [G]). This *Wiener–Pitt phenomenon* is equivalent to the *inversion problem* which states that the assumption $|\widehat{\mu}(n)| > c > 0$ for all $n \in \mathbb{Z}$ and constant c does not ensure the invertibility of μ as an element in the Banach algebra $M(\mathbb{T})$. Moreover, it is closely related to the asymmetry of the algebra $M(\mathbb{T})$, discussed in [R].

On the other hand there are classes of Borel measures for which the spectrum equals the closure of the set of Fourier coefficients. Such measures are said to have a *natural spectrum*. It is known that absolutely continuous and purely discrete measures have a natural spectrum (cf. [Za]). A natural question is how to recognize a measure with a natural spectrum using only the information about its Fourier coefficients. Motivated by this problem we introduce the notion of Wiener–Pitt sets. We say that a compact set $A \subset \mathbb{C}$ is a *Wiener–Pitt set* whenever $\widehat{\mu}(\mathbb{Z}) \subset A$ implies that μ has a natural spectrum.

Finite sets are easy examples of Wiener–Pitt sets. Indeed, if $A = \{a_1, \ldots, a_k\}$ then by Gelfand theory, the polynomial $P(z) = (z - a_1)\ldots(z - a_k)$

2010 *Mathematics Subject Classification*: Primary 43A10; Secondary 43A25.
Key words and phrases: natural spectrum, Wiener–Pitt phenomenon, Fourier–Stieltjes coefficients, convolution algebra, spectrum of measure.

DOI: 10.4064/sm221-2-2

satisfies $\widehat{P(\mu)}(n) = 0$ for every $n \in \mathbb{Z}$. Therefore $P(\mu) = 0$, which in turn yields $P(\psi(\mu)) = 0$ for every multiplicative linear functional ψ. Hence $\psi(\mu)$ is a root of P and therefore $\sigma(\mu) \subset A$. The finite sets are the only known class of Wiener–Pitt sets. The aim of this paper is to construct infinite (even uncountable) Wiener–Pitt compacta. Our construction gives a quite flexible family of zero-dimensional examples which are, moreover, stable under suitable small perturbations. Furthermore we construct an open subset $U \subset \mathbb{C}$ such that $0 \in \overline{U}$ and any continuous measure μ with $\widehat{\mu}(\mathbb{Z}) \subset U \cup \{0\}$ satisfies $\mu \in \mathrm{Rad}(L^1(\mathbb{T}))$; thanks to the theorem of Zafran (see Theorem 19 in Section 4), this gives

THEOREM 1. *There exists an open set $U \subset \mathbb{C}$ with $0 \in \overline{U}$ such that every continuous measure μ with $\widehat{\mu}(\mathbb{Z}) \subset U \cup \{0\}$ has a natural spectrum.*

In the general case we prove the following.

THEOREM 2. *There exists a set K homeomorphic to the Cantor set such that $0 \in K$ and every measure μ with $\widehat{\mu}(\mathbb{Z}) \subset K$ has a natural spectrum.*

To complete the above results we provide an example of a singular measure with spectrum contained in the set U of Theorem 1. The example, interesting in its own right, is given in Section 6; we combine the techniques of Riesz products with Rudin–Shapiro polynomials to get a singular measure with coefficients smaller than any sequence tending to 0 sufficiently fast taken in advance.

The construction is quite involved—it uses four main ingredients. The first is the Zafran characterization of measures with Fourier coefficients tending to zero with a natural spectrum. The second is the Katznelson–DeLeeuw theorem which is the main ingredient of their qualitative version of the Grużewska–Rajchman theorem (however, we use a stronger and more involved result from [GM], just to avoid unnecessary complications). The third ingredient is the Bożejko–Pełczyński theorem on the uniform invariant approximation property of $L^1(\mathbb{T})$. The fourth is the Littlewood conjecture proved by McGehee–Pigno–Smith and independently by Konyagin.

The main construction of the proof of Theorem 1, under the additional assumption that the relevant measure has Fourier coefficients tending to 0, is presented in Section 3. The aim of Section 4 is to prove that this additional assumption can be omitted. In Section 5 we complete the proof of Theorem 1 and we show how Theorem 2 can be derived from Theorem 1. Section 2 contains auxiliary lemmas. Here we combine two main analytical ingredients, the Bożejko–Pełczyński uniform invariant approximation property and the Littlewood conjecture, to derive Lemma 8, the main tool used in further inductions.

Remark 3. Some results of this paper are contained in the first author's MA thesis (see [O]).

2. Preparatory lemmas. In this section we prove some crucial lemmas which are also of independent interest. The following definition will be useful.

Definition 4. For $\mu \in M(\mathbb{T})$ and $\varepsilon > 0$ we write $\mu \in F(\varepsilon)$ iff $|\widehat{\mu}(n)| < \varepsilon$ for all $n \in \mathbb{Z}$. The set of trigonometric polynomials will be denoted $\mathscr{P}$. The abbreviation $\# f$ will be used for the number of elements in the support of $\widehat{f}$ (for $f \in \mathscr{P}$), i.e. $\# f = \# \operatorname{supp} \widehat{f} = \#\{n \in \mathbb{Z} : \widehat{f}(n) \neq 0\}$. Also, for $f \in \mathscr{P}$ and a positive number a we write $f \in G(a)$ when $|\widehat{f}(n)| \geq a$ for all integers n such that $\widehat{f}(n) \neq 0$.

We will use two powerful results. The first is the Littlewood conjecture (for a proof consult [MPS] and [Ko])

Theorem 5 (McGehee, Pigno, Smith; Konyagin). *For every $f \in \mathscr{P}$ of the form*

$$f(t) = \sum_{k=1}^{N} c_k e^{i n_k t},$$

where n_k is a sequence of increasing integers and $|c_k| \geq 1$ for $1 \leq k \leq N$, we have

$$\|f\|_{L^1(\mathbb{T})} > L \ln N,$$

where the constant $L > 0$ does not depend on N.

The second fact is the invariant uniform approximation property of $L^1(\mathbb{T})$ (proofs are contained in the papers [BP] and [B] and in the book [Wo]).

Theorem 6 (Bożejko, Pełczyński; Bourgain). *Let $\Lambda \subset \mathbb{N}$ be a finite set with $\#\Lambda = k$. Then for every $\varepsilon > 0$ there exists $f \in \mathscr{P}$ such that*

(i) $\widehat{f}(n) = 1$ *for* $n \in \Lambda$.

(ii) $\|f\|_{L^1(\mathbb{T})} \leq 1 + \varepsilon$.

(iii) $\#\{n \in \mathbb{N} : \widehat{f}(n) \neq 0\} \leq (\alpha/\varepsilon)^{2k}$ *for some* $\alpha > 0$.

We will write $\mathrm{BPB}_{\varepsilon}(\Lambda)$ *for the set of polynomials with properties* (i)–(iii).

It is now time to formulate our first lemma.

Lemma 7. *Let $f \in \mathscr{P}$. If $\# f \geq d$ for some positive number d, then there exists a two-sided arithmetical progression $\Gamma \subset \mathbb{Z}$ such that*

$$d \leq \#(\operatorname{supp} \widehat{f} \cap \mathbf{1}_{\Gamma}) < 2d.$$

Proof. If $d \leq \# f < 2d$, we take $\Gamma = \mathbb{Z}$. Otherwise $\# f \geq 2d$ and taking $\Gamma_1 = 2\mathbb{Z}$ and $\Gamma_2 = 2\mathbb{Z} + 1$ we get $d \leq \#(\operatorname{supp} \widehat{\mu} \cap \mathbf{1}_{\Gamma_i})$ for some $i = 1, 2$. If moreover $\#(\operatorname{supp} \widehat{f} \cap \mathbf{1}_{\Gamma_i}) < 2d$, we put $\Gamma = \Gamma_i$. Otherwise we repeat this procedure. ∎

The second lemma is much more sophisticated (since for absolutely continuous measures the norm in $M(\mathbb{T})$ is equal to the norm in $L^1(\mathbb{T})$ of the density with respect to the Lebesgue measure, we will use these two notations interchangeably).

LEMMA 8. *There exists a function $\varepsilon = \varepsilon(K,a)$ and $c > 0$ such that whenever $\|f+\nu\|_{M(\mathbb{T})} < K$ for some $f \in G(a)$ and $\nu \in F(\varepsilon)$, then $\#f < \exp(cK/a)$.*

Proof. Set $d = \exp\left(\frac{4K}{aL}\right)$ and define

$$\varepsilon = \varepsilon(K,a) = \frac{aL\ln d}{4\alpha^{2d}} = K\exp\left(-2\ln(\alpha)\exp\left(\frac{4K}{aL}\right)\right) \tag{1}$$

where α is the constant from Theorem 6. We will show that the assumption $\#f \geq d$ leads to a contradiction, which proves our lemma with constant $c = 4/L$.

By Lemma 7 there exists $\Gamma \subset \mathbb{Z}$ such that $d \leq \#(\operatorname{supp}\widehat{\nu} \cap \mathbf{1}_\Gamma) < 2d$. We define $f_1 \in \mathscr{P}$ and $\nu_1 \in M(\mathbb{T})$ by taking $\widehat{f_1} = \widehat{f} \cdot \mathbf{1}_\Gamma$ and $\widehat{\nu_1} = \widehat{\nu} \cdot \mathbf{1}_\Gamma$. Since multiplying Fourier sequences by the characteristic function of Γ corresponds to convolution with a measure of norm one, we have $\|f_1 + \nu_1\|_{M(\mathbb{T})} < K$. By the definition of Γ, $\#f_1 < 2d$. It follows from Theorem 5 that $\|f_1\|_{M(\mathbb{T})} \geq a \cdot L\ln d$. Let $\Theta \in \mathrm{BPB}_1(\operatorname{supp} f_1)$. Then $\|\Theta\|_{L^1(\mathbb{T})} < 2$, $\widehat{\Theta}(n) = 1$ for $n \in \operatorname{supp} f_1$, $\#\Theta < \alpha^{4d}$ for some $\alpha > 0$. By the triangle inequality,

$$2K > \|(f_1+\nu_1) * \Theta\|_{M(\mathbb{T})} = \|f_1 + \nu_1 * \Theta\|_{M(\mathbb{T})} \geq \|f\|_{L^1(\mathbb{T})} - \|\Theta * \nu_1\|_{M(\mathbb{T})}.$$

Estimating the L^1-norm by the L^2-norm we get

$$\|\Theta * \nu_1\|_{L^1(\mathbb{T})} \leq \|\Theta * \nu_1\|_{L^2(\mathbb{T})} \leq 2\varepsilon\alpha^{2d}.$$

Altogether this gives $2K > aL\ln d - 2\varepsilon\alpha^{2d}$. Hence the formula (1) for $\varepsilon = \varepsilon(K,a)$ leads to a contradiction. ∎

The next lemma gives more information.

LEMMA 9. *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, a, K)$ such that if $\|f+\nu\|_{M(\mathbb{T})} < K$ for $f \in G(a)$ and $\nu \in F(\delta)$, then $\|f\|_{L^1(\mathbb{T})} < K(1+\varepsilon)$.*

Proof. Let $\Theta \in \mathrm{BPB}_{\varepsilon/2}(\operatorname{supp}\widehat{f})$. Then $\|\Theta\|_{L^1(\mathbb{T})} < 1+\varepsilon/2$, $\widehat{\Theta}(n) = 1$ for $n \in \operatorname{supp}\widehat{f}$, $\#\Theta < (\lambda/\varepsilon)^{2\#f} = \exp(2\#f\ln(\lambda/\varepsilon))$ for some $\lambda > 0$. By Lemma 8 for sufficiently small δ we have

$$\#\Theta < \exp\left(2\ln\left(\frac{\lambda}{\varepsilon}\right)\exp(cKa^{-1})\right),$$

where c is as in Lemma 8. By the triangle inequality

$$\left(1+\frac{\varepsilon}{2}\right)K \geq \|\Theta * (f+\nu)\|_{L^1(\mathbb{T})} = \|f + \Theta * \nu\|_{L^1(\mathbb{T})} \geq \|f\|_{L^1(\mathbb{T})} - \|\Theta * \nu\|_{L^1(\mathbb{T})}.$$

Since $|\widehat{\Theta}| < 1 + \varepsilon/2$, we have $\Theta * \nu \in F((1+\varepsilon/2)\delta)$. Obviously

$$\#(\Theta * \nu) \le \#\Theta < \exp\biggl(2\ln\Bigl(\frac{\lambda}{\varepsilon}\Bigr)\exp(cKa^{-1})\biggr).$$

Estimating the L^1-norm by the L^2-norm we get

$$\|\Theta * \nu\|_{L^1(\mathbb{T})} \le \|\Theta * \nu\|_{L^2(\mathbb{T})} \le \Bigl(1+\frac{\varepsilon}{2}\Bigr)\delta\exp\biggl(\ln\Bigl(\frac{\lambda}{\varepsilon}\Bigr)\exp(cKa^{-1})\biggr).$$

Altogether we get

$$\Bigl(1+\frac{\varepsilon}{2}\Bigr)K > \|f\|_{L^1(\mathbb{T})} - \Bigl(1+\frac{\varepsilon}{2}\Bigr)\delta\exp\biggl(\ln\Bigl(\frac{\lambda}{\varepsilon}\Bigr)\exp(cKa^{-1})\biggr).$$

If we put

$$\delta < \frac{\varepsilon}{1+\varepsilon}K\exp\biggl(-\ln\Bigl(\frac{\lambda}{\varepsilon}\Bigr)\exp(cKa^{-1})\biggr),$$

then the assumption $\|f\|_{L^1(\mathbb{T})} > (1+\varepsilon)K$ leads to a contradiction. ■

3. The case of $M_0(\mathbb{T})$. We will make use of the following result (see [W, Proposition 1.9]).

Proposition 10. *Suppose A is a commutative Banach algebra with unit and $x \in A$ has a finite spectrum, $\sigma(x) = \{\lambda_1, \ldots, \lambda_n\}$. Put $\delta = \min_{i\neq j}|\lambda_i - \lambda_j|$. Then there exist orthogonal idempotents $x_1, \ldots, x_n \in A$ (i.e. $x_i^2 = x_i$ and $x_i x_j = 0$ for $i \ne j$, $i, j = 1, \ldots, n$) such that*

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n.$$

Moreover, $\|x_i\| \le \delta^{-n+1}2^{n-1}\|x\|^{n-1}$ for $i = 1, \ldots, n$.

One abbreviation is useful when it comes to manipulating with convolution powers. We will write $f^m = f^{*m} = f*\cdots*f$ (m-times) for a function (or a measure) f and to avoid any misunderstandings we will not use pointwise multiplication for functions up to the end of this section.

Now, we prove a simple corollary of the last proposition. From now on, unless otherwise stated, $\|\cdot\|$ denotes the $L^1(\mathbb{T})$ norm.

Lemma 11. *Let f be a trigonometric polynomial such that*

$$\widehat{f}(\mathbb{Z}) = \{0, \lambda_1, \ldots, \lambda_k\}.$$

Put $\lambda_0 = 0$ and define $\delta = \min_{i\neq j}|\lambda_i - \lambda_j|$, $\lambda_{\max} = \max\{|\lambda| : \lambda \in \widehat{f}(\mathbb{Z})\}$. Then, for every $m \in \mathbb{N}$,

$$\|f^m\| \le k\delta^{-k}2^k\|f\|^k\lambda_{\max}^m.$$

Proof. We easily see that, if f is a polynomial, its spectrum in the algebra $M(\mathbb{T})$ is equal to $\widehat{f}(\mathbb{Z})$. Hence, by Proposition 10, we have

$$f = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_k f_k$$

for some orthogonal idempotent polynomials f_k. A simple calculation shows that

$$f^m = \Big(\sum_{l=1}^{k} \lambda_l f_l\Big)^m = \sum_{l=1}^{k} \lambda_l^m f_l.$$

Applying the estimate from Proposition 10 we complete the proof:

$$\|f^m\| \le \sum_{l=1}^{k} |\lambda_l|^m \|f_l\| \le k\delta^{-k} 2^k \|f\|^k \lambda_{\max}^m. \blacksquare$$

The next theorem allows the strict spectral conditions of the previous lemma to be relaxed.

THEOREM 12. *Let $\Lambda \subset \mathbb{C}$ be a finite set with $\#\Lambda = m+1$ such that $0 \in \Lambda$ and denote $\lambda_{\max} = \max\{|\lambda| : \lambda \in \Lambda\}$. Then there exists $C = C(\Lambda)$ such that, for every $K > 0$ and every $k > m$, there exists $\varepsilon = \varepsilon(K, \Lambda)$ with the property: if a trigonometric polynomial f with $\|f\| \le K$ satisfies $\widehat{f}(\mathbb{Z}) \subset \Lambda + B(0, \varepsilon)$, then*

$$\|f^k\| \le C \lambda_{\max}^{k-m} \|f\|^m,$$

where $C = C(\Lambda) = C(t\Lambda)$ for any $t \in \mathbb{C} \setminus \{0\}$.

Proof. Let $\lambda_0 = 0$, $\Lambda = \{0, \lambda_1, \dots, \lambda_m\}$ and put $\delta = \min_{i \ne j} |\lambda_i - \lambda_j|$. We may assume that $\varepsilon < \delta/2$, which guarantees $B(\lambda_i, \varepsilon) \cap B(\lambda_j, \varepsilon) = \emptyset$ for $i \ne j$.

Fix $n \in \mathbb{Z}$. Then there exists a unique λ_{i_n} with $\min_{j=0,1,\dots,m} |\lambda_j - \widehat{f}(n)| = |\lambda_{i_n} - \widehat{f}(n)|$. Now, we define a polynomial f_0 by the condition $\widehat{f_0}(n) = \lambda_{i_n} \in \Lambda$ for every $n \in \mathbb{Z}$. It is obvious that $\widehat{f_0}(\mathbb{Z}) \subset \Lambda$. Moreover, $g = f - f_0$ satisfies $\widehat{g}(\mathbb{Z}) \subset B(0, \varepsilon)$.

We have to estimate $\|g\|$ (an upper bound for $\|f_0\|$ follows from Lemma 11). A simple observation is that, if $\widehat{f}(l) = 0$ for some $l \in \mathbb{Z}$, then $\widehat{g}(l) = 0$. Recall that, for any polynomial h, we write $\#h = \#\{n \in \mathbb{Z} : \widehat{h}(n) \ne 0\}$. Using Parseval's identity we obtain

$$\|g\|_{L^1(\mathbb{T})} \le \|g\|_{L^2(\mathbb{T})} \le \varepsilon\sqrt{\#g} \le \varepsilon\sqrt{\#f}.$$

Putting $\gamma = \min_{n\in\mathbb{Z}}\{|\widehat{f}(n)| : \widehat{f}(n) \ne 0\}$, from the McGehee–Pigno–Smith + Konyagin theorem we obtain $\gamma\|f\| \ge L\ln(\#f)$, which leads to

$$\sqrt{\#f} \le \exp\bigg(\frac{\gamma}{2L}\|f\|\bigg) \le \exp\bigg(\frac{\gamma}{2L}K\bigg).$$

Altogether we have $\|g\| \le \varepsilon \exp\big(\frac{\gamma}{2L}K\big)$. Finally

$$\|f^k\| \le \|f_0^k\| + \sum_{l=0}^{k-1} \binom{k}{l} \|f_0^l\|\,\|g\|^{k-l}.$$

Using Lemma 11 we obtain

$$\|f_0^l\| \le m\delta^{-m}2^m\lambda_{\max}^l\|f\|^m \quad \text{for } l = 1\ldots,k.$$

Moreover, $\|g\|^{k-l} \le \varepsilon^{k-l}\exp\big(\frac{\gamma(k-l)}{2l}K\big)$. Collecting these facts, we get

$$\|f^k\| \le m\delta^{-m}2^m\|f\|^m\bigg(\lambda_{\max}^k + \varepsilon\sum_{l=1}^{k-1}\binom{k}{l}\lambda_{\max}^l\varepsilon^{k-l-1}\exp\bigg(\frac{\gamma(k-l)}{2l}K\bigg) + \frac{1}{\|f\|^m}\varepsilon^{k-1}\exp\bigg(\frac{m\gamma}{2l}K\bigg)\bigg).$$

Taking ε so small that the expression in parentheses is smaller than $2\lambda_{\max}^k$ we finally get $\|f^k\| \le m\delta^{-m}2^{m+1}\|f\|^m\lambda_{\max}^k$. Putting $C = m2^{m+1}\lambda_{\max}^m/\delta^m$ we have $C = C(\Lambda) = C(t\Lambda)$ for every $t \in \mathbb{C}\setminus\{0\}$, which gives the desired estimate. ∎

We now introduce the following notation. Let $C > 0$ and $k \in \mathbb{N}$, $k \ge 2$. We say that a compact set $\Lambda \subset \mathbb{C}$ with $0 \notin \Lambda$ belongs to the class $U(C,k)$ provided for every $K > 0$ there exists an open neighborhood V_K of Λ such that for every $\mu \in M_0(\mathbb{T})$ satisfying $\|\mu\|_{M(\mathbb{T})} \le K$ and $\sigma(\mu) \subset V_K \cup \{0\}$ we have

$$\|\mu^k\|_{M(\mathbb{T})} \le C\|\mu\|_{M(\mathbb{T})}^{k-1}.$$

By Theorem 12, every finite set $\Lambda \subset \mathbb{C}$ with $0 \notin \Lambda$ and $\#\Lambda = k$ belongs to the class $U(C,k)$.

THEOREM 13. *Let $C > 0$ and $k \in \mathbb{N}$. Assume that $X, Y \in U(C,k)$ are such that $X \subset B(0,r)$ and $Y \subset \{z \in \mathbb{C} : |z| > R\}$ for some $R, r > 0$. Then for every $C' > C$, there exists $\varepsilon = \varepsilon(r,R,C') > 0$ such that $\varepsilon X \cup Y \in U(C',k)$.*

Proof. Fix $K > 0$ and take $\mu \in M_0(\mathbb{T})$ such that

$$\sigma(\mu) \subset (\varepsilon X \cup Y + B(0,\delta)) \cup \{0\}$$

and $\|\mu\| < K$. Since $\mu \in M_0(\mathbb{T})$, there are only finitely many $n \in \mathbb{Z}$ satisfying $\widehat{\mu}(n) \in Y + B(0,\delta)$. Hence, we may define a polynomial f by the conditions $\widehat{f}(n) = \widehat{\mu}(n)$ if $\widehat{\mu}(n) \in Y + B(0,\delta)$, and $\widehat{f}(n) = 0$ otherwise. Then the measure $\nu = \mu - f$ satisfies $\widehat{\nu}(\mathbb{Z}) \subset (\varepsilon X \cup B(0,\delta)) \cup 0$ and the equality $\mu = f + \nu$ holds.

Now, we apply Lemma 9. For sufficiently small $\varepsilon = \varepsilon(c)$ we get $\|f\| \le c\|\mu\|$, where c is any number greater than 1, and consequently $\|\nu\| \le \|\mu\| + \|f\| \le (1+c)\|\mu\|$. The measure $\varepsilon^{-1}\nu$ has Fourier coefficients in $(X + B(0,\varepsilon^{-1}\delta)) \cup \{0\}$. By the assumption there exists $\delta > 0$ such that $\|(\varepsilon^{-1}\nu)^k\| \le C\|\varepsilon^{-1}\nu\|^{k-1}$, which yields $\|\nu^k\| \le C\varepsilon\|\nu\|^{k-1}$. We also have $\widehat{f}(\mathbb{Z}) = \sigma(f) \subset (Y + B(0,\delta)) \cup \{0\}$ and $Y \in U(C,k)$. Hence, taking smaller δ if necessary, we get $\|f^k\| \le C\|f\|^{k-1}$. Clearly, we may assume that the sets $\varepsilon X + B(0,\delta)$

and $Y + B(0, \delta)$ are disjoint, which leads to $\nu * f = 0$. A simple calculation

$$\begin{aligned}\|\mu^k\| = \|f^k + \nu^k\| \le \|f^k\| + \|\nu^k\| \le C\|f\|^{k-1} + C\varepsilon\|\nu\|^{k-1}\\ \le C(c^{k-1} + \varepsilon(1+c)^{k-1})\|\mu\|^{k-1}.\end{aligned}$$

Since c can be chosen arbitrarily close to 1, and ε may be as small as we wish, the theorem follows. ■

In the formulation and the proof of the next theorem we will write $s(n)$ instead of s_n for a clearer display.

THEOREM 14. *Let $0 < a < b$ and $k \in \mathbb{N}$, $k \ge 2$. For every $C > 0$ there exist a sequence of continuous functions $\psi_l : \mathbb{R}_+ \to \mathbb{R}_+$, $l \in \mathbb{N}$, such that whenever a decreasing sequence $s(n)$ tending to zero satisfies*

$$\frac{s(2^l n + 2^{l-1} + 1)}{s(2^l n + 1)} < \psi_l\left(a \frac{s(2^l n + 2^{l-1})}{s(2^l n + 1)}\right)$$

for every $l \in \mathbb{N}$, $n = 0, 1, \ldots, 2^{l-1}$ and $A_n \in U(C, k)$, $A_n \subset B(0, b) \cap \{z \in \mathbb{C} : |z| > a\}$ and (r_n) tends to zero rapidly enough, then any measure $\mu \in M_0(\mathbb{T})$ with $\|\mu\|_{M(\mathbb{T})} \le K$ and with

$$\widehat{\mu}(\mathbb{Z}) \subset \bigcup (s(n)A_n + B(0, r_n)) \cup \{0\}$$

satisfies $\mu^k \in L^1(\mathbb{T})$.

Proof. For simplicity assume $s(1) = 1$. Let C_m be an increasing sequence of real numbers such that $C < C_m < \widetilde{C}$ for all $m \in \mathbb{N}$ and some $\widetilde{C} > 0$. Let us also define ψ_l by $\widetilde{\psi_l}(\cdot) = \varepsilon(b, \cdot, C_l)$ where $\varepsilon(\cdot,\cdot,\cdot)$ is as in the previous theorem for $C := C_{l-1}$.

We show first by induction that for every $m, n \ge 0$,

$$B_{m,n} = \bigcup_{j=n\cdot 2^m+1}^{(n+1)2^m} \frac{s(j)}{s(n\cdot 2^m + 1)} A_j \in U(C_m, k), \tag{2}$$

Indeed, for $m = 0$ clearly $B_{0,n} = A_{n+1}$. For $m > 0$ the interval of integers $[n\cdot 2^m + 1, (n+1)\cdot 2^m]$ is the disjoint union of $[(2n)\cdot 2^{m-1} + 1, (2n+1)\cdot 2^{m-1}]$ and $[(2n+1)\cdot 2^{m-1} + 1, (2n+2)\cdot 2^{m-1}]$, which implies

$$B_{m,n} = B_{m-1,2n} \cup \frac{s((2n+1)2^{m-1} + 1)}{s(n\cdot 2^m + 1)} B_{m-1,2n+1}. \tag{3}$$

It is easy to check that

$$B_{m-1,2n} \subset \left\{ z \in \mathbb{C} : |z| > a \frac{s(n\cdot 2^m + 2^{m-1})}{s(n\cdot 2^m + 1)} \right\}$$

and $B_{m-1,2n+1} \subset B(0, b)$. From the previous theorem it follows that if we take in (3) the coefficient $\frac{s((2n+1)2^{m-1}+1)}{s(n\cdot 2^m+1)}$ such that

$$\frac{s((2n+1)2^{m-1}+1)}{s(n\cdot 2^m+1)} \le \widetilde{\psi}_m\Big(a\frac{s(n\cdot 2^m+2^{m-1})}{s(n\cdot 2^m+1)}\Big),$$

then the resulting union belongs to $U(C_m,k)$ and hence to $U(\widetilde{C},k)$.

Applying (2) to $n=1$ and $n=0$, we see that for every $m=0,1,2,\ldots,$

$$B_{m-1,1}=\bigcup_{j=2^{m-1}+1}^{2^m}\frac{s(j)}{s(2^{m-1}+1)}A_j\in U(\widetilde{C},k)$$

and

$$B_{m,0}\setminus B_{m-1,0}=\bigcup_{j=2^{m-1}+1}^{2^m}s(j)A_j=s(2^{m-1}+1)B_{m-1,1}. \tag{4}$$

Then it follows from Theorem 13 that there exists r'_m such that for every measure $\gamma\in M(\mathbb{T})$ with

$$\|\gamma\|_{M(\mathbb{T})}\le\frac{4K}{s(2^{m-1}+1)}\quad\text{and}\quad \widehat{\gamma}(\mathbb{Z})\subset B_{m-1,1}+B(0,r'_m) \tag{5}$$

we have

$$\|\gamma^k\|_{M(\mathbb{T})}<\widetilde{C}\|\gamma\|^{k-1}_{M(\mathbb{T})}. \tag{6}$$

Let $\mu\in M_0(\mathbb{T})$ with $\|\mu\|_{M(\mathbb{T})}\le K$ satisfy

$$\widehat{\mu}(\mathbb{Z})\subset\bigcup_{m=1}^{\infty}((B_{m,0}\setminus B_{m-1,0})+B(0,r_m))$$

where $r_m=r'_m s(2^{m-1}+1)$. Since $\mu\in M_0(\mathbb{T})$, for any fixed $m\in\mathbb{N}$ there exist only finitely many $p\in\mathbb{N}$ such that $\widehat{\mu}(p)\in(B_{m,0}\setminus B_{m-1,0})+B(0,r_m)$ (we may assume that these sets are disjoint for different m's). Hence we can define polynomials f_m by the condition $\widehat{f_m}(p)=\widehat{\mu}(p)$ for $p\in\mathbb{Z}$ such that $\widehat{\mu}(p)\in(B_{m,0}\setminus B_{m-1,0})+B(0,r_m)$, and $\widehat{f_m}(p)=0$ for other p's. Then $\widehat{\mu}=\sum_{m=0}^{\infty}\widehat{f_m}$.

The polynomials $f'_m=f_m/s(2^{m-1}+1)$ satisfy (5). In fact, the second part of (5) follows directly from (4). For the first part it is enough to show that $\|f_m\|\le 4K$. This follows from the observation that

$$f_m=\Big(\sum_{j=0}^{m}f_j\Big)-\Big(\sum_{j=0}^{m-1}f_j\Big),$$

the triangle inequality, Lemma 9 and the properties

$$\sum_{j=0}^{m}f_j\in G(as(2^m))\quad\text{and}\quad\sum_{j=m+1}^{\infty}f_j\in F(bs(2^m+1)).$$

Indeed, let us put in Lemma 9

$$f = \sum_{j=0}^{m} f_j, \qquad \nu = \mu - f, \qquad \varepsilon = 1,$$

and define a sequence of functions ψ_m by the formula

$$\psi_m(\cdot) = \min\bigg(\widetilde{\psi}_m(\cdot), \frac{1}{b}\delta(1,\cdot,K)\bigg).$$

Then the assumptions of Lemma 9 are satisfied, which leads to $\|f\| < 2K$. The second term is estimated analogously.

Finally, by (6) we have

$$\|f_m^k\|_{L^1(\mathbb{T})} < s(2^{m-1}+1)\widetilde{C}\|f_m\|_{L^1(\mathbb{T})}^{k-1} < s(2^{m-1}+1)\widetilde{C}(4K)^{k-1}.$$

Consequently, the series $\sum_{m=0}^{\infty} f_m^k$ is absolutely convergent in $L^1(\mathbb{T})$ to μ^k, which finishes the proof. ∎

4. Reduction to the case of $M_0(\mathbb{T})$. The first result which will be used in this section is the following theorem taken from the book [GM], closely related to results from [DK].

Theorem 15. *Let $r \in \mathbb{N}$, $r \geq 2$ and $\mu \in M_c(\mathbb{T})$. Define*

$$Q = Q(\mu) = \{n \in \mathbb{Z} : |\widehat{\mu}(n)| \geq 1\}$$

and suppose that $|\widehat{\mu}(n)| \leq e^{-r}$ for $n \notin Q$.

- *If $\|\mu\| < r^{1/2}/4$, then Q is a finite set.*
- *If $\|\mu\| < r^{1/2}/4$ and $N \in \mathbb{N}$ is such that*

$$r \leq (\ln(N/4)\ln\ln N)^{1/2}$$

then $\#Q < N$.

Applying the above theorem we will prove that a continuous measure belongs to $M_0(\mathbb{T})$ if special assumptions are imposed on the range of its Fourier transforms.

In the next lemma we denote

$$L(r,t) = \{z \in \mathbb{C} : r < |z| < t\} \qquad (0 < r < t).$$

Lemma 16. *Let w_k, t_k with $w_k < t_k$ be sequences of positive real numbers such that*

- *$t_k \to 0$ as $k \to \infty$.*
- *$t_k\sqrt{\ln(t_k/w_k)}$ is increasing and divergent to ∞.*

Let $L_k = L(w_k,t_k)$. If $\mu \in M_c(\mathbb{T})$ satisfies $\widehat{\mu}(\mathbb{Z}) \cap L_k = \emptyset$ for all $k \in \mathbb{N}$, then $\mu \in M_0(\mathbb{T})$.

Proof. Define $x_k = \ln \frac{t_k}{w_k}$. Then the second condition on w_k, t_k implies that the sequence $t_k\sqrt{x_k}$ is increasing and divergent to ∞. Hence, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$,

$$\|\mu\| < t_k\sqrt{x_k}/4.$$

By the assumption, $\widehat{\mu}(\mathbb{Z}) \cap L_k = \emptyset$ for every $k > k_0$. That means

$$\widehat{\mu}(m)/t_k \notin L(e^{-x_k}, 1) \quad \text{for all } m \in \mathbb{Z},\ k > k_0.$$

From Theorem 15 we deduce that the set

$$A_k = \{n \in \mathbb{Z} : |\widehat{\mu}(n)/t_k| > 1\} = \{n \in \mathbb{Z} : |\widehat{\mu}(n)| > t_k\}$$

is finite for all $k > k_0$, which finishes the proof. ∎

REMARK 17. Obviously the assumption $\widehat{\mu}(\mathbb{Z}) \cap L_k = \emptyset$ from the preceding lemma can be replaced by $\widehat{\mu}(\mathbb{Z}) \cap L_{k_n} = \emptyset$ for any subsequence k_n.

This will be used in the proof of Theorem 1 in the next section. The second reduction, needed in the proof of Theorem 2, is as follows: having an arbitrary $\mu \in M(\mathbb{T})$ we split it as $\mu = \mu_c + \mu_d$ where μ_c is the continuous part and μ_d the discrete part. Then from the assumptions on the set $\widehat{\mu}(\mathbb{Z})$ we would like to extract information about $\widehat{\mu_c}(\mathbb{Z})$ which with the aid of the last lemma leads to the conclusion that $\mu_c \in M_0(\mathbb{T})$, and for measures from this class we shall apply Theorem 14.

One thing remains to be proved: if $\mu \in M_0(\mathbb{T})$ has a natural spectrum and ν is an arbitrary measure with a natural spectrum, then $\mu + \nu$ also has a natural spectrum. The key to obtain this fact is an important theorem of Zafran [Za]. To formulate it we introduce the following definition.

DEFINITION 18. Let $\mathscr{C}$ denote the set of measures with natural spectrum with Fourier–Stieltjes coefficients from c_0, i.e.

$$\mathscr{C} = \{\mu \in M_0(\mathbb{T}) : \sigma(\mu) = \overline{\widehat{\mu}(\mathbb{Z})} = \widehat{\mu}(\mathbb{Z}) \cup \{0\}\}.$$

For any commutative Banach algebra A we denote by $\mathfrak{M}(A)$ the space of maximal modular ideals of A, identified also as the set of all multplicative linear functionals on A (cf. [Ż]).

THEOREM 19 (Zafran). *The following hold true:*

(i) *If $h \in \mathfrak{M}(M_0(\mathbb{T})) \setminus \mathbb{Z}$, then $h(\mu) = 0$ for $\mu \in \mathscr{C}$.*
(ii) *$\mathscr{C}$ is a closed ideal in $M_0(\mathbb{T})$.*
(iii) *$\mathfrak{M}(\mathscr{C}) = \mathbb{Z}$.*

It is easy an elementary fact that $L^1(\mathbb{T}) \subset \mathscr{C}$ and from the preceding theorem we conclude that

$$\mathrm{Rad}(L^1(\mathbb{T})) = \{\mu \in M(\mathbb{T}) : \mu^{*k} \in L^1(\mathbb{T}) \text{ for some } k \in \mathbb{N}\} \subset \mathscr{C}.$$

In contrast to the second conclusion of Zafran's theorem, the sum of two measures with natural spectrum does not necessarily have a natural spectrum.

The proof of this fact is based on the construction of a measure supported on an independent Cantor set as in [R] (see also [Za] for details). However, as stated before, assuming more on one summand provides the desired property.

THEOREM 20. *The sum of two measures with natural spectrum has a natural spectrum if one of them has Fourier coefficients tending to zero.*

Proof. The spectrum of a measure is the image of its Gelfand transform. Hence using the result of Zafran for $\mu \in \mathscr{C}$ and ν with natural spectrum, we obtain

$$\begin{aligned}\sigma(\mu+\nu) &= \{\varphi(\mu+\nu) : \varphi \in \mathfrak{M}(M(\mathbb{T}))\} \\ &= (\widehat{\mu}+\widehat{\nu})(\mathbb{Z}) \cup \{\varphi(\nu) : \varphi \in \mathfrak{M}(M(\mathbb{T})) \setminus \mathbb{Z}\}.\end{aligned}$$

Since ν has a natural spectrum, for every $\varphi \in \mathfrak{M}(M(\mathbb{T}))$ we have $\varphi(\nu) \in \overline{\widehat{\nu}(\mathbb{Z})}$. Now we consider two cases:

1. $\varphi(\nu) \in \overline{\widehat{\nu}(\mathbb{Z})} \setminus \widehat{\nu}(\mathbb{Z})$.
2. $\varphi(\nu) \in \widehat{\nu}(\mathbb{Z})$.

In the first case there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of integers such that $\lim_{k\to\infty} \widehat{\nu}(n_k) = \varphi(\nu)$. Since $\mu \in M_0(\mathbb{T})$, we get $\varphi(\nu) = \lim_{k\to\infty} \widehat{\nu}(n_k) = \lim_{k\to\infty}(\widehat{\mu}+\widehat{\nu})(n_k)$. Hence $\varphi(\nu) \in \overline{(\widehat{\mu}+\widehat{\nu})(\mathbb{Z})}$, which completes the proof in this case.

In the second case $\varphi(\nu) = \widehat{\nu}(n_0)$ for some $n_0 \in \mathbb{Z}$. If $\widehat{\mu}(n_0) = 0$ then $\varphi(\nu) = \widehat{\nu}(n_0) + \widehat{\mu}(n_0) \in (\widehat{\mu}+\widehat{\nu})(\mathbb{Z})$ and the result follows. Assume now $\widehat{\mu}(n_0) \neq 0$. If $\widehat{\nu}(n_0)$ is an accumulation point of $\sigma(\nu) = \overline{\widehat{\nu}(\mathbb{Z})}$, then we proceed as in the first case. It remains to consider the case when $\widehat{\nu}(n_0)$ is an isolated point of $\overline{\widehat{\nu}(\mathbb{Z})}$. Because $\mu \in M_0(\mathbb{T})$ we see that also $\widehat{\nu}(n_0)$ is an isolated point of $\overline{(\widehat{\mu}+\widehat{\nu})(\mathbb{Z})}$. We will prove a stronger statement: $\widehat{\nu}(n_0)$ is an isolated point of $\sigma(\mu+\nu)$.

Indeed, suppose on the contrary that there exists a sequence of complex numbers $(\lambda_k)_{k\in\mathbb{N}} \subset \sigma(\mu+\nu)$ tending to $\widehat{\nu}(n_0)$. Since the spectrum of a measure is the image of its Gelfand transform, we can choose a sequence $\varphi \notin (h_k)_{k\in\mathbb{N}} \subset \mathfrak{M}(M(\mathbb{T}))$ such that $h_k(\mu+\nu) = \lambda_k$. Without losing generality, we may assume that for a sufficiently large k, the functionals h_k are not the Fourier coefficients (otherwise $\widehat{\nu}(n_0) \in \overline{(\widehat{\mu}+\widehat{\nu})(\mathbb{Z})}$ and the proof is finished). Using again the theorem of Zafran we get

$$\lim_{k\to\infty} h_k(\mu+\nu) = \lim_{k\to\infty} h_k(\nu) = \widehat{\nu}(n_0).$$

But $h_k(\nu) \in \sigma(\nu) = \overline{\widehat{\nu}(\mathbb{Z})}$. Hence $\widehat{\nu}(n_0)$ is not an isolated point of $\overline{\widehat{\nu}(\mathbb{Z})}$, which contradicts the assumption.

Since $\sigma(\mu+\nu)$ has a complex number $\widehat{\nu}(n_0)$ as an isolated point, we can find two open sets $A, B \subset \mathbb{C}$ such that $A \cap B = \emptyset$, $\sigma(\mu+\nu) \subset A \cup B$, $\widehat{\nu}(n_0) \in B$ and $\sigma(\mu+\nu) \setminus \widehat{\nu}(n_0) \subset A$. Let f be a holomorphic function defined on $A \cup B$ by putting $f(z) = z$ for $z \in A$ and $f \equiv \widehat{\nu}(n_0) + 1$ on B. By the spectral mapping theorem there exists a measure $\upsilon := f(\mu+\nu) \in M(\mathbb{T})$ satisfying

$$\widehat{\upsilon}(m) = f((\widehat{\mu}+\widehat{\nu})(m)) \quad \text{for all } m \in \mathbb{Z}.$$

By the definition of f we have $\widehat{\upsilon}(n) = (\widehat{\mu}+\widehat{\nu})(n)$ for $n \neq n_0$. Moreover, since we have assumed $\widehat{\mu}(n_0) \neq 0$ we have $\widehat{\mu}(n_0) + \widehat{\nu}(n_0) \neq \widehat{\nu}(n_0)$ and $\widehat{\mu}(n_0) + \widehat{\nu}(n_0) \in A$, which leads to

$$\widehat{\upsilon}(n_0) = (\widehat{\mu}+\widehat{\nu})(n_0).$$

Therefore, the measures $\upsilon = f(\mu+\nu)$ and $\mu+\nu$ have the same Fourier coefficients, which implies $f(\mu+\nu) = \mu+\nu$. From item (i) of Zafran's theorem we have $\varphi(\mu) = 0$, which by functional calculus implies

$$\begin{aligned}\widehat{\nu}(n_0) &= \varphi(\mu+\nu) = \varphi(f(\mu+\nu)) = f(\varphi(\mu+\nu)) \\ &= f(\varphi(\nu)) = f(\widehat{\nu}(n_0)) = \widehat{\nu}(n_0)+1.\end{aligned}$$

This contradiction completes the proof. ∎

5. Proofs of the main theorems. We begin with the proof of Theorem 1. Let t_k, w_k and L_k be as in Lemma 16 and let $b > 2$, $a < 1$. We put $A_n = A = \{-1,1\}$ for $n \in \mathbb{N}$ and take $C > 0$ from Theorem 12 such that $A \in U(C,2)$. Next, let us denote by ψ_n the sequence of functions from Theorem 14. We construct inductively a sequence ε_n as follows: $\varepsilon_0 = 1$ and if $\varepsilon_0, \ldots, \varepsilon_n$ are chosen, take the smallest k such that $t_k < \varepsilon_1 \cdot \ldots \cdot \varepsilon_n$ and rename it as k_n. Now let ε_{n+1} be any number which satisfies $0 < \varepsilon_{n+1} < \frac{1}{2} w_{k_n}$ and $\varepsilon_{n+1} < \psi_n(a\varepsilon_1 \cdot \ldots \cdot \varepsilon_n)$. Every $n \in \mathbb{N}$ has a unique binary expansion

$$n = \sum a_i 2^i, \qquad a_i \in \{0,1\}.$$

We put $s(n+1) = \prod \varepsilon_i^{a_i}$. We take the sequence $r(n)$ from Theorem 14 and modify it (if necessary) to guarantee that $r(2^{n-1}) < t_{k_n}$ and $r(2^n) < \frac{1}{2} w_{k_n}$. Finally, we put

$$U = \bigcup_{n\in\mathbb{N}} (s(n)A_n + B(0, r(n))).$$

Let $\mu \in M_c(\mathbb{T})$ be such that $\widehat{\mu}(\mathbb{Z}) \subset U \cup \{0\}$. Then, by the construction, $(U \cup \{0\}) \cap I_{k_n} = \emptyset$. By Lemma 16 and the remark following it, $\mu \in M_0(\mathbb{T})$. Finally, Theorem 14 yields $\mu^2 \in L^1(\mathbb{T})$. Hence, by Zafran's theorem $\mu \in \mathscr{C}$, i.e. μ has a natural spectrum.

We move on now to the proof of Theorem 2. We start from the following simple lemma whose proof is left to the reader.

LEMMA 21. *Let $S = \bigcup_{k=1}^{\infty} B_k \subset \mathbb{C}$ be a union of balls such that $0 \in \overline{S}$. Then there exists a (topological) Cantor set K such that $K - K \subset \overline{S \cup -S}$.*

Let $D = \{-1,1\}$ and $E = \{-2,-1,1,2\}$. By Theorem 12, $D \in U(C,2)$ and $E \in U(C,4)$ for some $C > 0$. Let $(s_n)_{n=1}^{\infty} := (s(n))_{n=0}^{\infty}$ and $(r_n)_{n=1}^{\infty}$

satisfy the conditions of Theorem 14 for $A_n = D$, $n = 1, 2, \ldots$, and $(s_n)_{n=1}^{\infty}$ and $(r'_n)_{n=1}^{\infty}$ satisfy the conditions of Theorem 14 for $A_n = E$, $n = 1, 2, \ldots$; suppose moreover that $2|s_m| + r'_m + r'_n < r_n$ for $m > n$. Let G_n, $n = 1, 2, \ldots$, be a Cantor set satisfying, by Lemma 21,

$$G_n - G_n \subset \overline{\bigcup_{k>n} B(s_k, r'_k) \cup \bigcup_{k>n} B(-s_k, r'_k)}.$$

We also put $A_0 = B_0 = G_0 = \{0\}$, $r_0 = r'_0 = 0$, $s_0 = 1$ and with a little abuse of notation $B(0,0) = \{0\}$. Then we define

$$X = \bigcup_{n=0}^{\infty} (s_n A_n + G_n).$$

Clearly the set X is closed. Suppose now that $\widehat{\mu}(\mathbb{Z}) \subset X$. We will use the following result from [GW].

Lemma 22. *Let $\mu \in M(\mathbb{T})$. Then $\overline{\widehat{\mu_d}(\mathbb{Z})} \subset \overline{\widehat{\mu}(\mathbb{Z})}$ where μ_d is the discrete part of μ.*

Therefore

$$\begin{aligned}
\widehat{\mu}_c(\mathbb{Z}) \subset \widehat{\mu}(\mathbb{Z}) - \widehat{\mu}_d(\mathbb{Z}) \subset X - X &\subset \Big(\bigcup_{n=0}^{\infty} (s_n A_n + G_n) \Big) - \Big(\bigcup_{n=0}^{\infty} (s_n A_n + G_n) \Big) \\
&\subset \bigcup_{n \neq k} (s_n A_n - s_k A_k + G_n - G_k) \cup \bigcup_{n} ((s_n A_n + G_n) - (s_n A_n + G_n)) \\
&\subset \bigcup_{n<k} (s_n A_n + B(0, 2|s_k| + r'_n + r'_k)) \\
&\quad \cup \bigcup_{n} (s_n B_n + G_n - G_n) \cup \bigcup_{n} (G_n - G_n) \\
&\subset \bigcup_{n<k} (s_n B_n + B(0, 2|s_k| + r'_n + r'_k)) \\
&\quad \cup \bigcup_{n} (s_n B_n + B(0, 2r'_n)) \cup \bigcup_{n} (s_n B_n + B(0, r'_n)) \\
&\subset \bigcup_{n} (s_n B_n + B(0, r_n)).
\end{aligned}$$

By Theorem 1, we derive that μ_c has a natural spectrum. Finally, since additionally $\mu_c \in M_0$, Theorem 20 shows that $\mu = \mu_c + \mu_d$ has a natural spectrum.

6. The example. In this section we construct an example of a singular measure satisfying the assumptions of Theorem 1 which we call the *Riesz–Rudin–Shapiro product*. Our construction is an instance of *generalized Riesz products*, which are elaborated in Chapter 5 of the book [HMP].

First, let us recall a few results concerning the usual Riesz products. They are continuous probability measures on the circle group given as weak-star limits of trigonometric polynomials of the form

$$\prod_{k=1}^{N}(1+a_k\cos(n_k t)),$$

where $-1 \le a_k \le 1$ and n_k is a sequence of natural numbers satisfying the lacunarity condition $n_{k+1}/n_k \ge 3$. We will write

$$R(a_k,n_k)=\prod_{k=1}^{\infty}(1+a_k\cos(n_k t))$$

for the Riesz product built on the sequences a_k and n_k satisfying the above conditions. One of the oldest results (see [Zy]) on Riesz products is that $R(1,3^k)$ is singular with respect to Lebesgue measure (we will write simply "singular"). However, a much more general theorem was proved in [BM]. In its formulation below, $\mu \perp \nu$ means that the measures $\mu,\nu \in M(\mathbb{T})$ are mutually singular and $\mu \sim \nu$ denotes equivalence of measures, i.e., μ is absolutely continuous with respect to ν and vice versa.

Theorem 23 (Brown and Moran). *If a_k, b_k satisfy $-1 \le a_k, b_k \le 1$ and the natural numbers n_k have the property $n_{k+1}/n_k \ge 3$ then*

$$R(a_k,n_k)\perp R(b_k,n_k) \;\Leftrightarrow\; \sum_{k=1}^{\infty}(a_k-b_k)^2=\infty,$$

$$R(a_k,n_k)\sim R(b_k,n_k) \;\Leftrightarrow\; \sum_{k=1}^{\infty}(a_k-b_k)^2<\infty.$$

As we stated in the introduction, Riesz products may be used for a simple proof of the Wiener–Pitt phenomenon (see [G]). Moreover, Zafran [Za] gave a necessary and sufficient condition for the Riesz product $R(a_k,n_k)$ to have a natural spectrum under the assumption that the sequence a_k converges to zero.

Theorem 24 (Zafran). *Let a_k be a sequence tending to zero such that $-1 \le a_k \le 1$ and n_k be a sequence of natural numbers with $n_{k+1}/n_k \ge 3$. Then the Riesz product $R(a_k,n_k)$ has a natural spectrum if and only if there exists $m \in \mathbb{N}$ such that*

$$\sum_{k=1}^{\infty}|a_k|^m<\infty.$$

It is also proven in [BBM] that in the case when the Riesz product has all powers mutually singular, its spectrum is the whole disc $\{z \in \mathbb{C} : |z| \le 1\}$.

However, usual Riesz products are not sufficient for our needs and we move on to the construction of a more general class of measures.

Let us start with some preliminary lemmas and notation. The first one is a very simple arithmetic argument needed in calculating the Fourier–Stieltjes coefficients of our measure.

LEMMA 25. *Let $(m_j)_{j=1}^{\infty}$, $(r_j)_{j=1}^{\infty}$ and $(n_j)_{j=1}^{\infty}$ be increasing sequences of positive integers with*

$$r_k > 2\sum_{j=1}^{k-1}((2^{n_j}-1)m_j + r_j) \quad \text{for } k \geq 2.$$

Moreover, let $(c_j)_{j=1}^{\infty}$ be a sequence of positive integers satisfying

$$c_j \in \{r_j, m_j + r_j, 2m_j + r_j, \ldots, (2^{n_j}-1)m_j + r_j\} \quad \text{for } j \in \mathbb{N}.$$

Assume that an integer s is expressible in the form

$$s = \sum_{j=1}^{N} b_j c_j \quad \text{where} \quad N \in \mathbb{N},\ b_j \in \{-1,0,1\} \text{ and } b_N \neq 0.$$

Then this expression is unique.

The proof is obvious and we omit it.

The fundamental ingredient in our construction are the Rudin–Shapiro polynomials (cf. [R]). We recall them in the next definition.

DEFINITION 26. Let $P_0 \equiv 1$ and $Q_0 \equiv 1$. We define inductively two sequences of polynomials by the formula

$$P_{n+1}(t) = P_n(t) + e^{i2^n t}Q_n(t), \qquad Q_{n+1}(t) = P_n(t) - e^{i2^n t}Q_n(t).$$

We will reserve the name 'Rudin-Shapiro polynomials' for the sequence $(P_n)_{n=0}^{\infty}$. Now, we collect the well-known properties of these polynomials.

PROPOSITION 27. *For every $n \in \mathbb{N}$ we have*

$$P_n(t) = \sum_{k=0}^{2^n-1} a_k e^{ikt} \quad \text{where} \quad a_k \in \{-1,1\} \text{ for } k \in \{0,\ldots,2^n-1\}.$$

Hence $\|P_n\|_{L^2(\mathbb{T})} = 2^{n/2}$. Also, $\|P_n\|_{C(\mathbb{T})} \leq 2^{(n+1)/2}$.

Using $(P_n)_{n=1}^{\infty}$ we define another sequence $(w_k)_{k=1}^{\infty}$ of polynomials.

DEFINITION 28. Let $(P_n)_{n=1}^{\infty}$ be the sequence of Rudin–Shapiro polynomials and $(r_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$, $(n_k)_{k=1}^{\infty}$ be increasing sequences of positive integers. Let $(\varepsilon_k)_{k=1}^{\infty}$ be a decreasing sequence of positive numbers vanishing at infinity. We define polynomials $(w_k)_{k=1}^{\infty}$ by the formula

$$w_k = \varepsilon_k P_{n_k}(m_k t)e^{ir_k t} + \varepsilon_k \overline{P_{n_k}(m_k t)}e^{-ir_k t}.$$

We summarize the properties of the polynomials w_k.

PROPOSITION 29. *The polynomials w_k are real-valued on $\mathbb{T}$ and have the following form* (*the prime denotes a sum without the term corresponding to $l = 0$*)*:*

(7)

$$w_k(t) = \varepsilon_k \sum_{l=-2^{n_k}+1}^{2^{n_k}-1}{}' a_{|l|} e^{it(lm_k + \operatorname{sgn}(l) r_k)} + a_0(e^{-itr_k} + e^{itr_k}) \text{ where } a_l \in \{-1, 1\}.$$

Hence, $\|w_k\|^2_{L^2(\mathbb{T})} = \varepsilon_k^2(2^{n_k+1} - 1)$. *Moreover,* $\|w_k\|_{C(\mathbb{T})} \le \varepsilon_k 2^{(n_k+3)/2}$.

Proof. The polynomials w_k are real-valued by Definition 28. Equation (7) is straightforward by Proposition 27. The remaining properties follow from (7) and Proposition 27. ■

Equation (7) reveals an important feature of the polynomials w_k, namely the sequence of their Fourier coefficients has gaps of lengths m_k.

We are now ready to construct the Riesz–Rudin–Shapiro products. The proof of the following proposition is standard, based on fact that the weak-star convergence of a bounded sequence of measures follows from the point-wise convergence of its Fourier transforms (see for example [HMP]).

PROPOSITION 30. *Let $(r_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$, $(n_k)_{k=1}^{\infty}$ be increasing sequences of positive integers. Let $(\varepsilon_k)_{k=1}^{\infty}$ be a decreasing sequence of positive numbers vanishing at infinity and $(w_k)_{k=1}^{\infty}$ be the corresponding sequence of polynomials. Assume that*

$$r_k > 2\sum_{j=1}^{k-1}((2^{n_j} - 1)m_j + r_j) \quad \text{for } k \ge 2 \tag{8}$$

and moreover $\varepsilon_k 2^{(n_k+3)/2} < 1$ for all $k \in \mathbb{N}$. Then the sequence of polynomials

$$f_N(t) = \prod_{k=1}^{N}(1 - w_k(t))$$

converges in the weak-star topology of $M(\mathbb{T})$ as $N \to \infty$ to some positive measure $\mu \in M_0(\mathbb{T})$ with $\|\mu\|_{M(\mathbb{T})} = 1$ with the additional property

$$\widehat{\mu}(\mathbb{Z}) \subset \Big\{ \pm \prod_{k=1}^{m} \varepsilon_k^{l_k} : l_k \in \{0, 1\},\, m \in \mathbb{N} \Big\} \cup \{0\}.$$

We will write

$$\mu = \prod_{k=1}^{\infty}(1 - w_k)$$

to denote the measure μ obtained by the procedure described above. Adding more restrictions on our sequences we get

PROPOSITION 31. *Let $(r_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$, $(n_k)_{k=1}^{\infty}$ be increasing sequences of positive integers. Let $(\varepsilon_k)_{k=1}^{\infty}$ be a decreasing sequence of positive numbers vanishing at infinity and $(w_k)_{k=1}^{\infty}$ be the corresponding sequence of polynomials. Assume that*

$$\begin{aligned} r_k &> 2\sum_{j=1}^{k-1}((2^{n_j}-1)m_j + r_j) \quad \text{for } k \geq 2, \\ m_k &> 2\sum_{j=1}^{k-1}((2^{n_j}-1)m_j + r_j) \quad \text{for } k \geq 2 \end{aligned} \tag{9}$$

and moreover $\varepsilon_k 2^{(n_k+3)/2} < 1$ for all $k \in \mathbb{N}$. Also, assume that there exists a constant $c > 0$ such that

$$\varepsilon_k^2(2^{n_k+1}-1) > c \quad \text{for all } k \in \mathbb{N}. \tag{10}$$

Then the measure $\mu = \prod_{k=1}^{\infty}(1-w_k)$ does not belong to $L^2(\mathbb{T})$.

Proof. The above assumptions yield the existence of a measure μ as in Proposition 30. Fix $N \in \mathbb{N}$ and consider the polynomial

$$f_N(t) = \prod_{k=1}^{N}(1-w_k(t)).$$

An easy application of Parseval's identity gives

$$\sum_{k=-\infty}^{\infty} |\widehat{\mu}(k)|^2 \geq \sum_{k=-\infty}^{\infty} |\widehat{f_N}(k)|^2 = \|f_N\|_{L^2(\mathbb{T})}^2.$$

Hence it is enough to show that $\|f_N\|_{L^2(\mathbb{T})}^2 \to \infty$ as $N \to \infty$. We calculate (using the normalized Lebesgue measure on $[0, 2\pi]$)

$$\|f_N\|_{L^2(\mathbb{T})}^2 = \int \prod_{k=1}^{N}(1-w_k(t))^2\,dt = \int \prod_{k=1}^{N}(1-2w_k(t)+w_k^2(t))\,dt.$$

Expanding the last product we get multiples of the expressions

$$\pm\int w_{i_1}^{l_1}(t)\cdot w_{i_2}^{l_2}(t)\cdot\dots\cdot w_{i_m}^{l_m}(t)\,dt = \pm\int h(t)\,dt,$$

where $1 \leq m \leq N$, $i_1 < \dots < i_m$ and $l_1, \dots, l_m \in \{1,2\}$. A simple arithmetical argument based on (9) shows that the integral equals 0 unless $l_1 = \dots = l_m = 2$. Indeed, it is equal to $\widehat{h}(0)$, and to prove this assertion let us assume on the contrary that $l_s = 1$ for some $1 \leq s \leq m$. If $\widehat{h}(0) \neq 0$, then there exist integers $j_1, j_1', \dots, j_m, j_m'$ satisfying $j_d, j_d' \in \{-2^{n_{i_d}}+1, \dots, 2^{n_{i_d}}-1\} \setminus \{0\}$ for all $d = 1, \dots, m$ except $d = s$, for which j_s belongs to the same set of integers but $j'_s = 0$, such that

$$0 = \sum_{k=1}^{m} \big(m_k(j_k + j'_k) + r_k(\operatorname{sgn}(j_k) + \operatorname{sgn}(j'_k))\big).$$

Condtions (9) imply that this is possible if and only if $j_k + j'_k = 0$ for all k. However, this is excluded by the assumption $l_s = 1$, which leads to the vanishing of j'_s. Putting all this together we obtain

$$\|f_N\|^2_{L^2(\mathbb{T})} = \int \prod_{k=1}^{N} (1 - w_k(t))^2 \, dt = \int \prod_{k=1}^{N} (1 + w_k^2(t)) \, dt.$$

Forgetting the terms of order higher than two we have, by Proposition 29,

$$\int \prod_{k=1}^{N} (1 + w_k^2(t)) \, dt \ge 1 + \sum_{k=1}^{N} \int w_k^2(t) \, dt = 1 + \sum_{k=1}^{N} 2\varepsilon_k^2(2^{n_k} - 1)).$$

Using the assumption (10) we finally get

$$\|f_N\|^2_{L^2(\mathbb{T})} \ge 1 + Nc \to \infty \quad \text{as } N \to \infty. \ \blacksquare$$

The main result of this section states that under additional assumptions on the sequences $(r_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ the resulting measure is singular. We also show how to satisfy these assumptions.

THEOREM 32. *Let $(\varepsilon_k)_{k=1}^{\infty}$ be a decreasing sequence of positive numbers vanishing at infinity such that $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$ for all $k \in \mathbb{N}$. Then there exist sequences $(r_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$, $(n_k)_{k=1}^{\infty}$ of positive integers satisfying the conditions*

$$\begin{aligned} & r_k > 2\sum_{j=1}^{k-1}((2^{n_j} - 1)m_j + r_j) \qquad \text{for } k \ge 2, \\ (11) \qquad & m_k > 2\sum_{j=1}^{k-1}((2^{n_j} - 1)m_j + r_j) \qquad \text{for } k \ge 2, \\ & 1/2 < \varepsilon_k 2^{(n_k+3)/2} < 1, \end{aligned}$$

such that the positive measure $\mu = \prod_{k=1}^{\infty}(1 - w_k) \in M_0(\mathbb{T})$ with norm 1 is singular and satisfies

$$\widehat{\mu}(\mathbb{Z}) \subset \Big\{ \pm \prod_{k=1}^{m} \varepsilon_k^{l_k} : l_k \in \{0, 1\}, \, m \in \mathbb{N} \Big\} \cup \{0\}.$$

Proof. We show first how to choose the sequence $(n_k)_{k=1}^{\infty}$. We define n_k as the smallest integer satisfying

$$2\log_2 \frac{1}{\varepsilon_k} - 5 < n_k < 2\log_2 \frac{1}{\varepsilon_k} - 3.$$

To satisfy $n_{k+1} > n_k$ it is enough to have $2\log_2(1/\varepsilon_k) - 3 < 2\log_2(1/\varepsilon_{k+1}) - 5$, which is equivalent to $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$. Now we define the sequences $(r_k)_{k=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$. We put $m_1 = r_1 = 1$ and then we choose inductively

$$r_k > 2\sum_{j=1}^{k-1}((2^{n_j} - 1)m_j + r_j) \quad \text{for } k \geq 2,$$
$$m_k > 2\sum_{j=1}^{k-1}((2^{n_j} - 1)m_j + r_j) \quad \text{for } k \geq 2.$$

We can apply Proposition 31 to derive that the measure μ exists and does not belong to $L^2(\mathbb{T})$ (in fact, the assumption $2^{n_k}\varepsilon_k^2 > 1/32$ leads to this result without referring to Proposition 31). The proof of singularity follows essentially the argument given in [BM] and [GM] for Riesz products.

We obviously have

$$\sum_{k=1}^{\infty} 2^{n_k}\varepsilon_k = \infty.$$

Hence, we may choose a sequence $(c_k)_{k=1}^{\infty} \in l^2(\mathbb{N})$ of real numbers and an increasing sequence $(l_k)_{k=1}^{\infty}$ of integers such that

$$\sum_{l=l_k+1}^{l_{k+1}} c_l 2^{n_l}\varepsilon_l = 1 \quad \text{for all } k \in \mathbb{N} \tag{12}$$

with the additional property

$$\sum_{l=1}^{\infty} c_l^2 2^{n_l} < \infty. \tag{13}$$

We set

$$A_l = \{r_l, r_l + m_l, r_l + 2m_l, \ldots, r_l + (2^{n_l} - 1)m_l\} \subset \mathbb{N} \quad \text{for } l \in \mathbb{N}.$$

Clearly, $A_l \cap A_k = \emptyset$ for $l \neq k$ and $|A_l| = 2^{n_l}$. Moreover, $\widehat{\mu}(n) = \operatorname{sgn}\widehat{\mu}(n)\varepsilon_l$ for $n \in A_l$. Consider polynomials f_k for $k \in \mathbb{N}$ defined by

$$f_k(t) = \sum_{l=l_k+1}^{l_{k+1}} c_l \sum_{n\in A_l} \operatorname{sgn}\widehat{\mu}(n)e^{int}. \tag{14}$$

By (13) we have

$$\|f_k\|_{L^2(\mathbb{T})}^2 = \sum_{l=l_k+1}^{l_{k+1}} 2^{n_l}c_l^2 \to 0 \quad \text{as } k \to \infty. \tag{15}$$

We now perform a crucial calculation of $\|f_k\|^2_{L^2(\mathbb{T},\mu)}$:

$$\begin{aligned}\|f_k\|^2_{L^2(\mathbb{T},\mu)} &= \int f_k(t)\overline{f_k(t)}\,d\mu \\ &= \sum_l 2^{n_l}c_l^2 + \sum_l c_l^2 \sum_{\substack{n,m\in A_l\\ n\neq m}} \operatorname{sgn}\widehat{\mu}(n)\operatorname{sgn}\widehat{\mu}(m)\,\widehat{\mu}(m-n) + \sum_{\substack{l,r\\ l\neq r}} c_l 2^{n_l}\varepsilon_l c_r 2^{n_r}\varepsilon_r.\end{aligned}$$

However, $\widehat{\mu}(m-n)=0$ for $m,n\in A_l$, $n\neq m$ (by (11) and the construction of μ in Proposition 30) and the second sum vanishes. Simple manipulations of the remaining terms give

$$\sum_l 2^{n_l}c_l^2 + \sum_{\substack{l,r\\ l\neq r}} c_l 2^{n_l}\varepsilon_l c_r 2^{n_r}\varepsilon_r = \sum_l 2^{n_l}c_l^2 + \Big(\sum_l 2^{n_l}\varepsilon_l c_l\Big)^2 - \sum_l c_l^2 2^{2n_l}\varepsilon_l^2.$$

By (12), the second term equals 1 and so

$$\|f_k\|^2_{L^2(\mathbb{T},\mu)} = 1 + \sum_{l=l_k+1}^{l_{k+1}} c_l^2 2^{n_l}(1-2^{n_l}\varepsilon_l^2).$$

The assumption $2^{n_l}\varepsilon_l^2 > 1/32$ leads to $1-2^{n_l}\varepsilon_l^2 < 31/32$, which, with the aid of (13), gives

$$\sum_{l=l_k+1}^{l_{k+1}} c_l^2 2^{n_l}(1-2^{n_l}\varepsilon_l^2) \to 0 \quad \text{as } k\to\infty.$$

Hence

$$\|f_k\|_{L^2(\mathbb{T},\mu)} \to 1 \quad \text{as } k\to\infty. \tag{16}$$

We shall now show that

$$\lim_{k\to\infty} \|f_k - 1\|_{L^1(\mathbb{T},\mu)} = 0 \tag{17}$$

Applying the Schwarz inequality we get

$$\begin{aligned}\Big(\int |f_k-1|\,d\mu\Big)^2 &\le \int |f_k-1|^2\,d\mu = \int |f_k|^2\,d\mu - 2\mathrm{Re}\int f_k\,d\mu + 1 \\ &= \int |f_k|^2\,d\mu - 2\mathrm{Re}\sum_{l=l_k+1}^{l_{k+1}} c_l 2^{n_l}\varepsilon_l + 1 \to 0 \quad \text{as } k\to\infty.\end{aligned}$$

The last assertion follows from (12) and (16). This proves (17).

Finally, by (15) and (17), we can find a subsequence (f_{k_j}) and a Borel set $F\subset\mathbb{T}$ of full Lebesgue measure and with $\mu(F)=1$ such that

$$\lim_{j\to\infty} f_{k_j}(t) = 0 \quad \text{for } t\in F, \qquad \lim_{j\to\infty} f_{k_j} = 1 \quad \mu\text{-a.e. on } F.$$

Both the normalized Lebesgue measure and μ are positive and have norm 1. Hence μ is singular with respect to the Lebesgue measure, which finishes the proof. ∎

7. Final remarks. 1. The components of the open set U constructed in Theorem 1 tend to 0 very fast. A closer look at the proof shows that the reciprocal of the distance of the nth component to 0 grows as fast as the Ackermann function $A(4, 2n)$, which is very fast. It is not easy to see in which places of the proof this growth could be optimized.

2. Our proof shows that the continuous part of a measure satisfying the assumptions of Theorem 1 has an absolutely continuous convolution square. We do not know, however, whether this could be improved. In particular it would be interesting to find an example of an open set U_1 such that a continuous measure is absolutely continuous if it has all its Fourier coefficients in U_1.

3. It would also be interesting to construct an open set U_k with the property that any function with Fourier coefficients from U_k has only the kth convolution power absolutely continuous, and such that there exists a measure with Fourier coefficients in U_k with all smaller convolution powers singular.

4. The above property uses the fact that the sum of measures with Fourier coefficients tending to 0 whose convolution powers are absolutely continuous belongs to the Zafran class $\mathscr{C}$. We also conjecture that the converse holds: any measure from $\mathscr{C}$ can be decomposed into an (infinite) sum of measures whose convolution powers are absolutely continuous.

5. The crucial element in our proof was the use of the Littlewood conjecture to estimate the number of repetitions of any specific value taken by the Fourier coefficients. In the case of the Cantor group, the Littlewood conjecture does not hold. This fact encourages us to ask whether any infinite Wiener–Pitt set exists for the convolution measure algebra on the Cantor group.

6. Our Lemma 8 is a stronger version of the second part of Theorem 15, which is taken from [GM]. Our proof uses the exact version of the Littlewood conjecture, which was not available when [GM] was written. But our proof differs in more respects: it uses Bożejko–Pełczyński's invariant local approximation property, which seems to be a simpler method.

7. While Lemma 8 does not hold for torsion abelian groups, because the Littlewood conjecture is false there, it seems likely that Lemma 9 may be extended to this case—but this would require completely different methods.

Acknowledgements. The authors express their sincere gratitude to the reviewer whose thorough remarks and comments contributed significantly to the current shape of the paper. This concerns especially the example which was constructed under the influence of his suggestion.

The research of Przemysław Ohrysko has been supported by the Polish Ministry of Science grant no. N N201 397737 (years 2009-2012).

The research of Michał Wojciechowski has been supported by NCN grant no. N N201 607840.

References

[B] J. Bourgain, *A remark on entropy of Abelian groups and the invariant uniform approximation property*, Studia Math. 86 (1987), 79–84.

[BBM] W. J. Bailey, G. Brown and W. Moran, *Spectra of independent power measures*, Proc. Cambridge Philos. Soc. 72 (1972), 27–35.

[BP] M. Bożejko and A. Pełczyński, *An analogue in commutative harmonic analysis of the uniform bounded approximation property of Banach space*, Séminaire d'Analyse Fonctionnelle 1978–1979, École Polytech., Palaiseau, exp. 9, 1979, 9 pp.

[BM] G. Brown and W. Moran, *On orthogonality of Riesz products*, Math. Proc. Cambridge Philos. Soc. 76 (1974), 173–181.

[DK] K. DeLeeuw and Y. Katznelson, *The two sides of a Fourier–Stieltjes transform and almost idempotent measures*, Israel J. Math. 8 (1970), 213–229.

[GW] I. Glicksberg and I. Wik, *The range of Fourier–Stieltjes transforms of parts of measures*, in: Lecture Notes in Math. 266, Springer, 1972, 73–77.

[G] C. C. Graham, *A Riesz product proof of the Wiener–Pitt theorem*, Proc. Amer. Math. Soc. 44 (1974), 312–314.

[GM] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Springer, New York, 1979.

[HMP] B. Host, J.-F. Méla et F. Parreau, *Analyse Harmonique des Mesures*, Astérisque 135-136 (1986).

[Ka] Y. Katznelson, *An Introduction to Harmonic Analysis*, Cambridge Univ. Press, 2004.

[Ko] S. V. Konyagin, *On a problem of Littlewood*, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), 243–265 (in Russian).

[MPS] O. C. McGehee, L. Pigno and B. Smith, *Hardy's inequality and the L^1 norm of exponential sums*, Ann. of Math. 113 (1981), 613–618.

[O] P. Ohrysko, *Around the Wiener–Pitt phenomenon*, MA thesis, http://pohrysko.jimdo.com/różne-pliki/ (in Polish).

[R] W. Rudin, *Fourier Analysis on Groups*, Wiley Classics Library, 1990.

[S] Yu. A. Šreĭder, *The structure of maximal ideals in rings of measures with convolution*, Mat. Sb. 27 (1950), 297–318 (in Russian); English transl.: Amer. Math. Soc. Transl., 1953, no. 81, 28 pp.

[WP] N. Wiener and H. R. Pitt, *Absolutely convergent Fourier–Stieltjes transforms*, Duke Math. J. 4 (1938), 420–436.

[W] M. Wojciechowski, *The non-equivalence between the trigonometric system and the system of functions with pointwise restrictions on values in the uniform and L^1 norms*, Math. Proc. Cambridge Philos. Soc. 150 (2011), 561–571.

[Wo] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Univ. Press, 1991.

[Za] M. Zafran, *On spectra of multipliers*, Pacific J. Math. 47 (1973), 609–626.

[Zy] A. Zygmund, *Trigonometrical Series*, Cambridge Univ. Press, 2002.
[Ż] W. Żelazko, *Banach Algebras*, Elsevier and PWN, 1973.

Przemysław Ohrysko, Michał Wojciechowski
Institute of Mathematics
Polish Academy of Sciences
00-956 Warszawa, Poland
E-mail: p.ohrysko@gmail.com
miwoj-impan@o2.pl

Received November 13, 2012
Revised version November 19, 2013 and March 12, 2014 (7685)