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Optimality of the range for which equivalence between certain measures of smoothness holds

by

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Abstract. Recently it was proved for $1 < p < \infty$ that $\omega^m(f,t)_p$, a modulus of smoothness on the unit sphere, and $\widetilde{K}_m(f,t^m)_p$, a K-functional involving the Laplace-Beltrami operator, are equivalent. It will be shown that the range $1 < p < \infty$ is optimal; that is, the equivalence $\omega^m(f,t)_p \approx \widetilde{K}_m(f,t^r)_p$ does not hold either for $p = \infty$ or for $p = 1$.

1. Introduction and notations. The moduli of smoothness $\omega^m(f,t)_p$ (see [Di,99]) are given by

$$(1.1)\qquad \omega^m(f,t)_{L_p(S^{d-1})} = \omega^m(f,t)_p = \sup_{\rho\in O_t} \|\Delta_\rho^m f\|_{L_p(S^{d-1})}$$

where $S^{d-1} = \{\boldsymbol{x} = (x_1,\ldots,x_d) : x_1^2 + \cdots + x_d^2 = 1\}$, $O_t = \{\rho \in SO(d) : \rho\boldsymbol{x}\cdot\boldsymbol{x} \ge \cos t$ for all $\boldsymbol{x} \in S^{d-1}\}$, $SO(d)$ is the group of orthogonal matrices whose determinant equals 1, $\Delta_\rho f(\boldsymbol{x}) \equiv f(\rho\boldsymbol{x}) - f(\boldsymbol{x})$ and $\Delta_\rho^m f(\boldsymbol{x}) \equiv \Delta_\rho(\Delta_\rho^{m-1} f(\boldsymbol{x}))$.

The K-functional $\widetilde{K}_m(f,t^m)_p$ is given by

$$(1.2)\qquad \widetilde{K}_m(f,t^m)_p = \widetilde{K}_m(f,t^m)_{L_p(S^{d-1})} = \inf(\|f-g\|_{L_p(S^{d-1})} + t^m\|(-\widetilde{\Delta})^{m/2} g\|_{L_p(S^{d-1})})$$

where the infimum is taken on all g such that $(-\widetilde{\Delta})^{m/2} g \in L_p(S^{d-1})$, and $\widetilde{\Delta}$ is the Laplace–Beltrami operator given by

$$(1.3)\qquad \begin{aligned} &\widetilde{\Delta} f(\boldsymbol{x}) = \Delta F(\boldsymbol{x}), \quad \boldsymbol{x} \in S^{d-1},\\ &F(\boldsymbol{x}) = f\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}. \end{aligned}$$

We recall that

$$(1.4)\qquad H_k = \{\varphi_k : \widetilde{\Delta}\varphi_k = -k(k+d-2)\varphi_k\}, \quad k = 0,1,\ldots,$$

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$P_k f$ is the projection of f on H_k (in the $L_2(S^{d-1})$ sense) and

$$(1.5)\qquad (-\widetilde{\Delta})^\alpha f = \sum_{k=1}^{\infty} (k(k+d-2))^{\alpha/2} P_k f \quad \text{for } \alpha \neq 0,\ \alpha \in \mathbb{R}.$$

It was proved in [Da-Di-Hu] (and for even m in [Di,07]) that $\omega^m(f,t)_p \approx \widetilde{K}_m(f,t^m)_p$ for $1 < p < \infty$; that is,

$$(1.6)\qquad C^{-1}\widetilde{K}_m(f,t^m)_p \le \omega^m(f,t)_p \le C\widetilde{K}_m(f,t^m)_p, \quad 1 < p < \infty.$$

Here we show that the second inequality of (1.6) does not hold for $p = \infty$ or $p = 1$. The first inequality of (1.6) was proved for even m and $1 \le p \le \infty$ in [Da-Di-Hu, Th. 9.1] (and for even d and m and many other spaces in [Da-Di]).

The main result of this paper is summarized by the next theorem.

THEOREM 1.1. *The inequality*

$$\omega^m(f,t)_p \le C\widetilde{K}_m(f,t^m)_p$$

fails for $p = 1$ *and* $p = \infty$ *for any* $m = 1, 2, \ldots$.

This failure means that for any integer m and any constant C there exist $f \in L_1(S^{d-1})$ (for $p = 1$) and $f \in L_\infty(S^{d-1})$ (for $p = \infty$) for which the inequality is not valid in the range $0 < t \le t_0$.

2. A counterexample for L_∞. For $L_\infty(S^{d-1})$, $d \ge 3$ and $m = 2$ we use the function

$$(2.1)\qquad f(x,y,u_1,\ldots,u_{d-3},z) = \begin{cases} (x^2-y^2)\log(x^2+y^2), & x \neq 0,\ y \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

which is clearly in $L_\infty(S^{d-1})$. We recall (see [Er, Chapter XI] and [Vi, Ch. IX, p. 494]) that

$$(2.2)\qquad \begin{aligned} r^{-2}\widetilde{\Delta} f &= \Delta f - r^{-d+1}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial f}{\partial r}\right), \\ r &= (x^2+y^2+u_1^2+\cdots+u_{d-3}^2+z^2)^{1/2}, \end{aligned}$$

where Δ is the Laplacian. Straightforward calculation yields

$$\begin{aligned} \Delta f &= \frac{10(x^2-y^2)}{x^2+y^2} - \frac{4(x^4-y^4)}{(x^2+y^2)^2} \\ &= \frac{6(x^2-y^2)}{x^2+y^2} \quad \text{and} \quad |\Delta f| \le 6. \end{aligned}$$

We express f in polar coordinates given by (see [Er, Ch. XI] and [Vi, Ch. IX, p. 435])

$$\begin{aligned} z &= r\cos\theta_1, \\ u_{d-3} &= r\sin\theta_1\cos\theta_2, \\ &\vdots \\ u_1 &= r\sin\theta_1\cdots\sin\theta_{d-3}\cos\theta_{d-2}, \\ x &= r\sin\theta_1\cdots\sin\theta_{d-2}\cos\varphi, \\ y &= r\sin\theta_1\cdots\sin\theta_{d-2}\sin\varphi, \end{aligned} \tag{2.3}$$

where $0 \le \theta_i \le \pi$ for $1 \le i \le d-2$ and $0 \le \varphi \le 2\pi$. (Clearly, for $d = 3$, $u_1, \ldots, u_{d-3}$ do not exist.) Hence

$$f(r, \theta_1, \ldots, \theta_{d-2}, \varphi) = r^2\cos 2\varphi \sin^2\theta_1\cdots\sin^2\theta_{d-2}\log r^2\sin^2\theta_1\cdots\sin^2\theta_{d-2}.$$

Straightforward computation implies that (for $r = 1$)

$$\left| r^{-d+1}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial f}{\partial r}\right)\right|$$

is smaller than

$$C(1 + \sin^2\theta_1\ldots\sin^2\theta_{d-2}|\log(\sin^2\theta_1\cdots\sin^2\theta_{d-2})|),$$

which is bounded for all θ_i. The above, together with (2.2), implies that $\widetilde{\Delta} f$ is bounded on S^{d-1} (when $r = 1$) and hence

$$\widetilde{K}_2(f, t^2)_\infty \le C_1 t^2 \quad \text{for } f \text{ of (2.1)}. \tag{2.4}$$

We will now show that f given in (2.1) satisfies

$$\omega^2(f, t)_\infty \ge C_2 t^2|\log t|. \tag{2.5}$$

Choosing the point $\boldsymbol{\zeta} = (x, y, \ldots, z) = (0, \ldots, 0, -1)$ and the transformation (rotation)

$$\rho = \begin{pmatrix} \cos t & & 0\ldots0 & & \sin t \\ & 1 & & 0 & \\ 0 & & \ddots & & \\ & & & 1 & \\ -\sin t & & 0\ldots0 & & \cos t \end{pmatrix}, \tag{2.6}$$

we have

$$f(\rho\boldsymbol{\zeta}) - 2f(\boldsymbol{\zeta}) + f(\rho^{-1}\boldsymbol{\zeta}) = 2\sin^2 t\log\sin^2 t,$$

which establishes (2.5). Therefore, for $L_\infty(S^{d-1})$, $d \ge 3$ and $m = 2$ the right hand inequality of (1.6) is not valid.

To show that the right hand inequality of (1.6) fails for $m = 1$ we assume that it does not fail and hence, for $f \in C^2$ and $\rho \in O_t$,

$$|\Delta_\rho f| \le Ct\|(-\widetilde{\Delta})^{1/2} f\|_{L_\infty(S^{d-1})}.$$

Iterating the above will cause a contradiction with (2.5). We note that, for $f \in C^2(S^{d-1})$,

$$\widetilde{K}_1(f,t)_\infty \le Ct\|(-\widetilde{\Delta})^{1/2} f\|_\infty \quad \text{and} \quad \widetilde{K}_2(f,t^2)_\infty \le Ct^2\|(-\widetilde{\Delta}) f\|_\infty.$$

To our knowledge the case $m = 2$ does not imply the failure of the right hand inequality of (1.6) for all m. For even m, we set $m = 2\ell$ and use the function

$$f_{2\ell}(x,y,u_1,\ldots,u_{d-3},z) = \begin{cases} P_\ell(x,y)\log(x^2+y^2), & x \ne 0,\ y \ne 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.7}$$

with

$$P_\ell(x,y) = \sum_{k=0}^{\ell} a_k x^{2(\ell-k)} y^{2k}, \qquad P_\ell(\cos\varphi, \sin\varphi) = \cos 2\ell\varphi, \tag{2.8}$$

where the coefficients a_k are determined by $P_\ell(\cos\phi, \sin\phi) = \cos 2\ell\phi$. In Section 4 we show that using the Taylor formula, we will obtain

$$\omega^{2\ell}(f_{2\ell},t)_\infty \ge C_{2\ell} t^{2\ell} |\log t| \quad \text{for } 0 < t < t_0, \tag{2.9}$$

and using iteration of (2.2) and some delicate computation, we will obtain

$$\widetilde{K}(f_{2\ell}, t^{2\ell})_\infty \le C^*_{2\ell} t^{2\ell}. \tag{2.10}$$

Combining the inequalities (2.9) and (2.10) implies

$$\omega^{2\ell}(f_{2\ell},t)_\infty \ge A_{2\ell} \widetilde{K}_{2\ell}(f_{2\ell}, t^{2\ell})_\infty |\log t| \quad \text{for } 0 < t < t_0. \tag{2.11}$$

For odd m we use (2.9) and (2.10) with $\ell = m$ and follow exactly the considerations for $m = 1$.

We note that for $L_\infty(\mathbb{R}^d)$ (or $L_\infty(T^d)$) one has

$$C^{-1} K_m(f,t^m)_p \le \omega^m(f,t)_p \le C K_m(f,t^m)_p, \qquad 1 < p < \infty, \tag{2.12}$$

where translations in $\mathbb{R}^d$ (not elements of $SO(d)$) are used in the definition of $\omega^m(f,t)_p$, and the Laplacian (instead of the Laplace–Beltrami operator) is used in the definition of $K_m(f,t^m)_p$. For $d \ge 2$ and $p = \infty$ the right hand inequality of (2.12) fails because of the failure of the estimate of the Riesz transform (see [St]). The example given in (2.1) or (2.7) can be modified by

$$\begin{aligned} &f^*_{2\ell}(x,y,u_1,\ldots,u_{d-3},z) \\ &\quad = f_{2\ell}(x,y,u_1,\ldots,u_{d-3},z)\psi(x^2+y^2+u_1^2+\cdots+u_{d-3}^2+z^2) \end{aligned} \tag{2.13}$$

where

$$\psi(r^2) = \begin{cases} 1, & |r^2| \le 1, \\ 0, & |r^2| \ge 2, \end{cases}$$

$\psi \in C^\infty$ and $r^2 = x^2 + y^2 + \cdots + z^2$. The function $f^*_{2\ell}$ will provide an example for the failure of (2.12) for $d \geq 2$, $p = \infty$ and $m = 2\ell$ (when $d = 2$, z is eliminated). Following previous arguments, a contradiction can establish the above contention (on the failure of (2.12)) for odd m and $p = \infty$.

3. The failure of the inequality for L_1. For $L_1(S^{d-1})$, $d \geq 3$, we prove the failure of the right hand inequality of (1.6) by contradiction. We assume $\omega^m(H,t)_1 \leq C\widetilde{K}_m(H,t^m)_1$ for all $H \in L_1(S^{d-1})$. Setting $H = (-\widetilde{\Delta})^{-m/2}g$ for $g \in L_1(S^{d-1})$ satisfying $P_0 g = 0$ (i.e. $\int_{S^{d-1}} g(x)\,dx = 0$), one has $\|\Delta^m_\rho\{(-\widetilde{\Delta})^{-m/2}g\}\|_{L_1(S^{d-1})} \leq Ct^m\|g\|_{L_1(S^{d-1})}$ for all $\rho \in O_t$ where $(-\widetilde{\Delta})^{-m/2}f$ is given by (1.5). We note that Δ_ρ is not a multiplier operator but that it still commutes with powers of $-\widetilde{\Delta}$, i.e. with $(-\widetilde{\Delta})^\alpha$ ($\alpha \in \mathbb{R}$). As established in the last section, for any $M > 0$ we have a function $f \in L_\infty(S^{d-1})$ (and in fact $f \in C^m(S^{d-1})$), $t > 0$ and $\rho \in O_t$ such that

$$\|\Delta^m_\rho f\|_{L_\infty(S^{d-1})} \geq Mt^m\|(-\widetilde{\Delta})^{m/2}f\|_{L_\infty(S^{d-1})}$$

and hence for $F = (-\widetilde{\Delta})^{m/2}f$ (for which $P_0F = 0$),

$$\|\Delta^m_\rho(-\widetilde{\Delta})^{-m/2}F\|_{L_\infty(S^{d-1})} \geq Mt^m\|F\|_{L_\infty(S^{d-1})}.$$

We may now choose $G \in L_1(S^{d-1})$ with $\|G\|_{L_1(S^{d-1})} = 1$ so that

$$\langle G, \Delta^m_\rho(-\widetilde{\Delta})^{-m/2}F\rangle \geq \|\Delta^m_\rho(-\widetilde{\Delta})^{-m/2}F\|_{L_\infty(S^{d-})} - \varepsilon$$

where $\langle\varphi,\psi\rangle = \int_{S^{d-1}} \varphi(x)\psi(x)\,dx$.

For $g = G - P_0G$ which satisfies $\|g\|_{L_1(S^{d-1})} \leq 2$ and $P_0 g = 0$ we have

$$\langle \Delta^m_{\rho^{-1}}\{(-\widetilde{\Delta})^{m/2}g\}, F\rangle \leq Ct^m\|g\|_{L_1(S^{d-1})}\|F\|_{L_\infty(S^{d-1})} \leq 2Ct^m\|F\|_{L_\infty(S^{d-1})}$$

as $\rho^{-1} \in O_t$ if $\rho \in O_t$. However,

$$\begin{aligned}\langle \Delta^m_{\rho^{-1}}\{(-\widetilde{\Delta})^{-m/2}g\}, F\rangle &= \langle g, \Delta^m_\rho\{(-\widetilde{\Delta})^{-m/2}F\}\rangle = \langle G, \Delta^m_\rho\{(-\widetilde{\Delta})^{-m/2}F\}\rangle \\ &\geq \|\Delta^m_\rho\{(-\widetilde{\Delta})^{-m/2}F\}\|_{L_\infty(S^{d-1})} - \varepsilon \\ &\geq Mt^m\|F\|_{L_\infty(S^{d-1})} - \varepsilon,\end{aligned}$$

and this causes a contradiction for $M > 3C$.

For $L_1(\mathbb{R}^d)$ or $L_1(T^d)$ ($d \geq 2$) the same argument for the corresponding failure of (2.12) follows and in fact in this case both $\Delta^m_h f$ and $(-\Delta)^{-m/2}f$ are multiplier operators which naturally commute.

4. Proof of the inequality (2.11) for $\ell \geq 2$. Using the description of $f_{2\ell}$ in polar coordinates, i.e.

$$f_{2\ell} = r^{2\ell}\cos 2\ell\varphi \sin^{2\ell}\theta_1 \cdots \sin^{2\ell}\theta_{d-1} \log r^2 \sin^2\theta_1 \cdots \sin^2\theta_{d-2},$$

we have

$$r^2 r^{-d+1}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}f_{2\ell}\right)$$
$$= 2\ell(2\ell+d-2)f_{2\ell}+[(2\ell+d-2)+2\ell]r^{2\ell}\cos 2\varphi\ell\sin^{2\ell}\theta_1\cdots\sin^{2\ell}\theta_{d-2}.$$

To compute $\widetilde{\Delta}$ we also calculate $\Delta f_{2\ell}$:

$$\Delta f_{2\ell}=\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)P_\ell(x,y)\right)\log(x^2+y^2)+2\frac{\partial}{\partial x}P_\ell(x,y)\frac{2x}{x^2+y^2}$$
$$+2\frac{\partial}{\partial y}P_\ell(x,y)\frac{2y}{x^2+y^2}+P_\ell(x,y)\frac{8}{x^2+y^2}.$$

We now observe that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)P_\ell(x,y)=0.$$

This is shown using the two-dimensional description, i.e. $x=\rho\cos\psi$, $y=\rho\sin\psi$,

$$P_\ell(x,y)=\rho^{2\ell}\cos 2\ell\psi\quad\text{and}\quad\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial\rho^2}+\rho^{-1}\frac{\partial}{\partial\rho}+\rho^2\frac{\partial^2}{\partial\psi^2},$$

which imply

$$\left(\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2\right)P_\ell(x,y)$$
$$=2\ell(2\ell-1)\rho^{2\ell-2}\cos 2\ell\psi+2\ell\rho^{2\ell-2}\cos 2\ell\psi-(2\ell)^2\rho^{2\ell-2}\cos 2\ell\psi.$$

As $x^2+y^2=r^2\sin^2\theta_1\cdots\sin^2\theta_{d-2}$ and $P_\ell(x,y)$ is a homogeneous polynomial of degree 2ℓ in x and y, we can write

$$r^2\Delta f_{2\ell}=r^{2\ell}Q_\ell(\cos\psi,\sin\varphi,\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2})$$
$$=r^{2\ell}Q_\ell^*(\cos\varphi,\sin\varphi)(\sin\theta_1\cdots\sin\theta_{d-2})^{2\ell-2}$$

where Q_ℓ^* is a polynomial in $\cos\varphi$ and $\sin\varphi$.

Therefore, $\widetilde{\Delta}^{\ell-1}r^{2\ell}Q_\ell(\cos\varphi,\sin\varphi,\sin\theta_1\cdots\sin\theta_d)$ is bounded using the description of $\widetilde{\Delta}$ in polar coordinates as given in [Er, Ch. XI] (see also [Da-Di-Hu, (2.6)] and [Vi, (6), p. 494]). Similarly, $\widetilde{\Delta}^{\ell-1}r^{2\ell}\cos 2\ell\varphi\sin^{2\ell}\theta_1\cdots\sin^{2\ell}\theta_{d-2}$ is also bounded. To examine $2\ell(2\ell+d-1)\widetilde{\Delta}^{\ell-1}f_{2\ell}$ we follow the above procedure and obtain, after $\ell-1$ iterations, a constant times $f_{2\ell}$ plus other terms which are bounded. We note that $f_{2\ell}$ is bounded (when $r=1$) and hence $\|\widetilde{\Delta}^\ell f_{2\ell}\|_{L_\infty(S^{d-1})}\le C$, which implies (2.10). We now use ρ of (2.6) and $\boldsymbol{\zeta}=(0,0,\dots,0,-1)$ and note that $\|\Delta_\rho^{2\ell}f_{2\ell}\|_{L_\infty(S^{d-1})}\ge|\Delta_\rho^{2\ell}f_{2\ell}(\rho\boldsymbol{\zeta})|$. Using $a_0=1$ (with a_j of (2.8)), which follows by setting $\varphi=0$ and then using the Taylor formula, we have

$$|\Delta_\rho^{2\ell} f_{2\ell}(\rho^{\ell+1}\boldsymbol{\zeta}) = \Big| \sum_{j=-\ell}^{\ell} (-1)^j \binom{2\ell}{\ell+j} f_{2\ell}(\rho^{j+1+\ell}\boldsymbol{\zeta}) \Big|$$

$$= \Big| \sum_{j=-\ell}^{\ell} (-1)^j \binom{2\ell}{\ell+j} (\sin^{2\ell}(j+1+\ell)t) \log \sin^2 (j+1+\ell)t \Big|$$

$$= C_1 t^{2\ell} \Big| \Big(\frac{\partial}{\partial t}\Big)^{2\ell} ((\sin^{2\ell} t) \log \sin^2 t)_{t=\eta} \Big|$$

where η is in $[t, (2\ell+1)t]$. Since $\sin t \log \sin^2 t$ is bounded, we have

$$\Big| \Big(\frac{\partial}{\partial t}\Big)^{2\ell} (\sin^{2\ell} t) \log \sin^2 t \Big|_{t=\eta} = (2\ell)! \cos^{2\ell} \eta \log \sin^2 \eta + g(\eta)$$

where $g(\eta)$ is bounded. Therefore, for small t, $g(\eta)$ is insignificant compared with $|\cos^{2\ell} \eta \log \sin^2 \eta|$. This concludes the proof of (2.9), which, together with (2.10), implies (2.11).

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