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A conditional quasi-greedy basis of l_1

by

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Abstract. We show that the Lindenstrauss basic sequence in l_1 may be used to construct a conditional quasi-greedy basis of l_1, thus answering a question of Wojtaszczyk. We further show that the sequence of coefficient functionals for this basis is not quasi-greedy.

1. Introduction. In what follows, $\{e_i\}_{i=1}^{\infty}$ denotes the standard unit vector basis of l_1. In [5], the following concept was studied:

Definition. Let $\mathfrak{X}$ be a Banach space with dual $\mathfrak{X}^*$ and let $(x_i, x_i^*)_{i\in F}$ be a fundamental biorthogonal system in $\mathfrak{X} \times \mathfrak{X}^*$ with $\inf_{i\in F} \|x_i\| > 0$ and $\sup_{i\in F} \|x_i^*\| < \infty$. For $m \in \mathbb{N}$, define the operator $\mathcal{G}_m$ by

$$\mathcal{G}_m(x) = \sum_{i\in A} x_i^*(x) x_i,$$

where $A \subset F$ is a set of cardinality m such that $|x_i^*(x)| \ge |x_k^*(x)|$ whenever $i \in A$ and $k \notin A$ (note that A depends on x and may not be unique; nevertheless, $\mathcal{G}_m$ is well defined). Then $(x_i, x_i^*)_{i\in F}$ is a *quasi-greedy system* provided that the operators $\mathcal{G}_m$ satisfy

$$\lim_m \mathcal{G}_m(x) = x \quad \text{for each } x \in \mathfrak{X}.$$

Equivalently, by [5, Theorem 1], $(x_i, x_i^*)_{i\in F}$ is a quasi-greedy system if there exists a constant C so that for every $x \in \mathfrak{X}$ and for every $m \in \mathbb{N}$, we have

$$\|\mathcal{G}_m(x)\| \le C\|x\|.$$

If $\{x_i\}_{i=1}^{\infty}$ is, in addition, basic, it will be said to be a *quasi-greedy basis.* A sequence $\{z_i\}_{i=1}^{\infty}$ is *unconditional for constant coefficients* if there exist positive numbers c and C such that

$$c\Big\|\sum_{i=1}^{m} z_i\Big\| \le \Big\|\sum_{i=1}^{m} \varepsilon_i z_i\Big\| \le C\Big\|\sum_{i=1}^{m} z_i\Big\|$$

for any positive integer m and for any sequence of signs $\{\varepsilon_i\}_{i=1}^{m}$.

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REMARK. By [5, Proposition 2] a quasi-greedy basis is necessarily unconditional for constant coefficients.

In [5], Wojtaszczyk shows that, for $1 < p < \infty$, the space l_p has a conditional quasi-greedy basis. Since, by the Remark, a quasi-greedy basis of l_1 is quite close to being equivalent to $\{e_i\}_{i=1}^{\infty}$, and since $\{e_i\}_{i=1}^{\infty}$ is the unique normalized unconditional basis of l_1 up to equivalence [3], it is of interest to find a conditional quasi-greedy basis of l_1. In particular, the existence of such a basis would show that l_1 does not have a unique (up to equivalence) normalized basis that is unconditional for constant coefficients. In this note, we show that indeed such a basis exists.

Our example is derived from the basic sequence constructed by Lindenstrauss in [2]. This is a monotone, conditional basic sequence in l_1 whose closed linear span is an $\mathcal{L}_1$-space with no unconditional basis. We denote this sequence by $\{x_i\}_{i=1}^{\infty}$ and its associated sequence of coefficient functionals by $\{x_i^*\}_{i=1}^{\infty}$. They are defined as follows: For $i \in \mathbb{N}$,

$$x_i = e_i - \tfrac{1}{2}(e_{2i+1} + e_{2i+2}).$$

Considering x_i^* as an element of $l_\infty/[x_i]^\perp$, we write $x_i^* = y_i^* + [x_i]^\perp$ where the $y_i^* \in l_\infty$ are as defined by Holub and Retherford (see [1]): For each $i \in \mathbb{N}$, let α_i be the finite sequence of positive integers defined by the following conditions:

(1) $\alpha_i(1) = i$.

(2) $\alpha_i(j) = \alpha_i(j-1) - ([\alpha_i(j-1)/2] + 1)$ for admissible $j < i$ (that is, such that $\alpha_i(j) > 0$), where $[k]$ denotes the greatest integer less than or equal to k.

Then y_i^* is defined by

$$y_i^* = \sum_{j=1}^{|\alpha_i|} \left(\tfrac{1}{2}\right)^{j-1} e_{\alpha_i(j)}.$$

Thus, for example,

$$\begin{aligned}
y_1^* &= (1, 0, \ldots),\\
y_2^* &= (0, 1, 0, \ldots),\\
y_3^* &= \left(\tfrac{1}{2}, 0, 1, 0, \ldots\right),\\
y_4^* &= \left(\tfrac{1}{2}, 0, 0, 1, 0, \ldots\right),\\
y_5^* &= \left(0, \tfrac{1}{2}, 0, 0, 1, 0, \ldots\right),\\
y_6^* &= \left(0, \tfrac{1}{2}, 0, 0, 0, 1, 0, \ldots\right),\\
y_7^* &= \left(\tfrac{1}{4}, 0, \tfrac{1}{2}, 0, 0, 0, 1, 0, \ldots\right),\\
y_8^* &= \left(\tfrac{1}{4}, 0, \tfrac{1}{2}, 0, 0, 0, 0, 1, 0, 0, \ldots\right).
\end{aligned}$$

The properties of $\{x_i\}_{i=1}^{\infty}$ which we shall use in what follows are summarized in the following fact (see [2], [3], and, also, [4]):

FACT. *The sequence $\{x_i\}_{i=1}^{\infty}$ satisfies*:

(1) $\{x_i\}_{i=1}^{\infty}$ *is a monotone basic sequence* ([4, p. 455]).

(2) $[x_i]$ *has no unconditional basis* ([4, p. 455]).

(3) *For $n \in \mathbb{N}$, there exists an isomorphism T_n from $[x_i : 1 \le i \le n]$ onto l_1^n satisfying $\|T_n\|\|T_n^{-1}\| \le 2$* ([3, Ex. 8.1]).

2. Results. We now construct the basis heralded in the Introduction. To do this, it suffices by [5, Proposition 3] to construct such a basis in a space isomorphic to l_1. Towards this end, define, for each $n \in \mathbb{N}$,

$$F_n = [x_i : 1 \le i \le n].$$

Let

$$\mathfrak{X} = \Big(\sum_{i=1}^{\infty} \oplus F_i\Big)_1.$$

We claim that the natural basis $\{\widetilde{x}_i\}_{i=1}^{\infty}$ of $\mathfrak{X}$ obtained from the x_i's is the desired sequence. Indeed, it follows from the Fact that $\{\widetilde{x}_i\}_{i=1}^{\infty}$ is a monotone, conditional basis of $\mathfrak{X}$, and that $\mathfrak{X}$ is isomorphic to l_1. Moreover, assuming for the moment that $\{x_i\}_{i=1}^{\infty}$ is quasi-greedy, for $m \in \mathbb{N}$ and $y = \sum y_i \in \mathfrak{X}$, we have

$$\|\mathcal{G}_m(y)\| \le \sup_{k \in K_m} \sum_{i=1}^{\infty} \|\mathcal{G}_{k(i)}(y_i)\| \le C \sum_{i=1}^{\infty} \|y_i\| = C\|y\|,$$

where C is as in the Definition, $K_m = \{k : \mathbb{N} \to \mathbb{N} \cup \{0\} : \sum_{i=1}^{\infty} k(i) = m\}$, and $\mathcal{G}_0(x) = 0$ for each $x \in \mathfrak{X}$ (note that the operators $\mathcal{G}_j$ appearing in the above inequality are defined with respect to two different sequences). Thus, the fact that $\{\widetilde{x}_i\}_{i=1}^{\infty}$ is quasi-greedy will be a consequence of the following theorem.

THEOREM. *The sequence $\{x_i\}_{i=1}^{\infty}$ satisfies*

$$3\Big\|\sum_{i \in S_1 \cup S_2} \alpha_i x_i\Big\| \ge \Big\|\sum_{i \in S_1} \alpha_i x_i\Big\|, \tag{1}$$

whenever S_1 and S_2 are disjoint finite subsets of $\mathbb{N}$ with

$$\min_{i \in S_1} |\alpha_i| \ge \max_{i \in S_2} |\alpha_i|. \tag{2}$$

Proof. For $A \subseteq \mathbb{N}$ and $x \in l_1$, we denote by $P_A x$ the vector in l_1 whose jth coordinate is $x(j)$ if $j \in A$ and zero otherwise. Let S_1, S_2, and $\{\alpha_i\}_{i \in S_1 \cup S_2}$ be as above. Set

$$x = \sum_{i \in S_1} \alpha_i x_i \quad \text{and} \quad y = \sum_{i \in S_2} \alpha_i x_i,$$

and define the sets

$$A_0 = \Big\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = 1 \Big\},$$
$$B_0 = \Big\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = -1/2 \Big\},$$
$$C_0 = \Big\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = 1/2 \Big\}.$$

First, we concentrate our attention on B_0. We define the sets

$$W_1 = \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_0 \},$$
$$A_1 = \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_1 \},$$
$$B_1 = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_1 \}.$$

Finally, set

$$y_0 = x \quad \text{and} \quad y_1 = \sum_{i \in W_1} \alpha_i x_i.$$

Note that A_1, B_1, B_0, and C_0 are mutually disjoint. We also see from the triangle inequality that

$$\|P_{B_0} y_0\| \leq \|P_{B_0}(y_0 + y_1)\| + \|P_{B_0} y_1\|. \tag{3}$$

But, since $\|P_{B_0} y_1\| = \|P_{A_1} y_1\| + \|P_{B_1} y_1\|$, we deduce from (3) that

$$\|P_{B_0}(y_0 + y_1)\| + \|P_{B_1} y_1\| \geq \|P_{B_0} y_0\| - \|P_{A_1} y_1\|. \tag{$*$}$$

Concentrating our attention now on the set B_1, we let

$$W_2 = \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_1 \},$$
$$A_2 = \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_2 \},$$
$$B_2 = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_2 \}.$$

Set

$$y_2 = \sum_{i \in W_2} \alpha_i x_i.$$

Then A_1, A_2, B_1, B_2, B_0, and C_0 are mutually disjoint and, as above, we have

$$\|P_{B_1}(y_1 + y_2)\| + \|P_{B_2} y_2\| \geq \|P_{B_1} y_1\| - \|P_{A_2} y_2\|. \tag{$*$}$$

In general, at the lth step of the induction, we set

$$W_l = \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_{l-1} \},$$
$$A_l = \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_l \},$$
$$B_l = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_l \}.$$

Set

$$y_l = \sum_{i \in W_l} \alpha_i x_i.$$

Then the sets A_j, B_j $(1 \le j \le l)$, B_0, and C_0 are mutually disjoint, and we have

$$(*) \qquad \|P_{B_{l-1}}(y_{l-1} + y_l)\| + \|P_{B_l} y_l\| \ge \|P_{B_{l-1}} y_{l-1}\| - \|P_{A_l} y_l\|.$$

This process must end at some stage k with $W_{k+1} = \emptyset$. Summing the inequalities $(*)$ so obtained and simplifying, we obtain

$$(4) \qquad \sum_{i=1}^{k} \|P_{B_{i-1}}(y_{i-1} + y_i)\| + \|P_{B_k} y_k\| \ge \|P_{B_0} y_0\| - \sum_{i=1}^{k} \|P_{A_i} y_i\|.$$

But, for each $i = 0, 1, \ldots, k$, we have $B_i \cap \operatorname{supp} y_j = \emptyset$ for $j \notin \{i, i+1\}$. Moreover, the mutually disjoint sets A_i $(i = 1, \ldots, k)$ are contained in A_0. Recalling that $y_0 = x$, by these two observations the inequality (4) reduces to

$$(5) \qquad \sum_{i=0}^{k} \|P_{B_i}(x + y)\| \ge \|P_{B_0} x\| - \|P_{A_0} y\|.$$

Since the sets $C_0, A_0, B_0, B_1, B_2, \ldots, B_k$ are mutually disjoint, we have

$$\|x + y\| \ge \|P_{A_0}(x + y)\| + \sum_{i=0}^{k} \|P_{B_i}(x + y)\| + \|P_{C_0}(x + y)\|;$$

and so, using (5), we obtain

$$(6) \qquad \|x + y\| \ge \|P_{A_0}(x + y)\| - \|P_{A_0} y\| + \|P_{B_0} x\| + \|P_{C_0} x\|.$$

Using (2) we have $\|P_{A_0}(x + y)\| \ge \|P_{A_0} y\|$, and thus (6) implies

$$(7) \qquad \|x + y\| \ge \|P_{B_0} x\| + \|P_{C_0} x\|.$$

Also, by (2), we have $2\|P_{A_0}(x + y)\| \ge \|P_{A_0} x\|$; thus,

$$(8) \qquad 2\|x + y\| \ge \|P_{A_0} x\|.$$

Since $\|x\| = \|P_{A_0} x\| + \|P_{B_0} x\| + \|P_{C_0} x\|$, we may now obtain (1) by adding the inequalities (7) and (8). ■

Remark. We note that the sequence of coefficient functionals for $\{\widetilde{x}_i\}_{i=1}^{\infty}$ is not quasi-greedy. To see this, it is enough to show that $\{x_i^*\}_{i=1}^{\infty}$ is not quasi-greedy. Towards this end, note that

$$(9) \qquad \Big\| \sum_{i=1}^{2^{n+1}-2} (-1)^i x_i^* \Big\| \le \Big\| \sum_{i=1}^{2^{n+1}-2} (-1)^i y_i^* \Big\| = 1.$$

However, defining

$$\{\alpha_n\}_{n=1}^{\infty} = \left\{1, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \ldots\right\}$$

and

$$z_n = \sum_{i=1}^{2^{n+1}-2} \alpha_i x_i = e_1 + e_2 + \sum_{i=1}^{2^{n+1}-2} \frac{1}{2^{n+1}} e_{2\cdot(2^{n+1}-2)-1+i},$$

we obtain

$$\Big\|\sum_{i=1}^{2^{n+1}-2} x_i^*\Big\| \ge \frac{1}{\|z_n\|}\Big|\sum_{i=1}^{2^{n+1}-2} x_i^*(z_n)\Big| = \frac{1}{4}\Big\langle \sum_{i=1}^{2^{n+1}-2} x_i^*,\, e_1 + e_2\Big\rangle = \frac{n}{2}. \tag{10}$$

It follows from (9), (10), and the Remark of the Introduction that $\{x_i^*\}_{i=1}^{\infty}$ is not quasi-greedy.

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