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The minimal operator and the geometric maximal operator in $\mathbb{R}^n$

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Abstract. We prove two-weight norm inequalities in $\mathbb{R}^n$ for the minimal operator

$$\mathcal{m}f(x) = \inf_{Q \ni x} \frac{1}{|Q|} \int_Q |f|\, dy,$$

extending to higher dimensions results obtained by Cruz-Uribe, Neugebauer and Olesen [8] on the real line. As an application we extend to $\mathbb{R}^n$ weighted norm inequalities for the geometric maximal operator

$$M_0 f(x) = \sup_{Q \ni x} \exp\left(\frac{1}{|Q|} \int_Q \log|f|\, dx\right),$$

proved by Yin and Muckenhoupt [27].

We also give norm inequalities for the centered minimal operator, study powers of doubling weights and give sufficient conditions for the geometric maximal operator to be equal to the closely related limiting operator $M_0^* f = \lim_{r\to 0} M(|f|^r)^{1/r}$.

1. Introduction

1.1. *The minimal operator.* Given a measurable function f, the minimal function of f, $\mathcal{m}f$, is defined by

$$\mathcal{m}f(x) = \inf_{Q \ni x} \frac{1}{|Q|} \int_Q |f|\, dy, \tag{1.1}$$

where the supremum is taken over all cubes Q whose sides are parallel to the co-ordinate axes. By the Lebesgue differentiation theorem, $\mathcal{m}f(x) \le |f(x)| \le Mf(x)$ almost everywhere; intuitively, the minimal operator controls where f is small, just as the Hardy–Littlewood maximal operator controls where f is large.

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The minimal operator was introduced by Cruz-Uribe and Neugebauer [5], who used it to study the fine structure of A_p weights. They also considered the weighted norm inequalities for the minimal operator. This question is complicated by the fact that if $f \in L^p(\mathbb{R}^n)$ then $\mathcal{m}f(x) \equiv 0$. One solution is to take inverses, making something small (the minimal operator) into something large (which can be measured by a norm inequality). In the one-weight case they proved the following result. Here and below by a weight we mean a non-negative, locally integrable function.

THEOREM 1.1. *Given a weight w, the following are equivalent*:

(i) $w \in A_\infty$;

(ii) *there exists a constant C such that for every p, $0 < p < \infty$, and every $t > 0$,*

$$w(\{x \in \mathbb{R}^n : \mathcal{m}f(x) < 1/t\}) \leq \frac{C}{t^p} \int_{\mathbb{R}^n} \frac{w}{|f|^p}\, dx$$

for every f such that $1/f \in L^p(w)$;

(iii) *there exists a constant C such that for every p, $0 < p < \infty$,*

$$\int_{\mathbb{R}^n} \frac{w}{(\mathcal{m}f)^p}\, dx \leq C \int_{\mathbb{R}^n} \frac{w}{|f|^p}\, dx$$

for every f such that $1/f \in L^p(w)$.

Theorem 1.1 follows from Hölder's inequality and the one-weight norm inequalities for the maximal operator. Two-weight norm inequalities are significantly more difficult: working on the real line, Cruz-Uribe, Neugebauer and Olesen [8] were able to prove the following result.

THEOREM 1.2. *Given a pair of weights (u, v) and p, $0 < p < \infty$, let $\sigma = v^{1/(p+1)}$. Then the following are equivalent*:

(i) $(u, v) \in W_p$: *there exists a constant C_1 such that for every interval I,*

$$\frac{1}{|I|}\int_I u\, dx \leq C_1 \left(\frac{1}{|I|}\int_I \sigma\, dx\right)^{p+1};$$

(ii) *there exists a constant C_2 such that for every $t > 0$,*

$$u(\{x \in \mathbb{R} : \mathcal{m}f(x) < 1/t\}) \leq \frac{C_2}{t^p}\int_{\mathbb{R}} \frac{v}{|f|^p}\, dx$$

for every f such that $1/f \in L^p(v)$;

(iii) $(u, v) \in W_p^*$: *there exists a constant C_3 such that for every interval I,*

$$\int_I \frac{u}{\mathcal{m}(\sigma/\chi_I)^p}\, dx \leq C_3 \int_I \sigma\, dx;$$

(iv) *there exists a constant* C_4 *such that*

$$\int_{\mathbb{R}} \frac{u}{(mf)^p}\,dx \le C_4 \int_{\mathbb{R}} \frac{v}{|f|^p}\,dx$$

for every f *such that* $1/f \in L^p(v)$.

REMARKS. (i) In Theorem 1.2, the constants C_2 and C_4 depend on C_1 and C_3, respectively, but do not depend on p. However, $C_3 \approx pC_1$.

(ii) Unlike the Hardy–Littlewood maximal operator, the strong and weak-type norm inequalities for the minimal operator are governed by the same pairs of weights.

(iii) In the one-weight case the W_p condition becomes a reverse Hölder inequality which is equivalent to the A_∞ condition. See [5] for details.

The proof of Theorem 1.2 in [8] depends heavily on the special covering lemmas for the real line, and does not extend in a satisfactory manner to higher dimensions. In particular, to prove the weak-type norm inequality in $\mathbb{R}^n$ we had to assume that u was a doubling weight, but to prove the strong-type norm inequality we had to assume (among other things) that $\sigma = v^{1/(p+1)}$ was a doubling weight.

The main result of this paper is an extension of Theorem 1.2 to $\mathbb{R}^n$, $n > 1$, with uniform assumptions on u and v.

THEOREM 1.3. *Given* p, $0 < p < \infty$, *let* (u, v) *be a pair of weights such that either* u *or* $\sigma = v^{1/(p+1)}$ *is a doubling weight. Then in* $\mathbb{R}^n$, $n > 1$, *conditions* (i)–(iv) *of Theorem* 1.2 *are equivalent*, *with intervals replaced by cubes in* (i) *and* (iii).

The relation between the constants C_i, $1 \le i \le 4$, *depends on whether* u *or* σ *is doubling. Let* B_n *be the constant in the Besicovitch–Morse covering lemma* (see Lemma 2.1 below). *If* u *is a doubling weight with constant* $D(u)$ *then*

$$\begin{aligned} C_2 &\le B_n D(u) C_1, \\ C_3 &\le C_1 + pB_n 2^{-n} D(u) C_1, \\ C_4 &\le 8D(u)^2 C_3 + 8 \cdot 3^n p B_n D(u) C_3. \end{aligned}$$

If σ *is a doubling weight with constant* $D(\sigma)$ *then*

$$\begin{aligned} C_2 &\le 2^{-np} B_n D(\sigma)^{p+1} C_1, \\ C_3 &\le C_1 + pB_n 2^{-n(p+1)} D(\sigma)^{p+1} C_1, \\ C_4 &\le 8D(\sigma)^2 C_3. \end{aligned}$$

1.2. *The geometric maximal operator.* As an application of Theorem 1.3 we prove two-weight norm inequalities in $\mathbb{R}^n$ for the geometric maximal operator

$$M_0 f(x) = \sup_{Q \ni x} \exp\left(\frac{1}{|Q|}\int_Q \log|f|\,dx\right),$$

and the closely related limiting operator

$$M_0^* f(x) = \lim_{r\to 0} M_r f(x) = \lim_{r\to 0} M(|f|^r)(x)^{1/r}.$$

Norm inequalities for the geometric maximal operator on the real line were studied by Yin and Muckenhoupt [27] and others. Cruz-Uribe and Neugebauer [6] used Theorem 1.2 to give a new proof of their results and to prove analogous results for M_0^*. A key step in their proof is showing that for a dense family of functions f,

$$M_0 f(x) = \lim_{r\to 0} m(|f|^{-r})(x)^{-1/r}, \qquad x \in \mathbb{R}^n.$$

They used this to derive norm inequalities for the geometric maximal operator from norm inequalities for the minimal operator via a limiting argument. This was possible since the constants C_2 and C_4 in Theorem 1.2 are independent of p. (See [6] for details of the proof and for the history of the problem.)

Using Theorem 1.3 we can immediately extend their results to higher dimensions—their proofs go through without change.

THEOREM 1.4. *Given a pair of weights (u, v), suppose that either u is doubling or $\sigma_q = v^{1/(q+1)}$ is doubling for all q sufficiently large and*

$$\limsup_{q\to\infty} 2^{-nq} D(\sigma_q)^{q+1} < \infty.$$

Then the following are equivalent:

(i) $(u, v) \in W_\infty$: *there exists a constant C such that for every cube Q,*

$$\frac{1}{|Q|}\int_Q u\,dx \le C \exp\left(\frac{1}{|Q|}\int_Q \log v\,dx\right);$$

(ii) *there exists a constant C such that for all p, $0 < p < \infty$,*

$$u(\{x \in \mathbb{R}^n : M_0 f(x) > t\}) \le \frac{C}{t^p}\int_{\mathbb{R}^n} |f|^p v\,dx$$

for every $f \in L^p(v)$.

THEOREM 1.5. *Given a pair of weights (u, v), suppose that $\sigma_q = v^{1/(q+1)}$ is doubling for all q sufficiently large and*

$$\limsup_{q\to\infty} D(\sigma_q) < \infty.$$

Then the following are equivalent:

(i) $(u,v) \in W_\infty^*$: *there exists a constant* C *such that for all cubes* Q,

$$\int_Q M_0(v^{-1}\chi_Q)u\,dx \le C|Q|;$$

(ii) *there exists a constant* C *such that for all* p, $0 < p < \infty$,

$$\int_{\mathbb{R}^n} (M_0 f)^p u\,dx \le C \int_{\mathbb{R}^n} |f|^p v\,dx$$

for every $f \in L^p(v)$.

THEOREM 1.6. *Theorems* 1.4 *and* 1.5 *remain true if* M_0 *is everywhere replaced by* M_0^*, *provided that* $v \in I_\infty$:

$$\limsup_{Q,\sigma} \frac{1}{|Q|}\left(\frac{1}{|Q|}\int_Q v^{-\sigma}\,dx\right)^{1/\sigma} < \infty,$$

where the upper limit is taken over all cubes Q *containing the origin and all* $\sigma > 0$ *as* $|Q|$ *tends to infinity and* σ *tends to* 0. *The* I_∞ *condition is necessary as well as sufficient.*

REMARKS. (i) Unlike the minimal operator, the two-weight, weak and strong-type norm inequalities for the geometric maximal operator are governed by different weight classes. See Yin and Muckenhoupt [27] or Cruz-Uribe and Neugebauer [6] for an example. We suspect that this is connected to the fact that doubling conditions on u are sufficient in Theorem 1.4 but not in Theorem 1.5, but we are uncertain as to what the exact relation is.

(ii) A serendipitous consequence of our work on the geometric maximal operator was the discovery that Theorem 1.3 could be re-interpreted as a result about a maximal operator. Define the harmonic maximal operator by

$$M_{-1}f(x) = \sup_{Q \ni x}\left(\frac{1}{|Q|}\int_Q |f|^{-1}\,dy\right)^{-1};$$

in other words, $M_{-1}f(x)$ is the supremum of the harmonic averages of f on cubes containing x. It is immediate that

$$M_{-1}f(x) = m(f^{-1})(x)^{-1},$$

and Theorem 1.3 can be restated in terms of M_{-1}.

1.3. *Doubling conditions.* While the doubling conditions we assume in Theorems 1.3—1.6 are not unreasonable (cf. Wheeden [23]) they are not necessary: in Example 5.8 below we construct a pair $(u,v) \in W_p$ such that neither u nor v is doubling, but the weak and strong-type inequalities hold. However, we have repeatedly and unsuccessfully attempted either to prove Theorem 1.3 without doubling conditions or to construct a non-doubling pair

(u, v) which satisfy the W_p condition but for which the norm inequalities do not hold. This question remains open.

As a corollary to our proofs, we can show that the strong W_p condition,

$$\frac{1}{|2Q|}\int_{2Q} u\,dx \le C_1'\left(\frac{1}{|Q|}\int_Q \sigma\,dx\right)^{p+1}, \tag{1.2}$$

and the strong W_p^* condition,

$$\int_{3Q} \frac{u}{\mathcal{m}(\sigma/\chi_{3Q})^p}\,dx \le C_3'\int_Q \sigma\,dx, \tag{1.3}$$

are equivalent and are sufficient—without additional doubling conditions—for the weak and strong-type norm inequalities for the minimal operator to hold. However, Example 5.8 below shows that they are not necessary.

The strong W_p condition also yields a sufficient condition for Theorem 1.4 and the corresponding part of Theorem 1.6 to hold:

$$\text{strong } W_\infty\colon\quad \frac{1}{|2Q|}\int_{2Q} u\,dx \le C\exp\left(\frac{1}{|Q|}\int_Q \log v\,dx\right). \tag{1.4}$$

Surprisingly, the strong W_p^* condition does not yield a satisfactory sufficient condition for Theorem 1.5 to hold. The best we can obtain is a Sawyer-type condition involving a related but larger maximal operator.

1.4. *The centered minimal operator.* Another approach to eliminating doubling conditions is to modify the operator by restricting the collection of cubes over which it is defined. For example, while the weighted maximal operator, M_w, is not necessarily bounded in higher dimensions unless w is doubling (see Sjögren [20]), the weighted centered maximal operator and the weighted dyadic maximal operator are for all weights w.

The proof of Theorem 1.3 below or the proof of Theorem 1.2 in [8] are readily adapted to the dyadic minimal operator. The results in [6] also carry over to the dyadic geometric maximal operator with little change. Details are left to the reader.

The centered minimal operator, $\mathcal{m}_c$ (defined as in equation (1.1) but with the infimum restricted to cubes centered at x), is both more interesting and more difficult. As was noted in [8], unlike the maximal operator, the minimal operator and the centered minimal operator are not equivalent. A simple example on the real line is given by e^x: $\mathcal{m}(e^x) \equiv 0$ but $\mathcal{m}_c(e^x) = e^x$. In [8] it was conjectured that a "centered W_p" condition is necessary and sufficient for the weak-type inequality for $\mathcal{m}_c$. Here we adapt the proof of Theorem 1.3 to $\mathcal{m}_c$ and show that this condition (and the corresponding Sawyer-type condition) are necessary and sufficient for both the strong and weak-type norm inequalities.

THEOREM 1.7. *Given weights* (u,v) *and* p, $0<p<\infty$, *let* $\sigma = v^{1/(p+1)}$. *Then the following are equivalent*:

(i) $(u,v)\in W_{p,c}$: *there exists a constant* D_1 *such that for every cube* Q,

$$\frac{1}{|Q|}\int_Q u\,dx \le D_1\bigg(\frac{1}{|2Q|}\int_{2Q}\sigma\,dx\bigg)^{p+1};$$

(ii) *there exists a constant* D_2 *such that for every* $t>0$,

$$u(\{x\in\mathbb{R}^n : m_c f(x) < 1/t\}) \le \frac{D_2}{t^p}\int_{\mathbb{R}^n}\frac{v}{|f|^p}\,dx$$

for every f *such that* $1/f\in L^p(v)$;

(iii) $(u,v)\in W^*_{p,c}$: *there exists a constant* D_3 *such that for every cube* Q,

$$\int_Q \frac{u}{m_c(\sigma/\chi_Q)^p}\,dx \le D_3\int_Q \sigma\,dx;$$

(iv) *there exists a constant* D_4 *such that*

$$\int_{\mathbb{R}^n}\frac{u}{(m_c f)^p}\,dx \le D_4\int_{\mathbb{R}^n}\frac{v}{|f|^p}\,dx$$

for every f *such that* $1/f\in L^p(v)$.

REMARKS. (i) Unfortunately, except in the case $n=1$ (for which case we have a special proof) the constants D_2 and D_4 are not independent of p, so we cannot use the limiting technique of Cruz-Uribe and Neugebauer to prove the analogues of Theorems 1.4–1.6 for the centered geometric maximal operator, $M_{0,c}$. Even in the case $n=1$ the proof breaks down, since the Sawyer-type condition

$$W^*_{\infty,c}:\qquad \int_Q M_{0,c}(v^{-1}\chi_Q)u\,dx \le C|Q|$$

does not seem to imply the $W^*_{p,c}$ condition. The question of weighted norm inequalities for the centered geometric maximal operator and their relationship to norm inequalities for the centered minimal operator remains open.

(ii) We do not have control of the constant D_2 since for the centered minimal operator we do not have a direct proof that the $W_{p,c}$ condition implies the weak-type inequality. It would be very interesting to have such a proof, as it may yield a proof of a weak-type inequality for $M_{0,c}$.

1.5. *Organization.* The remainder of this paper is organized as follows:

In Sections 2–4 we prove Theorem 1.3. In Section 2 we show that the W_p condition is equivalent to the weak-type inequality; in Section 3 we show that the W_p and W_p^* conditions are the same; and in Section 4 we show that the W_p^* condition is equivalent to the strong-type inequality. At the end of each section we derive from the proofs the non-doubling sufficient conditions for the minimal operator and geometric maximal operator discussed above.

In Section 5 we examine the doubling conditions imposed on v in Theorems 1.3–1.5. We give examples of weights for which they do hold, and construct an example to show that there exists a doubling weight v such that v^r is not doubling for any r, $0 < r < 1$. This answers in the negative questions posed in Cruz-Uribe [3, p. 561] and Cruz-Uribe, Neugebauer and Olesen [8]. We also construct a pair of functions (u, v) in $\mathbb{R}^n$ which satisfy the W_p condition, are not doubling, and for which the strong-type norm inequality for m holds.

In Section 6 we prove Theorem 1.7. The key to the proof is a slightly stronger version of the Besicovitch–Morse covering lemma. Further, we prove an analogue of Theorem 1.1 by showing that in the one-weight case the $W_{p,c}$ condition is equivalent to a weak A_∞ condition due to Sawyer [19]: given any cube Q and a measurable set $E \subset Q$, there exist constants C and δ such that

$$\text{(1.5)} \qquad \text{weak } A_\infty\text{:} \quad \frac{w(E)}{w(2Q)} \leq C\left(\frac{|E|}{|Q|}\right)^{\delta}.$$

Finally, in Section 7 we consider a problem first studied in detail by Cruz-Uribe and Neugebauer [6]: sufficient conditions on a function f for the equality $M_0 f(x) = M_0^* f(x)$ to hold. They showed that this equality does not hold in general, but showed that if $\log f$ is locally integrable and $f \in L^p$ for some $p > 0$ then it holds almost everywhere. We generalize this result considerably.

THEOREM 1.8. *Let $v \in I_\infty$. Suppose $f \in L^p(v)$ for some $p > 0$ and suppose there exists a cube Q_0 (possibly infinite) such that* $\operatorname{supp} f = Q_0$ *and* $\log f \in L^1_{\mathrm{loc}}(Q_0)$. *Then $M_0 f(x) = M_0^* f(x)$ almost everywhere. Further, equality almost everywhere is the best possible.*

Theorem 1.8 is a partial generalization of a recent result by Wik [26], and we discuss the relation between the two results. Also, as a corollary to Theorem 1.8 we show that the best constants in the one-weight, strong-type norm inequalities for M_0 and for the limiting operator M_0^* are the same. This proves a conjecture made by Wik.

Throughout this paper all notation is standard or will be defined as needed. For the convenience of the reader, the principal definitions will be repeated in the relevant sections. All cubes are assumed to have their sides parallel to the co-ordinate axes. Given a cube Q, $l(Q)$ will denote the length of its sides and for any $r > 0$, rQ will denote the cube with the same center as Q and such that $l(rQ) = rl(Q)$. By *weights* we will always mean non-negative, locally integrable functions which are positive on a set of positive measure. Given a Lebesgue measurable set E and a weight w, $|E|$ will denote the Lebesgue measure of E, $w(E) = \int_E w\,dx$ and w/χ_E will denote the function equal to w on E and infinity elsewhere. We say that a non-negative measure μ is a *doubling measure* if for any cube Q, $\mu(2Q) \le D(\mu)\mu(Q)$; $D(\mu)$ is called the *doubling constant* of μ. If μ satisfies the doubling condition for all Q such that $2Q \subset Q_0$, we say that μ is a doubling measure on Q_0. If μ is doubling and $d\mu = w\,dx$, we say that w is a *doubling weight*. Given $1 < p < \infty$, $p' = p/(p-1)$ will denote the conjugate exponent of p. Finally, C will denote a positive constant whose value may change at each appearance.

2. The weak-type norm inequality. In this section we prove that the W_p condition,

$$\frac{1}{|Q|}\int_Q u\,dx \le C_1\left(\frac{1}{|Q|}\int_Q \sigma\,dx\right)^{p+1}, \tag{2.1}$$

is equivalent to the weak-type inequality for the minimal operator in $\mathbb{R}^n$. The proof requires the Besicovitch–Morse covering lemma. We state the precise version we need; for a proof see de Guzmán [13].

LEMMA 2.1. *Given a bounded set E in $\mathbb{R}^n$, suppose that for each $x \in E$ there exists a cube Q_x such that $x \in \frac{1}{2}Q_x$. Then there exists a sequence $\{Q_k\}$ of Q_x's such that E is contained in the union of the Q_k's; further, there exists a constant B_n, depending only on the dimension n, such that every point of $\mathbb{R}^n$ is contained in at most B_n of the Q_k's.*

Throughout this section, let $\sigma = v^{1/(p+1)}$.

To show that the weak-type norm inequality implies (2.1), fix a cube Q and let $f = \sigma\chi_Q$. If $\sigma(Q) = \infty$ then (2.1) is immediate. If it is finite then for each $x \in Q$,

$$\mathcal{m}f(x) \le \frac{1}{|Q|}\int_Q \sigma\,dx.$$

If we let $1/t$ in the weak-type inequality equal the right-hand side we get

$$u(Q) \le C_2\left(\frac{1}{|Q|}\int_Q \sigma\,dx\right)^{p} \cdot \int_Q \sigma\,dx,$$

and this is the W_p condition.

To prove that (2.1) implies the weak-type inequality, fix a function f such that $1/f \in L^p(v)$. Without loss of generality we may assume that f is non-negative. Further, we first assume that there exists a cube P such that f is positive on P and $\operatorname{supp}(1/f) = P$.

Fix $t > 0$ and let $E_t = \{x \in \mathbb{R}^n : \mathcal{M}f(x) < 1/t\}$. Then for every $x \in E_t$ there exists a cube Q_x containing x such that

$$\frac{1}{|Q_x|} \int_{Q_x} f\,dy < 1/t.$$

Each Q_x is contained in the cube P and E_t is contained in their union. Therefore, by Lemma 2.1, there exists a sequence $\{Q_j\}$ of Q_x's such that $\bigcup_j 2Q_j$ also contains E_t and such that the Q_j's have overlap of at most B_n.

Since u or σ is doubling, (2.1) implies the strong W_p condition (1.2); if u is doubling then $C_1' = 2^{-n}D(u)C_1$; otherwise, if σ is doubling then $C_1' = 2^{-n(p+1)}D(\sigma)^{p+1}C_1$. Therefore,

$$u(E_t) \leq \sum_j u(2Q_j) \leq 2^n C_1' \sum_j |Q_j|^{-p} \Big(\int_{Q_j} \sigma\,dx \Big)^{p+1}.$$

By Hölder's inequality, for each j,

$$\int_{Q_j} \sigma\,dx \leq \Big(\int_{Q_j} \frac{v}{f^p}\,dx \Big)^{1/(p+1)} \Big(\int_{Q_j} f\,dx \Big)^{p/(p+1)}.$$

Hence, since $Q_j = Q_x$ for some $x \in E_t$,

$$\begin{aligned} 2^n C_1' \sum_j |Q_j|^{-p} \Big(\int_{Q_j} \sigma\,dx \Big)^{p+1} &\leq \frac{2^n C_1'}{t^p} \sum_j \Big(\int_{Q_j} \sigma\,dx \Big)^{p+1} \Big(\int_{Q_j} f\,dx \Big)^{-p} \\ &\leq \frac{2^n C_1'}{t^p} \sum_j \int_{Q_j} \frac{v}{f^p}\,dx \\ &\leq \frac{2^n B_n C_1'}{t^p} \int_{\mathbb{R}^n} \frac{v}{f^p}\,dx. \end{aligned}$$

This implies that the weak-type inequality holds with $C_2 = 2^n B_n C_1'$.

To complete the proof, fix any non-negative $f \in L^p(v)$ and define the sequence of functions $f_k = (f + 1/k)/\chi_{P_k}$, where P_k is the cube centered at the origin with $l(P_k) = k$. Then each f_k is strictly positive on P_k, $\operatorname{supp}(1/f_k) = P_k$ and $1/f_k \in L^p(v)$ since $1/f \in L^p(v)$. The sequence $\{f_k\}$ decreases to f, so the sequence $\{\mathcal{M}f_k\}$ also decreases and $\lim_{n\to\infty} \mathcal{M}f_k \geq \mathcal{M}f$. On the other hand, for a fixed $x \in \mathbb{R}^n$ and any $\varepsilon > 0$, there exists a cube Q such that for all k sufficiently large, $Q \subset P_k$ and

$$\mathcal{M}f(x) \geq \frac{1}{|Q|} \int_Q f\,dy - \varepsilon \geq \mathcal{M}f_k(x) - \varepsilon.$$

Therefore, $\{\mathcal{M}f_k\}$ converges to $\mathcal{M}f$, and by the monotone convergence theorem the strong-type inequality holds for all f.

REMARK. Even without a doubling condition, the W_p condition is necessary for the weak-type inequality to hold. In the proof that it is sufficient, we only used a doubling condition to prove that the strong W_p condition holds. Hence the strong W_p condition is itself a sufficient condition which does not involve doubling. However, Example 5.8 shows that this condition is not necessary. By Jensen's inequality, if (u,v) satisfy the strong W_∞ condition (1.4) then (u,v) satisfy the strong W_p condition for all $p>0$ with a constant independent of p. This is enough to show that the strong W_∞ condition is sufficient for the weak-type norm inequalities for M_0 and M_0^* to hold (for the latter provided $v\in I_\infty$). See [6] for details.

3. The equivalence of W_p and W_p^*. In this section we prove that the W_p condition,

$$\frac{1}{|Q|}\int_Q u\,dx \le C_1\bigg(\frac{1}{|Q|}\int_Q \sigma\,dx\bigg)^{p+1}, \tag{3.1}$$

and the W_p^* condition,

$$\int_Q \frac{u}{m(\sigma/\chi_Q)^p}\,dx \le C_3\int_Q \sigma\,dx, \tag{3.2}$$

are equivalent. Throughout this section, let $\sigma = v^{1/(p+1)}$.

We first show that (3.2) implies (3.1). Fix a cube Q. If $\sigma(Q)=\infty$ then the W_p condition holds trivially, so assume it is finite. Then for every $x\in Q$, $m(\sigma/\chi_Q)(x)\le\sigma(Q)$. If $\sigma(Q)=0$ then the left-hand side of the W_p^* condition is finite only if $u(Q)=0$, and again the W_p condition holds trivially. But if $\sigma(Q)>0$ then we can substitute this inequality into (3.2) and (3.1) follows immediately.

To prove that (3.1) implies (3.2), fix a cube Q. We may assume without loss of generality that $u(Q)>0$ and $\sigma(Q)<\infty$, since otherwise the W_p^* condition holds trivially. For each $t>0$ let $E_t=\{x\in Q : m(\sigma/\chi_Q)(x)<1/t\}$. Then

$$\int_Q \frac{u}{m(\sigma/\chi_Q)^p}\,dx = p\int_0^R t^{p-1}u(E_t)\,dt + p\int_R^\infty t^{p-1}u(E_t)\,dt, \tag{3.3}$$

where R will be chosen below. Since $u(E_t)\le u(Q)$, the first integral on the right-hand side is bounded by $u(Q)R^p$.

To estimate the second integral on the right-hand side of (3.3) we proceed as in the proof of the weak-type inequality. Given $t>0$, for each $x\in E_t$ there exists a cube $Q_x^t\subset Q$ containing x such that

$$\frac{1}{|Q_x^t|}\int_{Q_x^t}\sigma\,dx < 1/t. \tag{3.4}$$

The cubes $2Q_x^t$ cover E_t, so by Lemma 2.1 there exists a sequence of cubes $\{Q_j^t\}$ which have finite overlap and such that the cubes $2Q_j^t$ still cover E_t. If we combine this with the strong W_p condition (1.2) (which holds since u or σ is doubling) and inequality (3.4), it follows that

$$
\begin{aligned}
(3.5)\qquad p\int_R^\infty t^{p-1}u(E_t)\,dt &\le p\int_R^\infty t^{p-1}\sum_j u(2Q_j^t)\,dt \\
&\le 2^n pC_1'\int_R^\infty t^{p-1}\sum_j |Q_j^t|\bigg(\frac{1}{|Q_j^t|}\int_{Q_j^t}\sigma\,dx\bigg)^{p+1} \\
&\le 2^n pC_1'\int_R^\infty t^{-2}\sum_j |Q_j^t|\,dt \\
&\le 2^n pC_1' B_n\int_R^\infty t^{-2}|E_t|\,dtr \le 2^n pC_1' B_n R^{-1}|Q|.
\end{aligned}
$$

Therefore,

$$
(3.6)\qquad \int_Q \frac{u}{m(\sigma/\chi_Q)^p}\,dx \le u(Q)R^p + 2^n pC_1' B_n R^{-1}|Q|.
$$

Let $R = |Q|/\sigma(Q)$. Then by (3.1), $u(Q)R^p \le C_1\sigma(Q)$. Hence

$$
\int_Q \frac{u}{m(\sigma/\chi_Q)^p}\,dx \le (C_1 + 2^n pC_1' B_n)\sigma(Q),
$$

which is (3.2) with $C_3 = C_1 + 2^n pC_1' B_n$.

Remark. As with the weak-type inequality in Section 2, we only used a doubling condition to get the strong W_p condition (1.2). Notice, however, that C_3 contains a factor of p. Because of this, even though the weak and strong-type norm inequalities for the minimal operator are both governed by the same class of weights—W_p—this is no longer the case for the geometric maximal operator.

4. The strong-type norm inequality. In this section we complete the proof of Theorem 1.3 by showing that the W_p^* condition,

$$
(4.1)\qquad \int_Q \frac{u}{m(\sigma/\chi_Q)^p}\,dx \le C_3\int_Q \sigma\,dx,
$$

is equivalent to the strong-type norm inequality for the minimal operator in $\mathbb{R}^n$. Throughout this section, let $\sigma = v^{1/(p+1)}$.

The proof that the strong-type inequality implies (4.1) is essentially the same as the proof in Section 2 that the weak-type inequality implies the

W_p condition: fix a cube Q, let $f = \sigma/\chi_Q$ and substitute this into the strong-type inequality. The W_p^* condition follows at once.

The proof that (4.1) implies the strong-type inequality is based on the proof of the strong-type norm inequality for the maximal operator given by Sawyer [18], and adapts ideas from Cruz-Uribe [4], Cruz-Uribe and Neugebauer [7] and Cruz-Uribe, Neugebauer and Olesen [9] to higher dimensions. For the proof we need two lemmas.

LEMMA 4.1. *If $(u, v) \in W_p^*$ and either u or σ is doubling, then the strong W_p^* condition* (1.3) *holds. If u is doubling then $C_3' \le D(u)^2 C_1 + 6^n p C_1' B_n$; if σ is doubling then $C_3' \le D(\sigma)^2 C_3$.*

Proof. If σ is doubling then the strong W_p^* condition follows immediately from (4.1).

If u is doubling, then, as shown in Section 3, $(u, v) \in W_p$ with constant $C_1 \le C_3$. We now repeat the proof that the W_p condition implies the W_p^* condition, except that we replace Q with $3Q$. Then inequality (3.6) becomes

$$\begin{aligned}\int_{3Q} \frac{u}{m(\sigma/\chi_{3Q})^p}\,dx &\le u(3Q)R^p + 2^n p C_1' B_n R^{-1}|3Q| \\ &\le D(u)^2 u(Q) R^p + 6^n p C_1' B_n R^{-1}|Q|.\end{aligned}$$

The rest of the proof now goes through as before to yield the desired inequality with $C_3' = D(u)^2 C_1 + 6^n p C_1' B_n$. ∎

The second lemma is a classical covering theorem. For the convenience of the reader we sketch the proof.

LEMMA 4.2. *Given a finite collection $\{Q_i\}_{i=1}^N$ of cubes, there exists a disjoint subcollection $\{Q_{i_j}\}_{j=1}^k$ such that each cube Q_r is contained in $3Q_{i_s}$ for some s.*

Proof. We may assume that the cubes are ordered so that $l(Q_1) \ge \ldots \ge l(Q_N)$. Let $i_1 = 1$. If any cube intersects Q_{i_1} then it is contained in $3Q_{i_1}$. Let Q_{i_2} be the largest cube which is disjoint from Q_{i_1}. Repeating this argument we form the desired subcollection. ∎

To prove that (4.1) implies the strong-type inequality, we first assume that v is everywhere positive; we treat the general case at the end. Fix a function f such that $1/f \in L^p(v)$. Without loss of generality we may assume that f is non-negative.

Fix $\alpha > 1$. For each integer k, let $A_k = \{x \in \mathbb{R}^n : \alpha^{-(k+1)} \le mf(x) < \alpha^{-k}\}$. If $x \in A_k$ then there exists a cube Q_x^k containing x such that

$$\alpha^{-(k+1)} \le \frac{1}{|Q_x^k|} \int_{Q_x^k} f\,dy < \alpha^{-k}.$$

By the continuity of the integral we may choose each Q_x^k so that the coordinates of its center and $l(Q_x^k)$ are rational. But then the set $\{Q_x^k\}_{x\in A_k}$ is countable, so we may enumerate its elements as $\{Q_j^k\}$. Clearly A_k is contained in their union. By induction define the sequence of disjoint sets E_j^k:

$$E_1^k = Q_1^k \cap A_k, \quad E_2^k = (Q_2^k \cap A_k) \setminus E_1^k, \quad E_3^k = (Q_3^k \cap A_k) \setminus (E_1^k \cup E_2^k), \ \ldots$$

Then $A_k = \bigcup_j E_j^k$, and since the A_k's are disjoint, the E_j^k's are pairwise disoint for all j and k.

As shown in Section 3, since $(u,v) \in W_p^*$, $(u,v) \in W_p$. So by the weak-type inequality, $u(\{x : \mathcal{M}f(x) = 0\}) = 0$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{u}{(\mathcal{M}f)^p}\,dx &= \sum_k \int_{A_k} \frac{u}{(\mathcal{M}f)^p}\,dx \le \sum_k u(A_k)\alpha^{p(k+1)} \\ &\le \alpha^p \sum_{j,k} u(E_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} f\,dx\right)^{-p}. \end{aligned}$$

Since v is positive, $\sigma(Q_j^k) > 0$. Further, by Hölder's inequality,

$$\begin{aligned} \sigma(Q_j^k) &\le \left(\int_{\mathbb{R}^n} \frac{v}{f^p}\,dx\right)^{1/(p+1)} \left(\int_{Q_j^k} f\,dx\right)^{p/(p+1)} \\ &\le \left(\int_{\mathbb{R}^n} \frac{v}{f^p}\,dx\right)^{1/(p+1)} (\alpha^{-k}|Q_j^k|)^{p/(p+1)} < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} &\alpha^p \sum_{j,k} u(E_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} f\,dx\right)^{-p} \\ &\qquad = \alpha^p \sum_{j,k} u(E_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} \sigma\,dx\right)^{-p} \left(\frac{\int_{Q_j^k} (f/\sigma)\cdot\sigma\,dx}{\int_{Q_j^k}\sigma\,dx}\right)^{-p}. \end{aligned}$$

Let $X = \mathbb{N}\times\mathbb{Z}$ and define the measure ω on X by

$$\omega(j,k) = u(E_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}\sigma\,dx\right)^{-p}.$$

Given a non-negative, measurable function h, define the operators S and T by

$$Sh(j,k) = \frac{\int_{Q_j^k}\sigma\,dx}{\int_{Q_j^k} h\sigma\,dx} \quad \text{and} \quad Th(j,k) = \frac{\int_{Q_j^k} h\sigma\,dx}{\int_{Q_j^k}\sigma\,dx}.$$

By Hölder's inequality, for any $r > 1$,

$$Sh(j,k) \le T(h^{1-r'})(j,k)^{r-1}.$$

Set $r = 1 + 2/p$; then we may rewrite the above inequality as

$$\int_{\mathbb{R}^n} \frac{u}{(mf)^p}\,dx \le \alpha^p \int_X S(f/\sigma)^p\,d\omega \le \alpha^p \int_X T((f/\sigma)^{-p/2})^2\,d\omega.$$

If T were a bounded operator from $L^2(\sigma)$ to $L^2(X,\omega)$ then

$$\int_{\mathbb{R}^n} \frac{u}{(mf)^p}\,dx \le \alpha^p C \int_{\mathbb{R}^n} \frac{\sigma^p}{f^p}\sigma\,dx = \alpha^p C \int_{\mathbb{R}^n} \frac{v}{f^p}\,dx.$$

Since T is bounded on L^∞ with constant 1, by the Marcinkiewicz interpolation theorem it will suffice to show that T is weak $(1,1)$. For each $\lambda > 0$, let

$$E_\lambda = \{(j,k) \in X : Th(j,k) > \lambda\}.$$

Then

$$\omega(E_\lambda) = \sum_{(j,k)\in E_\lambda} u(E_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} \sigma\,dx\right)^{-p} \le \sum_{(j,k)\in E_\lambda} \int_{E_j^k} \frac{u}{m(\sigma/\chi_{Q_j^k})^p}\,dx.$$

For each $M > 0$, let $E_M = \{(j,k) \in E_\lambda : j + |k| \le M\}$. Then it will suffice to show that there is a constant C independent of M such that

$$\sum_{(j,k)\in E_M} \int_{E_j^k} \frac{u}{m(\sigma/\chi_{Q_j^k})^p}\,dx \le \frac{C}{\lambda}\int_{\mathbb{R}^n} h\sigma\,dx.$$

Since the set $\{Q_j^k : (j,k) \in E_M\}$ is finite, by Lemma 4.2 there exists a disjoint subcollection $\{Q_n\}$ such that each cube $Q_j^k \subset 3Q_n$ for some n. Therefore, since the E_j^k's are pairwise disjoint and $E_j^k \subset Q_j^k$,

$$\begin{aligned}
\sum_{(j,k)\in E_M} \int_{E_j^k} \frac{u}{m(\sigma/\chi_{Q_j^k})^p}\,dx &\le \sum_n \sum_{Q_j^k\subset 3Q_n} \int_{E_j^k} \frac{u}{m(\sigma/\chi_{3Q_n})^p}\,dx \\
&\le \sum_n \int_{3Q_n} \frac{u}{m(\sigma/\chi_{3Q_n})^p}\,dx \\
&\le C_3' \sum_n \int_{Q_n} \sigma\,dx.
\end{aligned}$$

The last inequality follows from Lemma 4.1. Since each $Q_n = Q_j^k$ for some $(j,k) \in E_\lambda$, and since the Q_n's are disjoint,

$$C_3' \sum_n \int_{Q_n} \sigma\,dx \le \frac{C_3'}{\lambda}\sum_n \int_{Q_n} h\sigma\,dx \le \frac{C_3'}{\lambda}\int_{\mathbb{R}^n} h\sigma\,dx.$$

Hence T is weak-type $(1,1)$ with constant C_3'. Therefore, T is strong-type $(2,2)$ with constant $8C_3'$ (see, for example, Sadosky [17]), and so the strong-type inequality for the minimal operator holds for positive v with constant

$C_4 \leq 8\alpha^p C_3'$. However, there are no restrictions on $\alpha > 1$, so we may take the limit as α tends to 1 to get $C_4 \leq 8C_3'$.

To complete the proof, fix an arbitrary pair $(u, v) \in W_p^*$ and fix any non-negative f such that $1/f \in L^p(v)$. For each $k > 0$ define $v_k = v + k^{-(n+p+1)}$ and $f_k = (f + 1/k)/\chi_{P_k}$, where P_k is the cube centered at the origin with $l(P_k) = k$. Then a calculation shows that $(u, v_k) \in W_p^*$ with the same constant as (u, v). Further, $1/f_k \in L^p(v_k)$. As we showed at the end of Section 2, $\mathcal{M} f_k$ decreases to $\mathcal{M} f$. Therefore, by the special case given above,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{u}{(\mathcal{M} f_k)^p}\, dx &\leq C_4 \int_{P_k} \frac{v_k}{f_k^p}\, dx \\ &\leq C_4 \int_{\mathbb{R}^n} \frac{v}{f^p}\, dx + C_4 k^{-(n+p+1)} k^p |P_k|. \end{aligned}$$

The strong-type norm inequality now follows from the monotone convergence theorem.

Remark. Even without a doubling condition the W_p^* condition is necessary for the strong-type inequality to hold. In the proof that it is sufficient, we only used a doubling condition to prove Lemma 4.1, so the strong W_p^* condition (1.3) is a sufficient condition which does not use doubling. The proof of Lemma 4.1 also shows that the strong W_p condition implies the strong W_p^* condition. Hence, combining this observation with the remarks at the end of Section 2, we see that the strong W_p condition is actually sufficient for both the weak and strong-type inequalities.

However, even though we were able to use the strong W_p condition to show that the strong W_∞ condition is a sufficient condition for the weak-type norm inequality for the geometric maximal operator, a similar argument fails to produce a sufficient condition for the strong-type inequality for M_0. To intuitively see why, note that while the strong W_∞ condition is the (formal) limit of the strong W_p condition as p tends to infinity, the (formal) limit of the strong W_p^* condition is

$$\int_{3Q} M_0(v^{-1}\chi_{3Q}) u\, dx \leq C|Q|.$$

Since $|3Q| = 3^n |Q|$, this is equivalent to the W_∞^* condition, and so nothing new is gained.

However, by introducing a larger maximal operator, we get a Sawyer-type condition which is sufficient. We define the "offset" geometric maximal operator by

$$M_0^o f(x) = \sup_{3Q \ni x} \exp\left(\frac{1}{|Q|} \int_Q \log f\, dx \right).$$

Then following the argument in [6], if (u, v) are such that for all cubes Q,

$$\int_Q M_0^o(v^{-1}\chi_Q)u\,dx \le C|Q|,$$

then (u, v) satisfy the strong W_p^* condition for all p with a constant independent of p. Hence this yields a sufficient condition for the strong-type norm inequality for M_0 that does not require a doubling condition, but one of limited utility.

5. Doubling conditions on v^r, $0 < r < 1$. In this section we examine the doubling conditions which appear in Theorems 1.3, 1.4 and 1.5. These results assume different but closely related doubling conditions on a weight v:

(D1) for all r, $0 < r < 1$, v^r is a doubling weight;

(D2) for all r sufficiently small, v^r is a doubling weight and

$$\limsup_{r\to 0} D(v^r) < \infty;$$

(D3) for all r sufficiently small, v^r is a doubling weight and

$$\limsup_{r\to 0} 2^{-n/r} D(v^r)^{1/r} < \infty.$$

Condition (D1) is equivalent to saying that $v = w^{1/r}$, where w is a doubling weight. Note that this does not imply that v is a doubling weight: a simple counter-example on the real line is $v(x) = \min(1, |x|^{-1})$.

Examples of doubling weights are well known: for instance, if $w \in A_\infty$ then w is a doubling weight. Examples of doubling weights which are not in A_∞ have been constructed by Fefferman and Muckenhoupt [10], Strömberg [22] and Wik [25]. However, it is not obvious *a priori* that there exist weights which satisfy conditions (D2) and (D3). We begin by showing that the A_∞ condition implies both.

LEMMA 5.1. *If $w \in A_\infty$ then conditions* (D1)–(D3) *hold.*

Proof. By carefully keeping track of constants we can show w^r is doubling for all $r < 1$ and that condition (D3) holds. (It is immediate that (D3) implies (D2).)

Fix $w \in A_\infty$; then $w \in A_p$ for some $p > 1$ with constant $A_p(w)$. Then, given any cube Q and $r < 1$, by Hölder's inequality,

$$\begin{aligned}\int_{2Q} w^r\,dx &\le |2Q|\bigg(\frac{1}{|2Q|}\int_{2Q} w\,dx\bigg)^r\\ &\le 2^n A_p(w)^r|Q|\bigg(\frac{1}{|2Q|}\int_{2Q} w^{1-p'}\,dx\bigg)^{-r(p-1)}\end{aligned}$$

$$\le 2^{n+nr(p-1)} A_p(w)^r |Q| \left(\frac{1}{|Q|} \int_Q (w^r)^{(1-p')/r}\, dx \right)^{-r(p-1)}$$
$$\le 2^{n+nr(p-1)} A_p(w)^r \int_Q w^r\, dx.$$

Hence $D(w^r) \le 2^{n+nr(p-1)} A_p(w)^r$ and condition (D3) follows immediately. ∎

The situation for doubling weights which are not in A_∞ is more complex. We will construct the following two examples. They are similar in spirit to the one of Fefferman and Muckenhoupt; we use an idea of Wik's to show they are not in A_∞.

EXAMPLE 5.2. *There exists a weight v on $\mathbb{R}$ such that v is a doubling weight but for any $r > 0$, $r \neq 1$, v^r is not a doubling weight.*

EXAMPLE 5.3. *There exists a weight w on $\mathbb{R}$ such that for every $r > 0$ w^r is a doubling weight and $D(w^r)$ is uniformly bounded for all $r \le 1$, but w^r is not in A_∞.*

We believe that Example 5.3 also satisfies condition (D3) but we cannot get a sufficiently sharp estimate on the doubling constant to show this. Further, after repeated attempts we were unable to construct a weight for which (D2) holds but (D3) does not. Thus the following questions remain open:

(i) Does there exist a weight v and $0 < a < b$ such that v^r is doubling for $a \le r \le b$, but not for $r < a$ or $r > b$?

(ii) Does there exist a weight v such that v^r is doubling for all $r \le 1$ but condition (D2) does not hold?

(iii) Are conditions (D2) and (D3) equivalent?

The basic building block for our examples is given in Theorem 5.5. Hereafter, by a dyadic step function we mean a function of the form $\sum a_i \chi_{J_i}$, where the J_i's are disjoint dyadic intervals.

DEFINITION 5.4. For $\alpha > 1$, define the operator T_α on dyadic step functions as follows: given a dyadic interval $I = [m2^k, (m+1)2^k)$, let

$$I_i = [m2^k + (i-1)2^{k-2}, m2^k + i2^{k-2}), \qquad 1 \le i \le 4,$$

be the four dyadic subintervals of length $|I|/4$. Then define

$$T_\alpha(\chi_I) = \chi_{I_1} + \alpha\chi_{I_2} + \alpha^{-1}\chi_{I_3} + \chi_{I_4}.$$

More generally, if $f = \sum a_i \chi_{J_i}$ is a dyadic step function, define

$$T_\alpha(f) = \sum a_i T_\alpha(\chi_{J_i}).$$

THEOREM 5.5. *For $\alpha > 1$, define the sequence $\{u_{\alpha,n}\}$ of functions by induction*: $u_{\alpha,0} = \chi_{[0,1)}$ *and* $u_{\alpha,n} = T_\alpha(u_{\alpha,n-1}) = T_\alpha^n(u_{\alpha,0})$. *Then*:

(i) *for each* α, n *and* $r>0$, $u^r_{\alpha,n} = u_{\alpha^r,n}$;

(ii) *given a dyadic interval* $I\subset[0,1]$, $|I|\geq 4^{-n}$, *then* $u_{\alpha,n+s}(I) = \gamma_\alpha^s u_{\alpha,n}(I)$, *for all* $s\geq 1$, *where* $\gamma_\alpha = 1/2+(\alpha+\alpha^{-1})/4$;

(iii) *for each* n *and* l, $-n\leq l\leq n$,

$$|\{x\in[0,1]:u_{\alpha,n}(x)=\alpha^l\}| = 4^{-n}\binom{2n}{n+l};$$

(iv) *for each* n, $u_{\alpha,n}$ *is doubling on* $[0,1]$ *and* $D(u_{\alpha,n})$ *depends on* α *but not on* n;

(v) *there does not exist* $p>1$ *such that for all* n, $u_{\alpha,n}$ *satisfies the* A_p *condition on* $[0,1]$ *with uniformly bounded constant.*

REMARK. For each n, $u_{\alpha,n}$ is bounded and bounded away from zero, so each $u_{\alpha,n}$ is a doubling weight and satisfies the A_∞ condition on $[0,1]$.

To prove Theorem 5.5 we need two lemmas.

LEMMA 5.6. *Given a weight* v, *suppose there exists a constant* $S(v)$ *such that if* I *and* J *are adjacent dyadic intervals* (*whose union need not be a dyadic interval*), *then* $v(I)\leq S(v)v(J)$. *Then* v *is a doubling weight and* $D(v)\leq P(S(v))$, *where* P *is a degree* 5 *polynomial.*

Proof. Fix an interval I; if j is such that $2^j<|I|\leq 2^{j+1}$ then I must contain a dyadic interval I_d of length 2^{j-1}. Since $|2I|\leq 2^{j+2}$, $2I$ must be contained in the union of 8 adjacent dyadic intervals of length 2^{j-1}. Then I_d must be either the third or fourth interval from the left or the right; repeatedly applying our hypothesis shows that

$$v(2I)\leq P(S(v))v(I_d)\leq P(S(v))v(I),$$

where

$$\begin{aligned}P(S(v)) = \max(&1+2S(v)+2S(v)^2+S(v)^3+S(v)^4+S(v)^5,\\ &1+2S(v)+2S(v)^2+2S(v)^3+S(v)^4). \ \blacksquare\end{aligned}$$

REMARK. Essentially the same proof shows that if I_0 is a dyadic interval and the hypothesis holds for all dyadic subintervals of I_0 then u is doubling on I_0.

The following result is due to Wik [25].

LEMMA 5.7. *A weight* w *is in* A_p, $p>1$, *if and only if there exist constants* $q>p$ *and* $C_w>0$ *such that*, *given any cube* Q *and measurable set* $E\subset Q$,

$$\frac{w(E)}{w(Q)}\geq C_w\left(\frac{|E|}{|Q|}\right)^q.$$

Proof of Theorem 5.5. By Definition 5.4,

$$u_{\alpha,n} = \sum_{i=1}^{4^n} a_{i,n}\chi_{J_i^n},$$

where the J_i^n's are disjoint dyadic intervals, $|J_i^n| = 4^{-n}$. Denote the four dyadic subintervals of J_i^n of length 4^{-n-1} by $J_{i,j}^n$, $1 \leq j \leq 4$.

We prove part (i) by induction on n. Fix α and r. It is immediate that $u_{\alpha^r,0} = u_{\alpha,0}^r$. Now suppose that for some n,

$$u_{\alpha^r,n} = u_{\alpha,n}^r = \sum a_{i,n}^r\chi_{J_i^n}.$$

Then

$$\begin{aligned}u_{\alpha^r,n+1} = T_{\alpha^r}(u_{\alpha^r,n}) &= \sum_i a_{i,n}^r\chi_{J_{i,1}^n} + \alpha^r a_{i,n}^r\chi_{J_{i,2}^n} + \alpha^{-r}a_{i,n}^r\chi_{J_{i,3}^n} + a_{i,n}^r\chi_{J_{i,4}^n}\\ &= u_{\alpha,n+1}^r.\end{aligned}$$

Hence this equality holds for all n.

To prove part (ii), fix α and n. It will suffice to prove that $u_{\alpha,n+1}(I) = \gamma_\alpha u_{\alpha,n}(I)$. There exist i and k such that $I = J_i^n \cup J_{i+1}^n \cup \ldots \cup J_{i+k}^n$. Hence,

$$\begin{aligned}u_{\alpha,n+1}(I) = T_\alpha(u_{\alpha,n})(I) &= \sum_{j=0}^k \int_{J_{i+j}^n} a_{i+j,n}T_\alpha(\chi_{J_{i+j}^n})\,dx\\ &= \gamma_\alpha\sum_{j=0}^k a_{i+j,n}|J_{i+j}^n| = \gamma_\alpha\sum_{j=0}^k u_{\alpha,n}(J_{i+j}^n)\\ &= \gamma_\alpha u_{\alpha,n}(I).\end{aligned}$$

We prove part (iii) by induction on n. For $n = 1$ it is immediate. Now suppose it holds for some n and all l. Since $u_{\alpha,n+1} = T_\alpha(u_{\alpha,n})$, for each l, the set $\{x \in [0,1] : u_{\alpha,n+1}(x) = \alpha^l\}$ will consist of three disjoint subsets: $x \in [0,1]$ such that $u_{\alpha,n}(x) = \alpha^l$ and whose "height" was unchanged by T_α; x such that $u_{\alpha,n}(x) = \alpha^{l-1}$ and whose height was raised to α^l by T_α; and x such that $u_{\alpha,n}(x) = \alpha^{l+1}$ and whose height was lowered to α^l by T_α. Therefore, by our induction hypothesis and by Pascal's identity,

$$\begin{aligned}&|\{x \in [0,1] : u_{\alpha,n+1}(x) = \alpha^l\}|\\ &\quad= \frac{1}{2}4^{-n}\binom{2n}{n+l} + \frac{1}{4}4^{-n}\binom{2n}{n+l-1} + \frac{1}{4}4^{-n}\binom{2n}{n+l+1}\\ &\quad= 4^{-n-1}\left[\binom{2n}{n+l-1} + \binom{2n}{n+l}\right] + 4^{-n-1}\left[\binom{2n}{n+l} + \binom{2n}{n+l+1}\right]\\ &\quad= 4^{-n-1}\left[\binom{2n+1}{n+l} + \binom{2n+1}{n+l+1}\right] = 4^{-n-1}\binom{2n+2}{n+l+1}.\end{aligned}$$

Hence part (iii) holds for all n.

To prove part (iv), by Lemma 5.6 it will suffice to show that if I and J are two adjacent dyadic subintervals of $[0,1]$, $|I|=|J|$, and I to the left of J, then for all k,

$$\alpha^{-2}u_{\alpha,k}(J)\le u_{\alpha,k}(I)\le\alpha^2u_{\alpha,k}(J). \tag{5.1}$$

We will show inequality (5.1) by induction. By inspection, it holds for $k=1$. Now suppose it holds for $k=n$. To show it holds for $k=n+1$ we consider three cases.

Case 1: I and J are subintervals of J_i^n for some i. On J_i^n, $u_{\alpha,n+1}=a_{i,n+1}T_\alpha(\chi_{J_i^n})$, so arguing as we did for $u_{\alpha,1}$ we see that in this case (5.1) holds for $k=n+1$.

Case 2: I and J are each the union of 2^m of the J_i^n's. Then by part (ii) above,

$$u_{\alpha,n+1}(J_i^n)=\gamma_\alpha u_{\alpha,n}(J_i^n),$$

and so it is immediate that in this case (5.1) holds for $k=n+1$.

Case 3: $I\subset J_i^n$, $J\subset J_{i+1}^n$ for some i. There are two possibilities. If $|I|\le|J_i^n|/4$, then $u_{\alpha,n+1}(I)=u_{\alpha,n}(I)$ and $u_{\alpha,n+1}(J)=u_{\alpha,n}(J)$, so (5.1) holds for $k=n+1$ in this case.

If $|I|=|J_i|/2$ then a direct computation shows that

$$u_{\alpha,n+1}(I)=\frac{\alpha^{-1}+1}{2}a_{i,n}|I|,\qquad u_{\alpha,n+1}(J)=\frac{\alpha+1}{2}a_{i+1,n}|J|.$$

Hence

$$u_{\alpha,n+1}(I)=\frac{\alpha^{-1}+1}{\alpha+1}\cdot\frac{a_i}{a_{i+1}}u_{\alpha,n+1}(J)=\frac{\alpha^{-1}+1}{\alpha+1}\cdot\frac{u_{\alpha,n}(J_i^n)}{u_{\alpha,n}(J_{i+1}^n)}u_{\alpha,n+1}(J).$$

Inequality (5.1) would follow in this case for $k=n+1$ if

$$\alpha^{-1}u_{\alpha,n}(J_{i+1}^n)\le u_{\alpha,n}(J_i^n)\le\alpha^2u_{\alpha,n}(J_{i+1}^n). \tag{5.2}$$

This inequality also follows by induction. It is immediate if $n=1$. To complete the induction, if it is true for some n then, given J_i^{n+1} and J_{i+1}^{n+1} there are two cases: for some j either they are both subintervals of some J_j^n or $J_i^{n+1}=J_{j,4}^n$ and $J_{i+1}^{n+1}=J_{j+1,1}^n$. In the first case we argue as we did when $n=1$. In the second case we note that $u_{\alpha,n+1}(J_i^{n+1})=u_{\alpha,n}(J_i^{n+1})$ and $u_{\alpha,n+1}(J_{i+1}^{n+1})=u_{\alpha,n}(J_{i+1}^{n+1})$. In either case we see that inequality (5.2) holds with n replaced by $n+1$. Hence it is true for all n, and this completes the proof of part (iv).

Finally, to prove part (v), by Lemma 5.7 it will suffice to construct an increasing sequence $\{n_k\}$ and a sequence of sets $E_k\subset Q=[0,1]$ such that for any $q > 1$,

$$\lim_{k\to\infty} \frac{u_{\alpha,n_k}(E_k)}{u_{\alpha,n_k}(Q)} \left(\frac{|E_k|}{|Q|}\right)^{-q} = 0. \tag{5.3}$$

Let $n_k = k^2$ and let $E_k = \{x \in [0,1] : u_{\alpha,n_k}(x) = \alpha^{-k}\}$. Then by part (iii) above,

$$|E_k| = 4^{-k^2} \binom{2k^2}{k^2 - k}, \qquad u_{\alpha,n_k}(E_k) = \alpha^{-k}|E_k|.$$

We estimate $|E_k|$ using Stirling's formula:

$$\begin{aligned} |E_k| &= \frac{(2k^2)!}{4^{k^2}(k^2-k)!(k^2+k)!} \approx \frac{(k^2)^{2k^2}}{(k^2-k)^{k^2-k}(k^2+k)^{k^2+k}(k^2-1)^{1/2}} \\ &\approx \left(\frac{k^4}{k^4-k^2}\right)^{k^2} \left(\frac{k^2-k}{k^2+k}\right)^{k} \frac{1}{k} \\ &\approx \left(1 + \frac{1}{k^2-1}\right)^{k^2-1} \left(1 - \frac{2}{k+1}\right)^{k+1} \frac{1}{k} \approx \frac{1}{ek}. \end{aligned}$$

By part (ii) above, $u_{\alpha,n_k}(Q) = \gamma_\alpha^{k^2}$. Since $\gamma_\alpha > 1$, for any $q > 1$,

$$\frac{u_{\alpha,n_k}(E_k)}{u_{\alpha,n_k}(Q)} \left(\frac{|Q|}{|E_k|}\right)^{q} \leq C\alpha^{-k}\gamma_\alpha^{-k^2} k^{q-1},$$

and (5.3) follows at once. ■

Remark. Since the doubling condition and the A_∞ condition are not affected by multiplication by constants, parts (iv) and (v) of Theorem 5.5 remain true if we replace $u_{\alpha,n}$ by $\gamma_\alpha^{-n} u_{\alpha,n}$.

Construction of Example 5.2. It will suffice to construct v on $[0,\infty)$, since by Lemma 5.6, if v is a doubling weight on $[0,\infty)$ and we extend it to $\mathbb{R}$ as an even function then it is doubling on $\mathbb{R}$.

Fix $\alpha > 1$, say $\alpha = 2$, and for $x \geq 0$ define

$$v(x) = \sum_{n=0}^{\infty} \gamma_\alpha^{-n} u_{\alpha,n}(x-n)\chi_{[n,n+1)}(x).$$

By Theorem 5.5(ii), for any $n \geq 0$,

$$\int_n^{n+1} v\,dx = 1. \tag{5.4}$$

To show that v is doubling on $[0,\infty)$ we will apply Lemma 5.6. By (5.4), if I and J are adjacent dyadic subintervals of $[0,\infty)$, $|I| = |J| \geq 1$, then $v(I) = v(J)$. Further, by Theorem 5.5, if I and J are both subintervals of $[n, n+1]$ for some n then $v(I) \leq \alpha^2 v(J)$. Therefore, it remains to check that this is the case for two intervals I and J such that $I \subset [n-1, n]$ and $J \subset [n, n+1]$. Let $I' = I - (n-1)$, $J' = J - n$. Fix $k \geq 1$ such that $4^{-k} \leq |I| < 4^{-k+1}$. If $|I| = 4^{-k}$ then $u_{\alpha,k}(I') = u_{\alpha,k}(J') = |I|$; if $|I| = 2 \cdot 4^{-k}$ then $u_{\alpha,k}(I') = (1+\alpha^{-1})|I|/2$, $u_{\alpha,k}(J') = (1+\alpha)|I|/2$. Further, by Theorem 5.5(ii), $u_{\alpha,n-1}(I') = \gamma_\alpha^{n-1-k} u_{\alpha,k}(I')$ and $u_{\alpha,n}(J') = \gamma_\alpha^{n-k} u_{\alpha,k}(J')$. Therefore, $v(I) = \gamma_\alpha^{-k} u_{\alpha,k}(I')$ and $v(J) = \gamma_\alpha^{-k} u_{\alpha,k}(J')$, so

$$\frac{1+\alpha^{-1}}{1+\alpha} v(J) \leq v(I) \leq v(J).$$

Therefore, by Lemma 5.6, v is a doubling weight.

Now fix $r \neq 1$. To see that v^r is not a doubling weight, note that by Theorem 5.5(i),

$$v(x)^r = \sum_{n=0}^{\infty} \gamma_\alpha^{-rn} u_{\alpha^r,n}(x-n)\chi_{[n,n+1)}(x).$$

Hence by Theorem 5.5(ii),

$$\int_n^{n+1} v^r\,dx = \gamma_{\alpha^r}^n / \gamma_\alpha^{nr}.$$

Let $\Gamma_r = \gamma_{\alpha^r}/\gamma_\alpha^r$; if $r < 1$ then $\Gamma_r < 1$; if $r > 1$ then $\Gamma_r > 1$. Now let $I = [0, n]$ and $J = [n, 2n]$. Then

$$v^r(I) = \sum_{k=0}^{n-1} \Gamma_r^k = \frac{\Gamma_r^n - 1}{\Gamma_r - 1} \quad \text{and} \quad v^r(J) = \sum_{k=n}^{2n-1} \Gamma_r^k = \frac{\Gamma_r^{2n} - 1}{\Gamma_r - 1} - \frac{\Gamma_r^n - 1}{\Gamma_r - 1}.$$

Therefore, if $r \neq 1$, then $v^r(I)/v^r(J)$ tends to either 0 or infinity as n tends to infinity, so v^r is not a doubling weight. ∎

Construction of Example 5.3. As in the construction of Example 5.2, we will construct w on $[0, \infty)$ and use Lemma 5.6 to show w^r is a doubling weight for all $r > 0$.

Again fix $\alpha > 1$, say $\alpha = 2$. Define the sequence $\{n_k\}$ as follows: let $n_0 = 0$, and for $k \geq 1$, if $2^n \leq k < 2^{n+1}$, n an integer, let $n_k = n$. Now define

$$w(x) = \sum_{k=0}^{\infty} \gamma_\alpha^{-n_k} u_{\alpha,n_k}(x-k)\chi_{[k,k+1)}(x).$$

To show that w^r is doubling and satisfies condition (D2), first note that if I and J are adjacent dyadic intervals with $|I| = |J| \leq 1$, then arguing as we did in the first half of Example 5.2, we obtain $w^r(I) \leq \alpha^{2r} w^r(J)$.

Now suppose that $|I| = |J| = 2^n$, $n \geq 1$, and I is to the left of J. Given any k, arguing as in the second half of Example 5.2 gives $w^r([k, k+1]) = \Gamma_r^{n_k}$. Let $I = [m2^n, (m+1)2^n]$. There are three cases. If $m \geq 2$ then by our choice of the n_k's, $w^r(I) = w^r(J)$. If $m = 1$ then $w^r(I) = (2\Gamma_r)^n = \Gamma_r w^r(J)$. If $m = 0$, then

$$w^r(I) = 1 + \sum_{k=0}^{n-1} 2^k \Gamma_r^k = 1 + \frac{(2\Gamma_r)^n - 1}{2\Gamma_r - 1} \quad \text{and} \quad w^r(J) = (2\Gamma_r)^n,$$

so

$$\frac{w^r(I)}{w^r(J)} = \frac{1}{2\Gamma_r - 1} + \frac{1}{(2\Gamma_r)^n} - \frac{1}{(2\Gamma_r)^n(2\Gamma_r - 1)}.$$

Therefore, we can conclude that for all $r > 0$, w^r is a doubling weight. Further, if $r \le 1$ there exists a constant R_α such that $1/2 < R_\alpha \le \Gamma_r \le 1$. (E.g. if $\alpha = 2$, $R_\alpha \approx 0.97$.) Hence if $r \ge 1$, $D(w^r)$ is uniformly bounded.

Finally, by Theorem 5.5(v), since n_k tends to infinity as k tends to infinity, w^r is not in A_∞ for any r. ∎

Remark. The bound given by Lemma 5.6 for $D(w^r)$ is extremely poor, but we have been unable to estimate it (or more precisely, $D(u_{\alpha,n})$) in any other way. For this reason we cannot show that w satisfies condition (D3).

Doubling not necessary. We conclude this section with an example to show that doubling conditions are not necessary for the W_p condition,

$$\frac{1}{|I|} \int_I u \, dx \le C_1 \left(\frac{1}{|I|} \int_I \sigma \, dx \right)^{p+1},$$

to govern the weighted norm inequalities for the minimal operator in higher dimensions.

Example 5.8. *There exists a pair of weights $(u, v) \in W_p(\mathbb{R}^n)$ such that neither u or $\sigma = v^{1/(p+1)}$ is doubling, but*

$$\int_{\mathbb{R}^n} \frac{u}{(mf)^p} \, dx \le C \int_{\mathbb{R}^n} \frac{v}{|f|^p} \, dx \tag{5.5}$$

for every f such that $1/f \in L^p(v)$.

Proof. For clarity we will construct our example in $\mathbb{R}^2$. The same argument extends to $\mathbb{R}^n$, $n \ge 3$.

Let $v_0(t) = e^{-1/|t|}$. Then v_0^r is not doubling for any $r > 0$. Further, in [8] it was shown that if $u_0(t)$ is defined by the integral equation

$$\frac{1}{t} \int_0^t u_0 \, ds = \left(\frac{1}{t} \int_0^t v_0^{1/(p+1)} \, ds \right)^{p+1},$$

then u_0 is also not a doubling weight but $(u_0, v_0) \in W_p(\mathbb{R})$. We now define the pair (u, v) by $u(x, y) = u_0(x)u_0(y)$, $v(x, y) = v_0(x)v_0(y)$. It follows immediately from Fubini's theorem that $(u, v) \in W_p(\mathbb{R}^2)$ and that neither

u or $\sigma = v^{1/(p+1)}$ is doubling. The same argument also shows that the pair (u, v) does not satisfy the strong W_p condition (1.2).

It therefore remains to show that inequality (5.5) holds. Define the minimal operator restricted to the first co-ordinate as follows:

$$\mathcal{m}_1 f(x,y) = \inf_I \frac{1}{|I|} \int_I |f(t,y)|\, dt,$$

where the infimum is taken over all intervals in $\mathbb{R}$ which contain x. We define $\mathcal{m}_2$, the minimal operator restricted to the second co-ordinate, similarly. Then, again by Fubini's theorem, $\mathcal{m}f(x,y) \geq \mathcal{m}_1(\mathcal{m}_2 f(x,y))$, and inequality (5.5) follows by applying Theorem 1.2 to each variable in turn. ∎

6. The centered minimal operator. In this section we prove Theorem 1.7. Throughout this section, let $\sigma = v^{1/(p+1)}$.

The proof requires two covering lemmas. The first is a Whitney-type decomposition of a cube. This result is a special case of a covering lemma given by Sawyer [19]; because this case is much simpler we sketch the proof as a convenience to the reader.

LEMMA 6.1. *Given an open cube Q, there exists a sequence $\{Q_k\}$ of closed cubes contained in Q such that: the Q_k's have disjoint interiors; $l(Q_k) = \operatorname{dist}(Q_k, \partial Q)$; $\bigcup_k Q_k = Q$; $2Q_k \subset Q$ and any point in Q is contained in at most $3 \cdot 2^{n-1}$ of the $2Q_k$'s.*

Proof. Let $Q_0 = \frac{1}{3}Q$. For $j \geq 1$, form the "shell"

$$S_j = (3 - 2^{-j+1})Q_0 \setminus (3 - 2^{-j+2})Q_0.$$

Each shell S_j can be divided into $m_j = 2^{jn}(3 - 2^{-j+1})^n - 2^{jn}(3 - 2^{-j+2})^n$ closed cubes with disjoint interiors. Further, each of these cubes Q_k is such that $l(Q_k) = \operatorname{dist}(Q_k, \partial Q) = 2^{-j}|Q_0|$. Clearly Q is the union of the Q_k's, $k \geq 0$, and $2Q_k \subset Q$. Finally, to see that the $2Q_k$'s have finite overlap, let $S_0 = Q_0$ and $S_{-1} = \emptyset$. Then, if $x \in S_j$, $j \geq 0$, we have $x \in 2Q_k$ for at most 2^{n-1} cubes Q_k in each of S_{j-1}, S_j and S_{j+1}. ∎

The second covering lemma is a variant of the Besicovitch–Morse covering lemma, which we prove in the next two lemmas. (Our proof is adapted from the proof of the Besicovitch–Morse covering lemma given by Wheeden and Zygmund [24].)

LEMMA 6.2. *Let $\{Q_k\}$ be a sequence of cubes such that for any $k > j$, $|Q_k| \leq \frac{3}{2}|Q_j|$ and $\frac{1}{5}Q_k \setminus Q_j$ is non-empty. Then there exists a constant C_n, depending only on the dimension n, such that every point $x \in \mathbb{R}^n$ is contained in at most C_n of the Q_k's.*

Proof. We will determine the degree of overlap of the Q_k's only at the origin; by translation the same value will hold for every point in $\mathbb{R}^n$. Let $\{Q_{k_i}\}$,

$k_i < k_{i+1}$, be the collection of cubes which contain the origin and whose centers lie in the "first" quadrant—the quadrant where the co-ordinates of points are all non-negative. Let $h_i = l(Q_{k_i})$. Then Q_{k_1} must contain the cube $[0, h_1/2)^n$. Since $|Q_{k_i}| \leq \frac{3}{2}|Q_{k_1}|$, for all $i > 1$, we must have $h_i \leq (3/2)^{1/n} h_1 \leq \frac{5}{4} h_1$. Therefore the center of Q_{k_i}, $i > 1$, must lie in the cube $\left[0, \frac{5}{8}h_1\right)^n$. Further, because $\frac{1}{5}Q_{k_i}$ cannot be contained in Q_{k_1}, the center of Q_{k_i} cannot be in the cube $\left[0, \frac{3}{8}h_1\right)^n$. To see this, consider the limiting case: $h_i = \frac{5}{4}h_1$ and $\frac{1}{5}Q_{k_i}$ is just contained in Q_{k_1}. Then the center of Q_{k_i} must lie within $h_1/8$ of the edge of Q_{k_1}.

Therefore, the centers of the cubes Q_{k_i}, $i > 1$, lie in $\left[0, \frac{5}{8}h_1\right)^n \setminus \left[0, \frac{3}{8}h_1\right)^n$. If a cube Q_{k_i} has its center in this region, then, since it contains the origin, $\frac{3}{4}h_1 \leq h_i \leq \frac{5}{4}h_1$. If two cubes Q_{k_i} and Q_{k_j}, $j > i$, both have their centers in this region then their centers must be at least $h_1/4$ apart. Otherwise, reasoning as we did before, we would have $\frac{1}{5}Q_{k_j} \subset Q_{k_i}$, a contradiction. There can be only a finite number, C_n, of such points, and C_n depends only on the dimension. If we rescale so that $h_1 = 8$, then we get a rough estimate for C_n as follows: count the number of unit cubes with integer co-ordinates which lie in the cube $[0, 5]^n$. Since each such cube can contain the center of at most one Q_{k_i}, there are 5^n such cubes. If we repeat this argument for the cubes whose centers lie in the other 2^{n-1} quadrants, we see that there can be at most 10^n cubes containing the origin. ∎

Remarks. (i) The constants $1/5$ and $3/2$ in Lemma 6.2 are not the only ones possible. We can replace them by any two positive constants $\delta < 1$ and $\gamma > 1$ such that $1/3 > \delta\gamma^{1/n}$.

(ii) The problem of finding the best value for the constant C_n is closely related to an open problem in finite point sets. See Croft, Falconer and Guy [2, p. 154].

Lemma 6.3. *Let $\{Q_\alpha\}$ be a collection of cubes in $\mathbb{R}^n$ whose union has finite measure, and for each α let $P_\alpha = \frac{1}{5}Q_\alpha$. Then there exists a sequence $\{Q_k\}$ of Q_α's such that for each α, $P_\alpha \subset Q_k$ for some Q_k, and such that each point $x \in \mathbb{R}^n$ is contained in at most C_n of the Q_k's, where C_n is the constant from Lemma* 6.2.

Proof. Let $F_0 = \emptyset$ and define

$$\beta_1 = \sup\{|Q_\alpha| : P_\alpha \notin F_0\}.$$

Since $\bigcup_\alpha Q_\alpha$ has finite measure, $\beta_1 < \infty$. Fix a cube Q_1 among the Q_α's such that $|Q_1| > \frac{2}{3}\beta_1$. Define $F_1 = \{P_\alpha \subset Q_1\}$ and

$$\beta_2 = \sup\{|Q_\alpha| : P_\alpha \notin F_1\}.$$

Clearly $\beta_2 \leq \beta_1$. If $\beta_2 = 0$ we are done. If not, continue this process. At the kth stage, if $\beta_k > 0$ fix a cube Q_k among those used to define β_k such that

$|Q_k| > \frac{2}{3}\beta_k$. define $F_k = \{P_\alpha \subset Q_k\}$ and

$$\beta_{k+1} = \sup\{|Q_\alpha| : P_\alpha \notin F_j,\ 1 \le j \le k\}.$$

Clearly the β_k's are decreasing, so if $k > j$, then $|Q_k| \le \beta_k \le b_j \le \frac{3}{2}|Q_j|$. Further, by our choice of Q_k, $P_k \setminus Q_j \ne \emptyset$. Therefore, by Lemma 6.2, the Q_k's have finite overlap.

It remains to show that each P_α is contained in some Q_k. If $\beta_k = 0$ for some k then this is immediate. Therefore, suppose that there are an infinite number of Q_k's, and suppose to the contrary that there exists an α such that P_α is not contained in any Q_k. Then for all k, $P_\alpha \notin F_{k-1}$, so $|Q_\alpha| \le \beta_k$. If we can show that β_k tends to 0 then we would have a contradiction. Since $\frac{2}{3}\beta_k \le |Q_k| \le \beta_k$, it will suffice to show that $|Q_k|$ tends to 0.

Suppose instead that there exists $\delta > 0$ such that $|Q_k| \ge \delta$. Then since there are an infinite number of Q_k's, and they have finite overlap, we have

$$\infty = \sum |Q_k| \le C_n \Big| \bigcup_\alpha Q_k \Big| < \infty,$$

a contradiction. Hence $|Q_k|$ tends to zero and we are done. ∎

Proof of Theorem 1.7. A standard argument shows that the strong-type inequality implies the weak-type inequality, and the arguments in Sections 2 and 4 show that the weak-type inequality implies the $W_{p,c}$ condition,

$$\frac{1}{|Q|}\int_Q u\,dx \le D_1\bigg(\frac{1}{|2Q|}\int_{2Q}\sigma\,dx\bigg)^{p+1}, \tag{6.1}$$

and the strong-type inequality implies the $W_{p,c}^*$ condition,

$$\int_Q \frac{u}{m_c(\sigma/\chi_Q)^p}\,dx \le D_3 \int_Q \sigma\,dx. \tag{6.2}$$

Therefore, to complete the proof it will suffice to show that (6.1) implies (6.2), and (6.2) implies the strong-type norm inequality.

Remark. We do not have a direct proof that the $W_{p,c}^*$ condition implies the $W_{p,c}$ condition. The argument in Section 3 fails in the same way that the proof that the W_∞^* condition implies the W_p^* condition fails in the centered case.

To prove that (6.1) implies (6.2), fix a cube Q and apply Lemma 6.1 to form the sequence $\{Q_k\}$. For each Q_k, let P_k be the union of all the cubes contained in Q whose centers are in Q_k. Since $\mathrm{dist}(Q_k, Q) = l(Q_k)$, P_k is a rectangle whose volume is at most $4 \cdot 5^{n-1}|Q_k|$. Let $G_n = 4 \cdot 5^{n-1}$.

We now argue as we did in Section 3. Fix a cube Q_k and for each $t > 0$ let $E_t = \{x \in Q_k : \mathcal{M}_c(\sigma/\chi_Q)(x) < 1/t\}$. Then for $R > 0$ to be fixed below,

$$\int_{Q_k} \frac{u}{\mathcal{M}_c(\sigma/\chi_Q)^p}\,dx = p\int_0^R t^{p-1}u(E_t)\,dt + p\int_R^\infty t^{p-1}u(E_t)\,dt. \tag{6.3}$$

The first integral on the right-hand side is bounded by $u(Q_k)R^p$. To estimate the second, note that if $x \in E_t$ there exists a cube $Q^t_{x,k}$ centered at x such that $2Q^t_{x,k} \subset P_k$ and

$$\frac{1}{|2Q^t_{x,k}|}\int_{2Q^t_{x,k}} \sigma\,dy < 1/t.$$

By Lemma 2.1 there exists a subcollection $\{Q^t_{j,k}\}$ which covers E_t and has finite overlap. We now repeat the argument in Section 3. With the same notation, inequality (3.5) becomes

$$\begin{aligned} p\int_R^\infty t^{p-1}u(E_t)\,dt &\le pD_1\int_R^\infty t^{-2}\sum_j |Q^t_{j,k}|\,dt \\ &\le pD_1B_n|P_k|/R = pD_1B_nG_n|Q_k|/R. \end{aligned}$$

Therefore the left-hand side of (6.3) is dominated by $u(Q_k)R^p + pC|Q_k|/R$. Fix R so that $R^p = \sigma(2Q_k)/u(Q_k)$. By the $W_{p,c}$ condition, $|Q_k|/R \le C\sigma(2Q_k)$, so

$$\int_{Q_k} \frac{u}{\mathcal{M}_c(\sigma/\chi_Q)^p}\,dx \le C\sigma(2Q_k).$$

Since the Q_k's are disjoint and the $2Q_k$'s have finite overlap, if we sum over k we get (6.2).

To prove that the $W^*_{p,c}$ condition implies the strong-type norm inequality, we will adapt the proof in Section 4. Fix a function f; without loss of generality we may assume that f is non-negative. Further, arguing as we did in Section 2, we may also assume that there exists a cube P and $\varepsilon > 0$ such that $f(x) > \varepsilon$ for all $x \in P$ and $1/f$ has support on P.

Fix $\alpha > 1$, and for each integer k let $A_k = \{x : \alpha^{-k-1} \le \mathcal{M}_c f(x) < \alpha^{-k}\}$. For each $x \in A_k$ there exists a cube Q^k_x contained in P and centered at x such that

$$\alpha^{-(k+1)} \le \frac{1}{|Q^k_x|}\int_{Q^k_x} f\,dy < \alpha^{-k}.$$

Let $P^k_x = \frac{1}{5}Q^k_x$. By the continuity of the integral there exists a small cube K^k_x centered at x and contained in P^k_x such that if $y \in K^k_x$ then

$$\frac{1}{|P^k_x|}\int_{P^k_x} \sigma\,dx \ge 2^{-1/p}\mathcal{M}_c(\sigma/\chi_{P^k_x})(y).$$

By Lemma 2.1 there exists a subcollection $\{K_j^k\}$ such that $A_k \subset \bigcup_j K_j^k$ and the K_j^k's have finite overlap. Let $E_j^k = A_k \cap K_j^k$. Then, since the A_k's are disjoint, the E_j^k's have finite overlap for all j and k.

Since $f(x) > \varepsilon$ on P, we have $u(\{x : \mathcal{m}_c f(x) = 0\}) = 0$. Therefore, we can proceed as we did in Section 4; using the same notation we get

$$\int_{\mathbb{R}^n} \frac{u}{(\mathcal{m}_c f)^p}\, dx \le \alpha^p \int_X S(f/\sigma)^p\, d\omega \le \alpha^p \int_X T((f/\sigma)^{-p/2})^2\, d\omega.$$

As before, it will suffice to show that T is weak $(1,1)$. For each $\lambda > 0$, let

$$E_\lambda = \{(j,k) \in X : Th(j,k) > \lambda\}.$$

Then

$$\begin{aligned}
\omega(E_\lambda) &= \sum_{(j,k)\in E_\lambda} u(E_j^k)\biggl(\frac{1}{|Q_j^k|}\int_{Q_j^k} \sigma\, dx\biggr)^{-p} \\
&\le 5^{np} \sum_{(j,k)\in E_\lambda} u(E_j^k)\biggl(\frac{1}{|P_j^k|}\int_{P_j^k} \sigma\, dx\biggr)^{-p} \\
&\le 2\cdot 5^{np} \sum_{(j,k)\in E_\lambda} \int_{E_j^k} \frac{u}{\mathcal{m}_c(\sigma/\chi_{P_j^k})^p}\, dx.
\end{aligned}$$

By our choice of the Q_j^k's, their union is bounded and so has finite measure. Therefore, by Lemma 6.3 there exists a subcollection $\{Q_n\}$ with finite overlap such that each P_j^k is contained in some Q_n. Since the E_j^k's have finite overlap, it follows that

$$\begin{aligned}
2\cdot 5^{np} \sum_{(j,k)\in E_\lambda} \int_{E_j^k} \frac{u}{\mathcal{m}_c(\sigma/\chi_{P_j^k})^p}\, dx &\le 2\cdot 5^{np} \sum_n \sum_{P_j^k \subset Q_n} \int_{E_j^k} \frac{u}{\mathcal{m}_c(\sigma/\chi_{Q_n})^p}\, dx \\
&\le 2B_n 5^{np} \sum_n \int_{Q_n} \frac{u}{\mathcal{m}_c(\sigma/\chi_{Q_n})^p}\, dx \\
&\le 2B_n D_3 5^{np} \sum_n \int_{Q_n} \sigma\, dx \\
&\le \frac{2B_n D_3 5^{np}}{\lambda} \sum_n \int_{Q_n} h\sigma\, dx \\
&\le \frac{2B_n C_n D_3 5^{np}}{\lambda} \int_{\mathbb{R}^n} h\sigma\, dx.
\end{aligned}$$

Therefore T is weak-type $(1,1)$ and our proof is complete. ■

REMARK. The weak $(1,1)$ constant of T is $2B_n C_n D_3 5^{np}$, and so the strong-type $(2,2)$ constant is $M = 16 B_n C_n D_3 5^{np}$. Hence $D_4 \le 8M\alpha^p$.

While we can take the limit as α tends to 1 to get $D_4 \leq 8M$, M itself still depends on p.

The proof on $\mathbb{R}$. If we restrict ourselves to the real line we can modify our proof to get the strong-type $(2,2)$ constant for T independent of p. The argument follows closely the proof of the main theorem in [4]; here we sketch the details.

Begin the proof as before, and choose the Q_x^k's to be open intervals. Omit the P_x^k's and choose each interval K_x^k to be concentric with the interval Q_x^k and such that if $y \in K_x^k$ then

$$\frac{1}{|Q_x^k|}\int_{Q_x^k} \sigma\,dx \geq 2^{-1/p} m_c(\sigma/\chi_{Q_x^k})(y).$$

Continue the above proof exactly the same until we have to show that T is weak-type $(1,1)$. Here we estimate as follows:

$$\omega(E_\lambda) = \sum_{(j,k)\in E_\lambda} u(E_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k}\sigma\,dx\right)^{-p} \leq 2\sum_{(j,k)\in E_\lambda}\int_{E_j^k}\frac{u}{m_c(\sigma/\chi_{Q_j^k})^p}\,dx.$$

Let $G = \bigcup Q_j^k$. Since the Q_j^k's are open, G is open and so it is the union of disjoint open intervals Q_n. Further, each Q_n is the union of Q_j^k's, so each set E_j^k is contained in a unique Q_n. Finally, by a lemma of Muckenhoupt [16] (also see [4]) we have

$$\int_{Q_n}\sigma\,dx \leq \frac{2}{\lambda}\int_{Q_n} h\sigma\,dx.$$

Therefore, if we use this collection $\{Q_n\}$ the above argument goes through, and we conclude that the weak $(1,1)$ constant for T in this case is $4D_3$, so the strong $(2,2)$ constant is $32D_3$, which does not depend on p.

The one-weight case. We conclude this section by proving the analogue of Theorem 1.1 for the centered minimal operator. For brevity, we write $w \in W_{p,c}$ instead of $(w,w) \in W_{p,c}$. Recall that a function w satisfies the weak A_∞ condition if, given any cube Q and a measurable set $E \subset Q$, there exist constants C and δ such that

$$\frac{w(E)}{w(2Q)} \leq C\left(\frac{|E|}{|Q|}\right)^\delta.$$

This condition was introduced by Sawyer [19] to generalize the good-λ inequality of Coifman and Fefferman [1]. He proved that $w \in$ weak A_∞ if and only if it satisfied the weak reverse Hölder inequality for some $s > 1$:

$$\text{weak } RH_s: \quad \left(\frac{1}{|Q|}\int_Q w^s\,dx\right)^{1/s} \leq C\frac{1}{|2Q|}\int_{2Q} w\,dx.$$

The weak RH_s condition plays an important role in the study of PDE's, potential theory and quasi-conformal mappings. (See, for example, Giaquinta [12], Iwaniec and Nolder [15] and Stredulinsky [21].) It has many properties in common with the reverse Hölder inequality; here we need two.

LEMMA 6.4. *If* $w \in$ *weak* RH_s *for some* $s > 1$ *then*:

(i) *there exists* $t > s$ *such that* $w \in$ *weak* RH_t;
(ii) *for each* r, $0 < r < 1$, $w^r \in$ *weak* $RH_{s/r}$.

The proof of (i) is found in [12] and [21]; the proof of (ii) is in [15] and Heinonen, Kilpeläinen and Martio [14, pp. 66–68].

We can now prove our result.

THEOREM 6.5. *Given a weight* w *and* p, $0 < p < \infty$, *the following are equivalent*:

(i) $w \in W_{p,c}$;
(ii) $w \in$ *weak* A_∞.

Proof. Suppose $w \in W_{p,c}$; then $w^{1/(p+1)} \in$ weak RH_{p+1}, so by Lemma 6.4, there exists $t > p + 1$ such that $w^{1/(p+1)} \in$ weak RH_t. Therefore, by Hölder's inequality, $w \in$ weak $RH_{t/(p+1)}$, so $w \in$ weak A_∞.

Now suppose that $w \in$ weak A_∞. Then for some $s > 1$, $w \in$ weak RH_s, so by Lemma 6.4, for each $p > 0$, $w^{1/(p+1)} \in$ weak $RH_{s(p+1)}$, which implies that $w \in W_{p,c}$. ■

7. The equality $M_0 f(x) = M_0^* f(x)$. In this section we prove Theorem 1.8. The proof uses two lemmas from [6]. Recall that $v \in I_\infty$ if

$$\limsup_{Q,\sigma} \frac{1}{|Q|}\left(\frac{1}{|Q|}\int_Q v^{-\sigma}\,dx\right)^{1/\sigma} < \infty,$$

where the upper limit is taken over all cubes Q containing the origin and all $\sigma > 0$ as $|Q|$ tends to infinity and σ tends to 0.

LEMMA 7.1. *Let* Q_0 *be a cube* (*possibly infinite*) *and suppose that* $\operatorname{supp} f = Q_0$. *If for some* p, $0 < p < \infty$, $f \in L^p(Q_0)$ *and* $\log f \in L^1_{\mathrm{loc}}(Q_0)$ *then* $M_0^* f(x) = M_0 f(x)$ *for almost every* x.

LEMMA 7.2. *Let* $v \in I_\infty$ *and suppose that for some* p, $0 < p < \infty$, $f \in L^p(v)$. *Let* Q_k *be the cube centered at the origin such that* $l(Q_k) = k$. *Then the following are true*:

(i) *for all* x, $M_0^* f(x) = \lim_{k\to\infty} M_0^*(f\chi_{Q_k})(x)$;
(ii) *there exists* $r > 0$ *such that* $f \in L^r_{\mathrm{loc}}$.

REMARKS. (i) In [6, Corollary 2.2], it was implicitly assumed that the cube Q_0 in Lemma 7.1 was finite; however the same argument works for all cubes.

(ii) In [6, Lemma 4.3], Lemma 7.2 is only proved for $p = 1$, but the proof adapts immediately to the case of arbitrary p. Further, (ii) is actually part of the proof of (i).

Proof of Theorem 1.8. Fix a function f such that $\log|f|$ is locally integrable and $f \in L^p(v)$ for some $p > 0$. Without loss of generality we may assume that f is non-negative. Then Lemmas 7.1 and 7.2 together imply that for almost every x,

$$M_0^* f(x) = \lim_{k\to\infty} M_0^*(f\chi_{Q_k})(x) = \lim_{k\to\infty} M_0(f\chi_{Q_k})(x).$$

Since $M_0(f\chi_{Q_k})(x) \le M_0 f(x)$, to show that the right-hand side equals $M_0 f(x)$, it will suffice to show that

$$M_0 f(x) \le \lim_{k\to\infty} M_0(f\chi_{Q_k})(x). \tag{7.1}$$

Fix $\varepsilon > 0$. Then there exists a cube Q containing x such that

$$M_0 f(x) - \varepsilon < \exp\left(\frac{1}{|Q|}\int_Q \log f\, dx\right).$$

But for all k sufficiently large, $Q \subset Q_k$, so the right-hand side is dominated by $M_0(f\chi_{Q_k})(x)$. Therefore,

$$M_0 f(x) - \varepsilon \le \lim_{k\to\infty} M_0(f\chi_{Q_k})(x),$$

and since $\varepsilon > 0$ is arbitrary, inequality (7.1) follows at once.

To complete the proof we will construct a non-negative function f on $\mathbb{R}$ such that f is bounded, $\operatorname{supp} f = [0,1]$, $\log f \in L^1[0,1]$ and $M_0^* f(0) > M_0 f(0)$.

For $n > 1$, let $I_n = [1/n - 1/n^4, 1/n]$ and let $I = \bigcup_n I_n$. Define

$$f(x) = \begin{cases} e^{-n^2}, & x \in I_n, \\ 1, & x \in [0,1]\setminus I, \\ 0, & x \in \mathbb{R}\setminus[0,1]. \end{cases}$$

Clearly f is bounded and $\operatorname{supp} f = [0,1]$. (Hence $f \in L^1(v)$ for any locally integrable v.) Second,

$$\int_0^1 |\log f|\, dx = \sum_{n=2}^{\infty} n^2 |I_n| = \sum_{n=2}^{\infty} n^{-2} < \infty.$$

Third, fix r, $0 < r < 1$. Then by a standard calculus argument, for any

$n > 1$, we have

$$\begin{aligned} M_r f(0) &\geq \Big(n \int_0^{1/n} f^r\,dx\Big)^{1/r} \geq (n|[0,1/n] \setminus I|)^{1/r} \\ &\geq \bigg(n\bigg(\frac{1}{n} - \sum_{k\geq n} \frac{1}{k^4}\bigg)\bigg)^{1/r} \geq \bigg(n\bigg(\frac{1}{n} - \frac{1}{3n^3}\bigg)\bigg)^{1/r} \\ &= \bigg(1 - \frac{1}{3n^2}\bigg)^{1/r}. \end{aligned}$$

The last term tends to 1 as n tends to infinity. Therefore $M_r f(0) = 1$ for all r, so $M_0^* f(0) = 1$.

Finally, fix t, $0 < t \leq 1$. For some $n \geq 1$, $(n+1)^{-1} < t \leq n^{-1}$. Then arguing as before, we obtain

$$\frac{1}{t}\int_0^t \log f\,dx \leq \frac{-1}{t} \sum_{k\geq n+1} k^2 |I_k| \leq -n \sum_{k\geq n+1} k^{-2} \leq \frac{-n}{n} = -1.$$

It follows that $M_0 f(0) \leq e^{-1} < 1 = M_0^* f(0)$. ∎

Remark. Wik [26] established the equality of $M_0 f(x)$ and $M_0^* f(x)$, f non-negative, given the following conditions:

(i) for some $r > 0$, $f \in L^r_{\mathrm{loc}}$;
(ii) $\log f \in L^1_{\mathrm{loc}}$;
(iii) for all $t > 0$, $|\{x : f(x) > t\}| < \infty$;
(iv) $\limsup_{k\to\infty} |Q_k|^{-1} \int_{Q_k} \log^+ f\,dx < \infty$, where Q_k is the cube centered at the origin with $l(Q_k) = k$.

Theorem 1.8 is a generalization of this result. To see this, first note that condition (ii) is a shared hypothesis. Second, we claim that conditions (i) and (iv) follow from the assumption that there exists $v \in I_\infty$ such that $f \in L^p(v)$ for some $p > 0$. Indeed, (i) follows from Lemma 7.2.

To show condition (iv): if $v \in I_\infty$, there exist constants N and M such that if $\sigma < 1/N$ and $|Q_k| > N$ then

$$\frac{1}{|Q_k|}\bigg(\frac{1}{|Q_k|}\int_{Q_k} v^{-\sigma}\,dx\bigg)^{1/\sigma} < M.$$

Fix k such that $|Q_k| > N$ and r such that $0 < r/(p-r) < 1/N$. Let $J_k = \{x \in Q_k : f(x) \geq 1\}$. Then $\log^+ f(x) = \log f(x)\chi_{J_k}(x)$ for $x \in Q_k$. Hence, by Jensen's inequality,

$$\exp\bigg(\frac{1}{|J_k|}\int_{Q_k} \log^+ f\,dx\bigg) = \exp\bigg(\frac{1}{|J_k|}\int_{J_k} \log f\,dx\bigg) \leq \bigg(\frac{1}{|J_k|}\int_{J_k} f^r\,dx\bigg)^{1/r}.$$

Then by Hölder's inequality,

$$\begin{aligned}\frac{1}{|J_k|}\int\limits_{J_k} f^r\,dx &\leq \frac{|Q_k|}{|J_k|}\Big(\int\limits_{\mathbb{R}^n} f^p v\,dx\Big)^{r/p}\\ &\qquad\times\bigg(\frac{1}{|Q_k|}\bigg(\frac{1}{|Q_k|}\int\limits_{Q_k} v^{-r/(p-r)}\,dx\bigg)^{(p-r)/r}\bigg)^{r/p}\\ &\leq \frac{|Q_k|}{|J_k|}\|f\|^r_{L^p(v)}M^r.\end{aligned}$$

Therefore,

$$\frac{1}{|Q_k|}\int\limits_{Q_k}\log^+ f\,dx \leq \frac{|J_k|}{|Q_k|}\log(M\|f\|_{L^p(v)}) + r^{-1}\frac{|J_k|}{|Q_k|}\log\bigg(\frac{|Q_k|}{|J_k|}\bigg).$$

Since $0 < |J_k|/|Q_k| \leq 1$ and the function $x\log(1/x)$ is bounded on $(0,1]$, this establishes condition (iv).

Finally, we note that condition (iii) need not hold for $f \in L^p(v)$. We construct a weight v on the real line as follows: for $n \geq 0$ let $I_n = [2^n-1, 2^n]$. For $x \geq 0$ define

$$v(x) = \begin{cases} 4^{-n}, & x \in I_n,\\ 1, & \text{otherwise.}\end{cases}$$

Extend v to $\mathbb{R}$ as an even function. A straightforward computation shows that $v \in I_\infty$. Now define a function f by

$$f(x) = \begin{cases} 2^n, & |x| \in I_n,\\ 1/x^2, & \text{otherwise.}\end{cases}$$

Then it is immediate that $f \in L^1(v)$, but

$$|\{x : f(x) > 1\}| = 2\sum_{n=1}^{\infty}|I_n| = \infty.$$

A conjecture of Wik. We conclude this section by proving a conjecture made by Wik [26]. Given a weight $w \in A_\infty$, let

$$\begin{aligned} m_\infty(w) &= \sup\Big\{\int\limits_{\mathbb{R}^n} M_0 f w\,dx : \int\limits_{\mathbb{R}^n}|f|w\,dx = 1\Big\},\\ m^*_\infty(w) &= \sup\Big\{\int\limits_{\mathbb{R}^n} M^*_0 f w\,dx : \int\limits_{\mathbb{R}^n}|f|w\,dx = 1\Big\}.\end{aligned}$$

Since $w \in A_\infty$, we have $w \in W^*_\infty$ and $w \in I_\infty$, so both supremums are finite. (See [6] for details.) Clearly $m_\infty(w) \leq m^*_\infty(w)$. Wik [26] showed that there exists a constant C such that $m^*_\infty(w) \leq Cm_\infty(w)$ and conjectured that in fact these two quantities are equal. Using Theorem 1.8 we show that this is true.

THEOREM 7.3. *Given* $w \in A_\infty$, $m_\infty(w) = m_\infty^*(w)$.

Proof. Fix $w \in A_\infty$ and fix a non-negative function $f \in L^1(w)$ such that $\int f w\,dx = 1$. We first construct a sequence $\{f_j\}$ that decreases to f and such that for each j, $f_j \in L^1(w)$ and $\log f_j \in L^1_{\mathrm{loc}}$. For each $k \geq 1$, let Q_k be the cube centered at the origin with $l(Q_k) = k$, and let $S_k = Q_k \setminus Q_{k-1}$. Choose a positive decreasing sequence $\{a_k\}$ such that

$$\sum_{k=1}^{\infty} a_k \int_{S_k} w\,dx < \infty.$$

For $j \geq 1$ and $x \in S_k$, define $f_j(x) = f(x) + a_k/j$. Clearly, $\{f_j\}$ decreases to f. Further, on each cube Q_k the functions f_j are bounded below, so $\log f_j \in L^1_{\mathrm{loc}}$. Finally, $f_j \in L^1(w)$, since

$$\begin{aligned}\int_{\mathbb{R}^n} f_j w\,dx &= \sum_{k=1}^{\infty} \int_{S_k} f_j w\,dx = \sum_{k=1}^{\infty} \int_{S_k} (f + a_k/j) w\,dx \\ &= \int_{\mathbb{R}^n} f w\,dx + \frac{1}{j} \sum_{k=1}^{\infty} a_k \int_{S_k} w\,dx < \infty.\end{aligned}$$

Since the f_j's are a decreasing sequence,

$$M_0^* f(x) \leq \lim_{j\to\infty} M_0^* f_j(x).$$

Therefore, by Fatou's lemma and Theorem 1.8,

$$\begin{aligned}\int_{\mathbb{R}^n} M_0^* f w\,dx &\leq \lim_{j\to 0} \int_{\mathbb{R}^n} M_0^* f_j w\,dx = \lim_{j\to 0} \int_{\mathbb{R}^n} M_0 f_j w\,dx \\ &= \lim_{j\to 0} \|f_j\|_{L^1(w)} \int_{\mathbb{R}^n} M_0(f_j/\|f_j\|_{L^1(w)}) w\,dx \\ &\leq m_\infty(w) \lim_{j\to 0} \|f_j\|_{L^1(w)} = m_\infty(w).\end{aligned}$$

Since this is true for all such f, $m_\infty^*(w) \leq m_\infty(w)$ and we are done. ■

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