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Boundedness of Fourier integral operators on Fourier Lebesgue spaces and affine fibrations

by

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Abstract. We study Fourier integral operators of Hörmander's type acting on the spaces $\mathcal{F}L^p(\mathbb{R}^d)_{\rm comp}$, $1 \le p \le \infty$, of compactly supported distributions whose Fourier transform is in L^p. We show that the sharp loss of derivatives for such an operator to be bounded on these spaces is related to the rank r of the Hessian of the phase $\Phi(x,\eta)$ with respect to the space variables x. Indeed, we show that operators of order $m = -r|1/2-1/p|$ are bounded on $\mathcal{F}L^p(\mathbb{R}^d)_{\rm comp}$ if the mapping $x \mapsto \nabla_x\Phi(x,\eta)$ is constant on the fibres, of codimension r, of an affine fibration.

1. Introduction. Consider the spaces $\mathcal{F}L^p(\mathbb{R}^d)_{\rm comp}$ of compactly supported distributions whose Fourier transform is in $L^p(\mathbb{R}^d)$, with the norm $\|f\|_{\mathcal{F}L^p} = \|\hat{f}\|_{L^p}$. In [3] we studied the boundedness on these spaces of Hörmander's type Fourier integral operators (FIOs) of the form

$$Tf(x) = \int e^{2\pi i\Phi(x,\eta)}\sigma(x,\eta)\hat{f}(\eta)\,d\eta. \tag{1.1}$$

Here the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is normalized to be $\hat{f}(\eta) = \int f(t)e^{-2\pi it\eta}\,dt$. The symbol σ is in $S^m_{1,0}$, Hörmander's class of order m, that is, $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ and

$$|\partial_x^\alpha\partial_\eta^\beta\sigma(x,\eta)| \le C_{\alpha,\beta}\langle\eta\rangle^{m-|\beta|}, \quad \forall(x,\eta) \in \mathbb{R}^{2d}, \tag{1.2}$$

where, as usual, $\langle\eta\rangle := (1+|\eta|^2)^{1/2}$. We also suppose that σ has compact support with respect to x.

The phase $\Phi(x,\eta)$ is real-valued, positively homogeneous of degree 1 in η, and smooth on $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. We assume $\Phi(x,\eta)$ is defined on an open subset $\Lambda \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, conic in dual variables, containing the closure of the set

$$\Lambda' = \{(x,\eta) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) : (x,\lambda\eta) \in \operatorname{supp}\sigma \text{ for some } \lambda > 0\}$$

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in $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. We also assume the non-degeneracy condition

$$\det\left(\frac{\partial^2 \Phi}{\partial x_i \partial \eta_l}\bigg|_{(x,\eta)}\right) \neq 0 \quad \forall (x,\eta) \in \Lambda. \tag{1.3}$$

It is easy to see that such an operator maps continuously the space $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions into the space $\mathcal{C}_0^\infty(\mathbb{R}^d)$ of test functions. See [8, 22] for the general theory of FIOs, and [19, 20] for results in L^p.

The main result of [3] states that if

$$m \leq -d\left|\frac{1}{2} - \frac{1}{p}\right|,$$

the operator T above, initially defined on $\mathcal{C}_0^\infty(\mathbb{R}^d)$, extends to a bounded operator on $\mathcal{F}L^p(\mathbb{R}^d)_{\mathrm{comp}}$ whenever $1 \leq p < \infty$. For $p = \infty$, T extends to a bounded operator on the closure of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ in $\mathcal{F}L^\infty(\mathbb{R}^d)_{\mathrm{comp}}$. Moreover this loss of derivatives was proved to be generally sharp in any dimension, even for phases linear in η. This result can be regarded as an investigation of the minimal loss of derivatives occurring in the Beurling–Helson theorem [1, 9] (see also [11, 16]).

However, it was shown in [16] that, in the case of phases linear in x, local $\mathcal{F}L^p$-boundedness holds without loss of derivatives, i.e. for $m = 0$. This suggests the possibility of intermediate thresholds, depending on the rank of the Hessian $d_x^2 \Phi(x,\eta)$.

A similar phenomenon is well-known for L^p-boundedness: a celebrated result by Seeger, Sogge and Stein [17] shows that T is bounded on L^p, $1 < p < \infty$, if $m \leq -(d-1)|1/2 - 1/p|$ (see also [20, 21]). Moreover, if the rank of the Hessian $d_\eta^2 \Phi(x,\cdot)$ is $\leq r$, then the threshold goes up to $-r|1/2 - 1/p|$, provided a certain smooth factorization condition is satisfied. Although that condition is not necessary for boundedness to hold, it turns out to be essential in the proof, given in [17], when the rank of $d_\eta^2 \Phi(x,\eta)$ is allowed to drop, and its relaxation is an open problem. The main reference on this topic is Ruzhansky's survey [13] and book [14], where the case of complex-valued phases is also considered. See also [12].

In this paper we present a variant of the smooth factorization condition which is relevant when dealing with $\mathcal{F}L^p$ spaces. Then we show, under that condition, that in fact the above threshold for local $\mathcal{F}L^p$-boundedness can go up.

Definition 1.1 (Spatial smooth factorization condition). Let $0 \leq r \leq d$ and suppose that for every $(x_0,\eta_0) \in \Lambda$ with $|\eta_0| = 1$, there exists an open neighbourhood Ω of x_0 and an open neighbourhood $\Gamma' \subset \mathbb{S}^{d-1}$ of η_0, with $\Omega \times \Gamma' \subset \Lambda$, satisfying the following condition. For every $\eta \in \Gamma'$ there exists a smooth fibration of Ω, smoothly depending on η and with affine fibres of codimension r, such that $\nabla_x\Phi(\cdot,\eta)$ is constant on every fibre ([1]).

Observe that this condition implies that the Hessian $d_x^2\Phi(x,\eta)$ has rank $\leq r$. Moreover the condition is always satisfied if $r = d$ or if $d_x^2\Phi(x,\eta)$ has constant rank r (in particular, for phases linear in x, corresponding to $r = 0$).

THEOREM 1.2. *Let σ and Φ satisfy the above assumptions. Moreover, assume that Φ satisfies the spatial smooth factorization condition* (*Definition* 1.1) *for some r. If*

$$m \leq -r\left|\frac{1}{2} - \frac{1}{p}\right|, \tag{1.4}$$

then the corresponding FIO T, initially defined on $\mathcal{C}_0^\infty(\mathbb{R}^d)$, extends to a bounded operator on $\mathcal{F}L^p(\mathbb{R}^d)_{\mathrm{comp}}$ whenever $1 \leq p < \infty$. For $p = \infty$, T extends to a bounded operator on the closure of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ in $\mathcal{F}L^\infty(\mathbb{R}^d)_{\mathrm{comp}}$.

The threshold in (1.4) is sharp in any dimension $d \geq 1$, even for phases $\Phi(x,\eta)$ which are linear in η. Indeed, in dimension d, consider the phase $\Phi(x,\eta) = \sum_{k=1}^r \varphi(x_k)\eta_k + \sum_{k=r+1}^d x_k\eta_k$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism with $\varphi(t) = t$ for $|t| \geq 1$ and whose restriction to $(-1,1)$ is non-linear. Consider then the symbol

$$\sigma(x,\eta) = G(x)\langle\eta\rangle^m \quad \text{with} \quad G \in \mathcal{C}_0^\infty(\mathbb{R}^d),\ G \equiv 1 \text{ on } [-1,1]^d.$$

Let moreover $1 \leq p \leq 2$. Then Theorem 1.2 and an easy variant of the arguments in [3, Section 6] show that the corresponding operator $T : \mathcal{F}L^p(\mathbb{R}^d)_{\mathrm{comp}} \to \mathcal{F}L^p(\mathbb{R}^d)_{\mathrm{comp}}$ is bounded if and only if m satisfies (1.4). By duality one can construct similar examples for $2 < p \leq \infty$.

The proof of the result in [3], corresponding to Theorem 1.2 with $r = d$ (when the spatial smooth factorization condition is automatically satisfied) used tools from time-frequency analysis, relying on our previous work [2]. Instead, the proof of Theorem 1.2 is inspired by more classical arguments in [17]. Indeed, we will conjugate the operator T above with the Fourier transform, obtaining the FIO

$$\tilde{T}f(x) = \mathcal{F} \circ T \circ \mathcal{F}^{-1} f(x) = \iint e^{2\pi i(\Phi(\eta,y) - x\eta)}\sigma(\eta,y)f(y)\,dy\,d\eta,$$

for which L^p-boundedness has to be proved. However, notice that we cannot apply to $\tilde{T}$ the well-known results of the classical (i.e. local) L^p-theory. Indeed, the corresponding symbol is no longer compactly supported with respect to y and the phase is no longer homogeneous with respect to η.

([1]) To be precise, by "a fibration of Ω, smoothly depending on $\eta \in \Gamma'$ and with fibres of codimension r" we mean that a smooth function $\Pi : \Omega \times \Gamma' \to \mathbb{R}^d$ is given, with $d_x\Pi$ having constant rank r. The fibres are the level sets of the mapping $\Pi(\cdot,\eta)$.

Among other things, it does not satisfy the (frequency) smooth factorization condition of [17]. In fact, for operators $\tilde{T}$ of this special type we will prove results in L^p in the limiting cases $p = 1, \infty$ too, which are generally false, for example, for the operator T above.

Our strategy consists in splitting $\tilde{T}$ into dyadic pieces via a Littlewood–Paley decomposition of the *physical* domain and then each dyadic operator is further split into a certain number of FIOs with symbols localized in thin boxes of the *frequency* domain, and phases essentially linear in η. By comparison, notice that in the classical L^p-theory one performs a dyadic decomposition and then a second decomposition, both in the frequency domain ([17, 20]); moreover, the geometry of our second decomposition is different from that in [17].

This discussion also shows that Theorem 1.2 can be read as a *global* boundedness result on L^p for the operator $\tilde{T}$, and hence it partially overlaps some recent results in [6] (see also some examples in [4] and, for the case $p = 2$, [5, 15]). In fact, sharp results are obtained there for general classes of operators, but no improvements upon them seem to be considered, under additional conditions like a smooth factorization.

Notation. We write $A \lesssim B$ if $A \le CB$ for some constant $C > 0$ which may depend on parameters, like Lebesgue exponents or the dimension d. We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$. Finally, for $R > 0$, $x_0 \in \mathbb{R}^d$, we denote by $B_R(x_0)$ the open ball in $\mathbb{R}^d$ with centre x_0 and radius R.

2. Preliminary results on FIOs. In the next section we will make use of the well-known composition formula for a pseudodifferential operator and a FIO. We collect here what is needed in the subsequent proofs.

First we recall that a *regularizing operator* is a pseudodifferential operator

$$Rf(x) = r(x, D)f = \int e^{2\pi i x\eta} r(x, \eta)\hat{f}(\eta)\, d\eta,$$

with a symbol r in the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ (equivalently, an operator with kernel in $\mathcal{S}(\mathbb{R}^{2d})$, which maps $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$). Then the composition formula for a pseudodifferential operator and a FIO is as follows (see, e.g., [8], [10, Theorem 4.1.1], [18, Theorem 18.2], [22]).

Theorem 2.1. *Let the symbol σ and the phase Φ satisfy the assumptions in the Introduction. Assume, in addition, $\sigma(x, \eta) = 0$ for $|\eta| \le 1$ if $\Phi(x, \eta)$ is not linear in η. Let $q(x, \eta)$ be a symbol in $S^{m'}_{1,0}$. Then*

$$q(x, D)T = S + R,$$

where S is a FIO with the same phase Φ and symbol $s(x, \eta)$ of order $m + m'$ satisfying

$$\operatorname{supp} s \subset \operatorname{supp} \sigma \cap \{(x, \eta) \in \Lambda : (x, \nabla_x \Phi(x, \eta)) \in \operatorname{supp} q\},$$

and R is a regularizing operator with symbol $r(x, \eta)$ satisfying

$$\Pi_\eta(\operatorname{supp} r) \subset \Pi_\eta(\operatorname{supp} \sigma),$$

where Π_η is the orthogonal projection on $\mathbb{R}^d_\eta$.

Moreover, the symbol estimates satisfied by s and the seminorm estimates of r in the Schwartz space are uniform when σ and q vary in a bounded subset of $S^m_{1,0}$ and $S^{m'}_{1,0}$ respectively.

3. Proof of the main result (Theorem 1.2). By means of a smooth cut off function near $\eta = 0$ we split the symbol σ of T into a symbol supported where $|\eta| \le 4$ and a symbol supported where $|\eta| \ge 2$. Now, the first symbol yields an operator which is bounded on all $\mathcal{F}L^p(\mathbb{R}^d)_{\text{comp}}$, $1 \le p < \infty$, as well as on the closure of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ in $\mathcal{F}L^\infty(\mathbb{R}^d)_{\text{comp}}$. This was shown in [3, Proposition 4.1], regardless of the order of the operator, and without assuming the condition (1.3), nor the spatial smooth factorization condition ([2]).

Hence we will prove the estimate

$$\|Tf\|_{\mathcal{F}L^p} \lesssim \|f\|_{\mathcal{F}L^p}, \quad \forall f \in \mathcal{C}_0^\infty(\mathbb{R}^d), \tag{3.1}$$

$1 \le p \le \infty$, for an operator satisfying the assumptions of Theorem 1.2 and whose symbol σ satisfies, in addition,

$$\sigma(x, \eta) = 0 \quad \text{for } |\eta| \le 2.$$

This clearly implies the conclusion of Theorem 1.2.

We first perform a further reduction. For every $(x_0, \eta_0) \in \Lambda'$ with $|\eta_0| = 1$, there exist an open neighbourhood $\Omega \subset \mathbb{R}^d$ of x_0, an open conic neighbourhood $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of η_0 and $\delta > 0$ such that

$$|\det \partial^2_{x,\eta} \Phi(x, \eta)| \ge \delta > 0, \quad \forall (x, \eta) \in \Omega \times \Gamma, \tag{3.2}$$

and

(3.3) for each $x \in \Omega$, the map $\Gamma \ni \eta \mapsto \nabla_x \Phi(x, \eta)$ is a diffeomorphism onto its range.

Hence, by a compactness argument and a finite partition of unity we can assume that σ itself is supported in a set of the type $\Omega' \times \Gamma$ for some open $\Omega' \subset\subset \Omega \subset \mathbb{R}^d$ and conic open $\Gamma \subset \mathbb{R}^d \setminus \{0\}$, with Φ satisfying the above conditions on $\Omega \times \Gamma$, as well as the spatial smooth factorization condition (Definition 1.1) for $x \in \Omega$ and $\eta \in \Gamma' := \Gamma \cap \mathbb{S}^{d-1}$.

Now, we will prove (3.1) with $p = 1, \infty$, for an operator T of order $m = -r/2$. Then the desired result when $1 < p < \infty$, for operators of order $m = -r|1/2 - 1/p|$, will follow by complex interpolation with the well-known case L^2 (see e.g. [20, p. 402]).

([2]) There the desired boundedness was proved on the so-called modulation spaces M^p. However, it was observed that the corresponding norm is equivalent to that of $\mathcal{F}L^p$ for distributions supported in a fixed compact set.

In detail, for the interpolation step we argue as follows. For $s \in \mathbb{R}$, denote by $\mathcal{F}L^p_s$ the space of tempered distributions f such that

$$\|f\|_{\mathcal{F}L^p_s} := \Big(\int \langle\eta\rangle^{ps} |\hat{f}(\eta)|^p \, d\eta\Big)^{1/p} < \infty,$$

with the obvious changes if $p = \infty$. For every $s \in \mathbb{R}$, the operator $\langle D\rangle^s$ defines an isomorphism of $\mathcal{F}L^p_s$ onto $\mathcal{F}L^p$. Hence, the operator $T = T\langle D\rangle^{-s}\langle D\rangle^s$ is bounded $\mathcal{F}L^p_s \to \mathcal{F}L^p$ if $T\langle D\rangle^{-s}$ is bounded on $\mathcal{F}L^p$. Observe moreover that $T\langle D\rangle^{-s}$ is a FIO with the same phase as T, and symbol $\sigma(x,\eta)\langle\eta\rangle^{-s}$, which has order $m - s$.

Suppose now that the desired result is already obtained for $p = 1, 2$. Take $1 < p < 2$ and consider a FIO T of order $m = -r(1/p - 1/2)$. Then, by the above remarks, T extends to a bounded operator $\mathcal{F}L^1_{m+r/2} \to \mathcal{F}L^1$ and $L^2_m \to L^2$. Hence, the boundedness on $\mathcal{F}L^p$ follows by complex interpolation, because, if $\theta \in (0,1)$ satisfies $(1-\theta)/1 + \theta/2 = 1/p$, then $(m + r/2)(1-\theta) + m\theta = 0$. The proof for $2 < p < \infty$ is similar.

Of course, when in (1.4) there is a strict inequality, the desired result follows from the equality case, for an operator with order $m' < m$ also has order m.

Hence, from now on, we assume $m = -r/2$ and prove (3.1) for $p = 1, \infty$.

The first step consists in conjugating T with the Fourier transform. The desired results will be proved if we verify that the operator $\tilde{T} = \mathcal{F} \circ T \circ \mathcal{F}^{-1}$ is continuous on L^1 and on the closure of $\mathcal{C}_0^\infty$ in L^∞. This operator has integral kernel

$$K(x,y) = \int e^{2\pi i(\Phi(\eta,y) - x\eta)} \sigma(\eta, y) \, d\eta,$$

which is smooth everywhere and supported in $\mathbb{R}^d \times \Gamma$. Indeed, as anticipated in the Introduction, the problem is the integrability at infinity.

Consider now the usual Littlewood–Paley decomposition, but on the physical domain. Namely, fix a smooth function $\psi_0(y)$ such that $\psi_0(y) = 1$ for $|y| \le 1$ and $\psi_0(y) = 0$ for $|y| \ge 2$. Set $\psi(y) = \psi_0(y) - \psi_0(2y)$, $\psi_j(y) = \psi(2^{-j}y)$, $j \ge 1$. Then

$$1 = \sum_{j=0}^{\infty} \psi_j(y), \qquad \forall y \in \mathbb{R}^d.$$

Notice that if $j \ge 1$, then ψ_j is supported where $2^{j-1} \le |y| \le 2^{j+1}$. Since $\sigma(\eta, y) = 0$ for $|y| \le 2$, we can write the kernel above as

$$K = \sum_{j \ge 1} K_j,$$

where

$$K_j(x,y) = \int e^{2\pi i(\Phi(\eta,y) - x\eta)} \sigma_j(\eta, y) \, d\eta,$$

and

$$\sigma_j(\eta, y) := \sigma(\eta, y)\psi_j(y).$$

Observe that η lies in the open neighbourhood Ω'. After shrinking Ω' and Γ if necessary, we see from the spatial smooth factorization condition that there exist an open neighbourhood $U \times V$ of $(0,0)$ in $\mathbb{R}^r \times \mathbb{R}^{d-r}$ and a smooth change of variables $U \times V \ni (u,v) \mapsto \eta_y(u,v) \in \Omega_y$, smoothly depending on the parameter $y \in \Gamma$ and homogeneous of degree 0 with respect to y, with $\Omega' \subset \Omega_y \subset \Omega$, such that the function $\eta \mapsto \nabla_1\Phi(\eta, y)$ is constant on each of the $(d-r)$-dimensional (pieces of) affine planes $u = \text{const}$. Here we denote by ∇_1 the gradient with respect to the first d variables, i.e. $\nabla_1\Phi(\eta, y) = \nabla_\eta(\Phi(\eta, y))$. Note that, because of the previous change of variables, we are using Definition 1.1 with the variables (x, η) replaced by (η, y).

For every $j \geq 1$ we choose u_j^ν, $\nu = 1, \ldots, N_r(j)$, such that $|u_j^\nu - u_j^{\nu'}| \geq C_0 2^{-j/2}$ for different ν, ν', and such that U is covered by balls with centre u_j^ν and radius $C_1 2^{-j/2}$. It is easy to see that $N_r(j) = O(2^{jr/2})$. Let then $\eta_j^\nu = \eta_y(u_j^\nu, 0)$. Consider moreover a smooth partition of unity tailored to the covering above, namely given by smooth functions $\chi_j^\nu(u)$, $\nu = 1, \ldots, N_r(j)$, supported in the above balls, and satisfying the estimate

$$|\partial_u^\alpha \chi_j^\nu(u)| \lesssim 2^{j|\alpha|/2}. \tag{3.4}$$

Accordingly we decompose the kernel K_j as $\sum_{\nu=1}^{N_r(j)} K_j^\nu$ with

$$K_j^\nu(x, y) = \int e^{2\pi i(\Phi(\eta, y) - x\eta)} \chi_j^\nu(u(y, \eta))\sigma_j(\eta, y)\, d\eta, \tag{3.5}$$

where the function $u = u(y, \eta)$ is obtained by the inverse change of variables. Consider now the second order Taylor expansion of $\Phi(\cdot, y)$ at η_j^ν:

$$\Phi(\eta, y) = \Phi(\eta_j^\nu, y) + \langle \nabla_1\Phi(\eta_j^\nu, y), \eta - \eta_j^\nu \rangle + R_j^\nu(\eta, y),$$

where

$$R_j^\nu(\eta, y) = \frac{1}{2}\int_0^1 (1-t)(d_1^2\Phi)(\eta_j^\nu + t(\eta - \eta_j^\nu), y)[\eta - \eta_j^\nu, \eta - \eta_j^\nu]\, dt. \tag{3.6}$$

Here $d_1^2\Phi$ stands for the Hessian of Φ with respect to the first d variables, regarded as a bilinear form.

For fixed j and ν, after a rotation we perform a splitting (η', η'') of η such that the vectors η'' are tangent to the pieces of plane $u = u_j^\nu$. We then consider the operator

$$L_j^\nu = (1 - \langle 2^{-j/2}\nabla_{\eta'}, 2^{-j/2}\nabla_{\eta'} \rangle)(1 - \langle \nabla_{\eta''}, \nabla_{\eta''} \rangle).$$

We have

$$(1+4\pi^2 2^{-j}|(\nabla_1\Phi(\eta_j^\nu,y)-x)'|^2)(1+4\pi^2|(\nabla_1\Phi(\eta_j^\nu,y)-x)''|^2)e^{2\pi i\langle\nabla_1\Phi(\eta_j^\nu,y)-x,\eta\rangle} = L_j^\nu e^{2\pi i\langle\nabla_1\Phi(\eta_j^\nu,y)-x,\eta\rangle}.$$

Moreover, we will prove in the Appendix that

$$|(L_j^\nu)^N(e^{2\pi i(\Phi(\eta_j^\nu,y)-\langle\nabla_1\Phi(\eta_j^\nu,y),\eta_j^\nu\rangle+R_j^\nu(\eta,y))}\chi_j^\nu(u(y,\eta))\sigma_j(\eta,y))|\le C_N 2^{-jr/2}. \tag{3.7}$$

Hence, a repeated integration by parts in (3.5) yields

$$|K_j^\nu(x,y)|\lesssim 2^{-jr}(1+2^{-j/2}|(\nabla_1\Phi(\eta_j^\nu,y)-x)'|)^{-2N}(1+|(\nabla_1\Phi(\eta_j^\nu,y)-x)''|)^{-2N},$$

where we take into account that we integrate on a set of measure $\lesssim 2^{-jr/2}$.

Hence one sees that, choosing $N>d/2$,

$$\int|K_j^\nu(x,y)|\,dx\lesssim 2^{-jr/2}. \tag{3.8}$$

To treat the integral of the kernel with respect to y, in view of (3.3) we can perform the change of variable $y\mapsto\nabla_1\Phi(\eta_j^\nu,y)$, whose inverse jacobian determinant is homogeneous of degree 0 and uniformly bounded because of (3.2). We therefore obtain

$$\int|K_j^\nu(x,y)|\,dy=\int_\Gamma|K_j^\nu(x,y)|\,dy\lesssim 2^{-jr/2}. \tag{3.9}$$

It follows from (3.8), (3.9) and the Schur test (see e.g. [7, Theorem 6.18]) that the operators with kernels K_j^ν are bounded on L^1 and L^∞, with norm $O(2^{-jr/2})$. Summing over $\nu=1,\dots,N_r(j)$ we see that the operators with kernels K_j satisfy the uniform estimates

$$\Big\|\int K_j(\cdot,y)f(y)\,dy\Big\|_{L^p}\lesssim\|f\|_{L^p},\qquad p=1,\infty,\qquad\forall u\in\mathcal{S}(\mathbb{R}^d).$$

Coming back to the $\mathcal{F}L^p$ spaces we see that the operators

$$T_jf(x):=\int e^{2\pi i\Phi(x,\eta)}\sigma(x,\eta)\psi_j(\eta)\hat{f}(\eta)\,d\eta$$

therefore satisfy the estimates

$$\|T_jf\|_{\mathcal{F}L^p}\lesssim\|f\|_{\mathcal{F}L^p},\qquad p=1,\infty,\qquad\forall u\in\mathcal{S}(\mathbb{R}^d). \tag{3.10}$$

Hence, in order to obtain the desired estimate for the original operator $T=\sum_{j=1}^\infty T_j$, we need to sum the estimates (3.10) over $j\ge1$.

Summing estimates (3.10) **for** $p=1$**.** Observe that if χ is a smooth function supported where $B_0^{-1}\le|\eta|\le B_0$ for some $B_0>0$, then trivially

$$\sum_{j=1}^\infty\|\chi(2^{-j}D)f\|_{\mathcal{F}L^1}\lesssim\|f\|_{\mathcal{F}L^1} \tag{3.11}$$

for every $f\in\mathcal{S}(\mathbb{R}^d)$, where $\chi(2^{-j}D)f=\mathcal{F}^{-1}[\chi(2^{-j}\cdot)\hat{f}]$.

Also, notice that, due to the frequency localization of T_j, the estimate (3.10) for $p=1$ can be refined as

$$\|T_j f\|_{\mathcal{F}L^1} = \|T_j(\chi(2^{-j}D)f)\|_{\mathcal{F}L^1} \lesssim \|\chi(2^{-j}D)f\|_{\mathcal{F}L^1},$$

where χ is a smooth function satisfying $\chi(\eta)=1$ for $1/2 \le |\eta| \le 2$ and $\chi(\eta)=0$ for $|\eta|\le 1/4$ and $|\eta|\ge 4$ (so that $\chi\psi=\psi$). Summing over j this last estimate with the aid of (3.11) we obtain

$$\|Tf\|_{\mathcal{F}L^1} \lesssim \|f\|_{\mathcal{F}L^1},$$

which is the desired estimate.

Summing estimates (3.10) **for** $p=\infty$**.** Here we use the following trivial remark. For $k\ge 0$, let $f_k \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\operatorname{supp}\hat{f}_0 \subset B_2(0)$ and

$$\operatorname{supp}\hat{f}_k \subset \{\eta\in\mathbb{R}^d : 2^{k-1}\le|\eta|\le 2^{k+1}\},\qquad k\ge 1.$$

If the sequence f_k is bounded in $\mathcal{F}L^\infty(\mathbb{R}^d)$, then the series $\sum_{k=0}^\infty f_k$ converges in $\mathcal{F}L^\infty(\mathbb{R}^d)$ and

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{\mathcal{F}L^\infty} \lesssim \sup_{k\ge 0}\|f_k\|_{\mathcal{F}L^\infty}. \tag{3.12}$$

Indeed, at each point in the frequency domain there are at most two non-zero terms in the sum. This yields

$$\begin{aligned}\|Tf\|_{\mathcal{F}L^\infty} &= \Big\|\sum_{k\ge 0}\psi_k(D)Tf\Big\|_{\mathcal{F}L^\infty} \lesssim \sup_{k\ge 0}\|\psi_k(D)Tf\|_{\mathcal{F}L^\infty}\\ &= \sup_{k\ge 0}\Big\|\sum_{j=1}^{\infty}\psi_k(D)T_j f\Big\|_{\mathcal{F}L^\infty}.\end{aligned} \tag{3.13}$$

Notice that the sequence of symbols $\sigma_j(x,\eta)$ is bounded in $S^{-r/2}_{1,0}$, whereas the sequence of symbols $\psi_k(\eta)$ is bounded in $S^0_{1,0}$.

Applying Theorem 2.1 to each product $\psi_k(D)T_j$, we have

$$\psi_k(D)T_j = S_{k,j} + R_{k,j}, \tag{3.14}$$

where $S_{k,j}$ are FIOs with the same phase Φ and symbols $\sigma_{k,j}$ belonging to a bounded subset of $S^{-d/2}_{1,0}$, supported in

$$\{(x,\eta)\in\Omega'\times\Gamma : |\nabla_x\Phi(x,\eta)|\le 2,\ 2^{j-1}\le|\eta|\le 2^{j+1}\}\quad \text{if } k=0, \tag{3.15}$$

and in

$$\{(x,\eta)\in\Omega'\times\Gamma : 2^{k-1}\le|\nabla_x\Phi(x,\eta)|\le 2^{k+1}, 2^{j-1}\le|\eta|\le 2^{j+1}\}\quad \text{if } k\ge 1. \tag{3.16}$$

The operators $R_{k,j}$ are smoothing operators whose symbols $r_{k,j}$ are in a bounded subset of $\mathcal{S}(\mathbb{R}^{2d})$ and supported where $2^{j-1}\le|\eta|\le 2^{j+1}$.

Observe that, by Euler's identity and (3.2),

$$|\nabla_x \Phi(x,\eta)| = |\partial^2_{x,\eta}\Phi(x,\eta)\eta| \asymp |\eta|, \qquad \forall (x,\eta) \in \Omega \times \Gamma.$$

Inserting this equivalence in (3.15) and (3.16), we find that there exists $N_0 > 0$ such that $\sigma_{k,j}$ vanishes identically if $|j-k| > N_0$.

On the other hand it is easily seen that the functions $r_k(x,\eta) := \sum_{j=1}^{\infty} r_{j,k}(x,\eta)$ are well defined and belong to a bounded subset of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. As a consequence, we have

$$\sum_{j=1}^{\infty} \psi_k(D) T_j = \sum_{j\geq 1\,:\,|j-k|\leq N_0} S_{k,j} + r_k(x,D).$$

It is then clear that the right-hand side in (3.13) can be dominated by the $\mathcal{F}L^\infty$ norm of f by using (3.14), (3.10) with $p=\infty$, and the fact that $\psi_k(D)$, $R_{j,k}$ and $r_k(x,D)$ are uniformly bounded on $\mathcal{F}L^\infty$.

This concludes the proof of Theorem 1.2.

Appendix. In this appendix we give some details of the proof of (3.7). We use the same notation as in the proof of Theorem 1.2. We also observe that all the formulae below are meant to hold on the support of $\chi_j^\nu(u(y,\eta))\sigma_j(\eta,y)$.

First we notice that the following key formula holds:

$$\nabla_{\eta''} = A(u,v)\nabla_v + O(2^{-j/2})\nabla_u. \tag{3.17}$$

Indeed,

$$\nabla_{\eta''} = A(u,v)\nabla_v + B(u,v)\nabla_u$$

for suitable smooth matrices A and B. On the other hand, because of our choice of the splitting η', η'', we have $B(u_j^\nu, v) = 0$ for every v. A Taylor expansion of $B(u,v)$ with respect to u, around $u = u_j^\nu$, then shows (3.17), because $|u - u_j^\nu| \leq C_1 2^{-j/2}$.

Similarly it turns out that

$$\nabla_v = A'(u,v)\nabla_{\eta''} + O(2^{-j/2})\nabla_{\eta'},$$

which implies the estimate

$$\frac{\partial \eta'}{\partial v} = O(2^{-j/2}) \tag{3.18}$$

for the Jacobian matrix $\partial\eta'/\partial v$. Moreover, using (3.18) and a Taylor expansion of η', as a function of (u,v), around $(u_j^\nu, 0)$, one deduces

$$(\eta - \eta_j^\nu)' = O(2^{-j/2}). \tag{3.19}$$

Now, we have to show that the repeated application of the first order operators

(3.20) $$2^{-j/2}\partial_{\eta'_k}, \quad \partial_{\eta''_l}, \quad k = 1, \dots, r,\ l = 1, \dots, d-r,$$

to

$$e^{2\pi i(\Phi(\eta_j^\nu, y) - \langle \nabla_1 \Phi(\eta_j^\nu, y), \eta_j^\nu\rangle + R_j^\nu(\eta, y))}\chi_j^\nu(u(y,\eta))\sigma_j(\eta, y)$$

yields expressions dominated by $C'2^{-jr/2}$.

This is of course true when these operators fall on $\sigma_j(\eta, y)$, since σ is a symbol of order $-r/2$.

When the operators $2^{-j/2}\partial_{\eta'_k}$ fall on $\chi_j^\nu(u(y,\eta))$ one obtains acceptable terms because the factor $2^{-j/2}$ in front of the derivative offsets the loss in (3.4). The same happens for the derivatives $\partial_{\eta''_l}$, because of (3.17) and the fact that $\chi_j^\nu(u(y,\eta))$ is constant on the (pieces of) planes $u = \text{const}$.

Hence it remains to prove that the repeated application of the operators in (3.20) on $R_j^\nu(\eta, y)$ yields uniformly bounded expressions. To this end, denote by P_ξ the orthogonal projection on the vector space parallel to the plane $u = \text{const}$ which contains $\xi := \eta_j^\nu + t(\eta - \eta_j^\nu)$, $0 \le t \le 1$. We observe that

$$(\eta - \eta_j^\nu)'' - P_\xi((\eta - \eta_j^\nu)'') = 0 \quad \text{if } (\eta - \eta_j^\nu)' = 0.$$

Hence

(3.21) $$(\eta - \eta_j^\nu)'' = E_j^\nu(t, y, \eta)(\eta - \eta_j^\nu)' + P_\xi((\eta - \eta_j^\nu)''),$$

for a suitable matrix $E_j^\nu(t, y, \eta)$ whose entries are positively homogeneous of degree 0 with respect to $y \in \Gamma$, and uniformly bounded, together with their derivatives. Now, in (3.6) we substitute

$$\eta - \eta_j^\nu = (\eta - \eta_j^\nu)' + (\eta - \eta_j^\nu)'' = (\eta - \eta_j^\nu)' + E_j^\nu(t, y, \eta)(\eta - \eta_j^\nu)' + P_\xi((\eta - \eta_j^\nu)'').$$

Since, by assumption, the gradient $\nabla_1\Phi(\cdot, y)$ is constant on the planes $u = \text{const}$, we have

$$P_\xi((\eta - \eta_j^\nu)'') \in \operatorname{Ker}(d_1^2\Phi(\xi, y)).$$

Using the bilinearity of the Hessian one sees that $R_j^\nu(\eta, y)$ can therefore be written in the form

(3.22) $$\langle G_j^\nu(\eta, y)(\eta - \eta_j^\nu)', (\eta - \eta_j^\nu)'\rangle,$$

where the matrix $G_j^\nu(\eta, y)$ has entries which are uniformly bounded, together with their derivatives, by $C2^j$ (because $\Phi(\eta, y)$ is positively homogeneous of degree 1 in y and here $|y| \asymp 2^j$).

In view of (3.19) we see that the repeated action of the operators (3.20) on the expression in (3.22) yields uniformly bounded terms.

This concludes the proof.

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