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Polaroid type operators and compact perturbations

by

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Abstract. A bounded linear operator T acting on a Hilbert space is said to be polaroid if each isolated point in the spectrum is a pole of the resolvent of T. There are several generalizations of the polaroid property. We investigate compact perturbations of polaroid type operators. We prove that, given an operator T and $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that $T+K$ is polaroid. Moreover, we characterize those operators for which a certain polaroid type property is stable under (small) compact perturbations.

1. Introduction. This paper is inspired by [1, 4, 5], where the stability of polaroid type properties under some commuting perturbations is studied. The purpose of this paper is to investigate the perturbations of polaroid type properties under small compact perturbations.

Throughout this paper, $\mathcal{H}$ denotes a complex separable infinite-dimensional Hilbert space. We let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{K}(\mathcal{H})$ the ideal of compact operators in $\mathcal{B}(\mathcal{H})$.

Let $T \in \mathcal{B}(\mathcal{H})$. We denote by $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{s}}(T)$ and $\sigma_{\mathrm{a}}(T)$ the spectrum, the point spectrum, the surjectivity spectrum and the approximate point spectrum of T respectively. Denote by $\ker T$ and $\operatorname{ran} T$ the kernel and the range of T respectively. The *ascent* of T is defined as the smallest non-negative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such an integer does not exist we define $p(T) = \infty$. Analogously, the *descent* of T is defined as the smallest non-negative integer $q := q(T)$ such that $\operatorname{ran} T^p = \operatorname{ran} T^{p+1}$. If such an integer does not exist we define $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are finite then $p(T) = q(T)$.

Recall that T is *Drazin invertible* if $p(T)$ and $q(T)$ are finite; this holds if and only if 0 is a pole of the resolvent of T (see [13, Proposition 50.2]). Moreover, T is *left Drazin invertible* if $p(T) < \infty$ and $\operatorname{ran}(T^{p(T)+1})$ is closed. Analogously, T is *right Drazin invertible* if $q(T) < \infty$ and $\operatorname{ran}(T^{q(T)})$ is

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closed. We say $\lambda \in \sigma_{\mathrm{a}}(T)$ is a *left pole* of T if $T-\lambda$ is left Drazin invertible, and $\lambda \in \sigma_{\mathrm{s}}(T)$ is a *right pole* of T if $T-\lambda$ is right Drazin invertible.

Given a subset σ of $\mathbb{C}$, we denote by $\operatorname{iso}\sigma$ the set of all isolated points of σ.

The notion of polaroid operators was first introduced in [11].

Definition 1.1. We say that $T \in \mathcal{B}(\mathcal{H})$ is *polaroid*, denoted by $T \in (\mathcal{P})$, if every $\lambda \in \operatorname{iso}\sigma(T)$ is a pole of the resolvent of T.

The polaroid property is often used as a basic condition to study Weyl's theorem for operators and its generalizations (see [2, 3, 4, 8, 9, 11]). Since people are interested in the stability of Weyl type theorems under perturbations, we are going to study small compact perturbations of polaroid properties.

Some other variants of the polaroid property are introduced in [2].

Definition 1.2. We say that $T \in \mathcal{B}(\mathcal{H})$ is *a-polaroid*, denoted by $T \in (\mathcal{AP})$, if every $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ is a pole of the resolvent of T; $T \in \mathcal{B}(\mathcal{H})$ is said to be *left polaroid*, denoted by $T \in (\mathcal{LP})$, if every $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ is a left pole of T; $T \in \mathcal{B}(\mathcal{H})$ is said to be *right polaroid*, denoted by $T \in (\mathcal{RP})$, if every $\lambda \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$ is a right pole of T.

It is easy to see that

$$T \text{ a-polaroid } \Rightarrow T \text{ left polaroid } \Rightarrow T \text{ polaroid},$$

and

$$T \text{ left polaroid } \Leftrightarrow T^* \text{ right polaroid}.$$

In [10], Duggal introduced the concept of hereditarily polaroid operators.

Definition 1.3. We say that $T \in \mathcal{B}(\mathcal{H})$ is *hereditarily polaroid*, denoted by $T \in (\mathcal{HP})$, if the restriction of T to each closed invariant subspace is polaroid.

The purpose of this paper is to investigate compact perturbations of Hilbert space operators with polaroid properties. We shall prove that given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T+K$ is polaroid. Moreover, we shall study the stability of polaroid properties under (small) compact perturbations. In order to state our main results, we first introduce some notations and terminology.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *semi-Fredholm* if $\operatorname{ran} T$ is closed and either $\operatorname{nul} T$ or $\operatorname{nul} T^*$ is finite, where $\operatorname{nul} T \triangleq \dim\ker T$ and $\operatorname{nul} T^* \triangleq \dim\ker T^*$; in this case, $\operatorname{ind} T \triangleq \operatorname{nul} T - \operatorname{nul} T^*$ is called the *index* of T. In particular, if $-\infty < \operatorname{ind} T < \infty$, then T is called a *Fredholm operator*. T is called a *Weyl operator* if it is Fredholm of index 0. The *Wolf spectrum* $\sigma_{\mathrm{lre}}(T)$, the *Weyl spectrum* $\sigma_{\mathrm{w}}(T)$ and the *essential approximate point* *spectrum* $\sigma_{\rm ea}(T)$ are defined by

$$\sigma_{\rm lre}(T) \triangleq \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\},$$
$$\sigma_{\rm w}(T) \triangleq \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$
$$\sigma_{\rm ea}(T) \triangleq \bigcap_{K \in \mathcal{K}(\mathcal{H})} \sigma_{\rm a}(T + K).$$

$\rho_{\text{s-F}}(T) \triangleq \mathbb{C} \setminus \sigma_{\rm lre}(T)$ is the *semi-Fredholm domain* of T. It is known that

$$\mathbb{C} \setminus \sigma_{\rm ea}(T) = \{\lambda \in \rho_{\text{s-F}}(T) : \operatorname{ind}(T - \lambda) \leq 0\}.$$

We denote

$$\rho^+_{\text{s-F}}(T) \triangleq \{\lambda \in \rho_{\text{s-F}}(T) : \operatorname{ind}(T - \lambda) > 0\},$$
$$\rho^-_{\text{s-F}}(T) \triangleq \{\lambda \in \rho_{\text{s-F}}(T) : \operatorname{ind}(T - \lambda) < 0\},$$
$$\rho^0_{\text{s-F}}(T) \triangleq \{\lambda \in \rho_{\text{s-F}}(T) : T - \lambda \text{ is Weyl}\}.$$

For $T \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \rho_{\text{s-F}}(T)$, the *minimal index* of $\lambda - T$ is defined by

$$\min \operatorname{ind}(\lambda - T) = \min\{\operatorname{nul}(\lambda - T), \operatorname{nul}(\lambda - T)^*\}.$$

It is well known that the function $\lambda \mapsto \min \operatorname{ind}(\lambda - T)$ is constant on every component of $\rho_{\text{s-F}}(T)$ except for an at most denumerable subset $\rho^{\rm s}_{\text{s-F}}(T)$ of $\rho_{\text{s-F}}(T)$ without limit points in $\rho_{\text{s-F}}(T)$. Furthermore, if $\mu \in \rho^{\rm s}_{\text{s-F}}(T)$ and λ is a point of $\rho_{\text{s-F}}(T)$ in the same component as μ but $\lambda \notin \rho^{\rm s}_{\text{s-F}}(T)$, then

$$\min \operatorname{ind}(\lambda - T) < \min \operatorname{ind}(\mu - T).$$

$\rho^{\rm s}_{\text{s-F}}(T)$ is called the set of *singular points* of the semi-Fredholm domain $\rho_{\text{s-F}}(T)$; $\rho^{\rm r}_{\text{s-F}}(T) = \rho_{\text{s-F}}(T) \setminus \rho^{\rm s}_{\text{s-F}}(T)$ is the set of *regular points*. For details, one can see [12, Corollary 1.14].

For $\lambda_0 \in \mathbb{C}$ and $\delta > 0$, we denote $B_\delta(\lambda_0) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta\}$.

We let $E(T) = \{\lambda \in \sigma_{\rm lre}(T) : \exists \delta > 0$ such that $\operatorname{ind}(T - \mu) < 0$ for $\mu \in B_\delta(\lambda) \setminus \{\lambda\}$ and $\min \operatorname{ind}(T - \mu) = 0$ for $\mu \in B_\delta(\lambda) \setminus [\{\lambda\} \cup \rho^{\rm s}_{\text{s-F}}(T)]\}$.

The main results of this paper are listed below.

Theorem 1.4 (Main Theorem 1). *Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (\mathcal{P})$.*

Theorem 1.5 (Main Theorem 2). *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent:*

(1) *Given $\varepsilon > 0$, there is $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T+K \notin (\mathcal{P})$.*
(2) *There exists $K \in \mathcal{K}(\mathcal{H})$ such that $T + K \notin (\mathcal{P})$.*
(3) $\operatorname{iso} \sigma_{\rm w}(T) \neq \emptyset$.

Theorem 1.6 (Main Theorem 3). *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent:*

(1) *Given $\varepsilon > 0$, there is $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T+K \in (\mathcal{AP})$.*
(2) $E(T) = \emptyset$.

THEOREM 1.7 (Main Theorem 4). *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

(1) *Given $\varepsilon > 0$, there is $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T+K \notin (\mathcal{AP})$.*
(2) *There exists $K \in \mathcal{K}(\mathcal{H})$ such that $T + K \notin (\mathcal{AP})$.*
(3) $\operatorname{iso}\sigma_{\mathrm{w}}(T) \neq \emptyset$ *or* $\rho^{-}_{\text{s-F}}(T) \neq \emptyset$.

The rest of this paper is organized as follows. In Section 2, we make some preparations. In Section 3, we give the proofs of the main results. In Section 4, we study compact perturbations of the left and right polaroid properties. Section 5 is devoted to investigating the compact perturbations of the hereditarily polaroid property.

2. Preparations. Let $T \in \mathcal{B}(\mathcal{H})$. If σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subset \Omega$ and $[\sigma(T)\setminus\sigma]\cap\overline{\Omega} = \emptyset$. We let $E(\sigma;T)$ denote the Riesz idempotent of T corresponding to σ, that is,

$$E(\sigma;T) = \frac{1}{2\pi i}\int_{\Gamma} (\lambda - T)^{-1}\, d\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we denote $\mathcal{H}(\sigma;T) = \operatorname{ran} E(\sigma;T)$. If $\lambda \in \operatorname{iso}\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$ and we simply write $\mathcal{H}(\lambda;T)$ instead of $\mathcal{H}(\{\lambda\};T)$; if, in addition, $\dim\mathcal{H}(\lambda;T) < \infty$, then λ is called a *normal eigenvalue* of T. The set of all normal eigenvalues of T will be denoted by $\sigma_0(T)$.

Obviously, each normal eigenvalue of T is a pole of the resolvent of T.

LEMMA 2.1 ([15, Theorem 2.10]). *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_i $(i = 1, 2)$ are clopen subsets of $\sigma(T)$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Then $\mathcal{H}(\sigma_1;T) + \mathcal{H}(\sigma_2;T) = \mathcal{H}$, $\mathcal{H}(\sigma_1;T) \cap \mathcal{H}(\sigma_2;T) = \{0\}$ and T admits the matrix representation*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1;T) \\ \mathcal{H}(\sigma_2;T) \end{matrix},$$

where $\sigma(T_i) = \sigma_i$ $(i = 1, 2)$.

LEMMA 2.2 ([12, Corollary 3.22]). *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that T admits the representation*

$$T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\sigma_{\mathrm{s}}(A) \cap \sigma_{\mathrm{a}}(B) = \emptyset$. Then $T \sim A \oplus B$.

Using the above lemma, we can obtain the following result, whose proof is left to the reader.

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that σ is a clopen subset of $\sigma(T)$. Then*

$$T = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^{\perp} \end{matrix} \sim \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^{\perp} \end{matrix},$$

where $\sigma(A) = \sigma$ and $\sigma(B) = \sigma(T) \setminus \sigma$.

If $S, T \in \mathcal{B}(\mathcal{H})$, then $S \sim T$ denotes that S and T are similar. By [6, Theorem 2.11] and Lemma 2.2, we can obtain the following lemma.

Lemma 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then:*

(1) *T is Drazin invertible if and only if*

$$T \sim \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix},$$

where T_1 is nilpotent and T_2 is invertible.

(2) *If $0 \in \mathrm{iso}_{\mathrm{a}}(T)$, then T is left Drazin invertible if and only if*

$$T \sim \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix},$$

where T_1 is nilpotent and T_2 is left invertible.

Lemma 2.5 ([7, Proposition 6.9]). *Let $T \in \mathcal{B}(\mathcal{H})$ and $\lambda_0 \in \mathrm{iso}\,\sigma(T)$. Then the following statements are equivalent:*

(1) $\lambda_0 \in \sigma_0(T)$.
(2) $\lambda_0 \in \rho^0_{\text{s-F}}(T)$.
(3) $\lambda_0 \in \rho_{\text{s-F}}(T)$.

Lemma 2.6 ([14, Lemma 3.2.6]). *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\emptyset \neq \varGamma \subset \sigma_{\mathrm{lre}}(T)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that*

$$T + K = \begin{bmatrix} N & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix},$$

where

(1) *N is a diagonal normal operator of uniformly infinite multiplicity, and $\sigma(N) = \sigma_{\mathrm{lre}}(N) = \overline{\varGamma}$,*
(2) *$\sigma(T) = \sigma(A)$, $\sigma_{\mathrm{lre}}(T) = \sigma_{\mathrm{lre}}(A)$ and $\mathrm{ind}(T-\lambda) = \mathrm{ind}(A-\lambda)$ for all $\lambda \in \rho_{\text{s-F}}(T)$.*

Lemma 2.7 ([16, Corollary 2.9]). *Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with*

$$\|K\| < \varepsilon + \max\{\mathrm{dist}[\lambda, \partial\rho_{\text{s-F}}(T)] : \lambda \in \sigma_0(T)\}$$

such that $\sigma_{\mathrm{p}}(T+K) = \rho^+_{\text{s-F}}(T)$.

3. Proof of the main theorems. For nonzero vectors $x, y \in \mathcal{H}$, we define the rank-one operator $x \otimes y \in \mathcal{B}(\mathcal{H})$ as $(x \otimes y)z = \langle z, y\rangle x$ for each $z \in \mathcal{H}$.

We first give a useful lemma.

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that* $\operatorname{iso}\sigma(T+K) = \sigma_0(T+K)$ *and*

$$\min \operatorname{ind}(T + K - \lambda) = \min \operatorname{ind}(T - \lambda)$$

for all $\lambda \in \rho^-_{\text{s-F}}(T)$.

Proof. If $\operatorname{iso}\sigma(T) \cap \sigma_{\text{lre}}(T) = \emptyset$, by Lemma 2.5 we have $\operatorname{iso}\sigma(T) = \sigma_0(T)$. In this case, we need to do nothing. If $\operatorname{iso}\sigma(T) \cap \sigma_{\text{lre}}(T) \neq \emptyset$, without loss of generality we assume that $\operatorname{iso}\sigma(T) \cap \sigma_{\text{lre}}(T) = \{\lambda_n\}_{n=1}^\infty$; the proof of the finite case is similar.

By Corollary 2.3, we have

$$T = \left[\begin{array}{cccc|c} A_1 & * & \cdots & & * \\ 0 & A_2 & \cdots & & * \\ 0 & 0 & \ddots & & \vdots \\ \hline 0 & 0 & \cdots & & B \end{array}\right] \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \\ \mathcal{H}_0 \end{matrix} = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}_0^\perp \\ \mathcal{H}_0 \end{matrix},$$

where $\sigma(A_n) = \{\lambda_n\}$, $\bigoplus_{k=1}^n \mathcal{H}_k = \sum_{k=1}^n \mathcal{H}(\lambda_k; T)$ for each $n \geq 1$ and $\mathcal{H}_0 = \mathcal{H} \ominus \bigoplus_{k=1}^\infty \mathcal{H}_n$. It is not difficult to see that $\sigma(B) \subset \sigma(T) \setminus \{\lambda_n\}_{n=1}^\infty$.

For $\varepsilon > 0$, by Lemma 2.6, there exists a compact operator $\overline{K_0}$ on $\mathcal{H}_0^\perp$ with $\|\overline{K_0}\| < \varepsilon/2$ such that

$$A + \overline{K_0} = \begin{bmatrix} \bigoplus_{n=1}^\infty \lambda_n I_n & * \\ 0 & A_0 \end{bmatrix} \begin{matrix} \bigoplus_{n=1}^\infty \mathcal{M}_n \\ \mathcal{H}_0^\perp \ominus (\bigoplus_{n=1}^\infty \mathcal{M}_n) \end{matrix},$$

where

- $\dim \mathcal{M}_n = \infty$ and I_n is the identity operator on $\mathcal{M}_n$ for each $n \geq 1$,
- $\sigma(A_0) = \sigma(A)$, $\sigma_{\text{lre}}(A_0) = \sigma_{\text{lre}}(A)$ and $\operatorname{ind}(A_0 - \lambda) = \operatorname{ind}(A - \lambda)$ for all $\lambda \in \rho_{\text{s-F}}(A)$.

Fix $n \geq 1$. Choose an ONB $\{e_k^{(n)}\}_{k=1}^\infty$ of $\mathcal{M}_n$ and define

$$K_n = \alpha_n \sum_{k=1}^\infty \frac{1}{k} e_k^{(n)} \otimes e_k^{(n)},$$

where $0 < \alpha_n < \varepsilon/2^{n+1}$ with $B_{\alpha_n}(\lambda_n) \cap \sigma(T) = \{\lambda_n\}$ and $B_{\alpha_n}(\lambda_n) \cap B_{\alpha_m}(\lambda_m) = \emptyset$ for $m \neq n$. We denote

$$K_0 = \begin{bmatrix} \overline{K_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_0^\perp \\ \mathcal{H}_0 \end{matrix}.$$

We let $K = \sum_{n=0}^{\infty} K_n$. Noting that $\sum_{n=0}^{\infty} \|K_n\| < \varepsilon$, we have $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \varepsilon$. It suffices to prove that $\operatorname{iso}\sigma(T+K) \subset \sigma_0(T+K)$ and $\min\operatorname{ind}(T+K-\lambda) = \min\operatorname{ind}(T-\lambda)$ for all $\lambda \in \rho_{\text{s-F}}^{-}(T)$.

Noting that

$$T+K = \begin{bmatrix} \bigoplus_{n=1}^{\infty}(\lambda_n I_n + K_n) & * & * \\ 0 & A_0 & * \\ 0 & 0 & B \end{bmatrix} \begin{matrix} \bigoplus_{n=1}^{\infty}\mathcal{M}_n \\ \mathcal{H}_0^{\perp} \ominus (\bigoplus_{n=1}^{\infty}\mathcal{M}_n) \\ \mathcal{H}_0 \end{matrix},$$

it is not difficult to see that

- $\operatorname{iso}\sigma(T+K) = \{\lambda_n + \alpha_n/k : n, k \geq 1\} \cup \sigma_0(B) = \sigma_0(T+K)$,
- for $\lambda \in \rho_{\text{s-F}}^{-}(T)$, we have $\lambda \notin \sigma(A_0) \cup \sigma(\bigoplus_{n=1}^{\infty}(\lambda_n I_n + K_n))$ and hence $\operatorname{nul}(T+K-\lambda) = \operatorname{nul}(B-\lambda) = \operatorname{nul}(T-\lambda)$.

If $\lambda_0 \in \rho_{\text{s-F}}^{-}(T+K)$, we have

$$\begin{aligned} \min\operatorname{ind}(T+K-\lambda_0) &= \operatorname{nul}(T+K-\lambda_0) = \operatorname{nul}(B-\lambda_0) \\ &= \operatorname{nul}(T-\lambda_0) = \min\operatorname{ind}(T-\lambda_0). \ \blacksquare \end{aligned}$$

Proof of Theorem 1.4. For $\varepsilon > 0$, by Lemma 3.1, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $\operatorname{iso}\sigma(T+K) = \sigma_0(T+K)$; then it is easy to see that $T+K \in (\mathcal{P})$. ∎

LEMMA 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\operatorname{iso}\sigma(T) \cap \sigma_{\text{lre}}(T) \neq \emptyset$. Then for each $\varepsilon > 0$ there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T+K \notin (\mathcal{P})$.*

Proof. Fix $\lambda_0 \in \operatorname{iso}\sigma(T) \cap \sigma_{\text{lre}}(T)$. By Corollary 2.3, T can be written as

$$T = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\lambda_0; T) \\ \mathcal{H}(\lambda_0; T)^{\perp} \end{matrix},$$

where $\sigma(A) = \{\lambda_0\}$ and $\sigma(B) = \sigma(T) \setminus \{\lambda_0\}$. Noting that $\lambda_0 \in \sigma_{\text{lre}}(T)$, we have $\dim \mathcal{H}(\lambda_0; T) = \infty$ and $\sigma_{\text{lre}}(A) = \{\lambda_0\}$.

For $\varepsilon > 0$, by Lemma 2.6, there exists a compact operator $\overline{K_1}$ on $\mathcal{H}(\lambda_0; T)$ with $\|\overline{K_1}\| < \varepsilon/2$ such that

$$A + \overline{K_1} = \begin{bmatrix} \lambda_0 I & * \\ 0 & A_0 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{H}(\lambda_0; T) \ominus \mathcal{M} \end{matrix},$$

where

- $\dim \mathcal{M} = \infty$ and I is the identity operator on $\mathcal{M}$,
- $\sigma(A_0) = \sigma_{\text{lre}}(A_0) = \{\lambda_0\}$.

Choose an ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{M}$ and define $K_2 \in \mathcal{K}(\mathcal{H})$ by

$$K_2 = \sum_{k=1}^{\infty} \frac{\varepsilon}{k+1} e_{k+1} \otimes e_k.$$

We denote

$$K_1 = \begin{bmatrix} \overline{K_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}(\lambda_0; T) \\ \mathcal{H}(\lambda_0; T)^\perp \end{matrix}.$$

In addition, we let $K = K_1 + K_2$; then $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \varepsilon$. Moreover, $T + K$ can be written as

$$T + K = \begin{bmatrix} \lambda_0 I + K_2 & * & * \\ 0 & A_0 & * \\ 0 & 0 & B \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{H}(\lambda_0; T) \ominus \mathcal{M} \\ \mathcal{H}(\lambda_0; T)^\perp \end{matrix} = \begin{bmatrix} A_1 & * \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\lambda_0; T) \\ \mathcal{H}(\lambda_0; T)^\perp \end{matrix}.$$

It is easy to see that $A_1 - \lambda_0$ is not nilpotent. Noting that $\sigma(\lambda_0 I + K_2) = \sigma(A_0) = \{\lambda_0\}$ and $\lambda_0 \notin \sigma(B)$, we have $\lambda_0 \in \operatorname{iso}\sigma(T + K)$. We claim that $T + K - \lambda_0$ is not Drazin invertible. Otherwise, by Lemma 2.4, we have

$$T + K - \lambda_0 \sim \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^\perp \end{matrix},$$

where T_1 is nilpotent and T_2 is invertible. Using a matrix calculation and [12, Theorem 3.19], it is easy to see that $A_1 - \lambda_0$ and T_1 are similar; this means that $A_1 - \lambda_0$ is nilpotent, a contradiction. Hence the claim follows and $T + K \notin (\mathcal{P})$. ∎

Proof of Theorem 1.5. (1)⇒(2). This relation is obvious.

(2)⇒(3). If $T + K \notin (\mathcal{P})$ for some $K \in \mathcal{K}(\mathcal{H})$, then there exists $\lambda_0 \in \operatorname{iso}\sigma(T+K)$ such that λ_0 is not a pole of the resolvent of $T+K$. By Lemma 2.5, we have $\lambda_0 \in \sigma_{\mathrm{lre}}(T + K) = \sigma_{\mathrm{lre}}(T)$. Noting that $\lambda_0 \in \operatorname{iso}\sigma(T + K)$, there exists $\delta > 0$ such that $T+K-\lambda$ is invertible for all $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$. Hence $\operatorname{ind}(T - \lambda) = 0$ for all $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$. Thus $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{w}}(T)$, so $\operatorname{iso}\sigma_{\mathrm{w}}(T) \neq \emptyset$.

(3)⇒(1). If $\operatorname{iso}\sigma_{\mathrm{w}}(T) \neq \emptyset$, we choose $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{w}}(T)$. For $\varepsilon > 0$, we denote

$$\sigma_1 = \{\lambda \in \sigma_0(T) : \operatorname{dist}[\lambda, \partial\rho_{\text{s-F}}(T)] \geq \varepsilon/2\} \quad \text{and} \quad \sigma_2 = \sigma(T) \setminus \sigma_1.$$

Then σ_1 is a finite clopen subset of $\sigma(T)$. By Corollary 2.3, T admits the representation

$$T = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_1; T)^\perp \end{matrix},$$

where $\sigma(A) = \sigma_1$, $\sigma(B) = \sigma_2$ and it is easy to verify that

$$\max\{\operatorname{dist}[\lambda, \partial\rho_{\text{s-F}}(B)] : \lambda \in \sigma_0(B)\} < \varepsilon/2.$$

By Lemma 2.7, there exists a compact operator $\overline{K_1}$ on $\mathcal{H}(\sigma_1; T)^\perp$ with

$\|\overline{K_1}\| < \varepsilon/2$ such that $\sigma_{\mathrm{p}}(B + \overline{K_1}) = \rho^+_{\text{s-F}}(B)$. We denote

$$K_1 = \begin{bmatrix} 0 & \\ & \overline{K_1} \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_1; T)^\perp \end{matrix};$$

then $K_1 \in \mathcal{K}(\mathcal{H})$ and $\|K_1\| < \varepsilon/2$.

Since $\lambda_0 \in \operatorname{iso} \sigma_{\mathrm{w}}(T)$, there exists $\delta > 0$ such that $\operatorname{ind}(T + K_1 - \lambda) = 0$ for all $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$. For fixed $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$, it is easy to see that $\operatorname{ind}(B + \overline{K_1} - \lambda) = 0$ and hence $\lambda \notin \sigma_{\mathrm{p}}(B + \overline{K_1})$. It follows that $B + \overline{K_1} - \lambda$ is invertible. Noting that $\lambda_0 \notin \sigma(A)$, we have $\lambda_0 \in \operatorname{iso} \sigma(T + K_1)$.

Since $\lambda_0 \in \operatorname{iso} \sigma_{\mathrm{w}}(T)$, we have $\lambda_0 \in \sigma_{\mathrm{lre}}(T)$ and hence $\lambda_0 \in \sigma_{\mathrm{lre}}(T + K_1) \cap \operatorname{iso} \sigma(T + K_1)$. By Lemma 3.2, there exists $K_2 \in \mathcal{K}(\mathcal{H})$ with $\|K_2\| < \varepsilon/2$ such that $T + K_1 + K_2 \notin (\mathcal{P})$. ■

As a corollary of [12, Theorem 3.47], we have the following result.

Lemma 3.3 ([12, Theorem 3.47]). *Let $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that:*

(1) *There exists no singular point in the components of $\rho^-_{\text{s-F}}(T+K)$ with minimum index zero.*
(2) $\min \operatorname{ind}(T + K - \lambda) = \min \operatorname{ind}(T - \lambda)$ *for all* $\lambda \in \rho^{\mathrm{r}}_{\text{s-F}}(T)$.

Proof of Theorem 1.6. (1)⇒(2). If $E(T) \neq \emptyset$, there exists $\lambda_0 \in \sigma_{\mathrm{lre}}(T)$ and $\delta > 0$ such that $\operatorname{ind}(T-\lambda) < 0$ for all $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$ and $\min \operatorname{ind}(T-\lambda) = 0$ for almost all $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$.

Fix $\mu_0 \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$ such that $\min \operatorname{ind}(T - \mu_0) = 0$. Then $T - \mu_0$ is bounded below. Hence there exists $\varepsilon_0 > 0$ such that

$$\|(T - \mu_0)x\| \geq 2\varepsilon_0$$

for all $x \in \mathcal{H}$ with $\|x\| = 1$.

We are going to show that $T + K \notin (\mathcal{AP})$ for any $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon_0$. Otherwise, there exists $K_0 \in \mathcal{K}(\mathcal{H})$ with $\|K_0\| < \varepsilon_0$ such that $T + K_0 \in (\mathcal{AP})$. We claim $\lambda_0 \notin \operatorname{iso} \sigma_{\mathrm{a}}(T + K_0)$. In fact, if $\lambda_0 \in \operatorname{iso} \sigma_{\mathrm{a}}(T + K_0)$, then since $T + K_0 \in (\mathcal{AP})$, the operator $T + K_0 - \lambda_0$ is Drazin invertible, which means that $\lambda_0 \in \operatorname{iso} \sigma(T + K_0)$, a contradiction. It follows that $\lambda_0 \notin \operatorname{iso} \sigma_{\mathrm{a}}(T + K_0)$ and hence there exists a sequence $\{\lambda_n\}_{n=1}^\infty \subset \sigma_{\mathrm{p}}(T + K_0) \cap (B_\delta(\lambda_0) \setminus \{\lambda_0\})$ such that $\lambda_n \to \lambda_0$.

Noting that $\|(T - \mu_0)x\| \geq 2\varepsilon_0$ for all $x \in \mathcal{H}$ with $\|x\| = 1$, we have

$$\|(T + K_0 - \mu_0)x\| \geq \|(T - \mu_0)x\| - \|K_0 x\| \geq \varepsilon_0$$

for all $x \in \mathcal{H}$ with $\|x\| = 1$.

This means that $\min \operatorname{ind}(T + K_0 - \lambda) = 0$ for all λ in an open neighborhood of μ_0, hence $\min \operatorname{ind}(T + K_0 - \lambda) = 0$ for almost all $\lambda \in B_\delta(\lambda_0)$. Now we can deduce that $\{\lambda_n\}_{n=1}^\infty$ are singular points in a component of $\rho^-_{\text{s-F}}(T + K_0)$ with minimum index zero.

Fix $\lambda_n \in B_\delta(\lambda_0)$. Then $\lambda_n \in \operatorname{iso}\sigma_{\mathrm{a}}(T+K_0)$, and since $T+K_0 \in (\mathcal{AP})$, it follows that $T+K_0-\lambda_n$ is Drazin invertible. Hence $\lambda_n \in \operatorname{iso}\sigma(T+K_0)$, a contradiction.

(2)⇒(1). For $\varepsilon > 0$, by Lemma 3.3, there exists $K_1 \in \mathcal{K}(\mathcal{H})$ with $\|K_1\| < \varepsilon/2$ such that

(a) there exists no singular point in the components of $\rho^-_{\text{s-F}}(T+K_1)$ with minimum index zero,

(b) $\min\operatorname{ind}(T+K_1-\lambda) = \min\operatorname{ind}(T-\lambda)$ for all $\lambda \in \rho^{\mathrm{r}}_{\text{s-F}}(T)$.

By Lemma 3.1, there exists $K_2 \in \mathcal{K}(\mathcal{H})$ with $\|K_2\| < \varepsilon/2$ such that

- $\operatorname{iso}\sigma(T+K_1+K_2) = \sigma_0(T+K_1+K_2)$,
- $\min\operatorname{ind}(T+K_1+K_2-\lambda) = \min\operatorname{ind}(T+K_1-\lambda)$ for all $\lambda \in \rho^-_{\text{s-F}}(T+K_1)$.

Let $K = K_1+K_2$. It suffices to show that $T+K \in (\mathcal{AP})$.

If $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T+K)$, there exists $\delta > 0$ such that $T+K-\lambda$ is bounded below for all $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$. We claim that $\operatorname{ind}(T+K-\lambda) = 0$ for all $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$. In fact, suppose that $\operatorname{ind}(T+K-\lambda) < 0$ for all $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$. By the construction of K_1 and K_2, we have $\min\operatorname{ind}(T-\lambda) = 0$ for almost all $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$. Noting that $E(T) = \emptyset$, we have $\lambda_0 \notin \sigma_{\mathrm{lre}}(T) = \sigma_{\mathrm{lre}}(T+K)$.

Now we have $\lambda_0 \in \rho_{\text{s-F}}(T+K)$ and hence $\operatorname{ind}(T+K-\lambda_0) < 0$. It follows that $0 < \operatorname{nul}(T+K-\lambda_0) < \infty$. By the construction of K_2, we have

$$\min\operatorname{ind}(T+K_1-\lambda_0) = \min\operatorname{ind}(T+K_1+K_2-\lambda_0) > 0,$$

and

$$\min\operatorname{ind}(T+K_1-\lambda) = \min\operatorname{ind}(T+K_1+K_2-\lambda) = 0$$

for all $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$.

This means that λ_0 is a singular point in a component of $\rho^-_{\text{s-F}}(T+K_1)$ with minimum index zero, which contradicts (a).

Hence $\operatorname{ind}(T+K-\lambda) = 0$ for all $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$. This means that $\lambda_0 \in \operatorname{iso}\sigma(T+K) = \sigma_0(T+K)$, hence λ_0 is a pole of $T+K$. ■

Proof of Theorem 1.7. (1)⇒(2). This is obvious.

(2)⇒(3). If $\operatorname{iso}\sigma_{\mathrm{w}}(T) = \rho^-_{\text{s-F}}(T) = \emptyset$, we are going to show that $T+K \in (\mathcal{AP})$ for all $K \in \mathcal{K}(\mathcal{H})$. For fixed $K \in \mathcal{K}(\mathcal{H})$ and $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T+K)$, there exists $\delta > 0$ such that $T+K-\lambda$ is bounded below for $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$. Hence $\operatorname{ind}(T-\lambda) \le 0$ for $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$. Since $\rho^-_{\text{s-F}}(T) = \emptyset$, it follows that $\operatorname{ind}(T-\lambda) = 0$ for $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$. Noting that $\operatorname{iso}\sigma_{\mathrm{w}}(T) = \emptyset$, we have $\lambda_0 \notin \sigma_{\mathrm{lre}}(T) = \sigma_{\mathrm{lre}}(T+K)$. Since $\operatorname{ind}(T+K-\lambda) = \operatorname{ind}(T-\lambda) = 0$ for $\lambda \in B_\delta(\lambda_0)\setminus\{\lambda_0\}$, we conclude that $\lambda_0 \in \operatorname{iso}\sigma(T+K)$. By Lemma 2.5, we have $\lambda_0 \in \sigma_0(T+K)$ and hence λ_0 is a pole of $T+K$.

(3)⇒(1). If $\operatorname{iso}\sigma_{\mathrm{w}}(T) \ne \emptyset$, by Theorem 1.5 there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T+K \notin (\mathcal{P})$, hence $T+K \notin (\mathcal{AP})$.

If $\rho^-_{\text{s-F}}(T) \neq \emptyset$, let Ω be a component of $\rho^-_{\text{s-F}}(T)$. Fix a $\lambda_0 \in \partial\Omega$; obviously, $\lambda_0 \in \sigma_{\text{lre}}(T)$. For $\varepsilon > 0$, by Lemma 2.6, there exists $K_1 \in \mathcal{K}(\mathcal{H})$ with $\|K_1\| < \varepsilon/2$ such that

$$T + K_1 = \begin{bmatrix} \lambda_0 & * \\ 0 & A \end{bmatrix} \begin{matrix} \bigvee\{e\} \\ \mathcal{H}_1 \end{matrix},$$

where $\|e\| = 1$ and $\mathcal{H}_1 = \{e\}^\perp$.

We denote

$$\sigma_1 = \{\lambda \in \sigma_0(A) : \operatorname{dist}[\lambda, \partial\rho_{\text{s-F}}(A)] \geq \varepsilon/4\} \quad \text{and} \quad \sigma_2 = \sigma(A) \setminus \sigma_1.$$

Then σ_1 is a finite clopen subset of $\sigma(A)$.

By Corollary 2.3, A can be written as

$$A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1(\sigma_1; A) \\ \mathcal{H}_1 \ominus \mathcal{H}_1(\sigma_1; A) \end{matrix},$$

where $\sigma(A_1) = \sigma_1$ and $\sigma(A_2) = \sigma_2$. It is easy to verify that

$$\max\{\operatorname{dist}[\lambda, \partial\rho_{\text{s-F}}(A_2)] : \lambda \in \sigma_0(A_2)\} < \varepsilon/4.$$

By Lemma 2.7, there exists a compact operator $\overline{K_2}$ on $\mathcal{H}_1 \ominus \mathcal{H}_1(\sigma_1; A)$ with $\|\overline{K_2}\| < \varepsilon/4$ such that $\sigma_{\text{p}}(A_2 + \overline{K_2}) = \rho^+_{\text{s-F}}(A_2)$. We denote

$$K_2 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & \overline{K_2} \end{bmatrix} \begin{matrix} \bigvee\{e\} \\ \mathcal{H}_1(\sigma_1; A) \\ \mathcal{H}_1 \ominus \mathcal{H}_1(\sigma_1; A) \end{matrix}.$$

Choose a $\lambda_1 \in \Omega$ such that $|\lambda_1 - \lambda_0| < \varepsilon/4$. We define a rank-one operator K_3 as $K_3 = (\lambda_1 - \lambda_0) e \otimes e$. Let $K = K_1 + K_2 + K_3$. Then $K \in \mathcal{K}(\mathcal{H})$, $\|K\| < \varepsilon$ and

$$T + K = \begin{bmatrix} \lambda_1 & * & * \\ 0 & A_1 & * \\ 0 & 0 & A_2 + \overline{K_2} \end{bmatrix} \begin{matrix} \bigvee\{e\} \\ \mathcal{H}_1(\sigma_1; A) \\ \mathcal{H}_1 \ominus \mathcal{H}_1(\sigma_1; A) \end{matrix}.$$

We claim that $T + K \notin (\mathcal{AP})$. Since $\operatorname{ind}(T + K - \lambda_1) < 0$, we have $\operatorname{ind}(A_2 + \overline{K_2} - \lambda_1) < 0$. Hence there exists $\delta > 0$ such that $\operatorname{ind}(A_2 + \overline{K_2} - \lambda) < 0$ for all $\lambda \in B_\delta(\lambda_1)$. Noting that $\sigma_{\text{p}}(A_2 + \overline{K_2}) = \rho^+_{\text{s-F}}(A_2)$, we conclude that $A_2 + \overline{K_2} - \lambda$ is bounded below for all $\lambda \in B_\delta(\lambda_1)$. Since $\sigma(A_1) \cap B_\delta(\lambda_1) = \emptyset$, it follows that $T + K - \lambda$ is bounded below for all $\lambda \in B_\delta(\lambda_1) \setminus \{\lambda_1\}$. Since $\lambda_1 \in \sigma_{\text{p}}(T + K)$, we have $\lambda_1 \in \operatorname{iso} \sigma_{\text{a}}(T + K)$.

On the other hand, since $\lambda_1 \notin \operatorname{iso} \sigma(T + K)$, we conclude that λ_1 is not a pole of the resolvent of $T + K$. Hence $T + K \notin (\mathcal{AP})$. ∎

4. Left (right) polaroid and compact perturbations. As a generalization of Theorem 1.4, we get the following result.

THEOREM 4.1. *Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (\mathcal{LP})$ and $T + K \in (\mathcal{RP})$.*

Proof. If $\operatorname{iso}\sigma_{\mathrm{ea}}(T) = \emptyset$, we shall show that $T \in (\mathcal{LP})$. In fact, if $\lambda_0 \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$, we can deduce that $\lambda_0 \in \rho_{\text{s-F}}(T)$ and hence $0 < \operatorname{nul}(T - \lambda_0) < \infty$. This means that λ_0 is a singular point in $\rho_{\text{s-F}}(T)$. By [12, Theorem 3.38], we have

$$T \sim \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix},$$

where $\mathcal{M}$ is a finite-dimensional Hilbert space, $\sigma(A) = \{\lambda_0\}$ and $\lambda_0 \in \rho^{\mathrm{r}}_{\text{s-F}}(B)$. It is easy to see that $A - \lambda_0$ is nilpotent and $B - \lambda_0$ is bounded below. By Lemma 2.4, we can deduce that λ_0 is a left pole of T and we have $T \in (\mathcal{LP})$. On the other hand, if $\operatorname{iso}\sigma_{\mathrm{ea}}(T^*) = \emptyset$, on can deduce that $T \in (\mathcal{RP})$.

We directly assume that $\operatorname{iso}\sigma_{\mathrm{ea}}(T) = \{\lambda_n\}_{n=1}^{\infty}$ and $\operatorname{iso}\sigma_{\mathrm{ea}}(T^*) = \{\overline{\mu_n}\}_{n=1}^{\infty}$. The proof of the other cases is similar or easier.

It is easy to see that $\{\lambda_n\}_{n=1}^{\infty} \cup \{\mu_n\}_{n=1}^{\infty} \subset \sigma_{\mathrm{lre}}(T)$. For $\varepsilon > 0$, by Lemma 2.6, there exists $K_0 \in \mathcal{K}(\mathcal{H})$ with $\|K_0\| < \varepsilon/4$ such that

$$T + K_0 = \begin{bmatrix} \bigoplus_{n=1}^{\infty} \lambda_n I_n & * \\ 0 & A \end{bmatrix} \begin{matrix} \bigoplus_{n=1}^{\infty} \mathcal{H}_n \\ \mathcal{H}_0 \end{matrix},$$

where

- $\dim \mathcal{H}_n = \infty$ and I_n is the identity operator on $\mathcal{H}_n$;
- $\mathcal{H}_0 = \mathcal{H} \ominus (\bigoplus_{n=1}^{\infty} \mathcal{H}_n)$;
- $\sigma(A) = \sigma(T)$, $\sigma_{\mathrm{lre}}(A) = \sigma_{\mathrm{lre}}(T)$ and $\operatorname{ind}(A - \lambda) = \operatorname{ind}(T - \lambda)$ for all $\lambda \in \rho_{\text{s-F}}(T)$.

One can easily deduce that $\{\overline{\mu_n}\}_{n=1}^{\infty} \subset \sigma_{\mathrm{lre}}(A^*)$. By Lemma 2.6 applied to A^*, there exists a compact operator $\overline{F_0}$ on $\mathcal{H}_0$ with $\|\overline{F_0}\| < \varepsilon/4$ such that

$$A + \overline{F_0} = \begin{bmatrix} B & * \\ 0 & \bigoplus_{n=1}^{\infty} \mu_n I'_n \end{bmatrix} \begin{matrix} \mathcal{H}_0 \ominus (\bigoplus_{n=1}^{\infty} \mathcal{M}_n) \\ \bigoplus_{n=1}^{\infty} \mathcal{M}_n \end{matrix},$$

where

- $\dim \mathcal{M}_n = \infty$ and I'_n is the identity operator on $\mathcal{M}_n$;
- $\sigma(B) = \sigma(A)$, $\sigma_{\mathrm{lre}}(B) = \sigma_{\mathrm{lre}}(A)$ and $\operatorname{ind}(B - \lambda) = \operatorname{ind}(A - \lambda)$ for all $\lambda \in \rho_{\text{s-F}}(A)$.

We choose $0 < \alpha_n < \varepsilon/2^{n+2}$ such that $B_{\alpha_n}(\lambda_n) \setminus \{\lambda_n\} \subset \rho_{\text{s-F}}(T)$ and $\{B_{\alpha_n}(\lambda_n)\}_{n=1}^{\infty}$ are pairwise disjoint. For fixed $n \geq 1$, choose an ONB $\{e_k^{(n)}\}_{k=1}^{\infty}$ of $\mathcal{H}_n$. We define

$$K_n = \alpha_n \sum_{k=1}^{\infty} \frac{1}{k} e_k^{(n)} \otimes e_k^{(n)}.$$

Also, choose $0 < \beta_n < \varepsilon/2^{n+2}$ such that $B_{\beta_n}(\mu_n) \setminus \{\mu_n\} \subset \rho_{\text{s-F}}(T)$ and $\{B_{\beta_n}(\mu_n)\}_{n=1}^{\infty}$ are pairwise disjoint. Select an ONB $\{f_k^{(n)}\}_{k=1}^{\infty}$ of $\mathcal{M}_n$. We define

$$F_n = \beta_n \sum_{k=1}^{\infty} \frac{1}{k} f_k^{(n)} \otimes f_k^{(n)}.$$

We denote

$$F_0 = \begin{bmatrix} 0 & 0 \\ 0 & \overline{F_0} \end{bmatrix} \begin{matrix} \bigoplus_{n=1}^{\infty} \mathcal{H}_n \\ \mathcal{H}_0 \end{matrix}.$$

Let $K = \sum_{i=0}^{\infty} K_i + \sum_{i=0}^{\infty} F_i$. Then $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \varepsilon$. It suffices to show that $T + K \in (\mathcal{LP}) \cap (\mathcal{RP})$. Now, $T + K$ can be written as

$$T + K = \begin{bmatrix} \bigoplus_{n=1}^{\infty}(\lambda_n I_n + K_n) & * & * \\ 0 & B & * \\ 0 & 0 & \bigoplus_{n=1}^{\infty}(\mu_n I'_n + F_n) \end{bmatrix} \begin{matrix} \bigoplus_{k=1}^{\infty} \mathcal{H}_n \\ \mathcal{H}_0 \ominus (\bigoplus_{k=1}^{\infty} \mathcal{M}_n) \\ \bigoplus_{k=1}^{\infty} \mathcal{M}_n \end{matrix}.$$

It is easy to check that each λ_n is a limit of eigenvalues of $T + K$ and each $\overline{\mu_n}$ is a limit of eigenvalues of $T^* + K^*$.

If $\lambda_0 \in \operatorname{iso} \sigma_{\mathrm{a}}(T + K)$, we claim that $\lambda_0 \notin \sigma_{\mathrm{lre}}(T + K)$. Indeed, if $\lambda_0 \in \sigma_{\mathrm{lre}}(T + K)$, it is easy to see that $\lambda_0 \in \operatorname{iso} \sigma_{\mathrm{ea}}(T)$ and hence λ_0 is a limit of eigenvalues of $T + K$, a contradiction. Hence $\lambda_0 \in \rho_{\text{s-F}}(T + K)$ and we have $0 < \operatorname{nul}(T + K - \lambda_0) < \infty$. This means that λ_0 is a singular point in $\rho_{\text{s-F}}(T + K)$. Using [12, Theorem 3.38] again, we see $T + K - \lambda_0$ is left Drazin invertible. If $\mu_0 \in \operatorname{iso} \sigma_{\mathrm{s}}(T + K)$, we consider $T^* + K^*$ and use a similar argument to deduce that $T + K - \mu_0$ is right Drazin invertible. ∎

THEOREM 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

(1) *Given $\varepsilon > 0$, there is $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \notin (\mathcal{LP})$.*
(2) *There exists $K \in \mathcal{K}(\mathcal{H})$ such that $T + K \notin (\mathcal{LP})$.*
(3) $\operatorname{iso} \sigma_{\mathrm{ea}}(T) \neq \emptyset$.

Proof. (1)⇒(2). This is obvious.

(2)⇒(3). Suppose that $\operatorname{iso} \sigma_{\mathrm{ea}}(T) = \emptyset$ and $K \in \mathcal{K}(\mathcal{H})$. Since $\operatorname{iso} \sigma_{\mathrm{ea}}(T+K) = \operatorname{iso} \sigma_{\mathrm{ea}}(T) = \emptyset$, as in the proof of Theorem 4.1, we can deduce that $T + K \in (\mathcal{LP})$.

(3)⇒(1). Suppose that $\operatorname{iso} \sigma_{\mathrm{ea}}(T) \neq \emptyset$ and choose $\lambda_0 \in \operatorname{iso} \sigma_{\mathrm{ea}}(T)$. Then $\lambda_0 \in \sigma_{\mathrm{lre}}(T)$ and there exists $\delta > 0$ such that $\operatorname{ind}(T - \lambda) \leq 0$ for all $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$.

For $\varepsilon > 0$, there exists $K_1 \in \mathcal{K}(\mathcal{H})$ with $\|K_1\| < \varepsilon/2$ such that

$$T + K_1 = \begin{bmatrix} \lambda_0 I & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{H}_1^{\perp} \\ \mathcal{H}_1 \end{matrix},$$

where $\dim \mathcal{H}_1^\perp = \infty$, $\sigma_{\rm lre}(A) = \sigma_{\rm lre}(T)$ and $\mathrm{ind}(A-\lambda) = \mathrm{ind}(T-\lambda)$ for all $\lambda \in \rho_{\text{s-F}}(T)$.

We let

$$\sigma_1 = \{\lambda \in \sigma_0(A) : \mathrm{dist}[\lambda, \partial\rho_{\text{s-F}}(A)] \geq \varepsilon/4\} \quad \text{and} \quad \sigma_2 = \sigma(A) \setminus \sigma_1.$$

Then σ_1 is a finite clopen subset of $\sigma(A)$. By Corollary 2.3, A can be written as

$$A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1(\sigma_1; A) \\ \mathcal{H}_1 \ominus \mathcal{H}_1(\sigma_1; A) \end{matrix},$$

where $\sigma(A_1) = \sigma_1$ and $\sigma(A_2) = \sigma_2$. It is easy to verify that

$$\max\{\mathrm{dist}[\lambda, \partial\rho_{\text{s-F}}(A_2)] : \lambda \in \sigma_0(A_2)\} < \varepsilon/4.$$

By Lemma 2.7, there exists a compact operator $\overline{K_2}$ on $\mathcal{H}_1 \ominus \mathcal{H}_1(\sigma_1; A)$ with $\|\overline{K_2}\| < \varepsilon/4$ such that

$$\sigma_{\rm p}(A_2 + \overline{K_2}) = \rho_{\text{s-F}}^+(A_2).$$

We denote

$$K_2 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & \overline{K_2} \end{bmatrix} \begin{matrix} \mathcal{H}_1^\perp \\ \mathcal{H}_1(\sigma_1; A) \\ \mathcal{H}_1 \ominus \mathcal{H}_1(\sigma_1; A) \end{matrix}.$$

Then $K_2 \in \mathcal{K}(\mathcal{H})$ and $\|K_2\| < \varepsilon/4$.

Choose an ONB $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}_1^\perp$. We let

$$K_3 = \sum_{k=1}^\infty \frac{\varepsilon}{k+3} e_{k+1} \otimes e_k.$$

Then $K_3 \in \mathcal{K}(\mathcal{H})$ and $\|K_3\| = \varepsilon/4$. We let $K = K_1 + K_2 + K_3$. Then $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ and $T + K$ admits the representation

$$T + K = \begin{bmatrix} \lambda_0 I + K_3 & * & * \\ 0 & A_1 & * \\ 0 & 0 & A_2 + \overline{K_2} \end{bmatrix} \begin{matrix} \mathcal{H}_1^\perp \\ \mathcal{H}_1(\sigma_1; A) \\ \mathcal{H}_1 \ominus \mathcal{H}_1(\sigma_1; A) \end{matrix}.$$

Since there exists $\delta > 0$ such that $\mathrm{ind}(A - \lambda) \leq 0$ for $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$, for fixed $\lambda \in B_\delta(\lambda_0) \setminus \{\lambda_0\}$ it is easy to verify that $\mathrm{ind}(A_2 - \lambda) \leq 0$, hence $\mathrm{ind}(A_2 + \overline{K_2} - \lambda) \leq 0$. Noting that $\sigma_{\rm p}(A_2 + \overline{K_2}) = \rho_{\text{s-F}}^+(A_2)$, it follows that $A_2 + \overline{K_2} - \lambda$ is bounded below.

It is easy to see that $\lambda_0 \notin \sigma(A_1)$ and $\sigma(\lambda_0 I + K_3) = \{\lambda_0\}$. Hence $\lambda_0 \in \mathrm{iso}\,\sigma_{\rm a}(T + K)$. On the other hand, since $\lambda_0 \in \sigma_{\rm lre}(A) = \sigma_{\rm lre}(A_2)$, we have $\lambda_0 \notin \sigma_{\rm p}(A_2 + \overline{K_2})$. Noting that $\sigma_{\rm p}(\lambda_0 I + K_3) = \emptyset$, we have $\lambda_0 \notin \sigma_{\rm p}(T + K)$. We claim that $T + K - \lambda_0$ is not left Drazin invertible. Otherwise, by Lemma

2.4, we have

$$T + K - \lambda_0 \sim \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^{\perp} \end{matrix},$$

where T_1 is nilpotent and T_2 is bounded below.

Since $T + K - \lambda_0$ is injective, it follows that T_1 is absent and hence $T + K - \lambda_0$ is bounded below. This means that K_3 is bounded below, a contradiction. Hence the claim follows and $T + K \notin (\mathcal{LP})$. ∎

We have the following result, dual to Theorem 4.2.

Theorem 4.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

(1) *Given $\varepsilon > 0$, there is $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \notin (\mathcal{RP})$.*
(2) *There exists $K \in \mathcal{K}(\mathcal{H})$ such that $T + K \notin (\mathcal{RP})$.*
(3) $\operatorname{iso} \sigma_{\mathrm{ea}}(T^*) \neq \emptyset$.

5. Hereditarily polaroid and compact perturbations. We first state the main results of this part.

Theorem 5.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\rho_{\text{s-F}}^{+}(T) = \emptyset$. Then given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (\mathcal{HP})$.*

Theorem 5.2. *Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \notin (\mathcal{HP})$.*

Recall that $T \in \mathcal{B}(\mathcal{H})$ is a *triangular operator* if it admits an upper triangular matrix representation, i.e.

$$T = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ 0 & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \end{matrix}, \tag{3.1}$$

with respect to a suitable ONB $\{e_n\}_{n=1}^{\infty}$.

Lemma 5.3 ([12, Theorem 6.4]). *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

(1) *Given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is triangular.*
(2) $\rho_{\text{s-F}}^{-}(T) = \emptyset$.

Lemma 5.4 ([12, Theorem 3.40]). *Let $T \in \mathcal{B}(\mathcal{H})$ be a triangular operator with matrix representation (3.1), and let $d(T) = \{a_{nn}\}_{n=1}^{\infty}$ be the diagonal sequence of T. If $\mathcal{M}$ is a non-zero invariant subspace of T^*, then the compression $T_{\mathcal{M}}$ of T to $\mathcal{M}$ is triangular with $d(T_M) \subset d(T)$. Furthermore,* $\operatorname{card}\{n : a_{nn} \in \sigma\} = \dim \mathcal{H}(\sigma; T)$ *for each clopen subset σ of $\sigma(T)$.*

Using a technique in the proof of [12, Theorem 3.40], we can now prove Theorem 5.1.

Proof of Theorem 5.1. If $\rho^{+}_{\text{s-F}}(T) = \emptyset$, for any given $\varepsilon > 0$, by Lemma 5.3, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T^* + K^*$ admits a representation

$$T^* + K^* = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ 0 & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \end{matrix}, \tag{3.2}$$

where $\{e_n\}_{n=1}^{\infty}$ is an ONB of $\mathcal{H}$. In addition, for suitable K, we can assume that $a_{ii} \neq a_{jj}$ for $i \neq j$. We claim that $T + K \in (\mathcal{HP})$.

If $T + K \notin (\mathcal{HP})$, there exists an invariant subspace $\mathcal{H}_1$ of $T + K$ such that $(T + K)|_{\mathcal{H}_1}$ is not polaroid. Hence $T + K$ admits a representation

$$T + K = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^{\perp} \end{matrix},$$

where A is not polaroid.

Since $T^* + K^*$ has form (3.2), where $a_{ii} \neq a_{jj}$ for $i \neq j$, there exists a linearly independent sequence $\{f_n\}_{n=1}^{\infty}$ with

$$f_n \in \ker(T^* + K^* - a_{nn}) \cap \Big(\bigvee_{k=1}^{n} \{e_k\}\Big) \quad \text{for } n \geq 1$$

such that $\bigvee_{n=1}^{\infty} \{f_n\} = \mathcal{H}$.

Noting that

$$T^* + K^* = \begin{bmatrix} B^* & * \\ 0 & A^* \end{bmatrix} \begin{matrix} \mathcal{H}_1^{\perp} \\ \mathcal{H}_1 \end{matrix},$$

we have

$$\mathcal{H}_1 = P_{\mathcal{H}_1}\mathcal{H} = P_{\mathcal{H}_1}\Big(\bigvee_{n=1}^{\infty} \{f_n\}\Big) = \bigvee_{n=1}^{\infty} \{P_{\mathcal{H}_1} f_n\} \subset \bigvee_{n=1}^{\infty} \ker(A^* - a_{nn}) \subset \mathcal{H}_1,$$

where $P_{\mathcal{H}_1}$ is the orthogonal projection with range $\mathcal{H}_1$.

It follows that $\mathcal{H}_1 = \bigvee_{n=1}^{\infty} \{P_{\mathcal{H}_1} f_n\}$, where $P_{\mathcal{H}_1} f_n \in \ker(A^* - a_{nn})$ for each $n \geq 1$. There exists a linearly independent subsequence $\{P_{\mathcal{H}_1} f_{n_k}\}_{k=1}^{\infty}$ such that $\mathcal{H}_1 = \bigvee_{k=1}^{\infty} \{P_{\mathcal{H}_1} f_{n_k}\}$. It is easy to see that A^* has an upper triangular matrix with respect to an ONB of $\mathcal{H}_1$ obtained by Gram–Schmidt orthonormalization of $\{P_{\mathcal{H}_1} f_{n_k}\}_{k=1}^{\infty}$, and $d(A^*) = \{a_{n_k n_k}\}_{k=1}^{\infty}$.

Since $A \notin (\mathcal{P})$, there exists $\lambda_0 \in \operatorname{iso} \sigma(A)$ such that $A - \lambda_0$ is not Drazin invertible. By Corollary 2.3, A admits a representation

$$A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1(\lambda_0; A) \\ \mathcal{H}_1 \ominus \mathcal{H}_1(\lambda_0; A) \end{matrix},$$

where $\sigma(A_1) = \{\lambda_0\}$ and $\sigma(A_2) = \sigma(A) \setminus \{\lambda_0\}$. Since $A - \lambda_0$ is not Drazin invertible, we have $\dim \mathcal{H}_1(\lambda_0; A) = \infty$.

By Lemma 5.4, we conclude that A_1^* is triangular with $d(A_1^*) \subset \{a_{n_k n_k}\}_{k=1}^{\infty}$, and

$$\dim \mathcal{H}_1(\overline{\lambda_0}; A^*) \leq \operatorname{card}\{k : a_{n_k n_k} = \overline{\lambda_0}\}.$$

Noting that $a_{ii} \neq a_{jj}$ for $i \neq j$, it follows that $\dim \mathcal{H}_1(\overline{\lambda_0}; A^*) \leq 1$ and hence $\dim \mathcal{H}_1(\lambda_0; A) \leq 1$, a contradiction. ∎

Now we are going to prove Theorem 5.2.

Proof of Theorem 5.2. For fixed $\varepsilon > 0$, choose a $\lambda_0 \in \sigma_{\text{lre}}(T)$. By Lemma 2.6, there exists $K_1 \in \mathcal{K}(\mathcal{H})$ with $\|K_1\| < \varepsilon/2$ such that

$$T + K_1 = \begin{bmatrix} \lambda_0 I & * \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^\perp \end{matrix},$$

where $\dim \mathcal{M} = \infty$ and I is the identity operator on $\mathcal{M}$. By Lemma 3.2, there exists a compact operator $\overline{K_2}$ on $\mathcal{M}$ with $\|\overline{K_2}\| < \varepsilon/2$ such that $\lambda_0 I + \overline{K_2}$ is not polaroid. We let

$$K_2 = \begin{bmatrix} \overline{K_2} & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^\perp \end{matrix} \quad \text{and} \quad K = K_1 + K_2.$$

Then $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ and $(T + K)|_{\mathcal{M}}$ is not polaroid, hence $T + K \notin (\mathcal{HP})$. ∎

Noting that Theorem 5.1 is established for $\rho_{\text{s-F}}^{+}(T) = \emptyset$, we conclude this paper with the following problem.

PROBLEM 5.5. *Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, can one find $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \in (\mathcal{HP})$?*

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