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Boundary value problems for linear operators in ordered Banach spaces

by

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Abstract. We study boundary value problems of the type $Ax = r$, $\varphi(x) = \varphi(b)$ $(\varphi \in M \subseteq E^*)$ in ordered Banach spaces.

1. Introduction. Let E be a real ordered Banach space, $A : E \to E$ a continuous linear operator and M a subset of E^*, the topological dual of E. We study Dirichlet type boundary value problems of the form

$$\begin{cases} Ax = r, \\ \varphi(x) = \varphi(b) \quad (\varphi \in M) \end{cases}$$

and we prove that under suitable assumptions on A there exists a natural choice of M such that this problem is uniquely solvable, and such that the solution x depends monotonically on r and b. Problems of this type emerge for example in discretization of linear elliptic boundary value problems [4, Chapter 4]. In this case the underlying space E is finite-dimensional and is ordered coordinatewise, in general.

To illustrate our results we consider the following example. Let $E = C([0,1],\mathbb{R})$ be endowed with the pointwise ordering, let B_n denote the Bernstein operator

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k},$$

and let $m \in \mathbb{N}$. We will see that the problem

$$\begin{cases} (B_n - \mathrm{id})^m f = g, \\ f(0) = \alpha, \quad f(1) = \beta \end{cases}$$

is uniquely solvable for each g in the range of $B_n - \mathrm{id}$ and each $\alpha, \beta \in \mathbb{R}$, and that the solution depends monotonically on α, β and g.

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2. Main result. Let E be a real Banach space ordered by a *cone* K. A cone K is a nonempty closed convex subset of E such that $\lambda K \subseteq K$ ($\lambda \geq 0$), and $K \cap (-K) = \{0\}$. As usual $x \leq y :\Leftrightarrow y - x \in K$. For $x \leq y$ let $[x, y]$ denote the order interval of all z with $x \leq z \leq y$. We assume in the following that K is normal (that is, $0 \leq x \leq y \Rightarrow \|x\| \leq \gamma\|y\|$ for some constant $\gamma \geq 1$), and has nonempty interior $\operatorname{int} K$. Let K^* denote the *dual cone* of K, that is, the set of all $\varphi \in E^*$ with $\varphi(x) \geq 0$ ($x \geq 0$). Let $L(E)$ denote the Banach algebra of all continuous linear operators $A : E \to E$, and for $A \in L(E)$ let A^* denote its adjoint.

An operator $A \in L(E)$ is called *quasimonotone increasing* [11] if

$$x \in K,\ \varphi \in K^*,\ \varphi(x) = 0 \ \Rightarrow\ \varphi(Ax) \geq 0.$$

It is well known [8] that $A \in L(E)$ is quasimonotone increasing if and only if $\exp(tA)(K) \subseteq K$ (that is, $\exp(tA)$ is a positive operator) for each $t \geq 0$. In this case also $\exp(tA^*)(K^*) \subseteq K^*$ ($t \geq 0$).

Next, if any $p \in \operatorname{int} K$ is fixed, then we may renorm E equivalently by the Minkowski functional of the order interval $[-p, p]$. This norm $\|\cdot\|$ satisfies

$$-cp \leq x \leq cp \ \Leftrightarrow\ \|x\| \leq c.$$

Under these settings we have

$$\|\varphi\| = \varphi(p) \qquad (\varphi \in K^*),$$

and we set

$$C^* := \{\varphi \in K^* : \varphi(p) = 1\} = \{\varphi \in E^* : \varphi(p) = \|\varphi\| = 1\}.$$

Moreover, we define a continuous sublinear functional $S : E \to \mathbb{R}$ by

$$S(x) = \min\{\lambda \in \mathbb{R} : x \leq \lambda p\}.$$

Note [6] that S is increasing with respect to the order given by K, that

$$S(x) = \max\{\varphi(x) : \varphi \in C^*\},$$

and that

$$\|x\| = \max\{S(-x), S(x)\} \qquad (x \in E).$$

We denote by $N(\cdot)$, $R(\cdot)$, and $\operatorname{ext}(\cdot)$ the kernel and range of a linear operator, and the set of all extremal points of a subset of a Banach space, respectively.

The aim of this paper is to prove the following results. Let $A \in L(E)$ be quasimonotone increasing with

$$N(A) \cap \operatorname{int} K \neq \emptyset$$

and fix $p \in N(A) \cap \operatorname{int} K$. Let E be normed with respect to this p. Moreover we assume that $t \mapsto \exp(tA)$ is strongly Cesàro integrable, that is,

$$\frac{1}{t}\int_0^t \exp(\tau A)x\,d\tau \tag{1}$$

is convergent in E as $t \to \infty$ for each $x \in E$.

Lemma 1. *Under the assumptions above we have: For each $x_0 \in E$ with $Ax_0 \geq 0$ there exists a unique $y_0 \in N(A)$ such that*

$$x_0 \leq y_0, \quad \varphi(x_0) = \varphi(y_0) \quad (\varphi \in \operatorname{ext}(N(A^*) \cap C^*)), \tag{2}$$

and

$$S(x_0) = \max\{\varphi(x_0) : \varphi \in \operatorname{ext}(N(A^*) \cap C^*)\}. \tag{3}$$

Theorem 1. *Let A be as in Lemma 1. Then for each $m \in \mathbb{N}$ the Dirichlet type boundary value problem*

$$\begin{cases} A^m x = r, \\ \varphi(x) = \varphi(b) \quad (\varphi \in \operatorname{ext}(N(A^*) \cap C^*)) \end{cases} \tag{4}$$

is uniquely solvable in E for each $r \in R(A^m)$ and $b \in E$, and the solution depends increasingly on b, decreasingly on r if m is odd, and increasingly on r if m is even. If in addition $R(A^m)$ is closed, then there exists a constant c such that

$$\|x\| \leq c\|r\| + \|b\| \quad (r \in R(A^m),\ b \in E). \tag{5}$$

Remark. Part (3) of Lemma 1 and Theorem 1 for $m = 1$ can be considered in analogy to the classical maximum principle and to the solution behaviour of linear second order BVPs [10], or corresponding BVPs for difference equations [4, Section 4.4].

3. Preliminaries. We make use of the following lemmata. We assume that $p \in \operatorname{int} K$, and that E is normed by the Minkowski functional of the order interval $[-p, p]$. We denote by $m_+[x, y]$ the right hand side directional derivative [9, Lemma II.5.6]:

$$m_+[x, y] = \lim_{h \to 0+} \frac{\|x + hy\| - \|x\|}{h}.$$

Lemma 2. *Let $A \in L(E)$ be quasimonotone increasing. Then*

$$\|\exp(tA)x\| \leq \exp(tm_+[p, Ap])\|x\| \quad (x \in E,\ t \geq 0).$$

For the proof of Lemma 2 see [5].

Lemma 3. *Let $A \in L(E)$ be quasimonotone increasing with $Ap = 0$. Let $x \in E$ and $u(t) = \exp(tA)x$ $(t \geq 0)$. Then*

1. $\|u(t)\| \leq \|x\|$ $(t \geq 0)$,
2. $t \mapsto S(u(t))$ *is decreasing on* $[0, \infty)$,

3. $Ax \geq 0 \Rightarrow S(u(t)) = S(x)$ $(t \geq 0)$.

Proof. 1. This follows by Lemma 2.

2. We have $x \leq S(x)p$, thus

$$u(t) = \exp(tA)x \leq S(x)\exp(tA)p = S(x)p \quad (t \geq 0).$$

Therefore $S(u(t)) \leq S(x)$ $(t \geq 0)$. Hence, if $0 \leq t_1 \leq t_2$ we obtain

$$S(u(t_2)) = S(\exp((t_2 - t_1)A)\exp(t_1 A)x) \leq S(\exp(t_1 A)x) = S(u(t_1)).$$

3. If $Ax \geq 0$, then u is increasing on $[0, \infty)$ (since $u'(t) = \exp(tA)Ax \geq 0$ in this case). Since S is increasing on E it follows that $t \mapsto S(u(t))$ is monotone increasing and (by item 2) $t \mapsto S(u(t))$ is monotone decreasing. ∎

4. Proof of Lemma 1 and Theorem 1. For $x \in E$ we set

$$Qx := \lim_{t\to\infty} \frac{1}{t}\int_0^t \exp(\tau A)x\,d\tau \quad (x \in E).$$

Then $Q \in L(E)$, and note that

$$(Q^*\varphi)(x) = \varphi(Qx) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(\exp(\tau A)x)\,d\tau$$

for each $x \in E$ and $\varphi \in E^*$.

By means of Lemma 2 we find that $\|\exp(tA)\| = 1$ $(t \geq 0)$. Therefore

$$\begin{aligned} QAx = AQx &= \lim_{t\to\infty} \frac{1}{t}\int_0^t A\exp(\tau A)x\,d\tau \\ &= \lim_{t\to\infty} \frac{1}{t}(\exp(tA)x - x) = 0 \quad (x \in E). \end{aligned}$$

Thus $Q(E) \subseteq N(A)$, and since $Q(x) = x$ $(x \in N(A))$ we see that Q is a projection onto $N(A)$. From the definition of Q we immediately get

$$Q(K) \subseteq K, \quad \|Q\| = 1.$$

Thus

$$Q^*(K^*) \subseteq K^*, \quad \|Q^*\| = 1.$$

Moreover Q^* is a projection onto $N(A^*)$. Indeed, if $\varphi = Q^*\psi$, then

$$(A^*\varphi)(x) = (A^*Q^*\psi)(x) = \psi(QAx) = 0 \quad (x \in E),$$

thus $Q^*(E^*) \subseteq N(A^*)$, and if $\varphi \in N(A^*)$, then

$$(Q^*\varphi)(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(\exp(\tau A)x)\,d\tau = \lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(x)\,d\tau = \varphi(x) \quad (x \in E),$$

therefore $Q^*\varphi = \varphi$.

We set $u(t) = \exp(tA)x_0$ $(t \geq 0)$, $g(0) = x_0$ and

$$g(t) = \frac{1}{t}\int_0^t u(\tau)\,d\tau \qquad (t > 0).$$

Then g is continuous and increasing on $[0,\infty)$ (since u is increasing), and therefore $y_0 := Qx_0 \geq x_0$. Moreover, by Lemma 3, we know that $S(u(t)) = S(x_0)$ $(t \geq 0)$. Since S is sublinear we have

$$S(g(t)) \leq \frac{1}{t}\int_0^t S(u(\tau))\,d\tau = S(x_0) \qquad (t > 0).$$

Thus $S(y_0) \leq S(x_0)$. Since S is increasing, in addition $S(x_0) \leq S(y_0)$, and so $S(x_0) = S(y_0)$. Since

$$\varphi(y_0) = \varphi(Qx_0) = (Q^*\varphi)(x_0) = \varphi(x_0) \qquad (\varphi \in N(A^*))$$

we have $y_0 \in N(A)$ satisfying (2). Uniqueness of y_0 will follow from unique solvability of (4) (with $m = 1$, $r = 0$ and $b = x_0$).

Next, we choose $\psi_0 \in C^*$ such that $\psi_0(x_0) = S(x_0)$, and we set $\varphi_0 := Q^*\psi_0$. Then $\varphi_0 \in N(A^*) \cap K^*$, and

$$\varphi_0(p) = (Q^*\psi_0)(p) = \psi_0(Qp) = \psi_0(p) = 1,$$

thus $\varphi_0 \in N(A^*) \cap C^*$.

We set $v(t) = \exp(tA^*)\psi_0$ $(t \geq 0)$, $h(0) = \psi_0$ and

$$h(t) = \frac{1}{t}\int_0^t v(\tau)\,d\tau \qquad (t > 0).$$

Now h is continuous, and with u also $\psi_0 \circ u$ and $(h(\cdot))(x_0)$ are increasing on $[0,\infty)$. For $t > 0$ we have

$$\begin{aligned}(h(t))(x_0) &= \frac{1}{t}\int_0^t (\exp(\tau A^*)\psi_0)(x_0)\,d\tau \\ &= \frac{1}{t}\int_0^t \psi_0(u(\tau))\,d\tau \to \psi_0(Qx_0) = \varphi_0(x_0) \qquad (t \to \infty).\end{aligned}$$

Thus $\psi_0(x_0) = (h(0))(x_0) \leq \varphi_0(x_0)$. Now

$$S(x_0) = \psi_0(x_0) \leq \varphi_0(x_0) \leq S(x_0),$$

and therefore $\varphi_0(x_0) = S(x_0)$. At this point we know that

$$S(x_0) = \max\{\varphi(x_0) : \varphi \in N(A^*) \cap C^*\}.$$

Since $N(A^*) \cap C^*$ is a convex and weak-* compact subset of E^*, and since $\varphi \mapsto \varphi(x_0)$ is an affine function on $N(A^*) \cap C^*$, its maximum is attained at an extremal point (see [2, Prop. 7.9]). Thus we have (3).

To prove that (4) has at most one solution first note that $N(A^2) = N(A)$. Indeed if $y \in N(A^2)$ then

$$\|\exp(tA)y\| = \|y + tAy\| \le \|y\| \quad (t \ge 0).$$

Thus $Ay = 0$. Consequently $N(A^n) = N(A)$ $(n \in \mathbb{N})$. Now, consider a solution $x \in E$ of the homogeneous problem

$$\begin{cases} A^m x = 0, \\ \varphi(x) = 0 \quad (\varphi \in \operatorname{ext}(N(A^*) \cap C^*)). \end{cases}$$

Then $x \in N(A^m) = N(A)$, and according to (3) we have $S(x) = S(-x) = 0$. Thus $x = 0$.

To prove the existence of a solution of (4) we consider

$$B : (I - Q)(E) \to R(A^m)$$

defined by $Bx = A^m x$ $(x \in (I-Q)(E))$. Then B is bijective, and $A^m B^{-1} r = r$. Moreover

$$\varphi(Qx) = (Q^*\varphi)(x) = \varphi(x) \quad (x \in E,\ \varphi \in N(A^*)).$$

Now,

$$x = B^{-1}r + Qb$$

satisfies $A^m x = r$, and for each $\varphi \in N(A^*)$ we have

$$\varphi(x) = \varphi(B^{-1}r) + \varphi(Qb) = 0 + \varphi(b) = \varphi(b).$$

In particular x is the solution of (4), and since Q is a positive operator, we see that the solution of (4) depends increasingly on b.

Next, let $r_1, r_2 \in R(A^m)$ with $r_1 \le r_2$, and let x_1, x_2 be the solutions of

$$\begin{cases} A^m x_i = r_i, \\ \varphi(x_i) = \varphi(b) \quad (\varphi \in \operatorname{ext}(N(A^*) \cap C^*)) \end{cases}$$

for $i = 1, 2$. Then $z = x_2 - x_1$ is the solution of

$$\begin{cases} A^m z = r_2 - r_1 \ge 0, \\ \varphi(z) = 0 \quad (\varphi \in \operatorname{ext}(N(A^*) \cap C^*)) \end{cases}$$

and

$$\varphi(A^{m-1}z) = ((A^*)^{m-1}\varphi)(z) = 0 \quad (\varphi \in \operatorname{ext}(N(A^*) \cap C^*)).$$

By means of (3) we have $S(A^{m-1}z) = 0$, so $A^{m-1}z \le 0$. Thus, $-z$ satisfies

$$\begin{cases} A^{m-1}(-z) \ge 0, \\ \varphi(-z) = 0 \quad (\varphi \in \operatorname{ext}(N(A^*) \cap C^*)) \end{cases}$$

and repeating this step we obtain

$$A^{m-2}(-z) \le 0, \quad A^{m-3}z \le 0, \quad \ldots, \quad (-1)^{m-1}z \le 0,$$

which means

$$(-1)^m x_1 \le (-1)^m x_2.$$

To prove (5) we assume in addition that $R(A^m)$ is closed. Since $(I-Q)(E)$ is a closed subspace of E, and since $B : (I-Q)(E) \to R(A^m)$ is bijective, we have a continuous inverse $B^{-1} : R(A^m) \to (I-Q)(E)$ in this case. Set $c := \|B^{-1}\|$. Now the solution $x = B^{-1}r + Qb$ of (4) satisfies

$$\|x\| \le \|B^{-1}r\| + \|Qb\| \le c\|r\| + \|b\|. \ \blacksquare$$

5. Fredholm operators. In special cases the convergence of the integral (1) for each $x \in E$ is automatically true under the remaining assumptions of Lemma 1. This is the case for example if E is reflexive [3, VIII.7.3]. For general E we can even prove a bit more if A is a Fredholm operator.

An operator $A \in L(E)$ is called a *Fredholm operator* if

$$\alpha(A) := \dim N(A) < \infty, \qquad \beta(A) := \operatorname{codim} R(A) < \infty.$$

In this case $R(A)$ is closed [7, Prop. 36.3], A^n is a Fredholm operator ($n \in \mathbb{N}$) [7, Prop. 25.3], and A^* is a Fredholm operator [7, Prop. 27.3]. Moreover $\operatorname{ind}(A) := \alpha(A) - \beta(A)$ is called the *index* of A.

For $A \in L(E)$ we define

$$a(A) := \min\{n \ge 0 : N(A^n) = N(A^{n+1})\},$$
$$d(A) := \min\{n \ge 0 : R(A^n) = R(A^{n+1})\},$$

with $\min \emptyset := \infty$. Now, A is called *chain-finite* if $a(A) < \infty$ and $d(A) < \infty$. In this case $a(A) = d(A)$ [7, Prop. 38.3]. We will use the following facts from the Riesz–Schauder theory of compact operators [7, Prop. 40.1]:

If $T \in L(E)$ is compact and $A = T - \mathrm{id}$, then A and A^* are Fredholm operators with

$$\alpha(A) = \beta(A) = \alpha(A^*) = \beta(A^*),$$
$$a(A) = d(A) = a(A^*) = d(A^*) < \infty.$$

If $T \in L(E)$, and T^k is compact for some $k \in \mathbb{N}$ (we call T *power compact* in this case), $A = T - \mathrm{id}$, $A_1 = T^k - \mathrm{id}$, and

$$A_2 = \mathrm{id} + T + \dots + T^{k-1},$$

then A_1 is a Fredholm operator of index 0, $a(A_1) = d(A_1) < \infty$, and

$$A_1 = AA_2 = A_2A.$$

Therefore

$$N(A) \subseteq N(A_1), \qquad R(A_1) \subseteq R(A),$$

and so

$$\alpha(A) \le \alpha(A_1) < \infty, \qquad \beta(A) \le \beta(A_1) < \infty.$$

Thus A is a Fredholm operator, and by [7, Exerc. 2, Sect. 38] we have

$$\operatorname{ind}(A) = 0, \qquad a(A) = d(A) < \infty.$$

The following results are stated for operators of the form $A = T - \mathrm{id}$, but also hold for $A = T - \lambda\,\mathrm{id}$ with $\lambda > 0$. We assume without loss of generality that $\lambda = 1$, since division by λ does not affect the following considerations.

Theorem 2. *Let $T \in L(E)$ be power compact, let $A = T - \mathrm{id}$ be quasimonotone increasing, and let $N(A) \cap \operatorname{int} K \neq \emptyset$. Then* (1) *is convergent as $t \to \infty$ for each $x \in E$. Moreover A is a Fredholm operator of index* 0 *and*

$$a(A) = d(A) = 1.$$

By Theorems 1 and 2 we get the following result on problem (4).

Theorem 3. *Let $T \in L(E)$ be power compact, let $A = T - \mathrm{id}$ be quasimonotone increasing, let $N(A) \cap \operatorname{int} K \neq \emptyset$, and let $m \in \mathbb{N}$. Then problem* (4) *is uniquely solvable in E for each $r \in R(A)$ and $b \in E$, and the solution depends increasingly on b, decreasingly on r if m is odd, and increasingly on r if m is even, and there exists a constant c such that*

$$\|x\| \le c\|r\| + \|b\| \quad (r \in R(A),\ b \in E).$$

Proof. By Theorem 2, A is a Fredholm operator of index 0, hence $R(A^m)$ is closed. Since $d(A) = 1$ we have $R(A) = R(A^m)$. The assertion now follows from Theorem 1. ∎

Remark. Theorem 3 applies to our introductory example. There $E = C([0,1],\mathbb{R})$ is ordered by the cone

$$K = \{f \in E : f(t) \ge 0\ (t \in [0,1])\}.$$

The operator B_n is compact, and since B_n is increasing, $A = B_n - \mathrm{id}$ is quasimonotone increasing. The function $p(t) = 1$ $(t \in [0,1])$ is in $\operatorname{int} K$, and $Ap = 0$. The norm induced by p is the maximum norm. Since $B_n^k \to B_1$ pointwise on E as $k \to \infty$ (cf. [1]) we see that $N(A) = \{1, t\}$. Hence $\alpha(A^*) = \alpha(A) = 2$. Let φ_0, φ_1 denote the functionals $\varphi_0(f) = f(0)$, $\varphi_1(f) = f(1)$ $(f \in E)$. Then $\varphi_0, \varphi_1 \in N(A^*)$, and since $\alpha(A^*) = 2$ we have $N(A^*) = \operatorname{span}\{\varphi_0, \varphi_1\}$. Thus

$$N(A^*) \cap C^* = \{\mu\varphi_0 + (1-\mu)\varphi_1 : \mu \in [0,1]\},$$

and therefore

$$\operatorname{ext}(N(A^*) \cap C^*)) = \{\varphi_0, \varphi_1\}.$$

6. Proof of Theorem 2. Fix $p \in N(A) \cap \operatorname{int} K$ and again let E be normed with respect to this p. Let $x \in E$. We have

$$\exp(tA)x = \exp(-t)\Big(\sum_{j=0}^{k-1} \frac{t^j T^j x}{j!} + T^k \sum_{j=k}^{\infty} \frac{t^j T^{j-k} x}{j!}\Big).$$

Since

$$\frac{d^k}{dt^k}\bigg(\sum_{j=k}^{\infty}\frac{t^jT^{j-k}x}{j!}\bigg) = \exp(tT)$$

we have

$$h(t) := \exp(-t)\bigg(\sum_{j=k}^{\infty}\frac{t^jT^{j-k}x}{j!}\bigg) = \int_0^t\int_0^{t_1}\dots\int_0^{t_{k-1}} \exp(t_kT - t\,\mathrm{id})x\,dt_k\dots dt_2\,dt_1.$$

Since $t_kT - t\,\mathrm{id}$ is quasimonotone increasing for $t_k, t \geq 0$, and since

$$m_+[p,(t_kT - t\,\mathrm{id})p] = m_+[p,(t_k - t)p] = t_k - t$$

we deduce by Lemma 2 that

$$\|\exp(t_kT - t\,\mathrm{id})x\| \leq \exp(t_k - t)\|x\| \quad (t_k, t \geq 0).$$

Thus

$$\|h(t)\| \leq \int_0^t\int_0^{t_1}\dots\int_0^{t_{k-1}} \exp(t_k - t)\|x\|\,dt_k\dots dt_2\,dt_1,$$

and evaluation of this integral proves that h is bounded on $[0,\infty)$. Now

$$\begin{aligned}\frac{1}{t}\int_0^t &\exp(\tau A)x\,d\tau \\ &= \frac{1}{t}\int_0^t \exp(-\tau)\bigg(\sum_{j=0}^{k-1}\frac{\tau^jT^jx}{j!}\bigg)\,d\tau + T^k\bigg(\frac{1}{t}\int_0^t h(\tau)\,d\tau\bigg) \quad (t>0)\end{aligned}$$

proves that

$$\bigg\{\frac{1}{t}\int_0^t \exp(\tau A)x\,d\tau : t>0\bigg\}$$

is relatively compact, and according to [3, VIII.7.1],

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \exp(\tau A)x\,d\tau$$

exists.

Next, we have already seen that A is a Fredholm operator with

$$\mathrm{ind}(A) = 0, \qquad a(A) = d(A) < \infty,$$

since T is power compact. In the proof of Theorem 1 we have seen that $N(A^2) = N(A)$. Thus $a(A) \leq 1$, and since A is not injective, $a(A) = 1$. Thus $d(A) = 1$ too. ∎

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