Rademacher functions in Cesàro type spaces

by

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Abstract. The Rademacher sums are investigated in the Cesàro spaces Ces_p ($1 \leq p \leq \infty$) and in the weighted Korenblyum–Kreǐn–Levin spaces $K_{p,w}$ on $[0,1]$. They span l_2 space in Ces_p for any $1 \leq p < \infty$ and in $K_{p,w}$ if and only if the weight w is larger than $t \log_2^{p/2}(2/t)$ on $(0,1)$. Moreover, the span of the Rademachers is not complemented in Ces_p for any $1 \leq p < \infty$ or in $K_{1,w}$ for any quasi-concave weight w. In the case when $p > 1$ and when w is such that the span of the Rademacher functions is isomorphic to l_2, this span is a complemented subspace in $K_{p,w}$.

1. Introduction and preliminaries. Consider the Rademacher functions on $[0,1]$ defined by $r_k(t) = \mathrm{sign}(\sin 2^k \pi t)$, $k \in \mathbb{N}$, $t \in [0,1]$, and the set of Rademacher sums

$$R_n(t) = \sum_{k=1}^{n} a_k r_k(t), \quad a_k \in \mathbb{R} \quad \text{for } k = 1, \dots, n, \text{ and } n \in \mathbb{N}.$$

The behaviour of Rademacher sums in $L_p = L_p[0,1]$ spaces is well known (cf. [6, p. 10]). In particular, it follows from the classical *Khintchine inequality* that there exist constants $A_p, B_p > 0$ such that for any sequence $\{a_k\}_{k=1}^{n}$ of real numbers and any $n \in \mathbb{N}$ we have

$$(1) \quad A_p \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2} \leq \|R_n\|_{L_p[0,1]} \leq B_p \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2}, \quad 0 < p < \infty,$$

and hence the Rademacher functions $\{r_n\}$ span an isomorphic copy of l_2 in L_p for every $0 < p < \infty$. Moreover, the subspace $[r_n] \simeq l_2$ is complemented in L_p for $1 < p < \infty$ and is not complemented in L_1 since no complemented infinite-dimensional subspace of L_1 can be reflexive. In L_∞ we have $\|R_n\|_{L_\infty[0,1]} = \sum_{k=1}^{n} |a_k|$ and so the Rademacher functions span an isometric copy of l_1, which is known to be uncomplemented in L_∞.

In the case of rearrangement invariant spaces X on $[0, 1]$, Rodin and Semenov [16] proved that $\|\sum_{k=1}^{\infty} a_k r_k\|_X \approx (\sum_{k=1}^{\infty} |a_k|^2)^{1/2}$ if and only if $G \subset X$, where G is the closure of $L_\infty[0, 1]$ in the Orlicz space $L_M[0, 1]$ generated by the function $M(u) = e^{u^2} - 1$. Note that this Orlicz space coincides with the Marcinkiewicz space $M(\varphi)$ generated by the function $\varphi(t) = t \log_2^{1/2}(2/t)$, $0 < t \leq 1$, where the norm is given by $\|f\|_{M(\varphi)} = \sup_{0 < x \leq 1} (1/\varphi(x)) \int_0^x f^*(t)\, dt$. Here, f^* is the non-increasing rearrangement of $|f|$. Moreover, Rodin–Semenov [17] and Lindenstrauss–Tzafriri [12, pp. 134–138] proved that $[r_n] \simeq l_2$ is complemented in X if and only if $G \subset X \subset G'$, where G' is the Köthe dual space to G.

By contrast, Astashkin [2] considered rearrangement invariant spaces X which are interpolation spaces between G and L_∞. It turns out that there exists a one-to-one correspondence between them and the sequence spaces F which are interpolation spaces for the couple (l_1, l_2); namely, $\|\sum_{k=1}^{\infty} a_k r_k\|_X \approx \|\{a_k\}_{k=1}^{\infty}\|_F$, where $X = (L_\infty, G)_\Phi^K$ and $F = (l_1, l_2)_\Phi^K$, with the same parameter space Φ.

Investigations of Rademacher sums in rearrangement invariant spaces are well presented in the books by Lindenstrauss–Tzafriri [12], Kreĭn–Petunin–Semenov [10] and Astashkin [3].

The main purpose of this paper is to investigate the behaviour of Rademacher functions in the *Cesàro function spaces* $\text{Ces}_p = \text{Ces}_p[0, 1]$, which are not rearrangement invariant (cf. [4] and [5], where also other properties are investigated). These spaces are the classes of all Lebesgue measurable real-valued functions f on $[0, 1]$ such that

$$\|f\|_{\text{Ces}_p} = \left[\int_0^1 \left(\frac{1}{x} \int_0^x |f(t)|\, dt \right)^p dx \right]^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|f\|_{\text{Ces}_\infty} = \sup_{0 < x \leq 1} \frac{1}{x} \int_0^x |f(t)|\, dt < \infty \quad \text{for } p = \infty.$$

We will see that there is an essential difference between the cases $p < \infty$ and $p = \infty$. Namely, for any $1 \leq p < \infty$, the Rademacher functions $\{r_n\}$ span an isomorphic copy of l_2 in $\text{Ces}_p[0, 1]$. At the same time, in $\text{Ces}_\infty[0, 1]$ this system does not contain an unconditional basic subsequence.

In fact, we will consider the Rademacher functions not only in the space $\text{Ces}_\infty =: K$, known as the *Korenblyum–Kreĭn–Levin space* [9], but also in more general $K_{p,w}$ spaces or *weighted Korenblyum–Kreĭn–Levin spaces*.

Let $1 \leq p < \infty$ and w be a quasi-concave function on $I = [0, 1]$, that is, $w(0) = 0$, w is non-decreasing, and $w(x)/x$ is non-increasing on $(0, 1]$. The spaces $K_{p,w} = K_{p,w}[0, 1]$ are the classes of all Lebesgue measurable

real-valued functions f on $[0,1]$ such that the norm

$$(2) \qquad \|f\|_{K_{p,w}} = \sup_{x \in I, x > 0} \left(\frac{1}{w(x)} \int_0^x |f(t)|^p \, dt \right)^{1/p}$$

is finite. These spaces are Banach ideal spaces on I. The *ideal* property means that if $|f| \leq |g|$ a.e. on I and $g \in K_{p,w}$, then $f \in K_{p,w}$ and $\|f\|_{K_{p,w}} \leq \|g\|_{K_{p,w}}$. In the special cases when $w(x) = 1$ and $w(x) = x$ we obtain the well-known spaces L_p and the *Korenblyum–Kreĭn–Levin spaces* K_p (see [9]), respectively. Moreover, $K_1 = K = \text{Ces}_\infty$.

For two Banach spaces X and Y the symbol $X \overset{C}{\hookrightarrow} Y$ means that the imbedding $X \subset Y$ is continuous with norm not greater than C, i.e., $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$.

2. Rademacher sums in Cesàro type spaces.
First we consider the Rademacher sums in the spaces $\text{Ces}_p[0,1]$ with $1 \leq p < \infty$.

THEOREM 1. *For any $1 \leq p < \infty$, the Rademacher functions $\{r_n\}$ span an isomorphic copy of l_2 in $\text{Ces}_p[0,1]$.*

Proof. We need to prove that there are positive constants A_p', B_p' such that

$$(3) \qquad A_p' \left(\sum_{k=1}^n a_k^2 \right)^{1/2} \leq \left\| \sum_{k=1}^n a_k r_k \right\|_{\text{Ces}_p} \leq B_p' \left(\sum_{k=1}^n a_k^2 \right)^{1/2}$$

for every $n \in \mathbb{N}$ and any real numbers $a_1, \ldots, a_n$.

Firstly, let us show that the following continuous embeddings hold:

$$(4) \qquad L_p \overset{p'}{\hookrightarrow} \text{Ces}_p \overset{1}{\hookrightarrow} \text{Ces}_1 = L_1(\ln 1/t) \overset{54}{\hookrightarrow} L_{1/3} \quad \text{if } 1 < p < \infty,$$

and

$$(5) \qquad L_2 \overset{\sqrt{2}}{\hookrightarrow} \text{Ces}_1 = L_1(\ln 1/t) \overset{54}{\hookrightarrow} L_{1/3} \quad \text{if } p = 1.$$

The first embedding in (4) is a consequence of the well-known Hardy inequality [7] and the second one follows directly from the fact that $L_p[0,1] \overset{1}{\hookrightarrow} L_1[0,1]$. Since

$$\int_0^1 \left(\frac{1}{x} \int_0^x |f(t)| \, dt \right) dx = \int_0^1 \left(\int_t^1 \frac{1}{x} \, dx \right) |f(t)| \, dt = \int_0^1 |f(t)| \ln \frac{1}{t} \, dt,$$

we have $\text{Ces}_1 = L_1(\ln 1/t)$. Assume now $f \in L_1(\ln 1/t)$ with $\|f\|_{L_1(\ln 1/t)} = 1$.

Since $\ln 1/t \geq 1 - t$ if $0 < t \leq 1$, it follows that

$$1 = \int_0^1 |f(t)| \ln 1/t \, dt \geq \int_0^1 |f(t)|(1-t) \, dt \geq \int_0^1 f^*(1-t)(1-t) \, dt$$

$$= \int_0^1 f^*(s)s \, ds \geq \int_0^t f^*(s)s \, ds \geq f^*(t)t^2/2,$$

or $f^*(t) \leq 2/t^2$. Thus,

$$\int_0^1 |f(t)|^{1/3} \, dt = \int_0^1 f^*(t)^{1/3} \, dt \leq \int_0^1 (2t^{-2})^{1/3} \, dt = 3 \cdot 2^{1/3}$$

and so $\|f\|_{L_{1/3}} \leq 54$, which finishes the proof of the last imbedding in (4) and (5).

For $p = 1$ the first embedding in (5) is a consequence of the Hölder–Rogers inequality

$$\int_0^1 |f(t)| \ln \frac{1}{t} \, dt \leq \|f\|_{L_2} \left(\int_0^1 \ln^2 \frac{1}{t} \, dt \right)^{1/2} = \sqrt{2} \, \|f\|_{L_2},$$

and the second one is obviously still true for $p = 1$.

In view of the Khintchine inequality (1) we obtain $\|R_n\|_{\mathrm{Ces}_p} \approx \|\{a_k\}_{k=1}^n\|_{l_2}$ or, more precisely,

$$\frac{A_{1/3}}{54} \|\{a_k\}_{k=1}^n\|_{l_2} \leq \frac{1}{54} \|R_n\|_{\mathrm{Ces}_p} \leq D_p \|R_n\|_{L_p} \leq D_p B_p \|\{a_k\}_{k=1}^n\|_{l_2},$$

where $D_p = p' := p/(p-1)$ if $p > 1$ and $D_p = \sqrt{2}$ if $p = 1$, and inequalities (3) are proved. ∎

The next result concerns the behaviour of the Rademacher sums in the space $\mathrm{Ces}_\infty := K$ and in its weighted version, i.e., the $K_{p,w}$ space. Our main result is the following:

THEOREM 2. *For every $1 \leq p < \infty$,*

$$(6) \qquad \|R_n\|_{K_{p,w}} \approx \|\{a_k\}_{k=1}^n\|_{l_2} + \max_{1 \leq m \leq n} \left(\frac{2^{-m}}{w(2^{-m})} \right)^{1/p} \Big| \sum_{k=1}^m a_k \Big|.$$

Proof. Firstly, we note that it is easy to see that for every $p \geq 1$ and all $m = 0, 1, \ldots, n$ we have

$$(7) \qquad \int_0^{2^{-m}} |R_n(t)|^p \, dt = 2^{-n} \sum_{\varepsilon_k = \pm 1} |a_1 + \cdots + a_m + \varepsilon_{m+1} a_{m+1} + \cdots + \varepsilon_n a_n|^p,$$

where the sum is taken over all choices of signs $\varepsilon_{m+1} = \pm 1, \ldots, \varepsilon_n = \pm 1$. Therefore, by (2) and by the Hölder–Rogers inequality, for every $m = 1, \ldots, n$,

we obtain

$$\|R_n\|_{K_{p,w}} \geq \left(\frac{1}{w(2^{-m})} \int\limits_0^{2^{-m}} |R_n(t)|^p \, dt \right)^{1/p}$$

$$\geq \frac{2^{m(1-1/p)}}{w(2^{-m})^{1/p}} \int\limits_0^{2^{-m}} |R_n(t)| \, dt$$

$$= \frac{2^{m(1-1/p)-n}}{w(2^{-m})^{1/p}} \sum_{\varepsilon_k = \pm 1} |a_1 + \cdots + a_m + \varepsilon_{m+1} a_{m+1} + \cdots + \varepsilon_n a_n|$$

$$\geq \frac{2^{m(1-1/p)-n}}{w(2^{-m})^{1/p}} \Big| \sum_{\varepsilon_k = \pm 1} (a_1 + \cdots + a_m + \varepsilon_{m+1} a_{m+1} + \cdots + \varepsilon_n a_n) \Big|$$

$$= \frac{2^{m(1-1/p)-n}}{w(2^{-m})^{1/p}} \, 2^{n-m} \Big| \sum_{k=1}^m a_k \Big| = \left(\frac{2^{-m}}{w(2^{-m})} \right)^{1/p} \Big| \sum_{k=1}^m a_k \Big|.$$

Hence, for every $m = 1, \ldots, n$,

$$(8) \qquad \|R_n\|_{K_{p,w}} \geq \left(\frac{2^{-m}}{w(2^{-m})} \right)^{1/p} \Big| \sum_{k=1}^m a_k \Big|.$$

On the other hand, by (2) and the classical Khintchine inequality (1), we obtain

$$\|R_n\|_{K_{p,w}} \geq \frac{1}{w(1)^{1/p}} \sup_{0 < x \leq 1} \left(\int\limits_0^x |R_n(t)|^p \, dt \right)^{1/p}$$

$$= \frac{1}{w(1)^{1/p}} \|R_n\|_{L_p[0,1]} \geq \frac{A_p}{w(1)^{1/p}} \|\{a_k\}_{k=1}^n\|_{l_2}.$$

This together with (8) implies that

$$\|R_n\|_{K_{p,w}} \geq C_{p,w} \left(\|\{a_k\}_{k=1}^n\|_{l_2} + \max_{1 \leq m \leq n} \left(\frac{2^{-m}}{w(2^{-m})} \right)^{1/p} \Big| \sum_{k=1}^n a_k \Big| \right),$$

where $C_{p,w} = \frac{1}{2} \max\{A_p w(1)^{-1/p}, 1\}$.

Let us prove the converse inequality. Since $R_n(t)$ is a constant on $[0, 2^{-n}]$, we have

$$\frac{1}{x} \int\limits_0^x |R_n(t)|^p \, dt = 2^n \int\limits_0^{2^{-n}} |R_n(t)|^p \, dt \quad (0 < x \leq 2^{-n}).$$

Moreover, by the quasi-concavity of w, one has $x/w(x) \leq 2^{-n}/w(2^{-n})$ if $0 < x \leq 2^{-n}$ and $w(2^{-j}) \leq 2w(x)$ if $x \geq 2^{-j-1}$ $(j = 1, 2, \ldots)$. Therefore, we

obtain

$$\sup_{x\in(0,1]} \frac{1}{w(x)} \int_0^x |R_n(t)|^p\, dt$$

$$= \max\left[\sup_{x\in(0,2^{-n}]} \frac{1}{w(x)} \int_0^x |R_n(t)|^p\, dt,\ \max_{1\le k\le n} \sup_{x\in[2^{-n+k-1},2^{-n+k}]} \frac{1}{w(x)} \int_0^x |R_n(t)|^p\, dt \right]$$

$$\le \max\left[\sup_{x\in(0,2^{-n}]} \frac{x}{w(x)}\frac{1}{x} \int_0^x |R_n(t)|^p\, dt,\ \max_{1\le k\le n} \frac{1}{w(2^{-n+k-1})} \int_0^{2^{-n+k}} |R_n(t)|^p\, dt \right]$$

$$\le \max\left[\frac{1}{w(2^{-n})} \int_0^{2^{-n}} |R_n(t)|^p\, dt,\ \max_{1\le k\le n} \frac{2}{w(2^{-n+k})} \int_0^{2^{-n+k}} |R_n(t)|^p\, dt \right]$$

$$\le 2\max\left[\frac{1}{w(2^{-n})} \int_0^{2^{-n}} |R_n(t)|^p\, dt,\ \max_{1\le k\le n} \frac{1}{w(2^{-n+k})} \int_0^{2^{-n+k}} |R_n(t)|^p\, dt \right]$$

$$= 2\max_{0\le k\le n} \frac{1}{w(2^{-n+k})} \int_0^{2^{-n+k}} |R_n(t)|^p\, dt,$$

and so

$$(9) \qquad \|R_n\|_{K_{p,w}} \le 2^{1/p} \max_{0\le k\le n} \left(\frac{1}{w(2^{k-n})} \int_0^{2^{-n+k}} |R_n(t)|^p\, dt \right)^{1/p}.$$

Now, taking into account (7) and using the Minkowski inequality together with the upper estimate from the Khintchine inequality (1), we see that for all $k = 0, 1, \ldots, n$,

$$\left(\int_0^{2^{k-n}} |R_n(t)|^p\, dt \right)^{1/p}$$

$$= \left(2^{-n} \sum_{\varepsilon_i=\pm 1} |a_1 + \cdots + a_{n-k} + \varepsilon_{n-k+1}a_{n-k+1} + \cdots + \varepsilon_n a_n|^p \right)^{1/p}$$

$$\le \left(2^{-n} \sum_{\varepsilon_i=\pm 1} |a_1 + \cdots + a_{n-k}|^p \right)^{1/p}$$

$$+ \left(2^{-n} \sum_{\varepsilon_i=\pm 1} |\varepsilon_{n-k+1}a_{n-k+1} + \cdots + \varepsilon_n a_n|^p \right)^{1/p}$$

$$= 2^{(k-n)/p}\left(\left| \sum_{i=1}^{n-k} a_i \right| + \left\| \sum_{j=1}^{k} a_{n-k+j} r_j \right\|_{L_p[0,1]} \right)$$

$$\le 2^{(k-n)/p}\left(\left| \sum_{i=1}^{n-k} a_i \right| + B_p \| \{a_{n-k+j}\}_{j=1}^k \|_{l_2} \right).$$

Therefore, using (9), we get

$$\|R_n\|_{K_{p,w}} \leq 2^{1/p}\left(B_p w(1)^{-1/p}\|\{a_k\}_{k=1}^n\|_{l_2} + \max_{1\leq m\leq n}\left(\frac{2^{-m}}{w(2^{-m})}\right)^{1/p}\Big|\sum_{k=1}^m a_k\Big|\right)$$

$$\leq 2^{1/p}\max\{B_p w(1)^{-1/p}, 1\}\left(\|\{a_k\}_{k=1}^n\|_{l_2} + \max_{1\leq m\leq n}\left(\frac{2^{-m}}{w(2^{-m})}\right)^{1/p}\Big|\sum_{k=1}^m a_k\Big|\right),$$

and the proof is complete. ∎

THEOREM 3. *The norms* $\|R_n\|_{K_{p,w}}$ *are uniformly equivalent to the norms* $\|\{a_k\}_{k=1}^n\|_{l_2}$ *with respect to* $n \in \mathbb{N}$ *if and only if there is a constant* $c > 0$ *such that*

$$(10) \qquad w(t) \geq ct\log_2^{p/2}(2/t) \quad \text{for all } 0 < t \leq 1.$$

Proof. If (10) holds, then for all $1 \leq m \leq n$ we have

$$(11) \qquad \left(\frac{2^{-m}}{w(2^{-m})}\right)^{1/p} m^{1/2} \leq c^{-1/p}.$$

Since, by the Cauchy–Schwarz inequality,

$$\Big|\sum_{k=1}^m a_k\Big| \leq m^{1/2}\|\{a_k\}_{k=1}^m\|_{l_2},$$

(6) and (10) give the required equivalence.

Conversely, suppose that condition (10) does not hold. Then, by the quasi-concavity of w, inequality (11) is not satisfied and hence there exists a sequence of natural numbers $m_k \to \infty$ such that

$$(12) \qquad \left(\frac{2^{-m_k}}{w(2^{-m_k})}\right)^{1/p} m_k^{1/2} \to \infty \quad \text{as } k \to \infty.$$

Consider the Rademacher sums $R_k(t)$ corresponding to the sequences of coefficients $a^k = (a_i^k)_{i=1}^{m_k}$, where $a_i^k = m_k^{-1/2}$, $1 \leq i \leq m_k$. We have $\|a^k\|_{l_2} = 1$ for all $k = 1, 2, \ldots$, but $\sum_{i=1}^m a_i^k = m_k^{1/2}$ $(k = 1, 2, \ldots)$, which together with (12) and (6) implies that $\|R_k\|_{K_{p,w}} \to \infty$ as $k \to \infty$. ∎

REMARK 1. Let $1 \leq p < \infty$ and let w be a quasi-concave function on $[0, 1]$. Consider the p-convexification $M_{p,w}$ of the Marcinkiewicz space $M_{1,w} = M(w)$ with the norm given by

$$\|f\|_{M_{p,w}} = \sup_{0<x\leq 1}\left(\frac{1}{w(x)}\int_0^x f^*(t)^p\,dt\right)^{1/p}.$$

The space $M_{p,w}$ can be treated as a rearrangement invariant version of $K_{p,w}$. It is not hard to check that the embedding $G \subset M_{p,w}$ (see Section 1) holds if and only if the condition (10) is satisfied. Therefore, by Theorem 3 and by

the result of Rodin–Semenov [16] (see also [12, Th. 2.b.4]), it follows that the Rademacher functions span l_2 in $M_{p,w}$ if and only if they span l_2 in $K_{p,w}$.

Moreover, it is instructive to compare the behaviour of Rademacher sums in the spaces $M_{1,w} = M(w)$ and $K_{1,w}$ in the case when $w(t) = t \log_2^{1/q}(2/t)$, where $q > 2$. Then (10) does not hold and, by (6),

$$\|R_n\|_{K_{1,w}} \approx \|\{a_k\}_{k=1}^n\|_{l_2} + \max_{1 \leq m \leq n} m^{-1/q} \Big| \sum_{k=1}^m a_k \Big|.$$

On the other hand, by the result of Rodin–Semenov [16, p. 221] and Pisier [15] (see also Marcus–Pisier [14, pp. 277–278]), the norm $\|R_n\|_{M(w)}$ is equivalent to the norm of the sequence of coefficients in the Marcinkiewicz sequence space $l_{q,\infty}$ given by

$$\|\{a_k\}\|_{l_{q,\infty}} = \sup_{m=1,2,\dots} m^{-1/q} \sum_{k=1}^m a_k^*,$$

where $\{a_k^*\}$ is the non-increasing rearrangement of $\{|a_k|\}_{k=1}^\infty$.

The following corollaries follow directly from Theorem 1 in the special case $w(x) = x$.

COROLLARY 1. *For any $1 \leq p < \infty$ we have the equivalence*

$$(13) \qquad \|R_n\|_{K_p} \approx \|\{a_k\}_{k=1}^n\|_{l_2} + \max_{1 \leq m \leq n} \Big| \sum_{k=1}^m a_k \Big|,$$

and, in particular, the same holds for the space $\mathrm{Ces}_\infty[0,1] = K_1$.

COROLLARY 2. *Let $1 \leq p < \infty$. The series $\sum_{k=1}^\infty a_k r_k$ is convergent in K_p if and only if $\sum_{k=1}^\infty a_k^2 < \infty$ and the series $\sum_{k=1}^\infty a_k$ is convergent.*

COROLLARY 3. *The sequence of Rademacher functions is equivalent in K_p to each of its subsequences.*

COROLLARY 4. *The system of Rademacher functions does not contain an unconditional basic subsequence in K_p.*

REMARK 2. From (13) we obtain

$$(14) \qquad \Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{K_p} \approx \|\{a_k\}_{k=1}^\infty\|_{l_2} + \sup_{m \in \mathbb{N}} \Big| \sum_{k=1}^m a_k \Big|.$$

A similar equivalence holds also for the BMO space on $[0,1]$. In 1985 Mikhail Leĭbov showed in his PhD dissertation [11] that

$$\Big\| \sum_{k=1}^\infty a_k r_k \Big\|_{\mathrm{BMO}} \approx \|\{a_k\}_{k=1}^\infty\|_{l_2} + \sup_{l,m \in \mathbb{N},\, l \leq m} \Big| \sum_{k=l}^m a_k \Big|.$$

Noting that

$$\sup_{m \in \mathbb{N}} \left| \sum_{k=1}^{m} a_k \right| \leq \sup_{l,m \in \mathbb{N},\, l \leq m} \left| \sum_{k=l}^{m} a_k \right|$$

$$= \sup_{l,m \in \mathbb{N},\, l \leq m} \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{l-1} a_k \right| \leq 2 \sup_{m \in \mathbb{N}} \left| \sum_{k=1}^{m} a_k \right|,$$

we obtain estimates which are similar to (14) but with BMO instead of K_p. Note that the spaces K_p and BMO are not comparable, that is, no one of them is embedded in the other.

3. Complementability of Rademacher subspaces in Cesàro type spaces. We first consider the problem of complementability of the closed linear span $[r_n]_{n=1}^{\infty}$ in the spaces $K_{p,w}$. We begin with the case when $p = 1$.

THEOREM 4. *For every quasi-concave function w, the subspace $[r_n]_{n=1}^{\infty}$ of the Banach space $K_{1,w}$ is not complemented in that space.*

Proof. We will consider only the case when $\lim_{x \to 0+} x/w(x) > 0$ (in other words, when $K_{1,w}$ coincides with the Korenblyum–Kreĭn–Levin space $K := \mathrm{Ces}_{\infty}[0,1]$ with equivalence of norms) because the proof in the remaining case when $\lim_{x \to 0+} x/w(x) = 0$ is completely similar and even a little easier.

Suppose that $[r_n]_{n=1}^{\infty}$ is a complemented subspace of K, and P is the corresponding bounded linear projector. Since, by Theorem 2, the Rademacher functions form a basic sequence in K, there exist functionals $\phi_n \in K^*$ ($n = 1, 2, \ldots$) such that

$$Pf(x) = \sum_{n=1}^{\infty} \phi_n(f) r_n(x) \quad (f \in K). \tag{15}$$

By [13], the Köthe dual K' equals $\tilde{L}_1$ with equality of norms, where

$$\|f\|_{\tilde{L}_1} = \|\tilde{f}\|_{L_1}, \quad \text{with} \quad \tilde{f}(x) = \operatorname*{ess\,sup}_{t \in [x,1]} |f(t)|.$$

Since $\tilde{L}_1$ is a total set in K, by [8, Ch. 10, Theorem 3.6, Russian edition],

$$K^* = K' \oplus (K')^d, \tag{16}$$

where $(K')^d$ is the set of all singular bounded linear functionals on K. In particular, if $\theta \in (K')^d$, then

$$\theta(f) = 0 \quad \text{for every } f \in K_0, \tag{17}$$

where K_0 is the separable part of K.

Equality (16) implies that $\phi_n = \psi_n + \theta_n$, where $\psi_n \in K'$ and $\theta_n \in (K')^d$ ($n = 1, 2, \ldots$). Moreover, since P is a projection onto $[r_n]_{n=1}^{\infty}$,

$$\psi_n(r_n) + \theta_n(r_n) = 1 \quad \text{and} \quad \psi_n(r_i) + \theta_n(r_i) = 0 \quad \text{if } i \neq n. \tag{18}$$

By (17), $\theta_n(r_i) = \theta_n(\chi_{[0,1]}) := c_n$ for all positive integers n and i. Therefore, (18) implies that $\psi_n(r_i) = -c_n$ for all $i \neq n$. On the other hand,

$$(19) \qquad \psi_n(f) = \int_0^1 g_n(t) f(t) \, dt,$$

where $g_n \in \tilde{L}_1$ $(n = 1, 2, \ldots)$. Taking into account that $\tilde{L}_1 \subset L_1$ and that $\{r_i\}$ is a uniformly bounded orthonormal sequence of functions we have $\psi_n(r_i) \to 0$ as $i \to \infty$ for every $n = 1, 2, \ldots$. Therefore, $c_n = 0$ $(n = 1, 2, \ldots)$, and, by (18) and (19), we conclude that

$$(20) \qquad \int_0^1 g_n(t) r_n(t) \, dt = 1 \quad \text{and} \quad \int_0^1 g_n(t) r_i(t) \, dt = 0 \quad \text{if } i \neq n.$$

Moreover, the restriction of the projection P from (15) to the separable part K_0 (we will denote it in the same way) may be represented in the form

$$Pf(x) = \sum_{n=1}^{\infty} \int_0^1 g_n(t) f(t) \, dt \, r_n(x) \qquad (f \in K_0).$$

Next, we note that there exist a small enough $h \in (0,1)$ and a positive integer n_0 such that for all $n \geq n_0$,

$$(21) \qquad \left| \int_h^1 g_n(t) r_n(t) \, dt \right| \geq \frac{1}{2}.$$

In fact, otherwise, (20) implies that there is a subsequence $\{g_{n_k}\} \subset \{g_n\}$ satisfying

$$(22) \qquad \int_0^{1/i} |g_{n_i}(t)| \, dt \geq \left| \int_0^{1/i} g_{n_i}(t) r_{n_i}(t) \, dt \right| \geq \frac{1}{2} \qquad (i = 1, 2, \ldots).$$

By Theorem 1, for every $f \in L_\infty \subset K$ we have

$$\sum_{n=1}^{\infty} \left(\int_0^1 g_n(t) f(t) \, dt \right)^2 < \infty,$$

which implies that $g_n \xrightarrow{w} 0$ in $L_1[0,1]$. Therefore, by the Dunford–Pettis criterion [1, Theorem 5.2.9], $\{g_n\}_{n=1}^{\infty}$ is an equi-integrable set in $L_1[0,1]$, which contradicts (22). Thus, inequalities (21) hold for all $n \geq n_0$.

Now consider the operator P_h defined by

$$P_h f(x) = \sum_{n=1}^{\infty} \int_h^1 g_n(t) f(t) \, dt \, r_n(x).$$

Since, by Theorem 2 and the classical Khintchine inequality (1),

$$\Big\| \sum_{n=1}^{\infty} a_n r_n \Big\|_{L_1[0,1]} \le C \Big\| \sum_{n=1}^{\infty} a_n r_n \Big\|_K,$$

P_h acts boundedly from K into $L_1[0,1]$. At the same time, if f is a measurable function on $[0,1]$ such that supp $f \subset [h,1]$, then

$$\|f\|_{L_1[h,1]} \le \|f\|_K = \sup_{h \le x \le 1} \frac{1}{x} \int_x^1 |f(t)|\, dt \le \frac{1}{h} \|f\|_{L_1[h,1]},$$

and therefore $P_h : L_1[h,1] \to L_1[0,1]$ is bounded. Moreover, by the definition of the operator P_h and inequality (1), we see that P_h is weakly compact. Since the space $L_1[h,1]$ has the Dunford–Pettis property [1, Theorem 5.4.5] it follows that P_h is weak-to-norm sequentially continuous. Therefore, taking into account that $r_n \chi_{[h,1]} \xrightarrow{w} 0$ in $L_1[h,1]$, we find that $\|P_h(r_n \chi_{[h,1]})\|_{L_1[0,1]} \to 0$. On the other hand, by (1) and (21),

$$\|P_h(r_n \chi_{[h,1]})\|_{L_1[0,1]} \asymp \Big(\sum_{i=1}^{\infty} \Big(\int_h^1 g_i(t) r_n(t)\, dt \Big)^2 \Big)^{1/2} \ge \Big| \int_h^1 g_n(t) r_n(t)\, dt \Big| \ge \frac{1}{2},$$

for all $n \ge n_0$. This contradiction concludes the proof. ∎

In the case $p > 1$ the situation is completely different.

THEOREM 5. *Let* $1 < p < \infty$. *If the weight* w *satisfies condition* (10), *then the subspace* $[r_n]_{n=1}^{\infty}$ *is complemented in* $K_{p,w}$.

Proof. By [17] (cf. [12, pp. 134–138]), the orthogonal projection

$$Qf(x) = \sum_{n=1}^{\infty} \int_0^1 r_n(t) f(t)\, dt\, r_n(x)$$

is bounded in $L_p[0,1]$ $(1 < p < \infty)$. Therefore, Theorem 3 yields

$$\|Qf\|_{K_{p,w}} \le C \|Qf\|_{L_p} \le C \|Q\|_{L_p \to L_p} \|f\|_{L_p} \le C \|Q\|_{L_p \to L_p} w(1)^{1/p} \|f\|_{K_{p,w}},$$

which implies that Q is bounded in $K_{p,w}$. ∎

REMARK 3. In contrast to the spaces $K_{1,w}$, the subspace $[r_n]_{n=1}^{\infty}$ is complemented in a Marcinkiewicz space $M_{1,w} = M(w)$ if and only if $G \subset M(w) \subset G'$ (cf. [17] and [12, pp. 134–138]). Note that the left hand embedding is equivalent to condition (10).

Finally, let us consider the case of Cesàro spaces with finite p. First we note that, in contrast to $K = \mathrm{Ces}_\infty[0,1]$, the spaces $\mathrm{Ces}_p[0,1]$ $(1 \le p < \infty)$ are separable. This implies that $\mathrm{Ces}_p[0,1]^*$ coincides with the Köthe dual $\mathrm{Ces}_p[0,1]'$ (described in [4]). It is also easy to see that for any $h \in (0,1)$ there is a constant $C_h > 0$ such that for every measurable function f on $[0,1]$ with supp $f \subset [h,1]$, we have $\|f\|_{\mathrm{Ces}_p} \le C_h \|f\|_{L_1}$. Therefore, by Theorem 1,

arguing in the same way as in the proof of Theorem 4, we can prove the following result:

THEOREM 6. *For any $1 \leq p < \infty$, the subspace $[r_n]$ is not complemented in* $\mathrm{Ces}_p[0, 1]$.

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