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Dual spaces of compact operator spaces and the weak Radon–Nikodým property

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Abstract. We deal with the weak Radon–Nikodým property in connection with the dual space of $\mathcal{K}(X,Y)$, the space of compact operators from a Banach space X to a Banach space Y. First, under the weak Radon–Nikodým property, we give a representation of that dual. Next, using this representation, we provide some applications to the dual spaces of $\mathcal{K}(X,Y)$ and $\mathcal{K}_{w^*w}(X^*,Y)$, the space of weak*-weakly continuous operators.

1. Introduction. Several contributions to the detection of copies of ℓ_1 were made in the sixties by A. Pełczyński [Pe]. After his works, many mathematicians tried to characterize $\ell_1 \not\subseteq X$ and find a nonseparable dual Banach space which contains no ℓ_1. In 1973, R. C. James constructed a Banach space JT (called James tree space) that is separable, contains no ℓ_1 and has nonseparable dual [Ja, LS]. In 1974, Rosenthal provided the true understanding of ℓ_1's absence by proving the following theorem (called *Rosenthal's ℓ_1 theorem*) [Ro].

THEOREM 1 (Rosenthal's ℓ_1 theorem). *Each bounded sequence in a Banach space X has a weakly Cauchy subsequence if and only if X contains no isomorphic copy of ℓ_1.*

Shortly after Rosenthal's ℓ_1 theorem, a number of classical characterizations were formulated. In particular, Musiał proved the following theorem: *a Banach space X contains no isomorphic copy of ℓ_1 if and only if for each complete finite measure space (Ω,Σ,μ) and each μ-continuous X^*-valued countably additive vector measure $\nu:\Sigma\to X^*$ of bounded variation, there exists a Pettis integrable function $f:\Omega\to X^*$ such that $\nu(E) = \text{P-}\int_E f\,d\mu$ for all $E\in\Sigma$.* A Banach space satisfying the latter condition in Musiał's theorem is said to have the *weak Radon–Nikodým property* [M1]. A great deal

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of research about the weak Radon–Nikodým property has been done in the last decades [F, M1, M2, RSU, SS1, SS2, SS3, T].

The spaces $\mathcal{K}(X,Y)$ and $\mathcal{K}_{w^*w}(X^*,Y)$ and their dual spaces have been studied by several mathematicians [FS, G, Jo, K, Ru]. In particular, Feder and Saphar proved the following theorem [FS].

Theorem 2. *Suppose X^{**} or Y^* has the Radon–Nikodým property. For every $\phi \in \mathcal{K}(X,Y)^*$ and $\varepsilon > 0$, there are $(x_n^{**}) \subset X^{**}$ and $(y_n^*) \subset Y^*$ such that $\phi(T) = \sum_{n=1}^{\infty} x_n^{**}T^*(y_n^*)$ for all $T \in \mathcal{K}(X,Y)$ and $\sum_{n=1}^{\infty} \|x_n^{**}\|\,\|y_n^*\| < \|\phi\| + \varepsilon$.*

This theorem characterizes $\mathcal{K}(X,Y)^*$ in the case that X^{**} or Y^* has the Radon–Nikodým property. Moreover, we observe the following corollary concerning the topology of $\mathcal{K}(X,Y)^*$: *if X^{**} or Y^* has the Radon–Nikodým property, then $X^{**} \otimes Y^*$ is w^*-sequentially dense in $\mathcal{K}(X,Y)^*$.*

Our aim in this paper is to apply the weak Radon–Nikodým property to the dual of the compact operator space. Then we provide a generalized representation of $\mathcal{K}(X,Y)^*$ related to Feder and Saphar's theorem, and a topological property of $\mathcal{K}(X,Y)^*$ in the case that X^{**} or Y^* has the weak Radon–Nikodým property. Moreover, we give similar results for $\mathcal{K}_{w^*w}(X^*,Y)$ in the case that X^* or Y^* has the weak Radon–Nikodým property.

2. Notation and preliminaries. Throughout this paper (Ω,Σ,μ) is a complete measure space and X and Y are Banach spaces.

Let $\mathcal{F}(X,Y)$ denote the space of finite rank linear operators from X into Y. Let $T \in \mathcal{F}(X,Y)$ and put

$$N^0(T) = \inf \sum_{i=1}^{n} \|x_i^*\|\,\|y_i\|,$$

where the infimum is taken over all finite representations $T(x) = \sum_{i=1}^{n} x_i^*(x)y_i$ for all $x \in X$. Then $N^0(T)$ is called a *finite nuclear norm* [Pi, p. 93].

Recall that T is an *integral operator* from X to Y if there exist a finite measure space (Ω,Σ,μ) and a pair of operators $S : X \to L_\infty(\mu)$ and $R : L_1(\mu) \to Y^{**}$ such that $Q_YT = RIS$ where Q_Y is the natural mapping of $Y \to Y^{**}$ and I is the canonical mapping from $L_\infty(\mu)$ into $L_1(\mu)$. Also the *integral norm* of an integral operator is defined by

$$\|T\|_{\mathrm{I}} = \inf \|S\|\,\|R\|\mu(\Omega),$$

where the infimum is taken over all such factorizations of T. The space of integral operators from X into Y with this norm will be denoted by $\mathcal{I}(X,Y)$. We observe that above, the space $L_\infty(\mu)$ can be replaced by a $C(K)$ space, where the measure μ is a positive regular Borel measure on K, and I is the canonical mapping from $C(K)$ into $L_1(\mu)$. Also recall that $T : X \to Y$

is called a *Pietsch integral operator* if there exist a finite measure space (Ω, Σ, μ) and a pair of operators $S : X \to L_\infty(\mu)$ and $R : L_1(\mu) \to Y$ such that $T = RIS$ where I is the canonical mapping from $L_\infty(\mu)$ into $L_1(\mu)$. The *Pietsch integral norm* of a Pietsch integral operator is defined by

$$\|T\|_{\mathrm{PI}} = \inf \|S\| \, \|R\| \mu(\Omega),$$

where the infimum is taken over all such factorizations of T. The space of Pietsch integral operators from X into Y with this norm will be denoted by $\mathcal{PI}(X, Y)$. Note that if T is a Pietsch integral operator, then T is an integral operator. Also it is known that $\mathcal{I}(X, Y) = \mathcal{PI}(X, Y)$ if Y is a dual space or an $L_1(\mu)$ space [Ry, p. 65].

3. The dual of the compact operator space and the weak Radon–Nikodým property. In this section, we provide a representation of $\mathcal{K}(X, Y)^*$ in the case that X^{**} or Y^* has the weak Radon–Nikodým property. Throughout this section K is a compact topological space and $\mathcal{B}_K$ is the σ-algebra of Borel subsets of K. Recall that if $T : C(K) \to X$ is a bounded linear operator, then $F_T : \mathcal{B}_K \to X^{**}$ is called the *representing measure* for T if $F_T(E) = T^{**}(\chi_E)$. It is well known that $\ell_1 \not\subseteq X$ if and only if each X^*-valued measure of σ-bounded variation has a relatively compact range [M1, Corollary 10]. Also the operator $T : C(K) \to X$ is compact if and only if the representing measure of T has a relatively compact vector range [Ry, Proposition 5.27].

We start by showing the following lemma.

Lemma 3.1. *X^* has the weak Radon–Nikodým property if and only if for all Banach spaces Y, every integral operator $T : Y \to X^*$ is compact.*

Proof. ($\Rightarrow$) First we show that for each compact topological space K, every Pietsch integral operator $T : C(K) \to X^*$ is compact. Since T is a weakly compact operator, the representing measure F of T is a regular countably additive vector measure. Since T is a Pietsch integral operator, F is of bounded variation. By assumption and [M2, Theorem 9.7], we have $\ell_1 \not\subseteq X$, hence F has a relatively compact range, from which we conclude that T is compact.

Now take $T \in \mathcal{I}(Y, X^*) = \mathcal{PI}(Y, X^*)$. Then there exist a compact Hausdorff space Ω with a positive regular Borel measure μ and a pair of operators $S : Y \to C(\Omega)$, $R : L_1(\mu) \to X^*$ such that $T = RIS$, where $I : C(\Omega) \to L_1(\mu)$ is the canonical mapping. Put $U = RI : C(\Omega) \to X^*$. Then U is a Pietsch integral operator (see Section 2). By the above argument, U is compact. Hence so is T.

($\Leftarrow$) The argument is standard. Suppose X^* does not have the weak Radon–Nikodým property. Then $\ell_1 \subseteq X$, so by Pełczyński's Theorem [Pe], we have $L_1([0,1]) \subseteq X^*$. Let $S : L_1([0,1]) \to X^*$ be an isomorphic embedding and let Σ be the σ-algebra of Lebesgue measurable sets on $[0,1]$. Define $G : \Sigma \to L_1([0,1])$ by $G(E) = \chi_E$. Clearly G is a countably additive vector measure. Moreover, it does not have compact range. Indeed, put $E_n = \{t \in [0,1] : \sin(2^n \pi t) > 0\}$. It is easily checked that $\|\chi_{E_m} - \chi_{E_n}\|_{L_1([0,1])} \ge 1/4$ for all $n \ne m \in \mathbb{N}$ so $(G(E_n))$ in $B_{L_1([0,1])}$ has no convergent subsequence. Let $I : L_\infty[0,1] \to L_1[0,1]$ be the inclusion operator. Put $T = SI$. Then T is an integral operator. By assumption, T is compact, so $\{T(\chi_E) : E \in \Sigma\}$ is a relatively compact set. Since $T(\chi_E) = SI(\chi_E) = S(\chi_E) = S(G(E))$ and S is an isomorphic embedding, this is a contradiction. ∎

Remark 3.2. If X^* has the weak Radon–Nikodým property, then every $T \in \mathcal{I}(X,Y)$ is compact. Indeed, T^* is then an integral operator from Y^* to X^*. By Lemma 3.1, T^* is compact, hence so is T. Thus we can regard $\mathcal{I}(X,Y)$ as a subset of $\mathcal{K}(X,Y)$ if X^* has the weak Radon–Nikodým property.

Also we need the following well known lemma and its proof [Ry, Proposition 4.12 and Corollary 4.13].

Lemma 3.3. *If either X^* or Y has the approximation property, then*

$$\mathcal{K}(X,Y) = X^* \hat{\otimes}_\varepsilon Y.$$

Proof. (a) Suppose that Y has the approximation property. Let $T : X \to Y$ be a compact operator and let $\varepsilon > 0$. Then there exists a finite rank operator $R : Y \to Y$ such that $\|y - Ry\| < \varepsilon$ for every y in the relatively compact subset $T(B_X)$ of Y. Let $S = RT$. Then S is a finite rank operator and $\|T - S\| < \varepsilon$.

(b) Suppose that X^* has the approximation property. Let $T : X \to Y$ be compact and $\varepsilon > 0$. Since $T^* : Y^* \to X^*$ is compact, there exists a finite rank operator $R : X^* \to X^*$ such that $\|\varphi - R\varphi\| < \varepsilon$ for every φ in the relatively compact set $T^*(B_{Y^*})$. Since T is compact, T^{**} maps into Y and so $T^{**}R^*$ is a finite rank operator from X^{**} into Y. Put $S = T^{**}R^*Q_X$ where Q_X is the natural mapping from X into X^{**}. Then S is a finite rank operator from X to Y. We claim that $\mathcal{S}$ approximates T. Indeed, if $x \in B_X$, then

$$\begin{aligned}
\|Tx - Sx\| &= \sup_{\psi \in B_{Y^*}} |\psi Tx - \psi T^{**} R^* Q_X(x)| \\
&= \sup_{\psi \in B_{Y^*}} |(T^*\psi)x - (RT^*)^* Q_X(x)\psi| \\
&= \sup_{\psi \in B_{Y^*}} |(T^*\psi)x - (RT^*\psi)x| \le \sup_{\psi \in B_{Y^*}} \|T^* - R(T^*\psi)\| < \varepsilon. \ ∎
\end{aligned}$$

Also we need the following lemma [Pi, p. 102, 6.8.4 Lemma 1].

LEMMA 3.4. *Let $T \in \mathcal{F}(X,Y)$ and $S \in \mathcal{I}(X,Y)$. Then*
$$N^0(ST) \le \|S\|_{\mathrm{I}}\|T\|.$$

Now we provide our main theorem.

THEOREM 3.5. *Let X and Y be Banach spaces such that X^{**} or Y^* has the weak Radon–Nikodým property. Then, for all $\phi \in \mathcal{K}(X,Y)^*$, there exist a sequence $(((x_i^n)^{**})_{i=1}^{m_n})_{n=1}^{\infty}$ in X^{**} and a sequence $(((y_i^n)^*)_{i=1}^{m_n})_{n=1}^{\infty}$ in Y^* such that*
$$\langle \phi, T\rangle = \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}(T^*((y_i^n)^*))$$
for all $T \in \mathcal{K}(X,Y)$. Moreover,
$$\limsup_{n\to\infty} \sum_{i=1}^{m_n} \|(x_i^n)^{**}\|\,\|(y_i^n)^*\| \le \|\phi\|.$$

Proof. (a) Suppose that X^{**} has the weak Radon–Nikodým property. Let $\Gamma = B_{Y^*}$. Define $i : Y \to \ell^\infty(\Gamma)$ by $i(y) = (y^*(y))_{y^*\in\Gamma}$. Then i is the canonical injection of Y into $\ell^\infty(\Gamma)$. Since $\ell^\infty(\Gamma)$ has the approximation property, we obtain $\mathcal{K}(X, \ell^\infty(\Gamma)) = \ell^\infty(\Gamma) \mathbin{\hat{\otimes}_\varepsilon} X^*$. So we can define $J : \mathcal{K}(X,Y) \to \ell^\infty(\Gamma) \mathbin{\hat{\otimes}_\varepsilon} X^*$ by $J(T) = iT$. Since
$$\|J(T)\| = \sup_{x\in B_X} \|i(T(x))\|_\infty = \sup_{x\in B_X} \|Tx\| = \|T\|$$
for all $T \in \mathcal{K}(X,Y)$, J is an isometry from $\mathcal{K}(X,Y)$ into $\ell^\infty(\Gamma) \mathbin{\hat{\otimes}_\varepsilon} X^*$. Take any $\phi \in \mathcal{K}(X,Y)^*$. Then there exists $\hat{\phi} \in (\ell^\infty(\Gamma) \mathbin{\hat{\otimes}_\varepsilon} X^*)^*$ such that $\langle \phi, T\rangle = \langle \hat{\phi}, J(T)\rangle$ for all $T \in \mathcal{K}(X,Y)$ and $\|\phi\| = \|\hat{\phi}\|$. Since $(\ell^\infty(\Gamma) \mathbin{\hat{\otimes}_\varepsilon} X^*)^* = \mathcal{PI}(\ell^\infty(\Gamma), X^{**})$ [Ry, Proposition 3.22], there exists $R \in \mathcal{PI}(\ell^\infty(\Gamma), X^{**})$ with $\|R\|_{\mathrm{PI}} = \|\hat{\phi}\|$ such that if $T \in \mathcal{K}(X,Y)$, then
$$\langle \phi, T\rangle = \langle \hat{\phi}, J(T)\rangle = \langle R, J(T)\rangle.$$
By Lemma 3.1, R is a compact operator from $\ell^\infty(\Gamma)$ to X^{**}. Since $\ell^\infty(\Gamma)^*$ has the metric approximation property, there exists a sequence (v_n) in $\ell^\infty(\Gamma)^* \otimes_\varepsilon X^{**}$ such that
$$\lim_{n\to\infty} \|v_n - R\| = 0. \tag{3.1}$$
By the proof of Lemma 3.3, each v_n is the form of $v_n = R^{**} s_n^* Q_{\ell^\infty(\Gamma)^*}$ where s_n is a finite rank operator on $\ell^\infty(\Gamma)$ and $\|s_n\| \le 1$. Also each v_n is a Pietsch integral operator from $\ell^\infty(\Gamma)$ to X^{**} because each s_n is a finite rank operator. Thus we obtain
$$\|v_n\|_{\mathrm{PI}} \le \|R^{**}\|_{\mathrm{PI}} \|s_n\| \le \|R\|_{\mathrm{PI}}.$$
Since $\|s_n\| \le 1$ for all $n \in \mathbb{N}$, by Lemma 3.4 we obtain
$$\|v_n\|_{N^0} \le \|R\|_{\mathrm{PI}}$$

for all $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, choose $((z_i^n)^*)_{i=1}^{m_n}$ in $\ell^\infty(\Gamma)^*$ and $((x_i^n)^{**})_{i=1}^{m_n}$ in X^{**} such that

$$v_n = \sum_{i=1}^{m_n} (z_i^n)^* \otimes (x_i^n)^{**} \quad \text{and} \quad \sum_{i=1}^{m_n} \|(z_i^n)^*\| \, \|(x_i^n)^{**}\| < \|R\|_{\mathrm{PI}} + 1/n.$$

Observe that if $\nu \in \ell^\infty(\Gamma) \otimes X^*$, then by (3.1) we have

$$\lim_{n\to\infty} |\langle R, \nu\rangle - \langle v_n, \nu\rangle| = 0. \tag{3.2}$$

Now we claim that if $T \in \mathcal{K}(X,Y)$, then, for each $n \in \mathbb{N}$,

$$\langle v_n, iT\rangle = \sum_{i=1}^{m_n} (x_i^n)^{**}((iT)^*((z_i^n)^*)). \tag{3.3}$$

Indeed, fix $n \in \mathbb{N}$. Since iT is in $\ell^\infty(\Gamma) \hat{\otimes}_\varepsilon X^*$, we can take a sequence (ω_k) in $\ell^\infty(\Gamma) \otimes X^*$ such that $\lim_{k\to\infty} \|\omega_k - iT\| = 0$. Each ω_k has a representation $\sum_{j=1}^{n_k} a_j^k \otimes b_j^k$. Then, for each $k \in \mathbb{N}$,

$$\begin{aligned}\langle v_n, \omega_k\rangle &= \sum_{j=1}^{n_k} v_n(a_j^k) b_j^k = \sum_{j=1}^{n_k}\sum_{i=1}^{m_n} (z_i^n)^*(a_j^k)(x_i^n)^{**}(b_j^k) \\ &= \sum_{i=1}^{m_n} (x_i^n)^{**}\Big(\sum_{j=1}^{n_k} (z_i^n)^*(a_j^k) b_j^k\Big).\end{aligned}$$

Since $\lim_{k\to\infty} \|\omega_k - iT\| = 0$, we have, for each $(z_i^n)^* \in \ell^\infty(\Gamma)^*$,

$$(iT)^*((z_i^n)^*) = \lim_{k\to\infty} \sum_{j=1}^{n_k} (z_i^n)^*(a_j^k) b_j^k.$$

Thus, for each $(x_i^n)^{**} \in X^{**}$,

$$(x_i^n)^{**}((iT)^*((z_i^n)^*)) = \lim_{k\to\infty} \sum_{j=1}^{n_k} (x_i^n)^{**}((z_i^n)^*(a_j^k) b_j^k).$$

Since $v_n \in (\ell^\infty(\Gamma) \hat{\otimes}_\varepsilon X^*)^*$, by the above argument we obtain (3.3):

$$\begin{aligned}\langle v_n, iT\rangle &= \lim_{k\to\infty} \langle v_n, \omega_k\rangle = \sum_{i=1}^{m_n} \lim_{k\to\infty} (x_i^n)^{**}\Big(\sum_{j=1}^{n_k} (z_i^n)^*(a_j^k) b_j^k\Big) \\ &= \sum_{i=1}^{m_n} (x_i^n)^{**}((iT)^*((z_i^n)^*).\end{aligned}$$

Now we show that if $T \in \mathcal{K}(X,Y)$, then

$$\langle R, iT\rangle = \lim_{n\to\infty} \langle v_n, iT\rangle.$$

Since $iT \in \ell^\infty(\Gamma) \hat{\otimes}_\varepsilon X^*$, there exists a sequence (ν_k) in $\ell^\infty(\Gamma) \otimes X^*$ such that

$$\lim_{k\to\infty} \|\nu_k - iT\| = 0.$$

Hence there exists $N_1 \in \mathbb{N}$ such that if $k > N_1$, then $\|\nu_k - iT\| < \varepsilon/\|R\|_{\mathrm{PI}}$. Now we choose $k_0 > N_1$. Then, by (3.2), there exists N_2 such that if $n > N_2$, then $|\langle R, \nu_{k_0}\rangle - \langle v_n, \nu_{k_0}\rangle| < \varepsilon$. Since, for all $n \in \mathbb{N}$, $\|v_n\|_{\mathrm{PI}} \le \|R\|_{\mathrm{PI}}$, we see that if $n > N_2$,

$$\begin{aligned}
&|\langle R, iT\rangle - \langle v_n, iT\rangle| \\
&\qquad \le |\langle R, iT\rangle - \langle R, \nu_{k_0}\rangle| + |\langle R, \nu_{k_0}\rangle - \langle v_n, \nu_{k_0}\rangle| + |\langle v_n, \nu_{k_0}\rangle - \langle v_n, iT\rangle| \\
&\qquad \le \|R\|_{\mathrm{PI}}\|iT - \nu_{k_0}\| + |\langle R, \nu_{k_0}\rangle - \langle v_n, \nu_{k_0}\rangle| + \|v_n\|_{\mathrm{PI}}\|iT - \nu_{k_0}\| < 3\varepsilon.
\end{aligned}$$

Now put $(y_i^n)^* = i^*((z_i^n)^*)$. Then, for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{m_n} \|i^*((z_i^n)^*)\| \, \|(x_i^n)^{**}\| < \|R\|_{\mathrm{PI}} + 1/n,$$

and so

$$\limsup_{n\to\infty} \sum_{i=1}^{m_n} \|i^*((z_i^n)^*)\| \, \|(x_i^n)^{**}\| \le \|R\|_{\mathrm{PI}}.$$

Finally we check that $\langle \phi, T\rangle = \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}(T^*((y_i^n)^*))$ for all $T \in \mathcal{K}(X, Y)$. By the above argument and (3.3),

$$\begin{aligned}
\langle \phi, T\rangle &= \langle \hat{\phi}, J(T)\rangle = \langle R, J(T)\rangle = \langle R, iT\rangle = \lim_{n\to\infty} \langle v_n, iT\rangle \\
&= \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}((iT)^*((z_i^n)^*)) = \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}(T^* i^*(z_i^n)^*) \\
&= \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}(T^*(y_i^n)^*).
\end{aligned}$$

(b) Suppose that Y^* has the weak Radon–Nikodým property. Then there exist a set Γ_1 and a canonical quotient map j of $\ell^1(\Gamma_1)$ onto X. Since $(\ell^1(\Gamma_1))^*$ has the approximation property, we obtain $\mathcal{K}(\ell^1(\Gamma_1), Y) = (\ell^1(\Gamma_1))^* \hat{\otimes}_\varepsilon Y$. So we can define $J : \mathcal{K}(X, Y) \to \ell^1(\Gamma_1)^* \hat{\otimes}_\varepsilon Y$ by $J(T) = Tj$. Since j is a quotient map, we obtain

$$\|J(T)\| = \sup_{(\alpha_\gamma)\in B_{\ell^1(\Gamma_1)}} \|T(j(\alpha_\gamma))\| = \|T\|$$

for all $T \in \mathcal{K}(X, Y)$. Thus J is an isometry from $\mathcal{K}(X, Y)$ into $\ell^1(\Gamma_1)^* \hat{\otimes}_\varepsilon Y$. Take any $\phi \in \mathcal{K}(X, Y)^*$. Since $\ell^1(\Gamma_1)^{**}$ has the metric approximation property, it follows from part (a) above that there exist $R \in \mathcal{PI}(\ell^1(\Gamma_1)^*, Y^*)$ and a sequence (v_n) in $\ell^1(\Gamma_1)^{**} \otimes_\varepsilon Y^*$ such that $\lim_{n\to\infty} \|v_n - R\| = 0$, and if $T \in \mathcal{K}(X,Y)$ then

$$\langle R, Tj\rangle = \lim_{n\to\infty} \langle v_n, Tj\rangle.$$

Now, for each $n \in \mathbb{N}$, choose $((z_i^n)^{**})_{i=1}^{m_n}$ in $\ell^1(\Gamma_1)^{**}$ and $((y_i^n)^*)_{i=1}^{m_n}$ in Y^* such that $v_n = \sum_{i=1}^{m_n} (z_i^n)^{**} \otimes (y_i^n)^*$ and $\sum_{i=1}^{m_n} \|(z_i^n)^{**}\|\, \|(y_i^n)^*\| \le \|\phi\| + 1/n$. Put $(x_i^n)^{**} = j^{**}((z_i^n)^{**})$. Then

$$\begin{aligned}\langle \phi, T\rangle = \langle R, Tj\rangle &= \lim_{n\to\infty} \langle v_n, Tj\rangle = \lim_{n\to\infty} \sum_{i=1}^{m_n} (z_i^n)^{**}((Tj)^*(y_i^n)^*)\\ &= \lim_{n\to\infty} \sum_{i=1}^{m_n} j^{**}((z_i^n)^{**})T^*(y_i^n)^* = \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}T^*(y_i^n)^*.\end{aligned}$$

Furthermore,

$$\limsup_{n\to\infty} \sum_{i=1}^{m_n} \|(x_i^n)^{**}\|\, \|(y_i^n)^*\| \le \|\phi\|. \ \blacksquare$$

REMARK 3.6. 1. In the proof of Theorem 3.5, we used the technique of Feder and Saphar [FS] as a guideline. However, our proof is quite different. Namely, in Feder and Saphar's proof, the crucial point is $\mathcal{PI}(\ell^\infty(\Gamma), X^{**}) = \ell^\infty(\Gamma)^* \hat{\otimes}_\pi X^{**}$ as Banach spaces under the Radon–Nikodým property of X^{**} and the metric approximation property of $\ell^\infty(\Gamma)^*$. On the other hand, the crucial point of our proofs is $\mathcal{PI}(\ell^\infty(\Gamma), X^{**}) \subset \ell^\infty(\Gamma)^* \hat{\otimes}_\varepsilon X^{**}$ as sets under the weak Radon–Nikodým property of X^{**} and the metric approximation property of $\ell^\infty(\Gamma)^*$.

2. Theorem 3.5, in a sense, provides a generalized representation in comparison with Feder and Saphar's. Namely, under the conditions of Theorem 2 in the Introduction, every $\phi \in \mathcal{K}(X,Y)^*$ can be represented as

$$\phi(T) = \lim_{n\to\infty} \sum_{i=1}^{n} x_i^{**} T^*(y_i^*)$$

for all $T \in \mathcal{K}(X,Y)$, where $\lim_{n\to\infty} \sum_{i=1}^n \|x_i^{**}\|\, \|y_i^*\| < \infty$.

3. Since the Radon–Nikodým property is stronger than the weak Radon–Nikodým property, one can ask whether the result of Feder and Saphar can be obtained by using Theorem 3.5 and its proof. However, we have not been able to do that.

4. Applications of the main theorem. In the Introduction, as a corollary of Feder and Saphar's theorem, we mentioned that if X^{**} or Y^* has the Radon–Nikodým property, then $X^{**} \otimes Y^*$ is w^*-sequentially dense in $\mathcal{K}(X,Y)^*$. We now derive a more general result from our representation theorem.

COROLLARY 4.1. *If X^{**} or Y^* has the weak Radon–Nikodým property, then $X^{**} \otimes Y^*$ is w^*-sequentially dense in $\mathcal{K}(X,Y)^*$ and $B_{X^{**}\otimes Y^*}$ is w^*-sequentially dense in $B_{\mathcal{K}(X,Y)^*}$.*

Proof. First, we assume that X^{**} has the weak Radon–Nikodým property. Let ϕ be in $\mathcal{K}(X,Y)^*$. By assumption and Theorem 3.5, there exist a sequence $(((x_i^n)^{**})_{i=1}^{m_n})_{n=1}^{\infty}$ in X^{**} and a sequence $(((y_i^n)^*)_{i=1}^{m_n})_{n=1}^{\infty}$ in Y^* such that

$$\langle \phi, T\rangle = \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}(T^*((y_i^n)^*))$$

for all $T \in \mathcal{K}(X,Y)$. Now we put $a_n = \sum_{i=1}^{m_n} (x_i^n)^{**} \otimes (y_i^n)^*$. Clearly, (a_n) is sequence in $X^{**} \otimes Y^*$. It is weak* convergent to ϕ because

$$\langle a_n, T\rangle = \sum_{i=1}^{m_n} (x_i^n)^{**}(T^*((y_i^n)^*)) \to \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^{**}(T^*((y_i^n)^*)) = \langle \phi, T\rangle$$

for each $T \in \mathcal{K}(X,Y)$. Moreover, $\|a_n\| \le \|\phi\|$ for all $n \in \mathbb{N}$. Indeed, by part (a) of the proof of Theorem 3.5,

$$a_n = \sum_{i=1}^{m_n} (x_i^n)^{**} \otimes (i^*(z_i^n)^*), \quad v_n = \sum_{i=1}^{m_n} (x_i^n)^{**} \otimes (z_i^n)^*$$

for all $n \in \mathbb{N}$, where i is the canonical injection of Y into $\ell^\infty(\Gamma)$, and $((z_i^n)^*)_{i=1}^{m_n}$ is in $\ell^\infty(\Gamma)^*$. Since $v_n \in (\ell^\infty(\Gamma) \hat{\otimes}_\varepsilon X^*)^*$, by (3.3) we have

$$\begin{aligned} |\langle a_n, T\rangle| &= \Big| \sum_{i=1}^{m_n} (x_i^n)^{**}(T^*(i^*(z_i^n)^*)) \Big| = \Big| \sum_{i=1}^{m_n} (x_i^n)^{**}((iT)^*(z_i^n)^*) \Big| \\ &= |\langle v_n, iT\rangle| \le \|v_n\|_{\mathrm{PI}} \le \|\phi\| \end{aligned}$$

for all $T \in B_{\mathcal{K}(X,Y)}$ and $n \in \mathbb{N}$. Hence $B_{X^{**}\otimes Y^*}$ is w^*-sequentially dense in $B_{\mathcal{K}(X,Y)^*}$.

Next, we assume that Y^* has the weak Radon–Nikodým property. Let ϕ be in $\mathcal{K}(X,Y)^*$. By Theorem 3.5, we have

$$\langle a_n, T\rangle \to \langle \phi, T\rangle$$

for all $T \in \mathcal{K}(X,Y)$ where $a_n = \sum_{i=1}^{m_n} (x_i^n)^{**} \otimes (y_i^n)^* \in X^{**} \otimes Y^*$. Hence $X^{**} \otimes Y^*$ is w^*-sequentially dense in $\mathcal{K}(X,Y)^*$. So, it is enough to show that $\|a_n\| \le \|\phi\|$ for all $n \in \mathbb{N}$. Indeed, by (b) in the proof of Theorem 3.5,

$$a_n = \sum_{i=1}^{m_n} j^{**}((z_i^n)^{**}) \otimes (y_i^n)^*, \qquad v_n = \sum_{i=1}^{m_n} (z_i^n)^{**} \otimes (y_i^n)^*$$

for all $n \in \mathbb{N}$ where j is the canonical quotient map of $\ell^1(\Gamma_1)$ onto X and

$((z_i^n)^{**})_{i=1}^{m_n}$ is in $\ell^1(\Gamma_1)^{**}$. Also, $\|T\| = \|Tj\|$ for all $T \in \mathcal{K}(X,Y)$. Therefore

$$|\langle a_n, T\rangle| = \Big|\sum_{i=1}^{m_n} j^{**}((z_i^n)^{**})(T^*(y_i^n)^*)\Big| = \Big|\sum_{i=1}^{m_n}(z_i^n)^{**}((Tj)^*(y_i^n)^*)\Big|$$
$$= |\langle v_n, Tj\rangle| \le \|v_n\|_{\mathrm{PI}} \le \|\phi\|$$

for all $T \in B_{\mathcal{K}(X,Y)}$ and $n \in \mathbb{N}$. ∎

Now we describe the dual of the space $\mathcal{K}_{w^*w}(X^*,Y)$ of weak*-weakly continuous compact operators from X^* to Y. By using Feder and Saphar's theorem Choi and Kim [CK] proved the following theorem.

Theorem 4.2. *Suppose X^* or Y^* has the Radon–Nikodým property. Then for every $\phi \in \mathcal{K}_{w^*w}(X^*,Y)$ and $\varepsilon > 0$, there are $(x_n^*) \subset X^*$ and $(y_n^*) \subset Y^*$ such that*

$$\phi(T) = \sum_{n=1}^{\infty} y_n^*(T^*x_n^*)$$

for all $T \in \mathcal{K}_{w^*w}(X^*,Y)$, and $\sum_{n=1}^{\infty}\|x_n^*\|\,\|y_n^*\| < \|\phi\| + \varepsilon$.

From the above theorem, we also derive that if X^* or Y^* has the Radon–Nikodým property, then $X^* \otimes Y^*$ is w^*-sequentially dense in $\mathcal{K}_{w^*w}(X^*,Y)$.

Now we are going to characterize $\mathcal{K}_{w^*w}(X^*,Y)^*$ in the case that X^* or Y^* has the weak Radon–Nikodým property. Since $\mathcal{K}(Y,X)$ is identified with $\mathcal{K}_{w^*w}(X^*,Y^*)$ by the bijection $T \leftrightarrow T^*$, Theorem 3.5 yields the following corollary.

Corollary 4.3. *Let X and Y be Banach spaces such that Y^{**} or X^* has the weak Radon–Nikodým property. Then, for all $\phi \in \mathcal{K}_{w^*w}(X^*,Y^*)^*$, there exist a sequence $(((x_i^n)^*)_{i=1}^{m_n})_{n=1}^{\infty}$ in X^* and a sequence $(((y_i^n)^{**})_{i=1}^{m_n})_{n=1}^{\infty}$ in Y^{**} such that*

$$\langle \phi, T\rangle = \lim_{n\to\infty}\sum_{i=1}^{m_n}(y_i^n)^{**}(T((x_i^n)^*))$$

for all $T \in \mathcal{K}_{w^*w}(X^*,Y^*)$, and

$$\limsup_{n\to\infty}\sum_{i=1}^{m_n}\|(x_i^n)^*\|\,\|(y_i^n)^{**}\| \le \|\phi\|.$$

Combining Corollary 4.3 and the proof of [CK, Theorem 1.2], we obtain the following corollary. For completeness, we provide a proof.

Corollary 4.4. *Suppose that X^* or Y^* has the weak Radon–Nikodým property. Then, for all $\phi \in \mathcal{K}_{w^*w}(X^*,Y)^*$, there exist a sequence*

$(((x_i^n)^*)_{i=1}^{m_n})_{n=1}^\infty$ *in* X^* *and a sequence* $(((y_i^n)^*)_{i=1}^{m_n})_{n=1}^\infty$ *in* Y^* *such that*

$$\langle \phi, T\rangle = \lim_{n\to\infty} \sum_{i=1}^{m_n} (y_i^n)^*(T((x_i^n)^*)$$

for all $T \in \mathcal{K}_{w^*w}(X^*, Y)$, *and*

$$\limsup_{n\to\infty} \sum_{i=1}^{m_n} \|(x_i^n)^*\| \, \|(y_i^n)^*\| \le \|\phi\|.$$

Proof. (a) Suppose that X^* has the weak Radon–Nikodým property. We define $\Psi : \mathcal{K}_{w^*w}(X^*, Y) \to \mathcal{K}_{w^*w}(X^*, Y^{**})$ by $\Psi(T) = Q_Y T$, where Q_Y is the natural mapping from Y into Y^{**}. We observe that Ψ is an isometry. Take $\phi \in \mathcal{K}_{w^*w}(X^*, Y)^*$. Then there exists $\hat{\phi} \in \mathcal{K}_{w^*w}(X^*, Y^{**})^*$ such that $\langle \phi, T\rangle = \langle \hat{\phi}, \Psi(T)\rangle$ for all $T \in \mathcal{K}_{w^*w}(X^*, Y)$ and $\|\phi\| = \|\hat{\phi}\|$. By Corollary 4.3, there exist a sequence $(((x_i^n)^*)_{i=1}^{m_n})_{n=1}^\infty$ in X^* and a sequence $(((y_i^n)^{***})_{i=1}^{m_n})_{n=1}^\infty$ in Y^{***} such that

$$\langle \hat{\phi}, S\rangle = \lim_{n\to\infty} \sum_{i=1}^{m_n} (y_i^n)^{***}(S((x_i^n)^*))$$

for all $S \in \mathcal{K}_{w^*w}(X^*, Y^{**})$, and $\limsup_n \sum_{i=1}^{m_n} \|(x_i^n)^*\| \, \|(y_i^n)^{***}\| \le \|\hat{\phi}\|$. We consider $(((x_i^n)^*)_{i=1}^{m_n})_{n=1}^\infty \subset X^*$ and $(((y_i^n)^{***} Q_Y)_{i=1}^{m_n})_{n=1}^\infty \subset Y^*$. Then

$$\langle \phi, T\rangle = \langle \hat{\phi}, \Psi(T)\rangle = \lim_{n\to\infty} \sum_{i=1}^{m_n} (y_i^n)^{***}(Q_Y T((x_i^n)^*))$$

$$= \lim_{n\to\infty} \sum_{i=1}^{m_n} (y_i^n)^{***} Q_Y (T((x_i^n)^*))$$

for all $T \in \mathcal{K}_{w^*w}(X^*, Y)$, and

$$\limsup_{n\to\infty} \sum_{i=1}^{m_n} \|(x_i^n)^*\| \, \|(y_i^n)^{***} Q_Y\| \le \|\phi\|.$$

(b) Suppose that Y^* has the weak Radon–Nikodým property. We define

$$\psi : \mathcal{K}_{w^*w}(X^*, Y) \to \mathcal{K}_{w^*w}(Y^*, X)$$

by $\psi(T) = Q_X^{-1} T^*$, where Q_X is the natural map from X into X^{**}. It is known that $T \in \mathcal{B}(X^*, Y)$ is w^*-to-w continuous if and only if $T^*(Y^*) \subset Q_X(X)$. Thus ψ is well defined and it is an isometry onto $\mathcal{K}_{w^*w}(Y^*, X)$.

Now let $\phi \in \mathcal{K}_{w^*w}(X^*, Y)^*$. We consider $\hat{\phi} : \mathcal{K}_{w^*w}(Y^*, X) \to \mathbb{F}$ given by

$$\hat{\phi}(S) = \phi(\psi^{-1}(S))$$

for all $S \in \mathcal{K}_{w^*w}(Y^*, X)$. Then $\hat{\phi} \in \mathcal{K}_{w^*w}(Y^*, X)^*$ and $\|\phi\| = \|\hat{\phi}\|$. By part (a), there exist a sequence $(((x_i^n)^*)_{i=1}^{m_n})_{n=1}^\infty$ in X^* and a sequence

$(((y_i^n)^*)_{i=1}^{m_n})_{n=1}^{\infty}$ in Y^* such that

$$\langle \hat{\phi}, S\rangle = \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^*(S((y_i^n)^*))$$

for all $S \in \mathcal{K}_{w^*w}(Y^*, X)$, and $\limsup_n \sum_{i=1}^{m_n} \|(x_i^n)^*\| \, \|(y_i^n)^*\| \le \|\hat{\phi}\| = \|\phi\|$. Hence

$$\begin{aligned}\langle \phi, T\rangle = \langle \phi, \psi^{-1}\psi T\rangle = \langle \hat{\phi}, \psi T\rangle &= \lim_{n\to\infty} \sum_{i=1}^{m_n} (x_i^n)^*(\psi T((y_i^n)^*)) \\ = \lim_{n\to\infty} \sum_{i=1}^{m_n} Q_X(\psi T((y_i^n)^*))(x_i^n)^* &= \lim_{n\to\infty} \sum_{i=1}^{m_n} (y_i^n)^*(T((x_i^n)^*))\end{aligned}$$

for all $T \in \mathcal{K}_{w^*w}(X^*, Y)$. ∎

Furthermore, the following can be obtained directly from our corollary.

COROLLARY 4.5. *If X^* or Y^* has the weak Radon–Nikodým property, then $X^* \otimes Y^*$ is w^*-sequentially dense in $\mathcal{K}_{w^*w}(X^*, Y)^*$, and $B_{X^*\otimes Y^*}$ is w^*-sequentially dense in $B_{\mathcal{K}_{w^*w}(X^*,Y)^*}$.*

Proof. This can be proved in the same manner as Corollary 4.1. For completeness, we just provide the proof in the case that X^* has the weak Radon–Nikodým property. Let ϕ be in $\mathcal{K}_{w^*w}(X^*, Y)^*$. Then, by assumption and Corollary 4.4, we have $\langle a_n, T\rangle \to \langle \phi, T\rangle$ for all $T \in \mathcal{K}_{w^*w}(X^*, Y)$, where $a_n = \sum_{i=1}^{m_n} (x_i^n)^* \otimes (y_i^n)^* \in X^* \otimes Y^*$. Hence $X^* \otimes Y^*$ is w^*-sequentially dense in $\mathcal{K}_{w^*w}(X^*, Y)^*$.

Now we claim that $\|a_n\| \le \|\phi\|$ for all $n \in \mathbb{N}$. Indeed, as in the proofs of Corollaries 4.1 and 4.4, we observe that

$$a_n = \sum_{i=1}^{m_n} (x_i^n)^* \otimes j^{**}((z_i^n)^{**})Q_Y, \qquad v_n = \sum_{i=1}^{m_n} (x_i^n)^* \otimes (z_i^n)^{**}$$

for all $n \in \mathbb{N}$ where j is the canonical quotient map of $\ell^1(\Gamma_1)$ onto Y^* and $((z_i^n)^{**})_{i=1}^{m_n}$ is in $\ell^1(\Gamma_1)^{**}$. Let T be in $B_{\mathcal{K}_{w^*w}(X^*,Y)^*}$. Since $Q_Y T \in \mathcal{K}_{w^*w}(X^*, Y^{**})$, there is $S \in \mathcal{K}(Y^*, X)$ such that $S^* = Q_Y T$. Also, since $\|S\| = \|Sj\|$, we have

$$\|T\| = \|Q_Y T\| = \|S^*\| = \|S\| = \|Sj\|.$$

Then

$$|\langle a_n, T\rangle| = \Big|\sum_{i=1}^{m_n} (y_i^n)^*(T((x_i^n)^*))\Big| = \Big|\sum_{i=1}^{m_n} j^{**}((z_i^n)^{**})Q_Y(T((x_i^n)^*))\Big|$$

$$= \Big|\sum_{i=1}^{m_n} j^{**}((z_i^n)^{**})Q_Y T((x_i^n)^*)\Big| = \Big|\sum_{i=1}^{m_n} j^{**}((z_i^n)^{**})S^*((x_i^n)^*)\Big|$$
$$= \Big|\sum_{i=1}^{m_n}(z_i^n)^{**}(Sj)^*((x_i^n)^*)\Big| = |\langle v_n, Sj\rangle| \le \|v_n\|_{\mathrm{PI}} \le \|\phi\|$$

for all $n \in \mathbb{N}$. ■

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