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On class A operators

by

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Abstract. We show that every class A operator has a scalar extension. In particular, such operators with rich spectra have nontrivial invariant subspaces. Also we give some spectral properties of the scalar extension of a class A operator. Finally, we show that every class A operator is nonhypertransitive.

1. Introduction. Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_{\rm ap}(T)$, and $\sigma_{\rm e}(T)$ for the spectrum, the approximate point spectrum, and the essential spectrum, respectively, and write $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ for the spectral radius of T. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(TT^*)^p \le (T^*T)^p$, where $0 < p < \infty$. In particular, 1-hyponormal operators and $\frac{1}{2}$-hyponormal operators are called hyponormal operators and semi-hyponormal operators, respectively.

An arbitrary operator $T \in \mathcal{L}(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is a partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$. Associated with T is the operator $|T|^{1/2}U|T|^{1/2}$ called the *Aluthge transform* of T, and denoted throughout this paper by $\widehat{T}$. For every $T \in \mathcal{L}(\mathcal{H})$, the sequence $\{\widehat{T}^{(n)}\}$ of Aluthge iterates of T is defined by $\widehat{T}^{(0)} = T$ and $\widehat{T}^{(n+1)} = \widehat{\widehat{T}^{(n)}}$ for every positive integer n (see [2], [15], and [16]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *w-hyponormal* if $|\widehat{T}| \ge |T| \ge |\widehat{T}^*|$ (see [3]), and *paranormal* if $\|Tx\|^2 \le \|T^2x\|\,\|x\|$ for all $x \in \mathcal{H}$. We say that $T \in \mathcal{L}(\mathcal{H})$ is *normaloid* if $\|T\| = r(T)$. It is well-known that every p-hyponormal operator is w-hyponormal and that every w-hyponormal operator is normaloid. Furuta–Ito–Yamazaki ([12]) introduced the following interesting class of Hilbert space operators.

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Definition 1.1. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to *class A* if it satisfies the condition $|T^2| \geq |T|^2$.

It is known from [12] that

$$\begin{aligned}\{\text{hyponormal operators}\} &\subset \{p\text{-hyponormal operators}\}\ (0 < p \leq 1)\\ &\subset \{\text{class A operators}\}\\ &\subset \{\text{paranormal operators}\}\\ &\subset \{\text{normaloid operators}\}.\end{aligned}$$

There is a vast literature concerning class A operators ([11]–[14], [27], [28], etc.). By a simple computation one can show that a weighted shift belongs to class A if and only if it is hyponormal. In [11], T. Furuta gives several examples of class A operators, including the following.

Example 1.2 ([11]). Let $A = \left(\begin{smallmatrix} 17 & 7 \\ 7 & 5 \end{smallmatrix}\right)^2$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right)^2$ be operators on $\mathbb{R}^2$, and let $\mathcal{H}_n = \mathbb{R}^2$ for all positive integers n. Consider the operator $T_{A,B}$ on $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ defined by

$$T_{A,B} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & B & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & B & \widehat{0} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & B & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & A & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & A & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the hat indicates the position of the $(0, 0)$ element in the matrix. Then $T_{A,B}$ is a class A operator, but is not p-hyponormal for any p.

An operator $S \in \mathcal{L}(\mathcal{H})$ is called *scalar* of order m if it possesses a spectral distribution of order m, i.e. a continuous unital morphism of topological algebras

$$u : C_0^m(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$$

such that $u(z) = S$, where as usual z stands for the identity function on C_0^m, the complex-valued continuously differentiable functions of order m, $0 \leq m \leq \infty$. An operator is said to be *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace.

In 1984, M. Putinar [24] showed that every hyponormal operator has a scalar extension. In 1987, his theorem was used to show that hyponormal operators with thick spectra have nontrivial invariant subspaces, a result due to S. Brown (see [7]). In this paper we generalize those theorems to the context of class A operators. In fact, we show that every class A operator is subscalar of order 12. In particular, every class A operator whose spectrum has nonempty interior has a nontrivial invariant subspace. Also we give some spectral properties of the scalar extension of a class A operator. Finally, we consider the *hypertransitive operator problem*, i.e., the question whether (NHT) $= \mathcal{L}(\mathcal{H})$ (defined later). In particular, we show that every class A operator is nonhypertransitive.

2. Preliminaries. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property at* z_0 if for every neighborhood D of z_0 and any analytic function $f : D \to \mathcal{H}$ with $(T - z)f(z) \equiv 0$, we have $f(z) \equiv 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property* (or SVEP) if it has the single-valued extension property at every z in $\mathbb{C}$. For an operator $T \in \mathcal{L}(\mathcal{H})$ with SVEP and for $x \in \mathcal{H}$ we can consider the set $\rho_T(x)$ of elements z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathcal{H}$, which satisfies $(T - z)f(z) \equiv x$. We denote $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ and $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $\mathcal{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well-known that

$$\text{Property } (\beta) \ \Rightarrow\ \text{Dunford's property } (C) \ \Rightarrow\ \text{SVEP}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ with SVEP is said to have the *decomposition property* (δ) (or simply the property (δ)) if $\mathcal{H} = \mathcal{H}_T(\overline{U}) + \mathcal{H}_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. It is well-known that the adjoint of a bounded linear operator on a Hilbert space with the property (β) has the property (δ) (see [1]).

Let z be the coordinate in $\mathbb{C}$, and let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let U be a bounded open subset of $\mathbb{C}$. We shall denote by $L^2(U, \mathcal{H})$ the Hilbert space of measurable functions $f : U \to \mathcal{H}$ such that

$$\|f\|_{2,U} = \Big(\int_U \|f(z)\|^2 \, d\mu(z)\Big)^{1/2} < \infty.$$

We denote the space $L^2(U, \mathcal{H}) \cap H(U, \mathcal{H})$ by $A^2(U, \mathcal{H})$, where $H(U, \mathcal{H})$ is the Fréchet space of analytic (holomorphic) $\mathcal{H}$-valued functions on U. Then $A^2(U, \mathcal{H})$ is a closed subspace of the $L^2(U, \mathcal{H})$, and the orthogonal projection of $L^2(U, \mathcal{H})$ onto this space will be denoted by P.

Now, we introduce a special Sobolev type space. Let U be a bounded open subset of $\mathbb{C}$ and m be a fixed nonnegative integer. Then the Sobolev space $W^m(U, \mathcal{H})$ is the space of functions $f \in L^2(U, \mathcal{H})$ whose derivatives

$\bar{\partial} f, \bar{\partial}^2 f, \ldots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, \mathcal{H})$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U},$$

$W^m(U, \mathcal{H})$ becomes a Hilbert space contained continuously in $L^2(U, \mathcal{H})$. The linear operator M of multiplication by z on $W^m(U, \mathcal{H})$ is continuous and it has a spectral distribution u of order m defined by the following relation: for $\varphi \in C_0^m(\mathbb{C})$ and $f \in W^m(U, \mathcal{H})$, $u(\varphi)f = \varphi f$. Hence M is a scalar operator of order m.

3. Main results. In this section, we show that every class A operator has a scalar extension. For this, we begin with the following lemma which is the key step to prove our main theorem.

LEMMA 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator and let D be any bounded disk containing $\sigma(T)$. Define the map $V : \mathcal{H} \to H(D)$ by*

$$Vh = \widetilde{1 \otimes h} \ (\equiv 1 \otimes h + \overline{(T - z)W^{12}(D, \mathcal{H})}),$$

where $H(D) = W^{12}(D, \mathcal{H})/\overline{(T - z)W^{12}(D, \mathcal{H})}$ and $1 \otimes h$ denotes the constant function sending $z \in D$ to h. Then V is one-to-one and has closed range.

Proof. Let $h_n \in \mathcal{H}$ and $f_n \in W^{12}(D, \mathcal{H})$ be sequences which satisfy

$$\lim_{n\to\infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^{12}} = 0. \tag{3.1}$$

Then by the definition of the norm of the Sobolev space, (3.1) implies that

$$\lim_{n\to\infty} \|(T - z)\bar{\partial}^i f_n\|_{2,D} = 0 \tag{3.2}$$

for $i = 1, \ldots, 12$. From (3.2) we get

$$\lim_{n\to\infty} \|(T^2 - z^2)\bar{\partial}^i f_n\|_{2,D} = 0 \tag{3.3}$$

for $i = 1, \ldots, 12$. Let $T^2 = U_2|T^2|$ and $\widehat{T^2} = V|\widehat{T^2}|$ be the polar decompositions of T^2 and $\widehat{T^2}$, respectively. Since $\widehat{T^2}|T^2|^{1/2} = |T^2|^{1/2}T^2$ and $\widehat{T^2}^{(2)}|\widehat{T^2}|^{1/2} = |\widehat{T^2}|^{1/2}\widehat{T^2}$, we have

$$\begin{cases} \lim_{n\to\infty} \|(\widehat{T^2} - z^2)\bar{\partial}^i |T^2|^{1/2} f_n\|_{2,D} = 0, \\ \lim_{n\to\infty} \|(\widehat{T^2}^{(2)} - z^2)\bar{\partial}^i |\widehat{T^2}|^{1/2}|T^2|^{1/2} f_n\|_{2,D} = 0, \end{cases} \tag{3.4}$$

for $i = 1, \ldots, 12$. Since T belongs to class A, from [13], T^2 is a w-hyponormal operator, and so $\widehat{T^2}$ is semi-hyponormal and $\widehat{T^2}^{(2)}$ is hyponormal by the definition of a w-hyponormal operator and [3]. Hence, it follows from (3.4) that

$$\lim_{n\to\infty} \|(\widehat{T^2}^{(2)} - z^2)^* \bar{\partial}^i |\widehat{T^2}|^{1/2} |T^2|^{1/2} f_n\|_{2,D} = 0 \tag{3.5}$$

for $i = 1, \ldots, 12$. By Theorem 3.1 of [18], there exists a constant C_D such that

$$\begin{aligned} &\|(I-P)\bar{\partial}^i |\widehat{T^2}|^{1/2} |T^2|^{1/2} f_n\|_{2,D} \\ &\qquad \le C_D \sum_{j=2+i}^{4+i} \|(\widehat{T^2}^{(2)} - z^2)^* \bar{\partial}^j |\widehat{T^2}|^{1/2} |T^2|^{1/2} f_n\|_{2,D} \end{aligned} \tag{3.6}$$

for $i = 0, 1, \ldots, 8$, where P denotes the orthogonal projection of $L^2(D, \mathcal{H})$ onto the Bergman space $A^2(D, \mathcal{H})$. From (3.5) and (3.6), we obtain

$$\lim_{n\to\infty} \|(I-P)\bar{\partial}^i |\widehat{T^2}|^{1/2} |T^2|^{1/2} f_n\|_{2,D} = 0 \tag{3.7}$$

for $i = 1, \ldots, 8$. Thus, by (3.4) and (3.7),

$$\lim_{n\to\infty} \|(\widehat{T^2}^{(2)} - z^2) P \bar{\partial}^i |\widehat{T^2}|^{1/2} |T^2|^{1/2} f_n\|_{2,D} = 0 \tag{3.8}$$

for $i = 1, \ldots, 8$. Since $\widehat{T^2}^{(2)}$ is hyponormal, it has the property (β). Hence

$$\lim_{n\to\infty} \|P \bar{\partial}^i |\widehat{T^2}|^{1/2} |T^2|^{1/2} f_n\|_{2,D_0} = 0 \tag{3.9}$$

for $i = 1, \ldots, 8$, where $\sigma(T) \subsetneq D_0 \subsetneq D$. From (3.7) and (3.9), we get

$$\lim_{n\to\infty} \|\, |\widehat{T^2}|^{1/2} |T^2|^{1/2} \bar{\partial}^i f_n\|_{2,D_0} = 0 \tag{3.10}$$

for $i = 1, \ldots, 8$. Since $\widehat{T^2} |T^2|^{1/2} = |T^2|^{1/2} T^2$, from (3.3) and (3.10) we obtain

$$\lim_{n\to\infty} \|z^4 \bar{\partial}^i f_n\|_{2,D_0} = 0 \tag{3.11}$$

for $i = 1, \ldots, 8$. By Theorem 3.1 of [18], there exists a constant C_{D_0} such that

$$\|(I-P) f_n\|_{2,D_0} \le C_{D_0} \sum_{i=4}^{8} \|z^4 \bar{\partial}^i f_n\|_{2,D_0}. \tag{3.12}$$

By (3.11) and (3.12), it follows that

$$\lim_{n\to\infty} \|(I-P) f_n\|_{2,D_0} = 0. \tag{3.13}$$

Combining (3.13) with (3.1), we have

$$\lim_{n\to\infty} \|(T-z) P f_n + 1 \otimes h_n\|_{2,D_0} = 0.$$

Let Γ be a curve in D_0 surrounding $\sigma(T)$. Then

$$\lim_{n\to\infty} \|P f_n(z) + (T-z)^{-1} (1 \otimes h_n)(z)\| = 0$$

uniformly for all $z \in \Gamma$. Applying the Riesz–Dunford functional calculus, we obtain

$$\lim_{n\to\infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} P f_n(z)\, dz + h_n \right\| = 0.$$

But by Cauchy's theorem, $\frac{1}{2\pi i}\int_{\Gamma} P f_n(z)\, dz = 0$. Hence

$$\lim_{n\to\infty} \|h_n\| = 0.$$

So, V is one-to-one and has closed range. ∎

Now we are ready to show that every class A operator has a scalar extension.

THEOREM 3.2. *Every class A operator in $\mathcal{L}(\mathcal{H})$ is subscalar of order* 12.

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator, let D be an arbitrary bounded open disk in $\mathbb{C}$ that contains $\sigma(T)$ and consider the quotient space

$$H(D) = W^{12}(D,\mathcal{H})/\overline{(T-z)W^{12}(D,\mathcal{H})}$$

endowed with the Hilbert space norm. The class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}$, respectively $\widetilde{S}$. Let M be multiplication by z on $W^{12}(D,\mathcal{H})$. As noted at the end of Section 2, M is a scalar operator of order 12 and has a spectral distribution u. Since the range of $T-z$ is invariant under M, $\widetilde{M}$ is well-defined. Moreover, consider the spectral distribution $u : C_0^{12}(\mathbb{C}) \to \mathcal{L}(W^{12}(D,\mathcal{H}))$ defined by the following relation: for $\varphi \in C_0^{12}(\mathbb{C})$ and $f \in W^{12}(D,\mathcal{H})$, $u(\varphi)f = \varphi f$. Then the spectral distribution u of M commutes with $T-z$, and so $\widetilde{M}$ is still a scalar operator of order 12 with $\widetilde{u}$ as a spectral distribution. Consider the operator $V : \mathcal{H} \to H(D)$ given by $Vh = \widetilde{1\otimes h}$ and denote the range of V by ran$V$. Since

$$VTh = \widetilde{1\otimes Th} = \widetilde{z\otimes h} = \widetilde{M}(\widetilde{1\otimes h}) = \widetilde{M}Vh$$

for all $h \in \mathcal{H}$, we have $VT = \widetilde{M}V$. In particular, ran$V$ is invariant under $\widetilde{M}$. Furthermore, it is closed by Lemma 3.1, and hence it is a closed invariant subspace of the scalar operator $\widetilde{M}$. Since T is similar to the restriction $\widetilde{M}|_{\mathrm{ran}V}$, and $\widetilde{M}$ is a scalar operator of order 12, T is subscalar of order 12. ∎

Theorem 3.2 has the following corollary.

COROLLARY 3.3.

(i) *Every p-hyponormal or w-hyponormal operator is subscalar.*
(ii) *If $T \in \mathcal{L}(\mathcal{H})$ is a class A operator, then $f(T)$ is subscalar for every function f analytic on a neighborhood of $\sigma(T)$.*

Proof. (i) Since every p-hyponormal and every w-hyponormal operator belongs to class A by Section 1, the assertion follows from Theorem 3.2.

(ii) Let T be a class A operator and let f be an analytic function on a neighborhood of $\sigma(T)$. With the same notations as in the proof of Theorem 3.2, we have $Vf(T) = f(\widetilde{M})V$. Thus $f(T)$ is subscalar. ∎

Recall from [6] that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *power regular* if $\lim_{n\to\infty} \|T^n h\|^{1/n}$ exists for every $h \in \mathcal{H}$.

COROLLARY 3.4.

(i) *Every class A operator satisfies the property* (β), *Dunford's property* (C), *and the single-valued extension property.*
(ii) *Every class A operator is power regular.*

Proof. (i) Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. It suffices to prove that T has the property (β). Since the property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 3.2 to the case of a scalar operator of order 12. Since every scalar operator has the property (β) (see [24]), T has the property (β).

(ii) Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. Since T is subscalar of order 12 from Theorem 3.2, it is the restriction of a scalar operator of order 12 to one of its closed invariant subspaces. Since a scalar operator is power regular and all restrictions of power regular operators to their invariant subspaces clearly remain power regular, T is power regular. ∎

Recall that an operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called *a quasiaffinity* if it has trivial kernel and dense range. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be a *quasiaffine transform* of $T \in \mathcal{L}(\mathcal{K})$ if there is a quasiaffinity $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $XS = TX$. Furthermore, S and T are *quasisimilar* if there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$.

COROLLARY 3.5. *Let C and D in $\mathcal{L}(\mathcal{H})$ belong to class A. If C and D are quasisimilar, then $\sigma(C) = \sigma(D)$ and $\sigma_{\mathrm{e}}(C) = \sigma_{\mathrm{e}}(D)$.*

Proof. Since C and D satisfy the property (β) from Corollary 3.4, the assertion follows from [25]. ∎

Next we will give some applications of Theorem 3.2 including a partial solution of the invariant subspace problem for class A operators. Moreover, the following theorem is a generalization of S. Brown's theorem and Berger's theorem (see [7] and [5]).

THEOREM 3.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator.*

(i) *If $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace.*
(ii) *There exists a positive integer K such that for all positive integers $k \geq K$, T^{2k} has a nontrivial invariant subspace.*

Proof. (i) This follows from Theorem 3.2 and [9].

(ii) From [13], T^2 is a w-hyponormal operator. Therefore, by [5] there exists a positive integer K such that for all positive integers $k \geq K$, T^{2k} has a nontrivial invariant subspace. ∎

Next we study some spectral properties of the scalar extension of a class A operator.

THEOREM 3.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. With the notation of the proof of Theorem* 3.2, $\sigma_{\widetilde{M}}(Vh) = \sigma_T(h)$ *for each* $h \in \mathcal{H}$.

Proof. Let $h \in \mathcal{H}$. If $\lambda_0 \in \rho_T(h)$, then there is an $\mathcal{H}$-valued analytic function g defined on a neighborhood U of λ_0 such that $(T-\lambda)g(\lambda) = h$ for all $\lambda \in U$. Then

$$(\widetilde{M} - \lambda)Vg(\lambda) = V(T-\lambda)g(\lambda) = Vh$$

for all $\lambda \in U$. Hence $\lambda_0 \in \rho_{\widetilde{M}}(Vh)$. That is, $\rho_{\widetilde{M}}(Vh) \supset \rho_T(h)$.

Conversely, suppose $\lambda_0 \in \rho_{\widetilde{M}}(Vh)$. Then there exists an $H(D)$-valued analytic function $\widetilde{f}$ on some neighborhood U of λ_0 such that $(\widetilde{M}-\lambda)\widetilde{f}(\lambda) = Vh$ for all $\lambda \in U$. Let $f \in H(U, W^{12}(D,\mathcal{H}))$ be a holomorphic lifting of $\widetilde{f}$ and fix $\zeta \in U$. Then $h-(\zeta-z)f(\zeta,z) \in \overline{(T-z)W^{12}(D,\mathcal{H})}$. Therefore, there is a sequence $\{g_n\} \subset H(U, W^{12}(D,\mathcal{H}))$ such that

$$\lim_{n\to\infty} \|h - (\zeta - z)f(\zeta,z) - (T-z)g_n(\zeta,z)\|_{W^{12}} = 0$$

with respect to $z \in U$. Then

$$\lim_{n\to\infty} \|h - (T-z)f_n\|_{W^{12}} = 0$$

where $f_n(z) := g_n(z,z)$ for $z \in U$. From the proof of Lemma 3.1 (cf. (3.13)), we obtain

$$\lim_{n\to\infty} \|(I-P)f_n\|_{2,U_0} = 0$$

where U_0 is an open neighborhood of λ_0 with $U_0 \subsetneq U$, and so

$$\lim_{n\to\infty} \|h - (T-z)Pf_n\|_{2,U_0} = 0.$$

This implies $h \in \overline{(T-z)H(U_0,\mathcal{H})}$. Since T has the property (β) from Corollary 3.4, the operator $T-z$ has closed range on $H(U_0,\mathcal{H})$. Thus $h \in (T-z)H(U_0,\mathcal{H})$, i.e., $\lambda_0 \in \rho_T(h)$. ∎

COROLLARY 3.8. *Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. With the notation of the proof of Theorem* 3.2, $\sigma(T) = \sigma(\widetilde{M})$.

Proof. Since $\sigma_T(h) = \sigma_{\widetilde{M}}(Vh)$ for all $h \in \mathcal{H}$ by Theorem 3.7, $\sigma_T(h) \subset \sigma(\widetilde{M})$ for all $h \in \mathcal{H}$. Hence $\bigcup\{\sigma_T(h) : h \in \mathcal{H}\} \subset \sigma(\widetilde{M})$. Since T has the single valued extension property by Corollary 3.4, it follows that $\sigma(T) = \bigcup\{\sigma_T(h) : h \in \mathcal{H}\} \subset \sigma(\widetilde{M})$.

Conversely, note that if $U \subset \mathbb{C}$ is any open disk containing $\sigma(T)$ and M is multiplication by z on $W^{12}(U, \mathcal{H})$, then $\sigma(\widetilde{M}) \subset \sigma(M) \subset \overline{U}$. From this property, if $\lambda \in \rho(T)$, then we can choose an open disk D so that $\widetilde{M} - \lambda$ is invertible. Since this algebraic property is independent of the choice of D, we get $\sigma(\widetilde{M}) \subset \sigma(T)$. ∎

Recall that a closed subspace of $\mathcal{H}$ is said to be *hyperinvariant* for T if it is invariant under every operator in the commutant $\{T\}'$ of T. An operator $T \in \mathcal{L}(\mathcal{H})$ is *decomposable* provided that, for each open cover $\{U, V\}$ of $\mathbb{C}$, there exist closed T-invariant subspaces Y, Z of $\mathcal{H}$ such that $\mathcal{H} = Y + Z$, $\sigma(T|_Y) \subset U$, and $\sigma(T|_Z) \subset V$. Here, $T|_Y$ denotes the restriction of T to Y.

THEOREM 3.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator and let $T \neq zI$ for all $z \in \mathbb{C}$. If S is a decomposable quasiaffine transform of T or $\lim_{n\to\infty} \|T^n h\|^{1/n} < \|T\|$ for some nonzero $h \in \mathcal{H}$, then T has a nontrivial hyperinvariant subspace.*

Proof. If S is a decomposable quasiaffine transform of T, then there exists a quasiaffinity X such that $XS = TX$ where S is decomposable. If T has no nontrivial hyperinvariant subspace, we may assume that $\sigma_{\mathrm{p}}(T) = \emptyset$ and $\mathcal{H}_T(F) = \{0\}$ for each closed set F proper in $\sigma(T)$ by Lemma 3.6.1 of [19]. Let $\{U, V\}$ be an open cover of $\mathbb{C}$ with $\sigma(T) \setminus \overline{U} \neq \emptyset$ and $\sigma(T) \setminus \overline{V} \neq \emptyset$. If $x \in \mathcal{H}_S(\overline{U})$, then $\sigma_S(x) \subset \overline{U}$. So there exists an analytic $\mathcal{H}$-valued function f defined on $\mathbb{C} \setminus \overline{U}$ such that $(S - z)f(z) \equiv x$ for all $z \in \mathbb{C} \setminus \overline{U}$. Hence $(T - z)Xf(z) = X(S - z)f(z) = Xx$ for all $z \in \mathbb{C} \setminus \overline{U}$. Thus $\mathbb{C} \setminus \overline{U} \subset \rho_T(Xx)$, which implies that $Xx \in \mathcal{H}_T(\overline{U})$, i.e., $X\mathcal{H}_S(\overline{U}) \subset \mathcal{H}_T(\overline{U})$. Similarly, $X\mathcal{H}_S(\overline{V}) \subset \mathcal{H}_T(\overline{V})$. Then since S is decomposable,

$$X\mathcal{H} = X\mathcal{H}_S(\overline{U}) + X\mathcal{H}_S(\overline{V}) \subseteq \mathcal{H}_T(\overline{U}) + \mathcal{H}_T(\overline{V}) = \{0\}.$$

But this is a contradiction. So T has a nontrivial hyperinvariant subspace.

Now suppose that $\lim_{n\to\infty} \|T^n h\|^{1/n} < \|T\|$ for some nonzero $h \in \mathcal{H}$. Since T is a class A operator,

$$\|Tx\|^2 = \langle |T|^2 x, x\rangle \le \langle |T^2| x, x\rangle \le \| \, |T^2| x\| \, \|x\| \le \|T^2 x\| \, \|x\|$$

for every $x \in \mathcal{H}$. This implies that

$$\|T^n x\|^2 = \|TT^{n-1}x\|^2 \le \|T^2 T^{n-1} x\| \, \|T^{n-1}x\| = \|T^{n+1}x\| \, \|T^{n-1}x\|$$

for every positive integer n and every $x \in \mathcal{H}$. Hence, Proposition 4.6 and a remark in [6] imply that T has a nontrivial hyperinvariant subspace. ∎

The following proposition provides the concrete structure of a compact class A operator.

PROPOSITION 3.10. *Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. If T is compact, then $T = B \oplus C \oplus (-C)$ where B and C are normal.*

Proof. If $T \in \mathcal{L}(\mathcal{H})$ is a class A operator, then T^2 is w-hyponormal from [13]. Since T^2 is compact, it is normal by [3]. Hence T is a square root of a normal operator, and so by [26] we get the following form:

$$T = B \oplus \begin{pmatrix} C & D \\ 0 & -C \end{pmatrix}$$

where B and C are normal and D is a positive one-to-one operator commuting with C. Since T is also normal by [14], D must be 0, completing the proof. ∎

If $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, then $\{T^n x\}_{n=0}^{\infty}$ is called the *orbit* of x under T, and is denoted by $\mathcal{O}(x,T)$. If $\mathcal{O}(x,T)$ is dense in $\mathcal{H}$, then x is called a *hypercyclic vector* for T. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hypertransitive* if every nonzero vector in $\mathcal{H}$ is hypercyclic for T. Denote the set of all nonhypertransitive operators in $\mathcal{L}(\mathcal{H})$ by (NHT). The *hypertransitive operator problem* is the question whether (NHT) $= \mathcal{L}(\mathcal{H})$. The following theorem shows that every class A operator belongs to (NHT).

THEOREM 3.11. *If $T \in \mathcal{L}(\mathcal{H})$ is a class A operator, then it is nonhypertransitive.*

Proof. If T is not a quasiaffinity, then $\sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{p}}(T^*) \neq \emptyset$. Hence T has a nontrivial invariant subspace, and so $T \in$ (NHT). On the other hand, suppose that T is a quasiaffinity. Then so is T^2. Since T^2 is w-hyponormal from [13], $\widehat{T^2}^{(2)}$ is hyponormal. Set $S = \widehat{T^2}$. Since $\widehat{S} = \widehat{T^2}^{(2)}$ is not hypercyclic from [17], there exists a nonzero vector $x \in \mathcal{H}$ such that $\mathcal{O}(x, \widehat{S})$ is not dense in $\mathcal{H}$. Let $S = U|S|$ be the polar decomposition of S. Since $U|S|^{1/2}\widehat{S} = SU|S|^{1/2}$,

$$S(U|S|^{1/2}\mathcal{O}(x,\widehat{S})) = U|S|^{1/2}(\widehat{S}\mathcal{O}(x,\widehat{S})) \subseteq U|S|^{1/2}\mathcal{O}(x,\widehat{S}).$$

Since T^2 is a quasiaffinity, so is S. Hence $|S|$ is a quasiaffinity and U is unitary. Therefore, $U|S|^{1/2}\mathcal{O}(x,\widehat{S})$ is not dense in $\mathcal{H}$. So $S \in$ (NHT). By the same argument as above, we can show that $T^2 \in$ (NHT). By [4] or [16], $T \in$ (NHT). ∎

COROLLARY 3.12. *If $T \in \mathcal{L}(\mathcal{H})$ is an invertible class A operator, then T and T^{-1} have a common nontrivial invariant closed set.*

Proof. This follows from the proof of Theorem 3.11 and [17]. ∎

The following theorem, based on the method of [10], gives a necessary and sufficient condition for hypercyclicity of the adjoint of a class A operator.

THEOREM 3.13. *If $T \in \mathcal{L}(\mathcal{H})$ belongs to class A, then T^* is hypercyclic if and only if $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for all nonzero $x \in \mathcal{H}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.*

Proof. Suppose that T^* is hypercyclic. Then by Proposition 2.3 of [10], it is enough to show that $\sigma(T)$ meets both $\mathbb{D}$ and $\mathbb{C}\setminus\overline{\mathbb{D}}$. Let $S = T|_{\mathcal{M}}$ for some closed T-invariant subspace $\mathcal{M}$ and let x be a hypercyclic vector for T^*. Since $(S^*)^n Px = P(T^*)^n x$ for each nonnegative integer n where P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, $\overline{\{(S^*)^n(Px)\}_{n=0}^{\infty}} = P(\overline{\{(T^*)^n x\}_{n=0}^{\infty}}) = P(\mathcal{H}) = \mathcal{M}$, i.e., Px is hypercyclic for S^*. Since S belongs to class A and S^* is hypercyclic, $r(S) = \|S\| = \|S^*\| > 1$ as mentioned in [23]. Hence, we have $\sigma(T)\cap(\mathbb{C}\setminus\overline{\mathbb{D}}) \neq \emptyset$. On the other hand, in order to prove $\sigma(S)\cap\mathbb{D} \neq \emptyset$, assume that $\sigma(S) \subset \mathbb{C}\setminus\mathbb{D}$. Since S^{-1} is a class A operator by [11] and $\sigma(S^{-1}) \subset \overline{\mathbb{D}}$, it follows that $\|S^{-1}\| = r(S^{-1}) \leq 1$. Since S^* is hypercyclic and invertible, $(S^*)^{-1}$ is hypercyclic by [23], and so $\|S^{-1}\| = \|(S^*)^{-1}\| > 1$ by [23], which is a contradiction. Therefore, $\sigma(S)\cap\mathbb{D} \neq \emptyset$.

Conversely, suppose that $\sigma_T(x)\cap\mathbb{D} \neq \emptyset$ and $\sigma_T(x)\cap(\mathbb{C}\setminus\overline{\mathbb{D}}) \neq \emptyset$ for all nonzero $x \in \mathcal{H}$. Then we get $\mathcal{H}_T(\mathbb{C}\setminus\mathbb{D}) = (0)$ and $\mathcal{H}_T(\overline{\mathbb{D}}) = (0)$. Since T has the property (β) by Corollary 3.4, T^* has the property (δ). Thus, by Proposition 2.5.14 in [20], we infer that both $\mathcal{H}_{T^*}(\mathbb{D})$ and $\mathcal{H}_{T^*}(\mathbb{C}\setminus\overline{\mathbb{D}})$ are dense in $\mathcal{H}$. By using Theorem 3.2 in [10], T^* is hypercyclic. ∎

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