On class A operators

by

SUNGEUN JUNG, EUNGIL KO and MEE-JUNG LEE (Seoul)

Abstract. We show that every class A operator has a scalar extension. In particular, such operators with rich spectra have nontrivial invariant subspaces. Also we give some spectral properties of the scalar extension of a class A operator. Finally, we show that every class A operator is nonhypertransitive.

1. Introduction. Let \( H \) be a complex separable Hilbert space and let \( \mathcal{L}(H) \) denote the algebra of all bounded linear operators on \( H \). If \( T \in \mathcal{L}(H) \), we write \( \sigma(T) \), \( \sigma_{\text{ap}}(T) \), and \( \sigma_e(T) \) for the spectrum, the approximate point spectrum, and the essential spectrum, respectively, and write \( r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} \) for the spectral radius of \( T \). An operator \( T \in \mathcal{L}(H) \) is said to be \( p \)-hyponormal if \( (TT^*)^p \leq (T^*T)^p \), where \( 0 < p < \infty \). In particular, \( 1 \)-hyponormal operators and \( \frac{1}{2} \)-hyponormal operators are called hyponormal operators and semi-hyponormal operators, respectively.

An arbitrary operator \( T \in \mathcal{L}(H) \) has a unique polar decomposition \( T = U|T| \), where \( |T| = (T^*T)^{1/2} \) and \( U \) is a partial isometry satisfying \( \ker U = \ker |T| = \ker T \) and \( \ker U^* = \ker T^* \). Associated with \( T \) is the operator \( |T|^{1/2}U|T|^{1/2} \) called the Aluthge transform of \( T \), and denoted throughout this paper by \( \widehat{T} \). For every \( T \in \mathcal{L}(H) \), the sequence \( \{\widehat{T}^{(n)}\} \) of Aluthge iterates of \( T \) is defined by \( \widehat{T}^{(0)} = T \) and \( \widehat{T}^{(n+1)} = \widehat{T}^{(n)} \) for every positive integer \( n \) (see [2], [15], and [16]).

An operator \( T \in \mathcal{L}(H) \) is said to be \( w \)-hyponormal if \( |\widehat{T}| \geq |T| \geq |\widehat{T}^*| \) (see [3]), and paranormal if \( \|Tx\|^2 \leq \|T^2x\|\|x\| \) for all \( x \in H \). We say that \( T \in \mathcal{L}(H) \) is normaloid if \( \|T\| = r(T) \). It is well-known that every \( p \)-hyponormal operator is \( w \)-hyponormal and that every \( w \)-hyponormal operator is normaloid. Furuta–Ito–Yamazaki ([12]) introduced the following interesting class of Hilbert space operators.

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Definition 1.1. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to class $A$ if it satisfies the condition $|T^2| \geq |T|^2$.

It is known from [12] that
\[
\{\text{hyponormal operators}\} \subset \{p\text{-hyponormal operators}\} \quad (0 < p \leq 1) \\
\subset \{\text{class A operators}\} \\
\subset \{\text{paranormal operators}\} \\
\subset \{\text{normaloid operators}\}.
\]

There is a vast literature concerning class $A$ operators ([11]–[14], [27], [28], etc.). By a simple computation one can show that a weighted shift belongs to class $A$ if and only if it is hyponormal. In [11], T. Furuta gives several examples of class $A$ operators, including the following.

Example 1.2 ([11]). Let
\[
A = \begin{pmatrix} 1 & 7 \\ 7 & 5 \end{pmatrix}^2 \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^2
\]
be operators on $\mathbb{R}^2$, and let $\mathcal{H}_n = \mathbb{R}^2$ for all positive integers $n$. Consider the operator $T_{A,B}$ on $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ defined by
\[
T_{A,B} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & B & 0 & 0 & 0 & \cdots \\ \vdots & 0 & B & \hat{0} & 0 & \cdots \\ \vdots & 0 & 0 & B & 0 & \cdots \\ \vdots & 0 & 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]
where the hat indicates the position of the $(0,0)$ element in the matrix. Then $T_{A,B}$ is a class $A$ operator, but is not $p$-hyponormal for any $p$.

An operator $S \in \mathcal{L}(\mathcal{H})$ is called scalar of order $m$ if it possesses a spectral distribution of order $m$, i.e. a continuous unital morphism of topological algebras
\[
u : C_0^m(\mathbb{C}) \to \mathcal{L}(\mathcal{H})
\]
such that $\nu(z) = S$, where as usual $z$ stands for the identity function on $C_0^m$, the complex-valued continuously differentiable functions of order $m$, $0 \leq m \leq \infty$. An operator is said to be subscalar of order $m$ if it is similar to the restriction of a scalar operator of order $m$ to an invariant subspace.

In 1984, M. Putinar [24] showed that every hyponormal operator has a scalar extension. In 1987, his theorem was used to show that hyponormal operators with thick spectra have nontrivial invariant subspaces, a result due to S. Brown (see [7]). In this paper we generalize those theorems to the context of class $A$ operators. In fact, we show that every class $A$ operator is
We denote \( \sigma_z \) in a neighborhood of \( f \) analytic function \( z \) valued extension property at operator is nonhypertransitive. Let \( L(NHT) = \) hypertransitive operator problem we consider the spectral properties of the scalar extension of a class A operator. Finally, has nonempty interior has a nontrivial invariant subspace. Also we give some subscalar of order 12. In particular, every class A operator whose spectrum on a Hilbert space with the property (SVEP) if it has the single-valued extension property at every \( z \) operator \( T \) (see [1]).

2. Preliminaries. An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to have the single-valued extension property at \( z_0 \) if for every neighborhood \( D \) of \( z_0 \) and any analytic function \( f : D \to \mathcal{H} \) with \( (T - z)f(z) \equiv 0 \), we have \( f(z) \equiv 0 \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to have the single-valued extension property (or SVEP) if it has the single-valued extension property at every \( z \) in \( \mathbb{C} \). For an operator \( T \in \mathcal{L}(\mathcal{H}) \) with SVEP and for \( x \in \mathcal{H} \) we can consider the set \( \rho_T(x) \) of elements \( z_0 \) in \( \mathbb{C} \) such that there exists an analytic function \( f(z) \) defined in a neighborhood of \( z_0 \), with values in \( \mathcal{H} \), which satisfies \( (T - z)f(z) \equiv x \). We denote \( \sigma_T(x) = \mathbb{C} \setminus \rho_T(x) \) and \( \mathcal{H}_T(F) = \{ x \in \mathcal{H} : \sigma_T(x) \subset F \} \), where \( F \) is a subset of \( \mathbb{C} \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to have Dunford’s property (\( \mathcal{C} \)) if \( \mathcal{H}_T(F) \) is closed for each closed subset \( F \) of \( \mathbb{C} \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to have the property (\( \beta \)) if for every open subset \( G \) of \( \mathbb{C} \) and every sequence \( f_n : G \to \mathcal{H} \) of \( \mathcal{H} \)-valued analytic functions such that \( (T - z)f_n(z) \) converges uniformly to 0 in norm on compact subsets of \( G \), \( f_n(z) \) converges uniformly to 0 in norm on compact subsets of \( G \). It is well-known that

\[
\text{Property (\( \beta \)) } \Rightarrow \text{ Dunford’s property (\( \mathcal{C} \)) } \Rightarrow \text{ SVEP.}
\]

An operator \( T \in \mathcal{L}(\mathcal{H}) \) with SVEP is said to have the decomposition property (\( \delta \)) (or simply the property (\( \delta \))) if \( \mathcal{H} = \mathcal{H}_T(U) + \mathcal{H}_T(V) \) for every open cover \( \{U, V\} \) of \( \mathbb{C} \). It is well-known that the adjoint of a bounded linear operator on a Hilbert space with the property (\( \beta \)) has the property (\( \delta \)) (see [II]).

Let \( z \) be the coordinate in \( \mathbb{C} \), and let \( d\mu(z) \), or simply \( d\mu \), denote the planar Lebesgue measure. Let \( U \) be a bounded open subset of \( \mathbb{C} \). We shall denote by \( L^2(U, \mathcal{H}) \) the Hilbert space of measurable functions \( f : U \to \mathcal{H} \) such that

\[
\|f\|_{2,U} = \left( \int_U \|f(z)\|^2 \, d\mu(z) \right)^{1/2} < \infty.
\]

We denote the space \( L^2(U, \mathcal{H}) \cap H(U, \mathcal{H}) \) by \( A^2(U, \mathcal{H}) \), where \( H(U, \mathcal{H}) \) is the Fréchet space of analytic (holomorphic) \( \mathcal{H} \)-valued functions on \( U \). Then \( A^2(U, \mathcal{H}) \) is a closed subspace of the \( L^2(U, \mathcal{H}) \), and the orthogonal projection of \( L^2(U, \mathcal{H}) \) onto this space will be denoted by \( P \).

Now, we introduce a special Sobolev type space. Let \( U \) be a bounded open subset of \( \mathbb{C} \) and \( m \) be a fixed nonnegative integer. Then the Sobolev space \( W^m(U, \mathcal{H}) \) is the space of functions \( f \in L^2(U, \mathcal{H}) \) whose derivatives
\( \bar{\partial} f, \bar{\partial}^2 f, \ldots, \bar{\partial}^m f \) in the sense of distributions still belong to \( L^2(U, \mathcal{H}) \). Endowed with the norm
\[
\|f\|_{W^m}^2 = \sum_{i=0}^{m} \|\bar{\partial}^i f\|_{2, U}^2,
\]
\( W^m(U, \mathcal{H}) \) becomes a Hilbert space contained continuously in \( L^2(U, \mathcal{H}) \). The linear operator \( M \) of multiplication by \( z \) on \( W^m(U, \mathcal{H}) \) is continuous and it has a spectral distribution \( u \) of order \( m \) defined by the following relation: for \( \varphi \in C_0^m(\mathbb{C}) \) and \( f \in W^m(U, \mathcal{H}) \), \( u(\varphi) f = \varphi f \). Hence \( M \) is a scalar operator of order \( m \).

3. Main results.

In this section, we show that every class A operator has a scalar extension. For this, we begin with the following lemma which is the key step to prove our main theorem.

**Lemma 3.1.** Let \( T \in L(\mathcal{H}) \) be a class A operator and let \( D \) be any bounded disk containing \( \sigma(T) \). Define the map \( V : \mathcal{H} \to H(D) \) by
\[
Vh = \widetilde{1} \otimes h \ (\equiv 1 \otimes h + (T - z)W^{12}(D, \mathcal{H})),
\]
where \( H(D) = W^{12}(D, \mathcal{H})/(T - z)W^{12}(D, \mathcal{H}) \) and \( 1 \otimes h \) denotes the constant function sending \( z \in D \to h \). Then \( V \) is one-to-one and has closed range.

**Proof.** Let \( h_n \in \mathcal{H} \) and \( f_n \in W^{12}(D, \mathcal{H}) \) be sequences which satisfy
\[
\lim_{n \to \infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^{12}} = 0.
\]
Then by the definition of the norm of the Sobolev space, (3.1) implies that
\[
\lim_{n \to \infty} \| (T - z)\bar{\partial}^i f_n \|_{2, D} = 0
\]
for \( i = 1, \ldots, 12 \). From (3.2) we get
\[
\lim_{n \to \infty} \| (T^2 - z^2)\bar{\partial}^i f_n \|_{2, D} = 0
\]
for \( i = 1, \ldots, 12 \). Let \( T^2 = U_2|T^2| \) and \( \widehat{T^2} = V|\widehat{T^2}| \) be the polar decompositions of \( T^2 \) and \( \widehat{T^2} \), respectively. Since \( \widehat{T^2}|T^2|^{1/2} = |T^2|^{1/2}T^2 \) and \( \widehat{T^2}^{(2)}|\widehat{T^2}|^{1/2} = |\widehat{T^2}|^{1/2}\widehat{T^2} \), we have
\[
\lim_{n \to \infty} \| (T^2 - z^2)\bar{\partial}^i |T^2|^{1/2} f_n \|_{2, D} = 0,
\]
\[
\lim_{n \to \infty} \| (\widehat{T^2}^{(2)} - z^2)\bar{\partial}^i |\widehat{T^2}|^{1/2} |T^2|^{1/2} f_n \|_{2, D} = 0,
\]
for \( i = 1, \ldots, 12 \). Since \( T \) belongs to class A, from \( [13] \), \( T^2 \) is a \( w \)-hyponormal operator, and so \( \widehat{T^2} \) is semi-hyponormal and \( \widehat{T^2}^{(2)} \) is hyponormal by the definition of a \( w \)-hyponormal operator and \( [3] \). Hence, it follows from (3.4)
that
\[(3.5) \lim_{n \to \infty} \| (\hat{T}^2(2) - z^2)^* \bar{\partial} |T^2|^{1/2} |T^2|^{1/2} f_n \|_{2,D} = 0 \]
for \( i = 1, \ldots, 12 \). By Theorem 3.1 of [18], there exists a constant \( C_D \) such that
\[(3.6) \| (I - P) \bar{\partial} |T^2|^{1/2} |T^2|^{1/2} f_n \|_{2,D} \]
\[\leq C_D \sum_{j=2+i}^{4+i} \| (\hat{T}^2(2) - z^2)^* \bar{\partial} |T^2|^{1/2} |T^2|^{1/2} f_n \|_{2,D} \]
for \( i = 0, 1, \ldots, 8 \), where \( P \) denotes the orthogonal projection of \( L^2(D, \mathcal{H}) \) onto the Bergman space \( A^2(D, \mathcal{H}) \). From (3.5) and (3.6), we obtain
\[(3.7) \lim_{n \to \infty} \| (I - P) \bar{\partial} |T^2|^{1/2} |T^2|^{1/2} f_n \|_{2,D} = 0 \]
for \( i = 1, \ldots, 8 \). Thus, by (3.4) and (3.7),
\[(3.8) \lim_{n \to \infty} \| (\hat{T}^2(2) - z^2) P \bar{\partial} |T^2|^{1/2} |T^2|^{1/2} f_n \|_{2,D} = 0 \]
for \( i = 1, \ldots, 8 \). Since \( \hat{T}^2(2) \) is hyponormal, it has the property (\( \beta \)). Hence
\[(3.9) \lim_{n \to \infty} \| P \bar{\partial} |T^2|^{1/2} |T^2|^{1/2} f_n \|_{2,D_0} = 0 \]
for \( i = 1, \ldots, 8 \), where \( \sigma(T) \subset D_0 \subset D \). From (3.7) and (3.9), we get
\[(3.10) \lim_{n \to \infty} \| \hat{T}^2 |T^2|^{1/2} |T^2|^{1/2} \bar{\partial} f_n \|_{2,D_0} = 0 \]
for \( i = 1, \ldots, 8 \). Since \( \hat{T}^2 |T^2|^{1/2} = |T^2|^{1/2} T^2 \), from (3.3) and (3.10) we obtain
\[(3.11) \lim_{n \to \infty} \| z^4 \bar{\partial} f_n \|_{2,D_0} = 0 \]
for \( i = 1, \ldots, 8 \). By Theorem 3.1 of [18], there exists a constant \( C_{D_0} \) such that
\[(3.12) \| (I - P) f_n \|_{2,D_0} \leq C_{D_0} \sum_{i=4}^{8} \| z^4 \bar{\partial} f_n \|_{2,D_0}. \]
By (3.11) and (3.12), it follows that
\[(3.13) \lim_{n \to \infty} \| (I - P) f_n \|_{2,D_0} = 0. \]
Combining (3.13) with (3.1), we have
\[\lim_{n \to \infty} \| (T - z) P f_n + 1 \otimes h_n \|_{2,D_0} = 0. \]
Let \( \Gamma \) be a curve in \( D_0 \) surrounding \( \sigma(T) \). Then
\[\lim_{n \to \infty} \| P f_n(z) + (T - z)^{-1} (1 \otimes h_n)(z) \| = 0\]
uniformly for all \( z \in \Gamma \). Applying the Riesz–Dunford functional calculus, we obtain
\[
\lim_{n \to \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z) \, dz + h_n \right\| = 0.
\]
But by Cauchy’s theorem, \( \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z) \, dz = 0 \). Hence
\[
\lim_{n \to \infty} \| h_n \| = 0.
\]
So, \( V \) is one-to-one and has closed range.

Now we are ready to show that every class A operator has a scalar extension.

**Theorem 3.2.** Every class A operator in \( \mathcal{L}(\mathcal{H}) \) is subscalar of order 12.

**Proof.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be a class A operator, let \( D \) be an arbitrary bounded open disk in \( \mathbb{C} \) that contains \( \sigma(T) \) and consider the quotient space
\[
H(D) = W^{12}(D, \mathcal{H})/(T - z)W^{12}(D, \mathcal{H})
\]
endowed with the Hilbert space norm. The class of a vector \( f \) or an operator \( S \) on \( H(D) \) will be denoted by \( \tilde{f} \), respectively \( \tilde{S} \). Let \( M \) be multiplication by \( z \) on \( W^{12}(D, \mathcal{H}) \). As noted at the end of Section 2, \( M \) is a scalar operator of order 12 and has a spectral distribution \( u \). Since the range of \( T - z \) is invariant under \( M \), \( \tilde{M} \) is well-defined. Moreover, consider the spectral distribution \( u : C_0^{12}(\mathbb{C}) \to \mathcal{L}(W^{12}(D, \mathcal{H})) \) defined by the following relation: for \( \varphi \in C_0^{12}(\mathbb{C}) \) and \( f \in W^{12}(D, \mathcal{H}) \), \( u(\varphi)f = \varphi f \). Then the spectral distribution \( u \) of \( M \) commutes with \( T - z \), and so \( \tilde{M} \) is still a scalar operator of order 12 with \( \tilde{u} \) as a spectral distribution. Consider the operator \( V : \mathcal{H} \to H(D) \) given by \( Vh = 1 \otimes h \) and denote the range of \( V \) by \( \text{ran} V \). Since
\[
VTh = 1 \otimes Th = z \otimes h = \tilde{M}(1 \otimes h) = \tilde{M}Vh
\]
for all \( h \in \mathcal{H} \), we have \( VT = \tilde{M}V \). In particular, \( \text{ran} V \) is invariant under \( \tilde{M} \). Furthermore, it is closed by Lemma 3.1 and hence it is a closed invariant subspace of the scalar operator \( \tilde{M} \). Since \( T \) is similar to the restriction \( \tilde{M}|_{\text{ran} V} \), and \( \tilde{M} \) is a scalar operator of order 12, \( T \) is subscalar of order 12.

Theorem 3.2 has the following corollary.

**Corollary 3.3.**

(i) Every \( p \)-hyponormal or \( w \)-hyponormal operator is subscalar.

(ii) If \( T \in \mathcal{L}(\mathcal{H}) \) is a class A operator, then \( f(T) \) is subscalar for every function \( f \) analytic on a neighborhood of \( \sigma(T) \).

**Proof.** (i) Since every \( p \)-hyponormal and every \( w \)-hyponormal operator belongs to class A by Section 1, the assertion follows from Theorem 3.2.
(ii) Let $T$ be a class A operator and let $f$ be an analytic function on a neighborhood of $\sigma(T)$. With the same notations as in the proof of Theorem 3.2, we have $Vf(T) = f(\tilde{M})V$. Thus $f(T)$ is subscalar. 

Recall from [6] that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be power regular if $\lim_{n \to \infty} \|T^n h\|^{1/n}$ exists for every $h \in \mathcal{H}$.

**Corollary 3.4.**

(i) Every class A operator satisfies the property $(\beta)$, Dunford’s property (C), and the single-valued extension property.

(ii) Every class A operator is power regular.

**Proof.** (i) Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. It suffices to prove that $T$ has the property $(\beta)$. Since the property $(\beta)$ is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 3.2 to the case of a scalar operator of order 12. Since every scalar operator has the property $(\beta)$ (see [24]), $T$ has the property $(\beta)$.

(ii) Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. Since $T$ is subscalar of order 12 from Theorem 3.2, it is the restriction of a scalar operator of order 12 to one of its closed invariant subspaces. Since a scalar operator is power regular and all restrictions of power regular operators to their invariant subspaces clearly remain power regular, $T$ is power regular. 

Recall that an operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be a quasi-affine transform of $T \in \mathcal{L}(\mathcal{K})$ if there is a quasiaffinity $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $XS = TX$. Furthermore, $S$ and $T$ are quasisimilar if there are quasiaffinities $X$ and $Y$ such that $XS = TX$ and $SY = YT$.

**Corollary 3.5.** Let $C$ and $D$ in $\mathcal{L}(\mathcal{H})$ belong to class A. If $C$ and $D$ are quasisimilar, then $\sigma(C) = \sigma(D)$ and $\sigma_e(C) = \sigma_e(D)$.

**Proof.** Since $C$ and $D$ satisfy the property $(\beta)$ from Corollary 3.4, the assertion follows from [25].

Next we will give some applications of Theorem 3.2 including a partial solution of the invariant subspace problem for class A operators. Moreover, the following theorem is a generalization of S. Brown’s theorem and Berger’s theorem (see [7] and [5]).

**Theorem 3.6.** Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator.

(i) If $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then $T$ has a nontrivial invariant subspace.

(ii) There exists a positive integer $K$ such that for all positive integers $k \geq K$, $T^{2k}$ has a nontrivial invariant subspace.
Proof. (i) This follows from Theorem 3.2 and [9].
(ii) From [13], $T^2$ is a $w$-hyponormal operator. Therefore, by [5] there exists a positive integer $K$ such that for all positive integers $k \geq K$, $T^{2k}$ has a nontrivial invariant subspace. □

Next we study some spectral properties of the scalar extension of a class A operator.

**Theorem 3.7.** Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. With the notation of the proof of Theorem 3.2, $\sigma_{\tilde{M}}(Vh) = \sigma_T(h)$ for each $h \in \mathcal{H}$.

**Proof.** Let $h \in \mathcal{H}$. If $\lambda_0 \in \rho_T(h)$, then there is an $\mathcal{H}$-valued analytic function $g$ defined on a neighborhood $U$ of $\lambda_0$ such that $(T - \lambda)g(\lambda) = h$ for all $\lambda \in U$. Then

$$(\tilde{M} - \lambda)Vg(\lambda) = V(T - \lambda)g(\lambda) = Vh$$

for all $\lambda \in U$. Hence $\lambda_0 \in \rho_{\tilde{M}}(Vh)$. That is, $\rho_{\tilde{M}}(Vh) \supset \rho_T(h)$.

Conversely, suppose $\lambda_0 \in \rho_{\tilde{M}}(Vh)$. Then there exists an $H(D)$-valued analytic function $\tilde{f}$ on some neighborhood $U$ of $\lambda_0$ such that $(\tilde{M} - \lambda)\tilde{f}(\lambda) = Vh$ for all $\lambda \in U$. Let $f \in H(U, W^{12}(D, \mathcal{H}))$ be a holomorphic lifting of $\tilde{f}$ and fix $\zeta \in U$. Then $h - (\zeta - z)f(\zeta, z) \in (T - z)W^{12}(D, \mathcal{H})$. Therefore, there is a sequence $\{g_n\} \subset H(U, W^{12}(D, \mathcal{H}))$ such that

$$\lim_{n \to \infty} ||h - (\zeta - z)f(\zeta, z) - (T - z)g_n(\zeta, z)||_{W^{12}} = 0$$

with respect to $z \in U$. Then

$$\lim_{n \to \infty} ||h - (T - z)f_n||_{W^{12}} = 0$$

where $f_n(z) := g_n(z, z)$ for $z \in U$. From the proof of Lemma 3.1 (cf. (3.13)), we obtain

$$\lim_{n \to \infty} ||(I - P)f_n||_{2, U_0} = 0$$

where $U_0$ is an open neighborhood of $\lambda_0$ with $U_0 \subset U$, and so

$$\lim_{n \to \infty} ||h - (T - z)Pf_n||_{2, U_0} = 0.$$
Conversely, note that if $U \subset \mathbb{C}$ is any open disk containing $\sigma(T)$ and $M$ is multiplication by $z$ on $W^{12}(U, \mathcal{H})$, then $\sigma(\tilde{M}) \subset \sigma(M) \subset U$. From this property, if $\lambda \in \rho(T)$, then we can choose an open disk $D$ so that $\tilde{M} - \lambda$ is invertible. Since this algebraic property is independent of the choice of $D$, we get $\sigma(\tilde{M}) \subset \sigma(T)$.

Recall that a closed subspace of $\mathcal{H}$ is said to be hyperinvariant for $T$ if it is invariant under every operator in the commutant $\{T\}'$ of $T$. An operator $T \in \mathcal{L}(\mathcal{H})$ is decomposable provided that, for each open cover $\{U, V\}$ of $\mathbb{C}$, there exist closed $T$-invariant subspaces $Y, Z$ of $\mathcal{H}$ such that $\mathcal{H} = Y + Z$, $\sigma(T|_{Y}) \subset U$, and $\sigma(T|_{Z}) \subset V$. Here, $T|_{Y}$ denotes the restriction of $T$ to $Y$.

**Theorem 3.9.** Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator and let $T \neq zI$ for all $z \in \mathbb{C}$. If $S$ is a decomposable quasiaffine transform of $T$ or $\lim_{n \to \infty} \|T^{n}h\|^{1/n} < \|T\|$ for some nonzero $h \in \mathcal{H}$, then $T$ has a nontrivial hyperinvariant subspace.

**Proof.** If $S$ is a decomposable quasiaffine transform of $T$, then there exists a quasiaffinity $X$ such that $XS = TX$ where $S$ is decomposable. If $T$ has no nontrivial hyperinvariant subspace, we may assume that $\sigma_{p}(T) = \emptyset$ and $\mathcal{H}_{T}(F) = \{0\}$ for each closed set $F$ proper in $\sigma(T)$ by Lemma 3.6.1 of [19]. Let $\{U, V\}$ be an open cover of $\mathbb{C}$ with $\sigma(T) \setminus U \neq \emptyset$ and $\sigma(T) \setminus V \neq \emptyset$. If $x \in \mathcal{H}_{S}(U)$, then $\mathcal{H}_{S}(x) \subset U$. So there exists an analytic $\mathcal{H}$-valued function $f$ defined on $\mathbb{C} \setminus U$ such that $(S - z)f(z) \equiv x$ for all $z \in \mathbb{C} \setminus U$. Hence $(T - z)xf(z) = X(S - z)f(z) = Xx$ for all $z \in \mathbb{C} \setminus U$. Thus $\mathbb{C} \setminus U \subset \rho_{T}(Xx)$, which implies that $Xx \in \mathcal{H}_{T}(U)$, i.e., $X\mathcal{H}_{S}(U) \subset \mathcal{H}_{T}(U)$. Similarly, $X\mathcal{H}_{S}(V) \subset \mathcal{H}_{T}(V)$. Then since $S$ is decomposable,

$$X\mathcal{H} = X\mathcal{H}_{S}(U) + X\mathcal{H}_{S}(V) \subseteq \mathcal{H}_{T}(U) + \mathcal{H}_{T}(V) = \{0\}.$$ 

But this is a contradiction. So $T$ has a nontrivial hyperinvariant subspace.

Now suppose that $\lim_{n \to \infty} \|T^{n}h\|^{1/n} < \|T\|$ for some nonzero $h \in \mathcal{H}$. Since $T$ is a class A operator,

$$\|Tx\|^{2} = \langle \|T^{2}x, x\rangle \leq \|T^{2}\| \|x\| \|x\| \leq \|T^{2}x\| \|x\|$$

for every $x \in \mathcal{H}$. This implies that

$$\|T^{n}x\|^{2} = \|TT^{n-1}x\|^{2} \leq \|T^{2}T^{n-1}x\| \|T^{n-1}x\| = \|TT^{n+1}x\| \|T^{n-1}x\|$$

for every positive integer $n$ and every $x \in \mathcal{H}$. Hence, Proposition 4.6 and a remark in [6] imply that $T$ has a nontrivial hyperinvariant subspace.

The following proposition provides the concrete structure of a compact class A operator.

**Proposition 3.10.** Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. If $T$ is compact, then $T = B \oplus C \oplus (-C)$ where $B$ and $C$ are normal.
Proof. If $T \in \mathcal{L}(\mathcal{H})$ is a class A operator, then $T^2$ is $w$-hyponormal from [13]. Since $T^2$ is compact, it is normal by [3]. Hence $T$ is a square root of a normal operator, and so by [26] we get the following form:

$$T = B \oplus \begin{pmatrix} C & D \\ 0 & -C \end{pmatrix}$$

where $B$ and $C$ are normal and $D$ is a positive one-to-one operator commuting with $C$. Since $T$ is also normal by [14], $D$ must be 0, completing the proof.

If $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, then $\{T^nx\}_{n=0}^\infty$ is called the orbit of $x$ under $T$, and is denoted by $O(x,T)$. If $O(x,T)$ is dense in $\mathcal{H}$, then $x$ is called a hypercyclic vector for $T$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called hypertransitive if every nonzero vector in $\mathcal{H}$ is hypercyclic for $T$. Denote the set of all nonhypertransitive operators in $\mathcal{L}(\mathcal{H})$ by $(NHT)$. The hypertransitive operator problem is the question whether $(NHT) = \mathcal{L}(\mathcal{H})$. The following theorem shows that every class A operator belongs to $(NHT)$.

Theorem 3.11. If $T \in \mathcal{L}(\mathcal{H})$ is a class A operator, then it is nonhypertransitive.

Proof. If $T$ is not a quasiaffinity, then $\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset$. Hence $T$ has a nontrivial invariant subspace, and so $T \in (NHT)$. On the other hand, suppose that $T$ is a quasiaffinity. Then so is $T^2$. Since $T^2$ is $w$-hyponormal from [13], $\hat{T}^2(2)$ is hyponormal. Set $S = \hat{T}^2$. Since $\hat{S} = \hat{T}^2(2)$ is not hypercyclic from [17], there exists a nonzero vector $x \in \mathcal{H}$ such that $O(x,\hat{S})$ is not dense in $\mathcal{H}$. Let $S = U|S|$ be the polar decomposition of $S$. Since $U|S|^{1/2}\hat{S} = SU|S|^{1/2}$,

$$S(U|S|^{1/2}O(x,\hat{S})) = U|S|^{1/2}(\hat{S}O(x,\hat{S})) \subseteq U|S|^{1/2}O(x,\hat{S}).$$

Since $T^2$ is a quasiaffinity, so is $S$. Hence $|S|$ is a quasiaffinity and $U$ is unitary. Therefore, $U|S|^{1/2}O(x,\hat{S})$ is not dense in $\mathcal{H}$. So $S \in (NHT)$. By the same argument as above, we can show that $T^2 \in (NHT)$. By [4] or [16], $T \in (NHT)$.

Corollary 3.12. If $T \in \mathcal{L}(\mathcal{H})$ is an invertible class A operator, then $T$ and $T^{-1}$ have a common nontrivial invariant closed set.

Proof. This follows from the proof of Theorem 3.11 and [17].

The following theorem, based on the method of [10], gives a necessary and sufficient condition for hypercyclicity of the adjoint of a class A operator.

Theorem 3.13. If $T \in \mathcal{L}(\mathcal{H})$ belongs to class A, then $T^*$ is hypercyclic if and only if $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \mathbb{D}) \neq \emptyset$ for all nonzero $x \in \mathcal{H}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. 
Proof. Suppose that $T^*$ is hypercyclic. Then by Proposition 2.3 of [10], it is enough to show that $\sigma(T)$ meets both $\mathbb{D}$ and $\mathbb{C} \setminus \mathbb{D}$. Let $S = T|_M$ for some closed $T$-invariant subspace $M$ and let $x$ be a hypercyclic vector for $T^*$. Since $(S^*)^n Px = P(T^*)^n x$ for each nonnegative integer $n$ where $P$ is the orthogonal projection of $\mathcal{H}$ onto $M$, $\{(S^*)^n(Px)\}^\infty_{n=0} = P(\{(T^*)^n x\}^\infty_{n=0}) = P(\mathcal{H}) = M$, i.e., $Px$ is hypercyclic for $S^*$. Since $S$ belongs to class A and $S^*$ is hypercyclic, $r(S) = \|S\| = \|S^*\| > 1$ as mentioned in [23]. Hence, we have $\sigma(T) \cap (\mathbb{C} \setminus \mathbb{D}) \neq \emptyset$. On the other hand, in order to prove $\sigma(S) \cap \mathbb{D} \neq \emptyset$, assume that $\sigma(S) \subset \mathbb{C} \setminus \mathbb{D}$. Since $S^{-1}$ is a class A operator by [11] and $\sigma(S^{-1}) \subset \mathbb{D}$, it follows that $\|S^{-1}\| = r(S^{-1}) \leq 1$. Since $S^*$ is hypercyclic and invertible, $(S^*)^{-1}$ is hypercyclic by [23], and so $\|S^{-1}\| = \|(S^*)^{-1}\| > 1$ by [23], which is a contradiction. Therefore, $\sigma(S) \cap \mathbb{D} \neq \emptyset$.

Conversely, suppose that $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \mathbb{D}) \neq \emptyset$ for all nonzero $x \in \mathcal{H}$. Then we get $\mathcal{H}_T(\mathbb{C} \setminus \mathbb{D}) = (0)$ and $\mathcal{H}_T(\mathbb{D}) = (0)$. Since $T$ has the property $(\beta)$ by Corollary 3.4, $T^*$ has the property $(\delta)$. Thus, by Proposition 2.5.14 in [20], we infer that both $\mathcal{H}_{T^*}(\mathbb{D})$ and $\mathcal{H}_{T^*}(\mathbb{C} \setminus \mathbb{D})$ are dense in $\mathcal{H}$. By using Theorem 3.2 in [10], $T^*$ is hypercyclic. 

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Sungeun Jung, Eungil Ko, Mee-Jung Lee
Department of Mathematics
Ewha Women’s University
120-750 Seoul, Korea
E-mail: ssung105@ewhain.net
eiko@ewha.ac.kr
meejung@ewhain.net

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