STUDIA MATHEMATICA 210 (3) (2012)

Almost ball remotal subspaces in Banach spaces

by

Tanmoy Paul (Bangalore)

Abstract. We investigate almost ball remotal and ball remotal subspaces of Banach spaces. Several subspaces of the classical Banach spaces are identified having these properties. Some stability results for these properties are also proved.

1. Introduction. We work with complex scalars. The closed unit ball and the unit sphere of a Banach space X are denoted by B_X and S_X respectively. By a subspace of a Banach space we always mean a closed subspace. For a closed and bounded set $M \subseteq X$, the farthest distance map ϕ_M is defined as $\phi_M(x) = \sup\{\|z - x\| : z \in M\}$, $x \in X$. For $x \in X$, let $F_M(x) = \{z \in M : \|z-x\| = \phi_M(x)\}$, the set of points of M farthest from x. Note that this set may be empty. Let $R(M) = \{x \in X : F_M(x) \neq \emptyset\}$.

We call a closed bounded set M *remotal* [Asp, DeZi, Ede] if $R(M) = X$, *densely remotal* if $\overline{R(M)} = X$, and *almost remotal* if $R(M)$ is a residual set, i.e., contains a dense G_δ set in X.

Definition 1.1. Let us call a subspace Y of X

(a) *ball remotal* (BR) in X if B_Y is remotal.
(b) *densely ball remotal* (DBR) in X if B_Y is densely remotal.
(c) *almost ball remotal* (ABR) in X if B_Y is almost remotal.

The notions (a) and (b) have been studied recently in [BLR, BP, BPR1, BPR2]. In this paper we concentrate on (c).

In [BLR, BP, BPR1, BPR2] the authors identified some other subspaces of Banach spaces which are BR/DBR. In this paper, we investigate whether any of these DBR subspaces are actually ABR.

In Section 2, we obtain necessary and/or sufficient conditions for a subspace to be ABR. In the process, we note that several subsets of certain Banach spaces are residual, many of which are of independent interest.

2010 *Mathematics Subject Classification*: Primary 46B20, 46E15; Secondary 41A99.
Key words and phrases: farthest points, densely ball remotal subspaces, Hahn–Banach smooth spaces, M-ideals, function spaces.

DOI: 10.4064/sm210-3-2

The main result of Section 2 is: If X is Hahn–Banach smooth and any extreme point of B_{X^*} is norm attaining, then X is ABR in X^{**}.

Among other things, we show that:

- c_0 is ABR in ℓ_∞.
- $\mathcal{K}(H)$ is ABR in $\mathcal{L}(H)$, where H is a Hilbert space.
- If X is a reflexive space having the approximation property then $\mathcal{K}(X, c_0)$ is ABR in $\mathcal{L}(X, \ell_\infty)$.

Section 3 is devoted to identifying some closed subspaces of classical Banach spaces which are ABR. A special attention is paid to M-ideals of function spaces. The main result of this section is: If $D \subseteq K$ is a closed set and $A \subseteq C(K)$ is a subspace such that (A, D) is an 'Urysohn pair' (Definition 3.1), then $\{f \in A : f|_D = 0\}$ is ABR in A. A few consequences of this result are also derived.

In Section 4, we discuss various stability results.

[HWW] is a standard reference for any unexplained terminology.

2. Almost ball remotality in Banach spaces. As noted in [BLR], it follows from known results that:

- A reflexive subspace is ABR [Lau, Theorem 2.3]. Hence in reflexive Banach spaces, all subspaces are ABR.
- If X^* is an Asplund space with a LUR dual norm, then any subspace of X^* is ABR [Ziz]. Such a subspace need not be reflexive.
- If X has the Radon–Nikodým Property (RNP), then w^*-closed subspace Y of X^* is ABR [DeZi, Proposition 3].

[Lau, Theorem 2.3] and [BPR1, Corollary 4.14] imply

THEOREM 2.1. *For a Banach space X, the following are equivalent:*

- *X is reflexive.*
- *X is an ABR subspace of any superspace.*
- *X is an ABR subspace of any superspace in which it embeds isometrically as a hyperplane.*

From [BLR, BP, BPR1, BPR2], it is clear that the problem of ball remotality becomes slightly simpler for so-called $(*)$-subspaces.

DEFINITION 2.2 ([BP]). Let Y be a subspace of X. We call Y a *$(*)$-subspace* of X if $\phi_{B_Y}(x) = \|x\| + 1$ for all $x \in X$.

A list of natural examples of $(*)$-subspaces can be found in [BLR]. A detailed discussion of $(*)$-subspaces is in [BP, Section 2] and [P, Chapter 2].

A major step in the proof of [Lau, Theorem 2.3] is the following result (see also [DGZ, Proposition II.2.7]).

Theorem 2.3. *For any bounded set $C \subseteq X$, the set*

$$G(C) := \Big\{x \in X : \sup_{z\in C} \operatorname{Re} x^*(x-z) = \phi_C(x) \text{ for all } x^* \in \partial\phi_C(x)\Big\}$$

is a dense G_δ set in X, where $\partial\phi_C(x)$ is the subdifferential of ϕ_C at x, i.e.,

$$\partial\phi_C(x) = \{x^* \in X^* : \operatorname{Re} x^*(z-x) \le \phi_C(z) - \phi_C(x) \text{ for all } z \in X\}.$$

Proposition 2.4. *Let Y be a subspace of X. Then $R(B_Y) \subseteq \{x \in X : \sup_{z\in B_Y} \operatorname{Re} x^*(x-z) = \phi_{B_Y}(x)$ for some $x^* \in \partial\phi_{B_Y}(x)\}$.*

Proof. Let $x \in R(B_Y)$, and let x^* and z be as in [BP, Theorem 2.5(b)]. Then, for any $y \in X$,

$$\begin{aligned}\operatorname{Re} x^*(y-x) + \phi_{B_Y}(x) &= \operatorname{Re} x^*(y-x) + \operatorname{Re} x^*(x+z)\\ &= \operatorname{Re} x^*(y+z) \le \|y+z\| \le \phi_{B_Y}(y).\end{aligned}$$

Therefore, $\operatorname{Re} x^*(y-x) \le \phi_{B_Y}(y) - \phi_{B_Y}(x)$. Hence, $x^* \in \partial\phi_{B_Y}(x)$. ∎

Remark 2.5. Clearly, the right hand set in the proposition contains $G(B_Y)$, and hence is residual.

We now find conditions on Y that make it ABR.

Definition 2.6. Let X be a Banach space. For $x \in X$, we set

$$D(x) = \{x^* \in S_{X^*} : x^*(x) = \|x\|\}.$$

For a subspace Y of X, we define

$$\begin{aligned}A_Y &:= \{x^* \in S_{X^*} : \|x^*|_Y\| = 1\},\\ N_Y &:= \{x^* \in S_{X^*} : x^*(y) = 1 \text{ for some } y \in S_Y\}.\end{aligned}$$

As noted in the proof of [BP, Theorem 2.3], for a $(*)$-subspace Y of X, $G(B_Y) = \{x \in X : D(x) \subseteq A_Y\}$. In fact, we have

Theorem 2.7 ([BP, Theorem 2.3]). *A subspace Y of X is a $(*)$-subspace if and only if the set $\{x \in X : D(x) \subseteq A_Y\}$ is a dense G_δ set in X.*

Now from [BLR, Proposition 2.10], it follows that if Y is a $(*)$-subspace of X, then

$$R(B_Y) = \{x \in X : D(x) \cap N_Y \ne \emptyset\} \subseteq \{x \in X : D(x) \cap A_Y \ne \emptyset\}. \tag{2.1}$$

Hence we have the following necessary and sufficient condition for a subspace to be $(*)$ and ABR.

Definition 2.8. For a Banach space X, let $\mathrm{NA}(X) = \{x^* \in X^* :$ there exists $x \in S_X$ such that $|x^*(x)| = \|x^*\|\}$ and $\mathrm{NA}_1(X) = \mathrm{NA}(X) \cap S_{X^*}$.

Theorem 2.9. *A subspace Y of X is $(*)$ and ABR if and only if $\{x \in X : D(x) \cap N_Y \ne \emptyset\}$ is a residual set.*

If Y is a $(*)$-subspace and $A_Y \cap \mathrm{NA}(X) = N_Y$, then Y is ABR.

Proof. To prove the second statement, it is enough to observe that $R(B_Y) = \{x \in X : D(x) \cap N_Y \neq \emptyset\} = \{x \in X : D(x) \cap A_Y \neq \emptyset\}$, which follows from the hypothesis. The last set contains $G(B_Y)$. ∎

When we consider X as a subspace of X^{**}, then

$$A_X = \{x^{***} \in S_{X^{***}} : \|x^{***}|_X\| = 1\} \supseteq S_{X^*}.$$

Therefore, X is a $(*)$-subspace of X^{**}.

DEFINITION 2.10. We call a Banach space X *Hahn–Banach smooth* (HBS for short) if every $x^* \in X^*$ has a unique norm-preserving extension to all of X^{**}; and *weakly Hahn–Banach smooth* (wHBS for short) if every $x^* \in \mathrm{NA}(X)$ has a unique norm-preserving extension to all of X^{**}.

The next theorem is well known but we recapture it in a completely different set-up, which leads to a few interesting observations.

THEOREM 2.11. *If X is HBS, then the set $\mathrm{NA}(X^*)$ is residual in X^{**}.*

Proof. Since X is HBS, we have $A_X = S_{X^*}$. Moreover, $G(B_X) = \{x^{**} \in X^{**} : D(x^{**}) \subseteq A_X\}$ is a dense G_δ in X^{**}. Now, if $x^{**} \in G(B_X)$, then $D(x^{**}) \subseteq S_{X^*}$, and hence $x^{**} \in \mathrm{NA}(X^*)$. It follows that $G(B_X) \subseteq \mathrm{NA}(X^*)$. ∎

REMARK 2.12. It is not necessary the case that $G(B_X) = \mathrm{NA}(X^*)$. For example, c_0 is HBS. Consider $\mathbf{1} \in \ell_\infty$. Clearly for any Banach limit $\Lambda \in \ell_\infty^*$, we have $\Lambda \in D(\mathbf{1})$, but $\Lambda \notin X^*$.

COROLLARY 2.13. *If X is HBS, then $\{x^{**} \in X^{**} : \|x^{**}\| = d(x^{**}, X)\}$ is of first category.*

The following observations come essentially from [BLR]. Let

$$\mathrm{NA}_2(X) = \{x^{**} \in X^{**} : x^{**}(x^*) = \|x^{**}\| \text{ for some } x^* \in \mathrm{NA}_1(X)\}.$$

Clearly, $X \subseteq \mathrm{NA}_2(X) \subseteq \mathrm{NA}(X^*)$.

PROPOSITION 2.14. *If X is wHBS, then $R(B_X) = \mathrm{NA}_2(X)$.*

Proof. If $x_0^{**} \in \mathrm{NA}_2(X)$, then there exist $x^* \in S_{X^*}$ and $x \in S_X$ such that $x_0^{**}(x^*) = \|x_0^{**}\|$ and $x^*(x) = 1$. Hence $-x \in B_X$ is farthest from x_0^{**}.

Conversely, suppose $x_0^{**} \in S_{X^{**}}$ has a farthest point $-x \in B_X$ so that $\|x_0^{**} + x\| = 2$. Let $x_1^{**} = (x_0^{**} + x)/2$ and $x^{***} \in D(x_1^{**})$. Let $x^* = x^{***}|_X$. Then $x^{***} \in D(x_0^{**})$ and $x^* \in D(x)$. Since X is wHBS, we have $x^{***} = x^*$, and hence $x_0^{**} \in \mathrm{NA}_2(X)$. ∎

REMARK 2.15. Since every wHBS space is Asplund, it is now an obvious consequence of Proposition 2.14 that if X is wHBS then X is ABR in X^{**} $\Leftrightarrow$ $\mathrm{NA}_2(X)$ is a residual $\Leftrightarrow$ $\mathrm{NA}_2(X) \cap \mathcal{Z}$ is a residual, where $\mathcal{Z} = \{x^{**} \in X^{**} :$ the norm is Fréchet differentiable at $x^{**}\}$. Also X^{**} being a w^*-Asplund space, if $x^{**} \in \mathrm{NA}_2(X) \cap \mathcal{Z}$ then $D(x^{**})$ is w^*-continuous and hence $x^{**} \in G(B_X)$. As X^* has RNP, it is now evident that $D(x^{**}) \in \mathrm{ext}(B_{X^*})$ and in addition $D(x^{**}) \in \mathrm{NA}(X)$.

COROLLARY 2.16. *If X is HBS as well as ball remotal in its bidual, then X is reflexive.*

THEOREM 2.17. *If X is HBS and any extreme point of B_{X^*} is in* $\mathrm{NA}(X)$, *then X is ABR in X^{**}.*

Proof. Combining Theorem 2.11 and Proposition 2.14, it suffices to show that $\mathrm{NA}_2(X) = \mathrm{NA}(X^*)$.

Since X is HBS, X^* has the RNP. If $x^{**} \in \mathrm{NA}(X^*)$, then $Z = \{x^* \in B_{X^*} : |x^{**}(x^*)| = \|x^{**}\|\}$ is a nonempty closed bounded convex set, which, by RNP, has a denting point, say z_0^*. But Z is a face of B_{X^*}, and hence z_0^* is an extreme point of B_{X^*}. By hypothesis, $z_0^* \in \mathrm{NA}(X)$. It follows that $x^{**} \in \mathrm{NA}_2(X)$. ∎

DEFINITION 2.18. A subspace Y of a Banach space X is an *M-ideal* in X if there is a projection P on X^* with $\ker(P) = Y^\perp$ and for all $x^* \in X^*$, $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$. A Banach space X is said to be *M-embedded* if X is an M-ideal in X^{**}; [HWW] is a well known reference for these concepts. It is well known that M-embedded spaces are HBS.

A Banach space X is an *L_1-predual* if X^* is isometrically isomorphic to $L_1(\mu)$ for some measure μ. It is known that if X is an *L_1-predual*, then any extreme point of B_{X^*} is in $\mathrm{NA}(X)$.

COROLLARY 2.19.

- *If X is M-embedded and any extreme point of B_{X^*} is in* $\mathrm{NA}(X)$, *then X is DBR in X^{**}.*
- *If X is HBS and an L_1-predual, then X is ABR in X^{**}.*

NOTATION. Let X, Y be Banach spaces.

- $X \check{\otimes} Y$ represents the injective tensor product of X and Y.
- $\mathcal{F}(X,Y)$ (resp. $\mathcal{K}(X,Y)$, $\mathcal{L}(X,Y)$) represents the space of all finite rank (resp. compact, bounded linear) operators from X to Y.

It is well known that $\mathcal{K}(H)^{**} = \mathcal{L}(H)$ and $\mathcal{K}(H)$ is an M-embedded space [HWW, Example III.1.4].

EXAMPLE 2.20.

(a) c_0 is ABR in ℓ_∞.
(b) $\mathcal{K}(H)$ is ABR in $\mathcal{L}(H)$.
(c) Let X be a reflexive Banach space with the approximation property (AP). Then $\mathcal{K}(X, c_0)$ is an M-embedded space which is ABR in its bidual $\mathcal{L}(X, \ell_\infty)$.
(d) Let X be a reflexive space with AP. Then $K(X, \ell_p)$, for $1 < p < \infty$, is HBS and also ABR in its bidual $L(X, \ell_p)$.

Proof. (a) c_0 is M-embedded as well as an L_1-predual.

(b) $X = \mathcal{K}(H)$ is M-embedded, and $X^* = B_1(H)$, the space of trace class operators. By [No], any extreme point of $B_{B_1(H)}$ is a rank one operator, and hence is in $\mathrm{NA}(X)$. This completes the proof.

(c) That $\mathcal{K}(X, c_0)$ is an M-ideal in $\mathcal{L}(X, \ell_\infty)$ follows from [Fin]. From the hypothesis, X^* has AP, hence $\mathcal{K}(X,Y) = \overline{\mathcal{F}(X,Y)}$ for any Banach space Y [LiCh, p. 17], and hence $\mathcal{K}(X,Y)^* = (X^* \check{\otimes} Y)^*$ [LiCh, p. 18]. Now due to a result Grothendieck [DU, p. 231], $\Phi \in (X^* \check{\otimes} Y)^*$ if and only if there exists a regular Borel measure μ on $B_X \times B_{Y^*}$ such that for all $x^* \in X^*$ and $y \in Y$,

$$\Phi(x^* \otimes y) = \int\limits_{B_X \times B_{Y^*}} \hat{x}(x^*) y^*(y)\, d\mu(x, y^*).$$

And $\|\Phi\| = |\mu|(B_X \times B_{Y^*})$.

Hence an extreme point of $(X^* \check{\otimes} Y)^*$ is of the form $\alpha\delta_x \otimes \beta\delta_{y^*}$ where $y^* \in \mathrm{ext}(B_{Y^*})$. Since in our case $Y = c_0$ and $\mathrm{ext}(B_{\ell_1}) \subseteq \mathrm{NA}(c_0)$, it follows that $\mathrm{ext}(B_{\mathcal{K}(X,c_0)^*}) \subseteq \mathrm{NA}(\mathcal{K}(X, c_0))$. It remains to prove that $\mathcal{K}(X, c_0)^{**} = \mathcal{L}(X, \ell_\infty)$. This follows from [Fab, Theorems 16.41, 16.42], since X has both RNP and AP.

(d) HBS-ness of $\mathcal{K}(X, \ell_p)$ follows from [HWW, p. 44]. Arguing similarly to (c), it can be proved that $\mathrm{ext}(B_{\mathcal{K}(X,\ell_p)^*}) \subseteq \mathrm{NA}(\mathcal{K}(X, \ell_p))$ and also $\mathcal{K}(X, \ell_p)^{**} = \mathcal{L}(X, \ell_p)$. ∎

Remark 2.21. Note that the spaces in (d) above are examples of the so-called HB-spaces [HWW, p. 44], which are HBS but not necessarily M-ideals.

For $X = \mathcal{K}(H)$, the set $G(B_X)$ is a dense G_δ subset of $\mathcal{L}(H)$, closed under scalar multiplication, contained in the set of norm attaining operators in $\mathcal{L}(H)$ and not containing unitary operators. The last observation follows from the fact that if $U \in \mathcal{L}(H)$ is a unitary, then $\mathrm{span}\{D(U)\} = \mathcal{L}(H)^*$ (see [AW]), hence $D(U) \not\subseteq S_{\mathcal{K}(H)^*}$.

3. Almost ball remotality in some classical Banach spaces. Let us begin with the space $C(K)$ of all complex-valued continuous functions on a compact Hausdorff space K, and its subspaces.

Let us recall the definition of an 'Urysohn pair' [BPR2].

Definition 3.1. Let A be a subspace of $C(K)$ and $D \subseteq K$ a closed set. We say that (A, D) is an *Urysohn pair* if the following property holds:

For any $t_0 \in K \setminus D$, there exists $f \in A$ such that $\|f\|_\infty = 1$, $f|_D \equiv 0$ and $f(t_0) = 1$.

Recall the following fact:

Proposition 3.2 ([BPR2, Proposition 2.2]). *Let (A, D) be an Urysohn pair and $Y = \{f \in A : f|_D = 0\}$. Then*

- *for any* $f \in A$, $\phi_{B_Y}(f) = \max\{\|f|_D\|_\infty, \|f|_{K\setminus D}\|_\infty + 1\}$,
- $f \in R(B_Y)$ *if and only if either* $\phi_{B_Y}(f) = \|f|_D\|_\infty$ *or there exists* $t \in K \setminus D$ *such that* $|f(t)| = \|f|_{K\setminus D}\|_\infty$.

We now prove

THEOREM 3.3. *If* (A, D) *is an Urysohn pair, then* $Y = \{f \in A : f|_D = 0\}$ *is an ABR subspace of* A.

Proof. By Theorem 2.3, it is enough to prove that $G(B_Y) \subseteq R(B_Y)$.

Let $f \in G(B_Y)$. By Proposition 3.2, it suffices to assume $\phi_{B_Y}(f) = \|f|_{K\setminus D}\|_\infty + 1 > \|f|_D\|_\infty$. Let $t \in \overline{K \setminus D}$ and $\alpha \in \mathbb{T}$ be such that $\alpha f(t) = |f(t)| = \|f|_{K\setminus D}\|_\infty$. By Proposition 3.2, it is enough to show that $t \in K \setminus D$.

CLAIM. $\alpha\delta_t \in \partial\phi_{B_Y}(f)$.

For $g \in A$,

$$\begin{aligned}\operatorname{Re}\alpha[g(t) - f(t)] &\le |g(t)| - |f(t)| \le \|g|_{K\setminus D}\|_\infty - \|f|_{K\setminus D}\|_\infty \\ &= \|g|_{K\setminus D}\|_\infty + 1 - (\|f|_{K\setminus D}\|_\infty + 1) \\ &\le \phi_{B_Y}(g) - \phi_{B_Y}(f).\end{aligned}$$

This proves the claim. By definition of $G(B_Y)$, it now follows that

$$\sup_{h\in B_Y} \operatorname{Re}\alpha[f(t) - h(t)] = \phi_{B_Y}(f).$$

If $t \in D$, then

$$\|f|_D\|_\infty \ge |f(t)| = \sup_{h\in B_Y} \operatorname{Re}\alpha[f(t) - h(t)] = \phi_{B_Y}(f) > \|f|_D\|_\infty,$$

a contradiction. Therefore, $t \in K \setminus D$. ∎

Combining Theorem 3.3 with the results of [BPR2], we get

COROLLARY 3.4. *In each of the following cases, every* M*-ideal in* X *is an ABR subspace.*

- $X = C(K)$.
- $X = C_0(L)$, *the space of all* $\mathbb{C}$*-valued continuous functions on a locally compact Hausdorff space* L *"vanishing at infinity".*
- $X = A$, *the disc algebra, i.e., the space of continuous functions on the closed unit disc* $\mathbb{D}$ *that are analytic on the open unit disc.*

Combining Theorem 3.3 with [BPR2, Theorem 2.11], we get

THEOREM 3.5. *Let* K *be a compact Hausdorff space and* $A \subseteq C(K)$ *a subspace such that every* $\mu \in A^\perp$ *is nonatomic. If* $D \subseteq K$ *is a closed set such that* $|\mu|(D) = 0$ *for all* $\mu \in A^\perp$, *then*

$$Y = \{a \in A : a|_D \equiv 0\}$$

is an M*-ideal as well as an ABR subspace of* A.

Let now Y be an arbitrary subspace of $C(K)$. We define

$$K_0 := \{t \in K : |g(t)| = 1 \text{ for some } g \in S_Y\},$$
$$K' := \Big\{t \in K : \sup_{g\in S_Y} |g(t)| = 1\Big\}.$$

THEOREM 3.6. *Let K be a compact metric space, Y a subspace of $C(K)$ and K_0, K' as defined above.*

(a) *If K_0 is residual, then Y is $(*)$ and ABR.*
(b) *If Y is a $(*)$ and DBR subspace of $C(K)$ and $K' \setminus K_0$ is at most countable, then Y is ABR.*
(c) *If Y is a $(*)$-subspace of $C(K)$ and $K_0 = K'$, then Y is ABR.*

Proof. (a) If K_0 is residual, then by [BPR1, Theorem 2.5], Y is a $(*)$-subspace of $C(K)$, and there are open dense sets U_n such that $\bigcap_n U_n \subseteq K_0$.

Since Y is a $(*)$-subspace, from [BPR1, Proposition 2.8] we have $R(B_Y) = \{f \in C(K) : f(t) = \|f\|_\infty \text{ for some } t \in K_0\}$.

For each n, let $Z_n = \{f \in C(K) : f|_{U_n^c} = 0\}$. Then Z_n is a $(*)$-subspace [BPR1, Theorem 2.5], an M-ideal [HWW, Example 1.4(a)], and hence an ABR subspace of $C(K)$ (Corollary 3.4) and we have

$$R(B_{Z_n}) = \{f \in C(K) : f(t) = \|f\|_\infty \text{ for some } t \in U_n\}.$$

Hence $T = \bigcap_n R(B_{Z_n})$ is a residual set.

Since K is metrizable, $C(K)$ is separable. Hence $G := \{f \in C(K) : \|\cdot\|_\infty$ is Gâteaux differentiable at $f\}$ is a dense G_δ subset of $C(K)$ [Ph, Theorem 1.20]. And if $f \in G$, then $\{t \in K : |f(t)| = \|f\|_\infty\}$ is a singleton [DGZ, Example 1.6(b)]. Let $W = G \cap T$. Clearly W is residual in $C(K)$.

CLAIM. $W \subseteq R(B_Y)$.

Let $f \in W$. There exists a unique $s \in K$ such that $|f(s)| = \|f\|_\infty$. It follows that $s \in U_n$ for all n, and hence $s \in K_0$. This proves that $f \in R(B_Y)$.

(b) Since Y is a $(*)$ and DBR subspace, from [BPR1, Theorems 2.5 and 2.13] it follows that K' is a residual set, and K_0 is dense in K. Let $K' \setminus K_0 = \{t_n\}$. Since K_0 is dense, the points t_n are not isolated. Therefore, $K_0 = K' \setminus \{t_n\}$ is also residual. The result now follows from (a). ■

EXAMPLE 3.7. If K is a countable compact space, then a $(*)$-subspace of $C(K)$ is DBR if and only if it is ABR.

THEOREM 3.8. *Let $\{\mu_n\}$ be a countable family of regular Borel measures on K. Let $S(\mu_n)$ denote the support of μ_n. Suppose that*

- *for each $n \geq 1$, $K \setminus S(\mu_n)$ is dense in K, and*
- *$\bigcup_n S(\mu_n)$ is a closed subset of K.*

Then $Y = \bigcap_n \ker \mu_n$ is an ABR subspace of $C(K)$.

Proof. Argue as in [BPR1, Theorem 5.7]. ■

THEOREM 3.9.

(a) *For a* $(*)$*-subspace of* c_0, *DBR* $\Leftrightarrow$ *ABR* $\Leftrightarrow$ *BR.*
(b) *For a* $(*)$*-subspace of* ℓ_∞, *DBR* $\Leftrightarrow$ *ABR.*

Proof. (a) follows from [BP, Corollary 3.3].

(b) It follows from the proof of [BP, Theorem 3.2] that if Y is $(*)$ and DBR in ℓ_∞, then $R := \{x \in \ell_\infty : \|x\|_\infty = |x_k| \text{ for some } k \in \mathbb{N}\} \subseteq R(B_Y)$.

It is easy to see that $R = R(B_{c_0})$ [BLR, Corollary 2.15] and we have already shown that c_0 is ABR in ℓ_∞ (Example 2.20(a)). ■

THEOREM 3.10. *Let a Banach space* X *be an* ℓ_1*-predual, that is,* X^* *is isometrically isomorphic to* ℓ_1. *Then* X *is ABR in* X^{**}.

Proof. Since $X^* = \ell_1$, X can be assumed to be a subspace of ℓ_∞. Thus, by Theorem 3.9 and [BP, Theorem 3.2], it is enough to prove that for each $n \in \mathbb{N}$, the coordinate functional e_n is in $\mathrm{NA}(X)$. But since $e_n \in \mathrm{ext}(B_{\ell_1})$ and X is an ℓ_1-predual, this is indeed the case. ■

It can be proved that there are uncountably many nonisometric Banach spaces X such that X^* is isometric to ℓ_1. Note that one such space, namely c, is not HBS.

PROPOSITION 3.11.

(a) *Any* w^**-closed subspace of* ℓ_∞ *is ABR.*
(b) *If* $c_0 \subseteq Y \subseteq \ell_\infty$, *then* Y *is* $(*)$ *and ABR in* ℓ_∞.
(c) *If* $\Lambda \in \mathrm{ext}(B_{\ell_\infty^*})$, *then* $Y = \ker \Lambda$ *is ABR in* ℓ_∞.
(d) *If* $\Lambda \in \ell_\infty^*$ *with* $|\Lambda(e_n)| < \frac{1}{2}\|\Lambda|_{c_0}\|$ *for all* n, *where* $\{e_n\} \subseteq c_0$ *are the canonical basis vectors, then* $Y = \ker \Lambda$ *is* $(*)$ *and ABR in* ℓ_∞.

Proof. To prove (a)–(c) argue as in [BP, Theorem 3.13].

(d) Simply use the arguments of [P, Theorem 3.2.20] and the results in Theorem 3.9. ■

4. Stability results

NOTATION. Let (X_n) be a family of Banach spaces. For $1 \le p \le \infty$, denote $\widetilde{X}_p = \bigoplus_p X_n$, $\widetilde{X}_0 = \bigoplus_{c_0} X_n$.

For any Banach space X, we simply write $c_0(X)$ or $\ell_\infty(X)$ in place of $\bigoplus_{c_0} X$ and $\bigoplus_{\ell_\infty} X$ respectively. We also denote by $c(X)$ the space of all convergent sequences in X.

Let K be a compact Hausdorff space and X a Banach space. Then $C(K, X)$ denotes the space of all X-valued continuous functions on K.

Let (Ω, Σ, μ) be a complete probability space. For $1 \leq p < \infty$, let $L_p(\mu, X)$ denote the space of all Bochner integrable functions f with $\int_\Omega \|f(t)\|^p \, d\mu(t) < \infty$ and define $\|f\|_p = (\int_\Omega \|f(t)\|^p \, d\mu(t))^{1/p}$.

If $p = \infty$, we say $f \in L_\infty(\mu, X)$ if $\inf\{a \geq 0 : \mu\{t \in \Omega : \|f(t)\| > a\} = 0\}$ is finite and in that case we define this infimum to be the norm of f.

We first prove the stability of the $(*)$-property under various sums of Banach spaces.

THEOREM 4.1. *Let $\{X_n : n \in \mathbb{N}\}$ be a family of Banach spaces and $Y_n \subseteq X_n$ be subspaces. Then the following are equivalent:*

(a) *Y_n is a $(*)$-subspace of X_n for all n.*
(b) *$\widetilde{Y}_0$ is a $(*)$-subspace of $\widetilde{X}_0$.*
(c) *$\widetilde{Y}_0$ is a $(*)$-subspace of $\widetilde{X}_\infty$.*
(d) *$\widetilde{Y}_\infty$ is a $(*)$-subspace of $\widetilde{X}_\infty$.*

Also (a) *implies $\widetilde{Y}_p$ is $(*)$ in $\widetilde{X}_p$, for $1 \leq p < \infty$.*

Proof. Equivalence of (a)–(d) is [P, Theorem 6.3.4].

That (a) implies $\widetilde{Y}_1$ is $(*)$ in $\widetilde{X}_1$, follows from [P, Theorem 6.3.1]. That (a) implies $\widetilde{Y}_p$ is $(*)$ in $\widetilde{X}_p$, for $1 < p < \infty$, follows from [P, Theorem 6.3.15]. ∎

THEOREM 4.2. *Let $Y \subseteq X$ be a subspace. The following are equivalent.*

(a) *Y is a $(*)$-subspace of X.*
(b) *$c(Y)$ is a $(*)$-subspace of $c(X)$.*
(c) *$L_p(\mu, Y)$ is a $(*)$-subspace of $L_p(\mu, X)$ for $p = 1$ and ∞.*
(d) *For every compact Hausdorff space K, $C(K, Y)$ is a $(*)$-subspace of $C(K, X)$.*

Proof. (a)$\Leftrightarrow$(b). If Y is a $(*)$-subspace in X, then by [P, Theorem 6.3.11], $c(Y)$ is a $(*)$-subspace in $c(X)$.

To prove the converse, let $x_0 \in X$ and define $(x_n) \in c(X)$ by $x_n = x_0$ for all n. Then given $\varepsilon > 0$ there exists $(y_n) \in B_{c(Y)}$ such that $\|(x_n) + (y_n)\|_\infty > \|(x_n)\|_\infty + 1 - \varepsilon = \|x_0\| + 1 - \varepsilon$. Pick an m such that $\|x_m + y_m\| > \|x_0\| + 1 - \varepsilon$. Then $\|x_0 + y_m\| > \|x_0\| + 1 - \varepsilon$ and hence the result follows.

(a)$\Rightarrow$(c) follows from Theorem 4.12 below together with the fact that $L_p(\mu, B_Y) \subseteq B_{L_p(\mu,Y)}$.

(c)$\Rightarrow$(a). Let $x \in X$ and define $f = \chi_\Omega x \in L_p(\mu, X)$. Then $\phi_{B_{L_p(\mu,Y)}}(f) = \|f\|_p + 1 = \|x\| + 1$.

Now from Theorem 4.12, for $p = 1$, $\|x\| + 1 = \int_\Omega \phi_{B_Y}(f(t)) \, d\mu(t) = \int_\Omega \phi_{B_Y}(x) \, d\mu(t) = \phi_{B_Y}(x)$. And for $p = \infty$,

$$\begin{aligned}\|x\| + 1 = \phi_{B_{L_\infty(\mu,Y)}}(f) &= \inf\{a \geq 0 : \mu\{t \in \Omega : \phi_{B_Y}(f(t)) > a\} = 0\} \\ &= \inf\{a \geq 0 : \mu\{t \in \Omega : \phi_{B_Y}(x) > a\} = 0\} = \phi_{B_Y}(x).\end{aligned}$$

(a)$\Leftrightarrow$(d) is [P, Corollary 6.4.2.]. ∎

Turning to natural summands of Banach spaces, using an argument somewhat similar to the one in [BLR, Lemma 3.1] we will prove

LEMMA 4.3. *For Banach spaces (X_i) suppose $X = \bigoplus_{i=1}^n X_i$ and there exists a monotone map $\varrho : (\mathbb{R}^+)^n \to \mathbb{R}^+$ such that if $x = (x_i)_{i=1}^n$ then $\|x\| = \varrho((\|x_i\|))$.*

Let $E_i \subseteq X_i$ be remotal (resp. densely remotal, almost remotal) in X_i. Then $\bigoplus E_i$ is remotal (resp. densely remotal, almost remotal) in X.

Proof. We only prove the statement for almost remotality. It is clear that if $E = \bigoplus_i E_i$, then $\bigoplus_i R(E_i) \subseteq R(E)$. Let $U_m^i \subseteq X_i$ be dense open sets with $\bigcap_{m=1}^\infty U_m^i \subseteq R(E_i)$. Then from the given condition, it follows that $p_i^{-1}(U_m^i)$ is dense open in X, where $p_i : X \to X_i$ is the canonical projection. Since $p_i^{-1}(U_m^i) = U_m^i \oplus \bigoplus_{j\neq i} X_j$, the set $\bigcap_{i=1}^n p_i^{-1}(U_m^i) = \bigoplus_{i=1}^n U_m^i$ is dense open in X. Now we have

$$\bigcap_{m\in\mathbb{N}} \Big(\bigcap_{i=1}^n p_i^{-1}(U_m^i)\Big) \subseteq \bigoplus_{i=1}^n R(E_i) \subseteq R(E).$$

This completes the proof. ∎

REMARK 4.4. Since any M-embedded space is an M-ideal in each of its even order duals, it is evident that if an M-embedded space X is BR (resp. DBR, ABR) in X^{**}, then X is BR (resp. DBR, ABR) in $X^{(n)}$ for any even n.

We call a subspace Y of X *factor reflexive* if X/Y is reflexive. Using identical arguments to those in [BLR, Theorem 3.6], with the help of Lemma 4.3 we have

THEOREM 4.5. *Let $\{X_i : i \in I\}$ be a family of reflexive Banach spaces. Let $X = \bigoplus_{c_0} X_i$. For any factor reflexive proximinal subspace Y of X, Y is ABR in X.*

Our next results generalizes the fact that c_0 is ABR in ℓ_∞.

THEOREM 4.6. *Let X be a reflexive Banach space and Y be a $(*)$-subspace of X. Then $c_0(Y)$ is $(*)$ and ABR in $\ell_\infty(X)$. In particular $c_0(X)$ is $(*)$ and ABR in $\ell_\infty(X)$.*

Proof. From Theorem 4.1 it follows that $c_0(Y)$ is a $(*)$-subspace of $\ell_\infty(X)$.

From [P, Lemma 6.3.5] it follows that

$$R(B_{c_0(Y)}) = \{(x_n) \in \ell_\infty(X) : \exists m \in \mathbb{N},\ \|(x_n)\|_\infty = \|x_m\| \text{ and } x_m \in R(B_Y)\}.$$

Let $A = \{(x_n) \in \ell_\infty(X) : (x_n)$ attains its norm at some $j \in \mathbb{N}\}$ and $B = \{(x_n) \in \ell_\infty(X) : x_n \in R(B_Y)$ for all $n\}$. Then $A \cap B \subseteq R(B_{c_0(Y)})$.

Being the dual of $\ell_1(X^*)$, a space having RNP, $\ell_\infty(X)$ is w^*-Asplund, and hence A contains a dense G_δ [Bo, Theorem 5.7.4].

Since Y is ABR, $\bigcap_n U_n \subseteq R(B_Y)$, where U_n's are dense open sets in X. Now $\bigcap_{n,j}\{(x_i) \in \ell_\infty(X) : x_j \in U_n\} \subseteq B$. Clearly, the left hand set is a dense G_δ. This shows both A and B are residual, and hence so is $A \cap B$. ∎

Remark 4.7. Essentially the same technique can be used to prove Theorem 4.6 for countable sum of reflexive spaces.

We do not know whether $c_0(Y)$ is ABR in $c_0(X)$ implies Y is ABR in X.

Let X be reflexive and Y a $(*)$-subspace of X. Since $c_0(Y) \subseteq c(Y) \subseteq \ell_\infty(Y) \subseteq \ell_\infty(X)$, all the intermediate spaces are $(*)$ and ABR in $\ell_\infty(X)$. Also from Theorem 4.6 it follows that $c_0(Y)$ is $(*)$ and ABR in $\ell_\infty(Y)$. Clearly $c(X)$ is $(*)$ and BR in $\ell_\infty(X)$ for any Banach space X.

Our next result shows that being a HBS space that is ABR in its bidual is stable under c_0-sums.

Theorem 4.8. *Let $\{X_i : i \in \mathbb{N}\}$ be a collection of HBS spaces which are also ABR in their biduals. Then $\widetilde{X}_0$ is also HBS and ABR in its bidual $\widetilde{X}_\infty^{**}$.*

Proof. From [BR, Corollary 2.8], it follows that $\widetilde{X}_0$ is an HBS space, hence $R(B_{\widetilde{X}_0}) = \mathrm{NA}_2(\widetilde{X}_0)$. For $i, m \in \mathbb{N}$, let $U_m^i \subseteq X_i^{**}$ be dense open sets such that $\bigcap_m U_m^i \subseteq R(B_{X_i})$.

Let $W_m^i = \{(x_n^{**}) \in \widetilde{X}_0^{**} : x_i \in U_m^i\}$. Then each W_m^i is a dense open set in $\widetilde{X}_0^{**}$. Let $W = \bigcap_{m,i} W_m^i$. Now from the proof of Theorem 2.11, the set $\mathcal{G} = \{x^{**} \in \widetilde{X}_0^{**} : D(x^{**}) \subseteq A_{\widetilde{X}_0}\}$ is a dense G_δ.

Claim. $\mathcal{G} \cap W \subseteq \mathrm{NA}_2(\widetilde{X}_0)$.

Let $x_0^{**} \in \mathcal{G} \cap W$. From the proof of Theorem 2.11, it follows that $x_0^{**} \in \mathrm{NA}(\widetilde{X}_0^*)$. Hence if $x_0^{**} = (x_n^{**})$, then there exists k such that $\|x_k^{**}\| = \|(x_n^{**})\|_\infty$. Also $x_k^{**} \in R(B_{X_k})$. Hence there exists $x_k \in B_{X_k}$ such that $\|x_k^{**} + x_k\| = \|x_k^{**}\| + 1$. Choose $x^{***} \in S_{X_k^{***}}$ such that $x^{***}(x_k^{**} + x_k) = \|x_k^{**} + x_k\|$. Then $x^{***} \in D(x_k^{**})$, which implies $x^{***} = J(x^*)$ for some $x^* \in S_{X_k^*}$, where $J : X^* \to X^{***}$ is the canonical map. The last statement follows from the fact that $\widetilde{X}_0$ is HBS and $D((x_n^{**})) \subseteq A_{\widetilde{X}_0}$. In fact, if $z^{***} = (z_n^{***}) \in S_{\widetilde{X}_0^{***}}$ is such that $z_k^{***} = x^{***}$ and all other coordinates are zero, then $z^{***} \in D((x_n^{**}))$.

Finally, define $z_0 = (z_n) \in \widetilde{X}_0$ by setting $z_k = x_k$ and all other coordinates zero, and define $z_0^* = (z_n^*) \in \widetilde{X}_0^*$ by setting $z_k^* = x_k^*$ and all other coordinates zero. Hence $|x_0^{**}(z_0^*)| = \|x_0^{**}\|$ and also $|z_0^*(z_0)| = 1$. This implies $x_0^{**} \in \mathrm{NA}_2(\widetilde{X}_0)$, completing the proof. ∎

Theorem 4.9. *If Y_n is $(*)$ and ABR in X_n for all $n \in \mathbb{N}$, then $\widetilde{Y}_\infty$ is $(*)$ and ABR in $\widetilde{X}_\infty$.*

Proof. As in the proof of Theorem 4.8, we define W_m^i and then $W = \bigcap_{m,i} W_m^i$. Next observe $\phi_{B_{\widetilde{Y}_\infty}}((x_j)) = \sup_j \phi_{B_{Y_j}}(x_j)$ [P, Theorem 6.3.8(b)]. Hence the result follows. ∎

Turning to other ℓ_p sums, $1 \le p < \infty$, we have

THEOREM 4.10. *For each $n \in \mathbb{N}$, let Y_n be a subspace of X_n. If at least one Y_n is $(*)$ and ABR in X_n, then $\widetilde{Y}_1$ is $(*)$ and ABR in $\widetilde{X}_1$.*

Proof. Suppose Y_k is $(*)$ and ABR in X_k. We have $\bigcap_{n\in\mathbb{N}} V_n^k \subseteq R(B_{Y_k})$, where $V_n^k \subseteq X_k$ are open and dense.

Define $W_n = \{(x_n) \in \widetilde{X}_1 : x_k \in V_n^k\}$. Then each W_n is open and dense. Hence $W = \bigcap_n W_n$ is a dense G_δ.

Now follow the proof of [P, Proposition 6.3.2]. ∎

THEOREM 4.11. *Let $\{X_n : n \in \mathbb{N}\}$ be a family of Banach spaces and $Y_n \subseteq X_n$ be subspaces. For $1 < p < \infty$, $\widetilde{Y}_p$ is a $(*)$ and BR/ABR subspace of $\widetilde{X}_p$ if and only if each Y_n is a $(*)$ and BR/ABR subspace of X_n.*

Proof. The BR part follows from [P, Theorem 6.3.17]. We now prove the ABR part.

Suppose each Y_n is $(*)$ and ABR in X_n. From Theorem 4.1, it follows that $\widetilde{Y}_p$ is a $(*)$-subspace of $\widetilde{X}_p$.

Now for each n, $R(B_{Y_n}, X_n) \supseteq \bigcap_m U_m^n$, where each U_m^n is a dense open subset of X_n. Define

$$W_m^n = \{(x_i) \in \widetilde{X}_p : x_n \in U_m^n\}.$$

Then W_m^n is a dense open set in $\widetilde{X}_p$, so $W = \bigcap_{m,n\in\mathbb{N}} W_m^n$ is dense in $\widetilde{X}_p$.

If $(x_i) \in W$ then $x_i \in R(B_{Y_i}, X_i)$ for all i, and hence by [P, Lemma 6.3.16], $(x_i) \in R(B_{\widetilde{Y}_p}, \widetilde{X}_p)$.

Conversely if $\widetilde{Y}_p$ is $(*)$ and ABR in $\widetilde{X}_p$, then Y_n is $(*)$ and DBR in X_n from [P, Theorem 6.3.17]. It remains to prove that Y_n's are ABR in X_n.

Let $R(B_{\widetilde{Y}_p}) \supseteq \bigcap_m V_m$, where $V_m \subseteq \widetilde{X}_p$ are dense open sets.

Let $p_n : \widetilde{X}_p \to X_n$ be the canonical projection. Then $p_n(V_m)$ is a dense open set. Since $\bigcap_m p_n(V_m) \subseteq p_n(\bigcap_m V_m) \subseteq R(B_{Y_n}, X_n)$ by [P, Lemma 6.3.16], the result follows. ∎

We now turn to spaces of Bochner integrable functions.

THEOREM 4.12. *Let $1 \le p \le \infty$, $f \in L_p(\mu, X)$ and Y be a subspace of X. Then*

(a) $\phi_{L_p(\mu,B_Y)}(f) = (\int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t))^{1/p}$ *for* $1 \le p < \infty$.

(b) $\phi_{L_\infty(\mu,B_Y)}(f) = \inf\{a \ge 0 : \mu\{t \in \Omega : \phi_{B_Y}(f(t)) > a\} = 0\}$.

Proof. (a) If $f \in L_p(\mu, X)$ and $g \in L_p(\mu, B_Y)$, then

$$\|f-g\|_p^p = \int_\Omega \|f(t)-g(t)\|^p \, d\mu(t) \le \int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t).$$

Hence,

$$\phi_{L_p(\mu,B_Y)}(f) \le \Big(\int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t)\Big)^{1/p}.$$

Now, let $f = \sum_{i=1}^n x_i\chi_{A_i} \in L_p(\mu, X)$ be a simple function. Without loss of generality, we may assume $\sum_{i=1}^n \mu(A_i) = 1$.

Observe that the map $x \mapsto \phi_{B_Y}^p(x)$ from X to $\mathbb{R}_{\ge 0}$ is continuous and given $\varepsilon > 0$ there exists $y \in B_Y$ such that $\|x-y\|^p > \phi_{B_Y}^p(x) - \varepsilon$.

Thus, given $\varepsilon > 0$ there exist $y_i \in B_Y$ such that $\|x_i - y_i\|^p > \phi_{B_Y}^p(x_i) - \varepsilon$ for $1 \le i \le n$.

Let $g = \sum_{i=1}^n y_i\chi_{A_i}$. Then $g \in L_p(\mu, B_Y)$ and

$$\begin{aligned}\|f-g\|_p^p &= \sum_{i=1}^n \|x_i - y_i\|^p \mu(A_i) > \sum_{i=1}^n (\phi_{B_Y}^p(x_i) - \varepsilon)\mu(A_i)\\ &= \sum_{i=1}^n \phi_{B_Y}^p(x_i)\mu(A_i) - \varepsilon = \int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t) - \varepsilon.\end{aligned}$$

Since ε is arbitrary, we have $\phi_{L_p(\mu,B_Y)}^p(f) \ge \int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t)$, i.e.,

$$\phi_{L_p(\mu,B_Y)}(f) \ge \Big(\int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t)\Big)^{1/p}.$$

Hence $\phi_{L_p(\mu,B_Y)}(f) = (\int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t))^{1/p}$ if f is simple.

Now the map $f \mapsto \int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t)$ from $L_p(\mu, X)$ to $\mathbb{R}_{\ge 0}$ is continuous, and since the simple functions are dense in $L_p(\mu, X)$, the result follows.

(b) We have

$$\begin{aligned}\phi_{L_\infty(\mu,B_Y)}(f) &= \sup_{g\in L_\infty(\mu,B_Y)} \|f-g\|_\infty\\ &= \sup_{g\in L_\infty(\mu,B_Y)} \inf\{a \ge 0 : \mu\{t \in \Omega : \|(f-g)(t)\| > a\} = 0\}\\ &= \inf\{a \ge 0 : \mu\{t \in \Omega : \phi_{B_Y}(f(t)) > a\} = 0\}. \ \blacksquare\end{aligned}$$

THEOREM 4.13. *For $1 \le p \le \infty$, if $f \in L_p(\mu, X)$, then $g \in L_p(\mu, B_Y)$ is farthest from f if and only if $g(t) \in B_Y$ is farthest from $f(t)$ a.e. $[\mu]$.*

Proof. CASE 1: $1 \le p < \infty$. Let $g \in L_p(\mu, B_Y)$ be farthest from f. From Theorem 4.12,

$$\int_\Omega \|f(t)-g(t)\|^p \, d\mu(t) = \int_\Omega \phi_{B_Y}^p(f(t))\, d\mu(t).$$

Since $\|f(t)-g(t)\|^p \le \phi^p_{B_Y}(f(t))$ a.e. $[\mu]$, we have $\|f(t)-g(t)\|^p = \phi^p_{B_Y}(f(t))$ a.e. $[\mu]$. Hence $g(t) \in F_{B_Y}(f(t))$ a.e. $[\mu]$.

Conversely, if $g \in L_p(\mu, B_Y)$ is such that $\|f(t) - g(t)\| = \phi_{B_Y}(f(t))$ a.e. $[\mu]$, then $\|f-g\|_p^p = \int_\Omega \|f(t)-g(t)\|^p\,d\mu(t) = \int_\Omega \phi^p_{B_Y}(f(t))\,d\mu(t) = \phi^p_{L_p(\mu,B_Y)}(f)$. That is, g is farthest from f.

Case 2: $p=\infty$. We have $f \in R(L_\infty(\mu, B_Y))$ if and only if there exists $g \in L_\infty(\mu, B_Y)$ such that $\|f-g\|_\infty = \phi_{L_\infty(\mu,B_Y)}(f)$. Hence

$$\begin{aligned}\inf\{b\ge 0 : \mu\{t\in\Omega : \|f(t)-g(t)\| > b\} = 0\}\\ = \inf\{a\ge 0 : \mu\{t\in\Omega : \phi_{B_Y}(f(t)) > a\} = 0\}.\end{aligned}$$

Consequently, $\|f(t)-g(t)\| = \phi_{B_Y}(f(t))$ a.e. $[\mu]$. ∎

Theorem 4.14. *Let Y be a separable ball remotal subspace of X. Then $L_p(\mu, B_Y)$ is remotal in $L_p(\mu, X)$ for $1 \le p \le \infty$. The converse is true for any subspace Y of X.*

Proof. An identical technique used in [BLR, Theorem 3.8] can be used to prove that there is a measurable selection of the set $\{(t, F_{B_Y}(f(t))) : t \in \Omega\}$. Let $g(t) \in F_{B_Y}(f(t))$ be the corresponding measurable selection.

It remains to prove that $g \in L_p(\mu, B_Y)$. For $p=\infty$, this is immediate, and for $1 \le p < \infty$,

$$\begin{aligned}\int_\Omega \|g(t)\|^p\,d\mu(t) &\le \int_\Omega \|g(t)-f(t)\|^p\,d\mu(t) + \int_\Omega \|f(t)\|^p\,d\mu(t)\\ &= \int_\Omega \phi^p_{B_Y}(f(t))\,d\mu(t) + \int_\Omega \|f(t)\|^p\,d\mu(t) < \infty.\end{aligned}$$

Here $\int_\Omega \phi^p_{B_Y}(f(t))\,d\mu(t) < \infty$, since $t \mapsto \phi^p_{B_Y}(f(t))$ is measurable.

To prove the converse, let $x \in X$ and set $f = x\chi_\Omega \in L_p(\mu, X)$. There exists $g \in L_1(\mu, B_Y)$ such that $\|f-g\|_p = \phi_{L_p(\mu,B_Y)}(f)$. From Theorem 4.13, it follows that for some $t\in\Omega$, $g(t)$ is farthest from x. ∎

Turning to spaces of continuous functions, we have the following

Theorem 4.15. *Let Y be a ABR subspace of X, and K a compact Hausdorff space. Then $C(K,Y)$ is an ABR subspace of $C(K,X)$.*

Proof. Let $\bigcap_n V_n \subseteq R(B_Y)$, where V_n's are open dense subsets of X.

Let $T_n = \{f \in C(K,X) : f(K) \subseteq V_n\}$. From [P, Lemma 6.4.5], it follows that T_n is dense in $C(K,X)$. We now show that T_n is open.

Let $f \in T_n$. There exists $\delta > 0$ such that $B(f(t),\delta) \subseteq V_n$ for each $t \in K$. In fact, otherwise we can find sequences $(t_m) \subseteq K$ and $(z_m) \subseteq V_n^c$ such that $\|f(t_m) - z_m\| < 1/m$. Now passing to a subnet of (t_m), we can find a $t_0 \in K$ such that $t_{m_i} \to t_0$, and hence $\|z_{m_i} - f(t_0)\| \to 0$. But z_m's are in V_n^c, a closed set, so $f(t_0) \notin V_n$, a contradiction.

Claim. *If* $\|f-g\| < \delta/2$, *then* $g \in T_n$.

For $t \in K$ we have $\|f(t)-g(t)\| < \delta/2$. Hence $g(t) \in V_n$. Since t is arbitrary, $g(K) \subseteq V_n$, proving the claim.

Let $f \in \bigcap_n T_n$. Then $f(K) \subseteq \bigcap_n V_n \subseteq R(B_Y)$. Hence $f \in R(B_{C(K,Y)})$ [P, Theorem 6.4.1]. ∎

Acknowledgements. The author would like to thank Pradipta Bandyopadhyay for many useful comments that improved the final version of the present paper. The author also thanks the referee for suggesting the necessary modifications of the earlier version. This research was supported by the NBHM Research Fellowship and the author would like to thank the NBHM for their financial support.

References

[AW] C. Akemann and N. Weaver, *Geometric characterization of some classes of operators in C^*-algebras and von Neumann algebras*, Proc. Amer. Math. Soc. 130 (2002), 3033–3037.

[Asp] E. Asplund, *Farthest points in reflexive locally uniformly rotund Banach spaces*, Israel J. Math. 4 (1966), 213–216.

[BLR] P. Bandyopadhyay, B.-L. Lin and T. S. S. R. K. Rao, *Ball remotal subspaces of Banach spaces*, Colloq. Math. 114 (2009), 119–133.

[BP] P. Bandyopadhayay and T. Paul, *Ball remotality in Banach spaces*, J. Math. Anal. Appl. 369 (2010), 525–536.

[BPR1] P. Bandyopadhayay, T. Paul and A. K. Roy, *Densely ball remotal subspaces of* $C(K)$, Indag. Math. (N.S.) 20 (2009), 381–395.

[BPR2] P. Bandyopadhayay, T. Paul and Ashoke K. Roy, *Ball remotality of M-ideals in some function spaces and function algebras*, Positivity 14 (2010), 459–471.

[BR] S. Basu and T. S. S. R. K. Rao, *Some stability results for asymptotic norming properties of Banach spaces*, Colloq. Math. 75 (1998), 271–284.

[Bo] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodým Property*, Lecture Notes in Math. 993, Springer, Berlin, 1983.

[DGZ] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monogr. Surveys Pure Appl. Math. 64, Longman Sci. & Tech., Harlow, 1993.

[DU] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.

[DeZi] R. Deville and V. E. Zizler, *Farthest points in w^*-compact sets*, Bull. Austral. Math. Soc. 38 (1988), 433–439.

[Ede] M. Edelstein, *Farthest points of sets in uniformly convex Banach spaces*, Israel J. Math. 4 (1966), 171–176.

[Fab] M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach Space Theory*, CMS Books in Math./Ouvrages de Math. de la SMC, Springer, New York, 2011.

[Fin] C. Finet and L. Quarta, *Some remarks on M-ideals and strong proximinality*, Bull. Korean Math. Soc. 40 (2003), 503–508.

[HWW] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[Lau] K.-S. Lau, *Farthest points in weakly compact sets*, Israel J. Math. 22 (1975), 168–174.

[LiCh] W. A. Light and E. W. Cheney, *Approximation Theory in Tensor Product Spaces*, Lecture Notes in Math. 1169, Springer, Berlin, 1985.

[No] K. Nourouzi, *Strongly exposed points in the unit ball of trace-class operators*, Int. J. Math. Math. Sci. 30 (2002), 393–397.

[P] T. Paul, *Ball remotality in Banach spaces and related topics*, Ph.D. Thesis, Indian Statistical Institute, 2011; http://sites.google.com/site/paultanmoysite.

[Ph] R. R. Phelps, *Convex Functions Monotone Operators and Differentiability*, Lecture Notes in Math. 1364, Springer, Berlin, 1993.

[Ziz] V. Zizler, *On some extremal problems in Banach spaces*, Math. Scand. 32 (1973), 214–224.

Tanmoy Paul
Stat–Math Unit
Indian Statistical Institute
Bangalore center, 8th Mile Mysore Road
R. V. College Post, Bangalore 560059, India
E-mail: tanmoy.iam@gmail.com

Received December 12, 2011
Revised version May 16, 2012 (7378)