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Operators with absolute continuity properties: an application to quasinormality

by

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Abstract. An absolute continuity approach to quasinormality which relates the operator in question to the spectral measure of its modulus is developed. Algebraic characterizations of some classes of operators that emerge in this context are found. Various examples and counterexamples illustrating the concepts of the paper are constructed by using weighted shifts on directed trees. Generalizations of these results that cover the case of q-quasinormal operators are established.

1. Introduction. The notion of a quasinormal operator, i.e., an operator with commuting factors in its polar decomposition, has been introduced by Arlen Brown in [2] (the unbounded case has been taken up in [21]). Such operators form a bridge between normal and subnormal operators. Quasinormal operators have been found to be useful in many constructions of operator theory, e.g., when dealing with the question of subnormality (see [4, 9, 21] for the general case and [10] for the case of composition operators).

In the present paper we develop an absolute continuity approach to quasinormality of unbounded operators. On the way we characterize wider classes of operators that seem to be of independent interest. First we prove that a closed densely defined Hilbert space operator A is quasinormal if and only if $\langle E(\cdot)Af, Af\rangle \ll \langle E(\cdot)|A|f, |A|f\rangle$ for every vector f in the domain $\mathcal{D}(A)$ of A (the symbol $\ll$ means absolute continuity), where E is the spectral measure of the modulus $|A|$ of A (see Theorem 3.1).

One may ask whether reversing the above absolute continuity implies the quasinormality of A. In general the answer is negative (cf. Examples 8.2–8.4).

Another question is: assuming more, namely that the Radon–Nikodym derivative of $\langle E(\cdot)|A|f, |A|f\rangle$ with respect to $\langle E(\cdot)Af, Af\rangle$ is bounded by a constant c which does not depend on $f \in \mathcal{D}(A)$, is it true that A is quasi-

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normal? We shall prove that the answer is affirmative for $c \leq 1$ and negative for $c > 1$ (cf. Theorem 5.1 and Section 8). The case of $c > 1$ leads to a new class of operators, called weakly quasinormal, which are characterized by means of the strong commutant of their moduli (cf. Theorem 4.3). Let us remark that operators which satisfy the reversed absolute continuity condition can be completely characterized in the language of operator theory (cf. Theorem 4.4).

The absolute continuity approach is implemented in the context of weighted shifts on directed trees (cf. Theorem 7.2; the concept of a weighted shift on a directed tree has been developed in [5]). This enables us to illustrate the theme of this article by various examples and to show that there is no relationship between the hyponormality class and the classes of operators studied in the present paper (cf. Section 8). Finally, we will provide some generalizations of our main results which cover the case of the so-called q-quasinormal operators (cf. Section 9).

2. Notation and terminology. In what follows, $\mathbb{C}$ stands for the set of all complex numbers. We denote by $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{R}_+$ the sets of nonnegative integers, positive integers and nonnegative real numbers, respectively. The symbol $\mathfrak{B}(\mathbb{R}_+)$ stands for the σ-algebra of all Borel subsets of $\mathbb{R}_+$. Given two finite positive Borel measures μ and ν on $\mathbb{R}_+$, we write $\mu \ll \nu$ if μ is absolutely continuous with respect to ν; if this is the case, then $d\mu/d\nu$ stands for the Radon–Nikodym derivative of μ with respect to ν. We denote by χ_Y the characteristic function of a set Y. The symbol $\sqcup$ stands for disjoint union of sets.

Let A be an operator in a complex Hilbert space $\mathcal{H}$ (all operators considered in this paper are assumed to be linear). Denote by $\mathcal{D}(A)$, $\mathcal{R}(A)$, A^* and $\bar{A}$ the domain, the range, the adjoint and the closure of A (in case they exist). If A is closed and densely defined, then $|A|$ stands for the square root of the positive selfadjoint operator A^*A (for the necessary facts concerning unbounded operators we refer the reader to [1, 24]). For two operators S and T in $\mathcal{H}$, we write $S \subseteq T$ if $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $Sf = Tf$ for all $f \in \mathcal{D}(S)$. The C^*-algebra of all bounded operators A in $\mathcal{H}$ such that $\mathcal{D}(A) = \mathcal{H}$ is denoted by $\boldsymbol{B}(\mathcal{H})$. The symbol $I_{\mathcal{H}}$ stands for the identity operator on $\mathcal{H}$. We write $\mathrm{L{\scriptstyle IN}}\,\mathcal{F}$ for the linear span of a subset $\mathcal{F}$ of $\mathcal{H}$.

We now recall a description of the strong commutant of a normal operator.

THEOREM 2.1 ([1, Theorem 6.6.3]). *Let A be a normal operator in $\mathcal{H}$, i.e., A is closed densely defined and $A^*A = AA^*$. If $T \in \boldsymbol{B}(\mathcal{H})$, then $TA \subseteq AT$ if and only if T commutes with the spectral measure of A.*

A densely defined operator S in $\mathcal{H}$ is said to be *subnormal* if there exists a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Sh = Nh$ for all $h \in \mathcal{D}(S)$. A densely defined operator T in $\mathcal{H}$ is said to be *hyponormal* if $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ and $\|T^*f\| \leq \|Tf\|$ for all $f \in \mathcal{D}(T)$. It is well-known that subnormal operators are hyponormal, but not conversely. Recall that subnormal (hyponormal) operators are closable and their closures are subnormal (hyponormal). We refer the reader to [15, 7, 19] and [20, 21, 22, 23] for elements of the theory of unbounded hyponormal and subnormal operators, respectively.

3. An absolute continuity approach to quasinormality. Following [21] (see also [2] for the case of bounded operators), we say that a closed densely defined operator A in a complex Hilbert space $\mathcal{H}$ is *quasinormal* if A *commutes* with the spectral measure E of $|A|$, i.e., $E(\sigma)A \subseteq AE(\sigma)$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. The ensuing fact is well-known (use [21, Proposition 1] and Theorem 2.1).

(3.1) A closed densely defined operator A in $\mathcal{H}$ is quasinormal if and only if $U|A| \subseteq |A|U$, where $A = U|A|$ is the polar decomposition of A.

Note that quasinormal operators are hyponormal (indeed, since $A \subseteq |A|U$, we get $U^*|A| \subseteq A^*$, which implies hyponormality). In fact, quasinormal operators are always subnormal (see [21, Theorem 2] for the general case; the bounded case can be deduced from [2, Theorem 1]). The reverse implication does not hold. For more information on quasinormal operators we refer the reader to [2, 3] (bounded operators) and [21, 11] (unbounded operators).

Now we show that quasinormality can be characterized by means of absolute continuity.

THEOREM 3.1. *Let A be a closed densely defined operator in $\mathcal{H}$ and E be the spectral measure of $|A|$. Then the following three conditions are equivalent:*

(i) *A is quasinormal,*
(ii) *$\langle E(\sigma)Af, Af\rangle = \langle E(\sigma)|A|f, |A|f\rangle$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ and $f \in \mathcal{D}(A)$,*
(iii) *$\langle E(\cdot)Af, Af\rangle \ll \langle E(\cdot)|A|f, |A|f\rangle$ for every $f \in \mathcal{D}(A)$.*

Proof. (i)⇒(ii). Let $A = U|A|$ be the polar decomposition of A. By (3.1) and Theorem 2.1, we have $UE(\cdot) = E(\cdot)U$. Since $P := U^*U$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(|A|)}$, we see that $P|A| = |A|$. Combining all this together, we get

$$\begin{aligned}\langle E(\sigma)Af, Af\rangle &= \langle E(\sigma)|A|f, P|A|f\rangle \\ &= \langle E(\sigma)|A|f, |A|f\rangle, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ f \in \mathcal{D}(A).\end{aligned}$$

(ii)⇒(iii). Obvious.

(iii)$\Rightarrow$(i). Fix finite systems $\sigma_1,\ldots,\sigma_n \in \mathfrak{B}(\mathbb{R}_+)$ and $f_1,\ldots,f_n \in \mathcal{D}(A)$. Then there exist finite systems $J_1,\ldots,J_n \subseteq \{1,\ldots,m\}$ and $\sigma_1',\ldots,\sigma_m' \in \mathfrak{B}(\mathbb{R}_+)$ such that $\sigma_k' \cap \sigma_l' = \emptyset$ for all $k \neq l$, and $\sigma_i = \bigcup_{j\in J_i} \sigma_j'$ for all $i \in \{1,\ldots,n\}$. Set $f_j' = \sum_{i=1}^n \chi_{J_i}(j) f_i$ for $j \in \{1,\ldots,m\}$. Then we have

$$\begin{aligned}
\text{(3.2)} \qquad \Big\|\sum_{i=1}^n E(\sigma_i)Af_i\Big\|^2 &= \Big\|\sum_{i=1}^n \sum_{j\in J_i} E(\sigma_j')Af_i\Big\|^2 \\
&= \Big\|\sum_{i=1}^n \sum_{j=1}^m \chi_{J_i}(j)E(\sigma_j')Af_i\Big\|^2 \\
&= \Big\|\sum_{j=1}^m E(\sigma_j')A\Big(\sum_{i=1}^n \chi_{J_i}(j)f_i\Big)\Big\|^2 \\
&= \sum_{j=1}^m \langle E(\sigma_j')Af_j', Af_j'\rangle.
\end{aligned}$$

Arguing as above, we get

$$\text{(3.3)} \qquad \Big\|\sum_{i=1}^n E(\sigma_i)|A|f_i\Big\|^2 = \sum_{j=1}^m \langle E(\sigma_j')|A|f_j', |A|f_j'\rangle.$$

Since $E(\sigma)|A| \subseteq |A|E(\sigma)$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, we have

$$\text{(3.4)} \qquad \mathcal{R}(|A|) = \textsc{Lin}\{E(\sigma)|A|f\colon \sigma \in \mathfrak{B}(\mathbb{R}_+),\ f \in \mathcal{D}(A)\}.$$

Combining (3.2) and (3.3) with (iii), we deduce that for all finite systems $\sigma_1,\ldots,\sigma_n \in \mathfrak{B}(\mathbb{R}_+)$ and $f_1,\ldots,f_n \in \mathcal{D}(A)$ the following implication holds:

$$\sum_{i=1}^n E(\sigma_i)|A|f_i = 0 \ \Rightarrow\ \sum_{i=1}^n E(\sigma_i)Af_i = 0.$$

This together with (3.4) implies that the map $\widetilde{T}_0\colon \mathcal{R}(|A|) \to \mathcal{H}$ given by

$$\text{(3.5)} \qquad \widetilde{T}_0\Big(\sum_{i=1}^n E(\sigma_i)|A|f_i\Big) = \sum_{i=1}^n E(\sigma_i)Af_i, \qquad \sigma_i \in \mathfrak{B}(\mathbb{R}_+),\ f_i \in \mathcal{D}(A),\ n \in \mathbb{N},$$

is well-defined and linear. Substituting $n = 1$ and $\sigma_1 = \mathbb{R}_+$ into (3.5), we see that $\widetilde{T}_0|A| = A$. This yields

$$\|\widetilde{T}_0(|A|f)\| = \|Af\| = \big\||A|f\big\|, \quad f \in \mathcal{D}(A),$$

which means that $\widetilde{T}_0$ is an isometry. Let $T_0\colon \overline{\mathcal{R}(|A|)} \to \mathcal{H}$ be a unique isometric and linear extension of $\widetilde{T}_0$. Define the operator $T \in \boldsymbol{B}(\mathcal{H})$ by $Tf = T_0Pf$ for $f \in \mathcal{H}$, where $P \in \boldsymbol{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(|A|)}$.

In view of (3.5), T is an extension of T_0 such that

$$A = T|A|, \tag{3.6}$$

$$TE(\sigma)|A| = E(\sigma)A, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{3.7}$$

Since T_0 is an isometry, we infer from the definition of T that $\mathcal{N}(T) = \mathcal{N}(|A|) = \mathcal{N}(A)$. This implies that T is a partial isometry and $A = T|A|$ is the polar decomposition of A. Since

$$TE(\sigma)(|A|f) \overset{(3.7)}{=} E(\sigma)Af \overset{(3.6)}{=} E(\sigma)T(|A|f), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ f \in \mathcal{D}(A),$$

we deduce that $TE(\sigma)|_{\overline{\mathcal{R}(|A|)}} = E(\sigma)T|_{\overline{\mathcal{R}(|A|)}}$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. As $\mathcal{N}(|A|)$ reduces E and $T|_{\mathcal{N}(|A|)} = 0$, we conclude that T commutes with the spectral measure E of $|A|$. By Theorem 2.1, we have $T|A| \subseteq |A|T$, which together with (3.1) completes the proof. ∎

4. A characterization of weak quasinormality. We say that a closed densely defined operator A in a complex Hilbert space $\mathcal{H}$ is *weakly quasinormal* if there exists $c \in \mathbb{R}_+$ such that

$$\langle E(\sigma)|A|f, |A|f\rangle \leq c\langle E(\sigma)Af, Af\rangle, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ f \in \mathcal{D}(A), \tag{4.1}$$

where E is the spectral measure of $|A|$ (or equivalently: for every $f \in \mathcal{D}(A)$, $\langle E(\cdot)|A|f, |A|f\rangle \ll \langle E(\cdot)Af, Af\rangle$ and $d\langle E(\cdot)|A|f, |A|f\rangle / d\langle E(\cdot)Af, Af\rangle \leq c$ almost everywhere with respect to $\langle E(\cdot)Af, Af\rangle$). The smallest such c will be denoted by $\mathfrak{c}_A$. It is worth mentioning that the constant $\mathfrak{c}_A$ is always greater than or equal to 1 whenever the operator A is nonzero. As proved in Theorem 5.1, a nonzero closed and densely defined operator A is quasinormal if and only if it is weakly quasinormal with $\mathfrak{c}_A = 1$.

Our goal in this section is to characterize weak quasinormality of unbounded operators. We begin with a technical lemma.

Lemma 4.1. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a contraction whose restriction to a closed linear subspace $\mathcal{K}$ of $\mathcal{H}$ is isometric. Then $T^*Tk = k$ for all $k \in \mathcal{K}$.*

Proof. Denote by $T|_{\mathcal{K}} \colon \mathcal{K} \to \mathcal{H}$ the restriction of T to $\mathcal{K}$. Since $(T|_{\mathcal{K}})^* = P_{\mathcal{K}}T^*$ and $T|_{\mathcal{K}}$ is an isometry, we have $P_{\mathcal{K}}T^*T|_{\mathcal{K}} = I_{\mathcal{K}}$. This and $\|T\| \leq 1$ yield

$$\|k\| = \|P_{\mathcal{K}}T^*Tk\| \leq \|T^*Tk\| \leq \|k\|, \quad k \in \mathcal{K},$$

which implies that $T^*Tk \in \mathcal{K}$ and thus $T^*Tk = k$ for all $k \in \mathcal{K}$. ∎

Below we show that Lemma 4.1 is not true if T is not a contraction.

Example 4.2. Let $\mathcal{K}$ be a nonzero complex Hilbert space and let $\mathcal{H} := \mathcal{K} \oplus \mathcal{K}$. Take $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \boldsymbol{B}(\mathcal{H})$ with $A, B, C, D \in \boldsymbol{B}(\mathcal{K})$. Then $T|_{\mathcal{K}\oplus\{0\}}$ is an isometry if and only if $A^*A + C^*C = I_{\mathcal{K}}$. It is also easily seen that $T^*T(\mathcal{K} \oplus \{0\}) \subseteq \mathcal{K} \oplus \{0\}$ if and only if $B^*A + D^*C = 0$. Substituting $\mathcal{K} = \mathbb{C}$

and $A = C = 1/\sqrt{2}$, and taking $B, D \in \mathbb{C}$ such that $B + D \neq 0$, we see that $T|_{\mathcal{K}\oplus\{0\}}$ is an isometry and $T^*T(\mathcal{K}\oplus\{0\}) \not\subseteq \mathcal{K}\oplus\{0\}$. Hence, by Lemma 4.1, T is not a contraction, independently of whether B and D are large or small numbers.

Now we are ready to characterize weak quasinormality. We will show in Section 8 that there exist weakly quasinormal operators A with $\mathfrak{c}_A > 1$ which are not hyponormal (and thus not quasinormal).

THEOREM 4.3. *Let A be a closed densely defined operator in $\mathcal{H}$ and let $c \in \mathbb{R}_+$. Then the following two conditions are equivalent:*

(i) *A is weakly quasinormal with $\mathfrak{c}_A \leq c$,*
(ii) *there exists $T \in \boldsymbol{B}(\mathcal{H})$ such that*

$$TA = |A|, \quad T|A| \subseteq |A|T \quad \text{and} \quad \|T\| \leq \sqrt{c}. \tag{4.2}$$

Moreover, the following assertions are valid:

(iii) *if A is weakly quasinormal and $A \neq 0$, then $\mathfrak{c}_A \geq 1$,*
(iv) *if A is weakly quasinormal, then the operator T in* (ii) *can be chosen so that $\mathcal{R}(T) = \overline{\mathcal{R}(|A|)}$ and $\|T\| = \sqrt{\mathfrak{c}_A}$,*
(v) *if $T \in \boldsymbol{B}(\mathcal{H})$ satisfies* (4.2)*, then $T|_{\overline{\mathcal{R}(A)}}\colon \overline{\mathcal{R}(A)} \to \mathcal{H}$ is an isometry, $T(\overline{\mathcal{R}(A)}) = \overline{\mathcal{R}(|A|)}$ and the partial isometry U in the polar decomposition of A takes the form $U = PT^*$, where $P \in \boldsymbol{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$.*

Proof. Let E be the spectral measure of $|A|$.

(i)⇒(ii). Without loss of generality we can assume that $c = \mathfrak{c}_A$. Arguing as in the proof of the implication (iii)⇒(i) of Theorem 3.1, we deduce that for all finite systems $\sigma_1, \ldots, \sigma_n \in \mathfrak{B}(\mathbb{R}_+)$ and $f_1, \ldots, f_n \in \mathcal{D}(A)$,

$$\Big\| \sum_{i=1}^n E(\sigma_i)|A|f_i \Big\|^2 \leq c \Big\| \sum_{i=1}^n E(\sigma_i)Af_i \Big\|^2. \tag{4.3}$$

Define the closed vector space $\mathcal{H}_0$ by

$$\mathcal{H}_0 = \overline{\mathrm{LIN}\{E(\sigma)Af\colon \sigma \in \mathfrak{B}(\mathbb{R}_+),\ f \in \mathcal{D}(A)\}}.$$

It follows from (4.3) that there exists a unique bounded linear map $T_0\colon \mathcal{H}_0 \to \mathcal{H}$ such that $\|T_0\| \leq \sqrt{c}$ and

$$T_0E(\sigma)A = E(\sigma)|A|, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{4.4}$$

Define the operator $T \in \boldsymbol{B}(\mathcal{H})$ by $Tf = T_0Qf$ for $f \in \mathcal{H}$, where $Q \in \boldsymbol{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$. Then T is an extension of T_0 such that $\|T\| \leq \sqrt{c}$. Substituting $\sigma = \mathbb{R}_+$ into (4.4), we get

$$TA = |A|. \tag{4.5}$$

Applying (4.4) twice yields

$$\begin{aligned}(4.6)\qquad TE(\sigma)(E(\tau)Af) &= TE(\sigma\cap\tau)Af = E(\sigma\cap\tau)|A|f \\ &= E(\sigma)E(\tau)|A|f = E(\sigma)T(E(\tau)Af), \quad f\in\mathcal{D}(A),\ \sigma,\tau\in\mathfrak{B}(\mathbb{R}_+).\end{aligned}$$

Hence $TE(\sigma)|_{\mathcal{H}_0} = E(\sigma)T|_{\mathcal{H}_0}$ for all $\sigma\in\mathfrak{B}(\mathbb{R}_+)$. Since $\mathcal{H}_0$ reduces the spectral measure E and $T|_{\mathcal{H}\ominus\mathcal{H}_0} = 0$, we obtain $TE(\cdot) = E(\cdot)T$. Applying Theorem 2.1, we get $T|A| \subseteq |A|T$. By (4.4), the definition of T and $E(\cdot)|A| \subseteq |A|E(\cdot)$, we have

$$(4.7)\qquad \mathcal{R}(T) = \mathcal{R}(T_0) \subseteq \overline{\mathcal{R}(|A|)}.$$

It follows from (4.5) that

$$(4.8)\qquad \|T(Af)\| = \big\||A|f\big\| = \|Af\|, \quad f\in\mathcal{D}(A).$$

Thus the operator $T|_{\overline{\mathcal{R}(A)}}\colon \overline{\mathcal{R}(A)}\to\mathcal{H}$ is an isometry. Since $TA = |A|$, we see that

$$\overline{\mathcal{R}(|A|)} = T(\overline{\mathcal{R}(A)}) \subseteq \mathcal{R}(T) \overset{(4.7)}{\subseteq} \overline{\mathcal{R}(|A|)},$$

which means that $\mathcal{R}(T) = \overline{\mathcal{R}(|A|)}$.

(ii)$\Rightarrow$(i). Let $P\in\boldsymbol{B}(\mathcal{H})$ be the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$ and let $U := PT^*$. If $h\in\mathcal{H}$, then

$$\begin{aligned} h\in\mathcal{N}(U) &\Leftrightarrow \langle PT^*h, Af\rangle = 0 \text{ for all } f\in\mathcal{D}(A)\\ &\Leftrightarrow \langle h, TAf\rangle = 0 \text{ for all } f\in\mathcal{D}(A)\\ &\overset{(4.2)}{\Leftrightarrow} \langle h, |A|f\rangle = 0 \text{ for all } f\in\mathcal{D}(A)\\ &\Leftrightarrow h\in\mathcal{H}\ominus\overline{\mathcal{R}(|A|)} = \mathcal{N}(|A|) = \mathcal{N}(A),\end{aligned}$$

which shows that $\mathcal{N}(U) = \mathcal{N}(A)$. Using the equality $TA = |A|$ and arguing as in (4.8), we see that the operator $T|_{\overline{\mathcal{R}(A)}}\colon \overline{\mathcal{R}(A)}\to\mathcal{H}$ is an isometry and $T(\overline{\mathcal{R}(A)}) = \overline{\mathcal{R}(|A|)}$. This implies that $I_{\overline{\mathcal{R}(A)}} = PT^*T|_{\overline{\mathcal{R}(A)}}$ (see the proof of Lemma 4.1). Thus

$$A = PT^*TA \overset{(4.2)}{=} PT^*|A| = U|A|.$$

This and the equalities $\mathcal{N}(U) = \mathcal{N}(A) = \mathcal{N}(|A|)$ imply that $A = U|A|$ is the polar decomposition of A, which proves (v).

It follows from (4.2) and Theorem 2.1 that

$$\begin{aligned}(4.9)\qquad \langle E(\sigma)|A|f, |A|f\rangle &= \langle E(\sigma)TAf, TAf\rangle = \|TE(\sigma)Af\|^2\\ &\le c\langle E(\sigma)Af, Af\rangle, \quad \sigma\in\mathfrak{B}(\mathbb{R}_+),\ f\in\mathcal{D}(A),\end{aligned}$$

which shows that A is weakly quasinormal and $\mathfrak{c}_A \le c$.

(iii) If $A \neq 0$ satisfies (4.1), then substituting $\sigma = \mathbb{R}_+$ into (4.1) yields

$$\big\||A|f\big\|^2 \leq c\|Af\|^2 = c\big\||A|f\big\|^2, \quad f \in \mathcal{D}(A),$$

which gives (iii).

It only remains to prove (iv). Assume that (i) holds. It follows from the proof of (i)⇒(ii) that there exists $T \in \boldsymbol{B}(\mathcal{H})$ such that (4.2) holds with $c = \mathfrak{c}_A$, i.e., $\|T\| \leq \sqrt{\mathfrak{c}_A}$, and $\mathcal{R}(T) = \overline{\mathcal{R}(|A|)}$. Now applying the reverse implication (ii)⇒(i) with $c = \|T\|^2$, we get $\mathfrak{c}_A \leq \|T\|^2$. This completes the proof. ∎

Reversing the absolute continuity in Theorem 3.1(iii) leads to a new class of operators that is essentially wider than the class of weakly quasinormal operators (cf. Section 8). The new class can be characterized as follows.

THEOREM 4.4. *Let A be a closed densely defined operator in $\mathcal{H}$ and E be the spectral measure of $|A|$. Then the following two conditions are equivalent:*

(i) $\langle E(\cdot)|A|f, |A|f\rangle \ll \langle E(\cdot)Af, Af\rangle$ *for every* $f \in \mathcal{D}(A)$,
(ii) *there exists a (unique) linear map* $T_0\colon \mathscr{H}_0 \to \mathcal{R}(|A|)$ *such that* [1] $T_0A = |A|$ *and* $T_0E(\sigma)|_{\mathscr{H}_0} = E(\sigma)T_0$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, *where*

$$\mathscr{H}_0 = \mathrm{LIN}\,\big\{E(\sigma)Af\colon \sigma \in \mathfrak{B}(\mathbb{R}_+),\ f \in \mathcal{D}(A)\big\}.$$

Moreover, the following assertion holds for any $c \in \mathbb{R}_+$:

(iii) *A is weakly quasinormal with $\mathfrak{c}_A \leq c$ if and only if T_0 is bounded and $\|T_0\| \leq \sqrt{c}$, where T_0 is as in* (ii).

Proof. (i)⇒(ii). Arguing as in the proof of the implication (iii)⇒(i) of Theorem 3.1, we show that the following implication holds for all finite systems $\sigma_1, \ldots, \sigma_n \in \mathfrak{B}(\mathbb{R}_+)$ and $f_1, \ldots, f_n \in \mathcal{D}(A)$:

$$\sum_{i=1}^n E(\sigma_i)Af_i = 0 \ \Rightarrow\ \sum_{i=1}^n E(\sigma_i)|A|f_i = 0.$$

This, combined with the fact that $E(\sigma)|A| \subseteq |A|E(\sigma)$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, implies that the map $T_0\colon \mathscr{H}_0 \to \mathcal{R}(|A|)$ given by

$$T_0\Big(\sum_{i=1}^n E(\sigma_i)Af_i\Big) = \sum_{i=1}^n E(\sigma_i)|A|f_i, \qquad \sigma_i \in \mathfrak{B}(\mathbb{R}_+),\ f_i \in \mathcal{D}(A),\ n \in \mathbb{N}, \tag{4.10}$$

is well-defined and linear. Substituting $n = 1$ and $\sigma_1 = \mathbb{R}_+$ into (4.10), we see that $T_0A = |A|$. Arguing as in (4.6) with T_0 in place of T, we verify that $T_0E(\sigma)|_{\mathscr{H}_0} = E(\sigma)T_0|_{\mathscr{H}_0}$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$.

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[1] Note that $\mathcal{R}(A) \subseteq \mathscr{H}_0$ and $E(\sigma)\mathscr{H}_0 \subseteq \mathscr{H}_0$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$.

It is clear that any linear map $T_0\colon \mathscr{H}_0 \to \mathcal{R}(|A|)$ with the properties specified by (ii) must satisfy (4.10), and as such is unique.

(ii)⇒(i). If $f \in \mathcal{D}(A)$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ are such that $\langle E(\sigma)Af, Af\rangle = 0$, then $E(\sigma)Af = 0$, and thus

$$\langle E(\sigma)|A|f, |A|f\rangle = \langle E(\sigma)T_0Af, T_0Af\rangle = \|T_0E(\sigma)Af\|^2 = 0,$$

which gives (i).

Now we justify the "moreover" part of the conclusion. If A is weakly quasinormal with $\mathfrak{c}_A \leq c$ and $T \in \boldsymbol{B}(\mathcal{H})$ is as in Theorem 4.3(ii), then clearly (use Theorem 2.1) $T|_{\mathscr{H}_0} = T_0$, which implies the boundedness of T_0 and gives $\|T_0\| \leq \sqrt{c}$. Conversely, if T_0 is bounded and $\|T_0\| \leq \sqrt{c}$, then by mimicking the argument used in (4.9) with T_0 in place of T, we see that A is weakly quasinormal with $\mathfrak{c}_A \leq c$. This completes the proof. ∎

Remark 4.5. It is worth mentioning that if A is a closed densely defined operator in $\mathcal{H}$ which is not weakly quasinormal and which satisfies the condition (i) of Theorem 4.4 (see Section 8 for constructions of such operators), then the operator T_0 appearing in (ii) of Theorem 4.4 is unbounded and it extends the isometric operator $T_0|_{\mathcal{R}(A)}$ (for the latter, consult (4.8)).

5. Quasinormality revisited. In this short section we show that quasinormality is completely characterized by the inequality (4.1) with $c = 1$. Recall that if A is a nonzero weakly quasinormal operator, then $\mathfrak{c}_A \geq 1$ (see Theorem 4.3(iii)).

Theorem 5.1. *Let A be a nonzero closed densely defined operator in $\mathcal{H}$. Then the following two conditions are equivalent:*

(i) *A is quasinormal,*
(ii) *A is weakly quasinormal with $\mathfrak{c}_A = 1$.*

Proof. (i)⇒(ii). Apply Theorem 3.1.

(ii)⇒(i). By Theorem 4.3 there exists an operator $T \in \boldsymbol{B}(\mathcal{H})$ that satisfies (4.2) with $c = 1$. Then, by the assertion (v) of Theorem 4.3, the operator $T|_{\overline{\mathcal{R}(A)}}\colon \overline{\mathcal{R}(A)} \to \mathcal{H}$ is an isometry. Since $\|T\| \leq 1$, we infer from Lemma 4.1 that $I_{\overline{\mathcal{R}(A)}} = T^*T|_{\overline{\mathcal{R}(A)}}$. This and the equality $TA = |A|$ yield

$$A = T^*TA = T^*|A|. \tag{5.1}$$

By (4.2) and Theorem 2.1, the operator T^* commutes with the spectral measure E of $|A|$. Hence, the fact that $E(\cdot)|A| \subseteq |A|E(\cdot)$ yields

$$E(\sigma)A \overset{(5.1)}{=} E(\sigma)T^*|A| = T^*E(\sigma)|A| \subseteq T^*|A|E(\sigma) \overset{(5.1)}{=} AE(\sigma)$$

for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$, which means that A is quasinormal. ∎

6. Boundedness of weakly quasinormal operators. Taking a quick look at Corollary 7.4 would suggest that if A is a quasinormal operator, then $\mathcal{R}(A) \subseteq \mathcal{D}(|A|^\alpha)$ for every positive real α. However, as shown below, this is not necessarily the case. In fact, if A is an unbounded quasinormal operator, then $\mathcal{R}(A) \nsubseteq \mathcal{D}(|A|^\alpha)$ for every positive real α. The particular case of $\alpha \geq 1$ can be deduced from [18, Lemma A.1] and [12, Theorem 3.3] (because $\mathcal{D}(|A|^\alpha) \subseteq \mathcal{D}(|A|) = \mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ for $\alpha \geq 1$). It is worth pointing out that there are unbounded closed densely defined Hilbert space operators A such that $\mathcal{R}(A) \subseteq \mathcal{D}(|A|)$ (cf. [12]).

Proposition 6.1. *If A is a weakly quasinormal operator in $\mathcal{H}$ such that $\mathcal{R}(A) \subseteq \mathcal{D}(|A|^\alpha)$ for some positive real α, then $A \in \boldsymbol{B}(\mathcal{H})$.*

Proof. First we show that

$$\mathcal{D}(|A|) = \mathcal{D}(|A|^{1+\alpha}). \tag{6.1}$$

The inclusion "$\supseteq$" is always true (cf. [18, Lemma A.1]). To prove the reverse inclusion, take $f \in \mathcal{D}(|A|)$. Denote by E the spectral measure of $|A|$. Since A is weakly quasinormal, we get

$$\int_\sigma x^2 \langle E(dx)f, f\rangle = \langle E(\sigma)|A|f, |A|f\rangle \leq \mathfrak{c}_A \langle E(\sigma)Af, Af\rangle, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

This and the assumption $\mathcal{R}(A) \subseteq \mathcal{D}(|A|^\alpha)$ imply that

$$\begin{aligned}\int_0^\infty x^{2(1+\alpha)} \langle E(dx)f, f\rangle &= \int_0^\infty x^{2\alpha} \langle E(dx)|A|f, |A|f\rangle \\ &\leq \mathfrak{c}_A \int_0^\infty x^{2\alpha} \langle E(dx)Af, Af\rangle = \mathfrak{c}_A \big\||A|^\alpha Af\big\|^2 < \infty.\end{aligned}$$

Hence $f \in \mathcal{D}(|A|^{1+\alpha})$, which completes the proof of (6.1). Now, by applying [18, Lemma A.1] to (6.1), we conclude that $|A| \in \boldsymbol{B}(\mathcal{H})$. This in turn implies that $A \in \boldsymbol{B}(\mathcal{H})$, which completes the proof. ∎

7. Weakly quasinormal weighted shifts on directed trees. The basic facts on directed trees and weighted shifts on directed trees can be found in [5]. We refer the reader to [6] for recent applications of this idea to general operator theory.

Let $\mathscr{T} = (V, E)$ be a directed tree (V and E stand for the sets of vertices and edges of $\mathscr{T}$, respectively). Set $V^\circ = V \setminus \{\mathsf{root}\}$ if $\mathscr{T}$ has a root and $V^\circ = V$ otherwise. For every vertex $u \in V^\circ$ there exists a unique vertex, denoted by $\mathsf{par}(u)$, such that $(\mathsf{par}(u), u) \in E$. Set $\mathsf{Chi}(u) = \{v \in V \colon (u, v) \in E\}$ for $u \in V$. If $W \subseteq V$, we put $\mathsf{Chi}(W) = \bigcup_{v \in W} \mathsf{Chi}(v)$ and $\mathsf{Des}(W) = \bigcup_{n=0}^\infty \mathsf{Chi}^{\langle n\rangle}(W)$, where $\mathsf{Chi}^{\langle 0\rangle}(W) = W$ and $\mathsf{Chi}^{\langle n+1\rangle}(W) = \mathsf{Chi}(\mathsf{Chi}^{\langle n\rangle}(W))$

for all integers $n \geq 0$. For $u \in V$, we set $\mathsf{Chi}^{\langle n\rangle}(u) = \mathsf{Chi}^{\langle n\rangle}(\{u\})$ and $\mathsf{Des}(u) = \mathsf{Des}(\{u\})$.

Denote by $\ell^2(V)$ the Hilbert space of all square summable complex functions on V with the standard inner product. The set $\{e_u\}_{u\in V}$, where $e_u := \chi_{\{u\}}$, is an orthonormal basis of $\ell^2(V)$. Put $\mathscr{E}_V = \mathrm{L{\small IN}}\{e_u \colon u \in V\}$.

Given $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$, we define the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ by

$$\begin{aligned} \mathcal{D}(S_{\boldsymbol{\lambda}}) &= \{f \in \ell^2(V) \colon \varLambda_{\mathscr{T}} f \in \ell^2(V)\},\\ S_{\boldsymbol{\lambda}} f &= \varLambda_{\mathscr{T}} f, \qquad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}), \end{aligned}$$

where $\varLambda_{\mathscr{T}}$ is the map defined on functions $f\colon V \to \mathbb{C}$ via

$$(\varLambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ,\\ 0 & \text{if } v = \mathsf{root}. \end{cases} \tag{7.1}$$

Such $S_{\boldsymbol{\lambda}}$ is called a *weighted shift* on the directed tree $\mathscr{T}$ with weights $\{\lambda_v\}_{v\in V^\circ}$. Let us recall that weighted shifts on directed trees are always closed (cf. [5, Proposition 3.1.2]).

Before characterizing weak quasinormality of a densely defined weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$, we describe the spectral measure of $|S_{\boldsymbol{\lambda}}|^\alpha$ for $\alpha \in (0,\infty)$.

LEMMA 7.1. *If $S_{\boldsymbol{\lambda}}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$, $\alpha \in (0,\infty)$ and E is the spectral measure of $|S_{\boldsymbol{\lambda}}|^\alpha$, then*

$$(E(\sigma)f)(v) = \chi_\sigma(\|S_{\boldsymbol{\lambda}} e_v\|^\alpha) f(v), \qquad v \in V,\ f \in \ell^2(V),\ \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

Proof. By [5, Proposition 3.4.3], $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol{\lambda}}) \cap \mathcal{D}(|S_{\boldsymbol{\lambda}}|^\alpha)$ and $|S_{\boldsymbol{\lambda}}|^\alpha e_u = \|S_{\boldsymbol{\lambda}} e_u\|^\alpha e_u$ for all $u \in V$. Hence, by [5, (2.2.1)], we have

$$E(\sigma)f = \sum_{u\in V} \chi_\sigma(\|S_{\boldsymbol{\lambda}} e_u\|^\alpha)\langle f, e_u\rangle e_u, \quad f \in \ell^2(V),\ \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

which implies that

$$(E(\sigma)f)(v) = \langle E(\sigma)f, e_v\rangle = \chi_\sigma(\|S_{\boldsymbol{\lambda}} e_v\|^\alpha) f(v)$$

for all $v \in V$, $f \in \ell^2(V)$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. This completes the proof. ∎

Now we characterize weak quasinormality of weighted shifts on directed trees.

THEOREM 7.2. *Let $S_{\boldsymbol{\lambda}}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$, and E be the spectral measure of $|S_{\boldsymbol{\lambda}}|$. Then the following assertions are valid.*

(i) *For any $c \in \mathbb{R}_+$, $S_{\boldsymbol{\lambda}}$ is weakly quasinormal with $\mathfrak{c}_{S_{\boldsymbol{\lambda}}} \leq c$ if and only if* (2)

(2) We adhere to the convention that $\sum_{v\in\emptyset} |\lambda_v|^2 = 0$. Note also that $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol{\lambda}})$ (cf. [5, Proposition 3.1.3(v)]), which means that the expression (7.2) makes sense.

$$\|S_{\boldsymbol\lambda} e_u\|^2 \leq c \sum_{v \in \mathsf{Chi}_{\,\mathrm{e}}(u)} |\lambda_v|^2, \quad u \in V, \tag{7.2}$$

where $\mathsf{Chi}_{\,\mathrm{e}}(u) := \{v \in \mathsf{Chi}(u) \colon \|S_{\boldsymbol\lambda} e_v\| = \|S_{\boldsymbol\lambda} e_u\|\}$.

(ii) $\langle E(\cdot)|S_{\boldsymbol\lambda}|f, |S_{\boldsymbol\lambda}|f\rangle \ll \langle E(\cdot)S_{\boldsymbol\lambda} f, S_{\boldsymbol\lambda} f\rangle$ *for all* $f \in \mathcal{D}(S_{\boldsymbol\lambda})$ *if and only if*

$$\forall u \in V \colon \ \|S_{\boldsymbol\lambda} e_u\| \neq 0 \ \Rightarrow \ \mathsf{Chi}'(u) \neq \emptyset, \tag{7.3}$$

where $\mathsf{Chi}'(u) := \{v \in \mathsf{Chi}_{\,\mathrm{e}}(u) \colon \lambda_v \neq 0\}$.

It is worth noting that if (7.2) holds, then according to our summation convention (see footnote 2) we have

$$\forall u \in V \colon \ \|S_{\boldsymbol\lambda} e_u\| \neq 0 \ \Rightarrow \ \mathsf{Chi}_{\,\mathrm{e}}(u) \neq \emptyset. \tag{7.4}$$

Proof of Theorem 7.2. It follows from Lemma 7.1 that

$$\langle E(\sigma)f, f\rangle = \sum_{u \in V} \chi_\sigma(\|S_{\boldsymbol\lambda} e_u\|)|f(u)|^2, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), f \in \ell^2(V).$$

Since $(|S_{\boldsymbol\lambda}|f)(u) = \|S_{\boldsymbol\lambda} e_u\| f(u)$ for all $u \in V$ and $f \in \mathcal{D}(S_{\boldsymbol\lambda})$ (cf. [5, Proposition 3.4.3]), we deduce that

$$\langle E(\sigma)|S_{\boldsymbol\lambda}|f, |S_{\boldsymbol\lambda}|f\rangle = \sum_{u \in V} \chi_\sigma(\|S_{\boldsymbol\lambda} e_u\|)\|S_{\boldsymbol\lambda} e_u\|^2 |f(u)|^2 \tag{7.5}$$

for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ and $f \in \mathcal{D}(S_{\boldsymbol\lambda})$. In view of the equality $V^\circ = \bigsqcup_{u \in V} \mathsf{Chi}(u)$ (cf. [5, Proposition 2.1.2]), we have

$$\begin{aligned} \langle E(\sigma)S_{\boldsymbol\lambda} f, S_{\boldsymbol\lambda} f\rangle &= \sum_{u \in V^\circ} \chi_\sigma(\|S_{\boldsymbol\lambda} e_u\|)|\lambda_u|^2 |f(\mathsf{par}(u))|^2 \\ &= \sum_{u \in V} \Big(\sum_{v \in \mathsf{Chi}(u)} \chi_\sigma(\|S_{\boldsymbol\lambda} e_v\|)|\lambda_v|^2 \Big) |f(u)|^2 \end{aligned} \tag{7.6}$$

for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ and $f \in \mathcal{D}(S_{\boldsymbol\lambda})$.

(i) Since, by [5, Proposition 3.1.3(v)], $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol\lambda})$, we infer from (7.5) and (7.6) that the inequality (4.1) holds with $A = S_{\boldsymbol\lambda}$ if and only if

$$\chi_\sigma(\|S_{\boldsymbol\lambda} e_u\|)\|S_{\boldsymbol\lambda} e_u\|^2 \leq c \sum_{v \in \mathsf{Chi}(u)} \chi_\sigma(\|S_{\boldsymbol\lambda} e_v\|)|\lambda_v|^2, \quad u \in V, \ \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{7.7}$$

First we show that (7.7) implies (7.2). Suppose (7.7) holds. Fix $u \in V$ and define the set $\Omega_u = \{\|S_{\boldsymbol\lambda} e_v\| \colon v \in \mathsf{Chi}(u)\} \subseteq \mathbb{R}_+$. We may assume that $\|S_{\boldsymbol\lambda} e_u\| \neq 0$. Then, by (7.1), $\mathsf{Chi}(u) \neq \emptyset$. If $\Omega_u = \mathbb{R}_+$, then clearly $\mathsf{Chi}_{\,\mathrm{e}}(u) \neq \emptyset$. If $\Omega_u \neq \mathbb{R}_+$, then substituting $\sigma = \{t\}$ with $t \in \mathbb{R}_+ \setminus \Omega_u$ into (7.7), we deduce that $\|S_{\boldsymbol\lambda} e_u\| \neq t$. Hence $\mathbb{R}_+ \setminus \Omega_u \subseteq \mathbb{R}_+ \setminus \{\|S_{\boldsymbol\lambda} e_u\|\}$, which yields $\mathsf{Chi}_{\,\mathrm{e}}(u) \neq \emptyset$. This proves (7.4). By substituting $\sigma = \{\|S_{\boldsymbol\lambda} e_u\|\}$ into (7.7), we obtain (7.2).

Now we show that (7.2) implies (7.7). Fix $u \in V$ and $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Without loss of generality we can assume that $\|S_{\boldsymbol\lambda} e_u\| \neq 0$ and $\|S_{\boldsymbol\lambda} e_u\| \in \sigma$. Then, by (7.2),

$$\begin{aligned}\chi_\sigma(\|S_{\boldsymbol\lambda} e_u\|)\|S_{\boldsymbol\lambda} e_u\|^2 &\le c \sum_{v \in \mathsf{Chi}_{\mathrm{e}}(u)} \chi_\sigma(\|S_{\boldsymbol\lambda} e_v\|)|\lambda_v|^2 \\ &\le c \sum_{v \in \mathsf{Chi}(u)} \chi_\sigma(\|S_{\boldsymbol\lambda} e_v\|)|\lambda_v|^2,\end{aligned}$$

which shows that (7.7) holds. This completes the proof of (i).

(ii) One can argue as in the proof of (i). We leave the details to the reader. ∎

The following characterization of quasinormality of weighted shifts on directed trees generalizes that of [5, Proposition 8.1.7] to the case of unbounded operators. The present proof is quite different from that for bounded operators.

Corollary 7.3. *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then the following two conditions are equivalent:*

(i) *$S_{\boldsymbol\lambda}$ is quasinormal,*
(ii) *$\|S_{\boldsymbol\lambda} e_u\| = \|S_{\boldsymbol\lambda} e_v\|$ for all $u \in V$ and $v \in \mathsf{Chi}(u)$ such that $\lambda_v \neq 0$.*

Moreover, if $V^\circ \neq \emptyset$ and $\lambda_v \neq 0$ for all $v \in V^\circ$, then $S_{\boldsymbol\lambda}$ is quasinormal if and only if $\|S_{\boldsymbol\lambda}\|^{-1} S_{\boldsymbol\lambda}$ is an isometry.

Proof. (i)⇒(ii). By Theorem 3.1, $S_{\boldsymbol\lambda}$ is weakly quasinormal with $\mathfrak{c}_{S_{\boldsymbol\lambda}} \le 1$. One can deduce from [5, Proposition 3.1.3] and Theorem 7.2(i), applied to $c = 1$, that $\lambda_v = 0$ for all $v \in \mathsf{Chi}(u) \setminus \mathsf{Chi}_{\mathrm{e}}(u)$ and $u \in V$. This implies (ii).

(ii)⇒(i). By our present assumption, $\lambda_v = 0$ for all $v \in \mathsf{Chi}(u) \setminus \mathsf{Chi}_{\mathrm{e}}(u)$ and $u \in V$. This implies that (7.2) holds with $c = 1$. Hence, by Theorem 7.2(i), $S_{\boldsymbol\lambda}$ is weakly quasinormal with $\mathfrak{c}_{S_{\boldsymbol\lambda}} \le 1$. Applying Theorems 4.3(iii) and 5.1 yields (i).

Arguing as in the proof of [5, Proposition 8.1.7], we deduce the "moreover" part of the conclusion from the equivalence (i)⇔(ii). ∎

We will show by example that there are unbounded quasinormal weighted shifts on directed trees (cf. Example 8.1).

The following corollary is closely related to Proposition 6.1.

Corollary 7.4. *If $S_{\boldsymbol\lambda}$ is a quasinormal weighted shift on a directed tree $\mathscr{T}$, then $S_{\boldsymbol\lambda}(\mathscr{E}_V) \subseteq \mathfrak{D}(|S_{\boldsymbol\lambda}|^\alpha)$ for every positive real α.*

Proof. By Corollary 7.3 and [5, Proposition 3.1.3], for every $u \in V$,

$$\sum_{v\in V} \|S_{\boldsymbol\lambda} e_v\|^{2\alpha} |(S_{\boldsymbol\lambda} e_u)(v)|^2 \overset{(7.1)}{=} \sum_{v\in V^\circ} \|S_{\boldsymbol\lambda} e_v\|^{2\alpha} |\lambda_v|^2 e_u(\mathsf{par}(v)) \tag{7.8}$$
$$= \sum_{v\in \mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^{2\alpha} |\lambda_v|^2 = \|S_{\boldsymbol\lambda} e_u\|^{2\alpha} \sum_{v\in \mathsf{Chi}(u)} |\lambda_v|^2 = \|S_{\boldsymbol\lambda} e_u\|^{2(\alpha+1)}.$$

Since, by [5, Proposition 3.4.3], $\mathscr{E}_V \subseteq \mathcal{D}(|S_{\boldsymbol\lambda}|^\alpha)$ and $|S_{\boldsymbol\lambda}|^\alpha e_u = \|S_{\boldsymbol\lambda} e_u\|^\alpha e_u$ for every $u \in V$, we deduce that a function $f \in \ell^2(V)$ belongs to $\mathcal{D}(|S_{\boldsymbol\lambda}|^\alpha)$ if and only if $\sum_{v\in V} \|S_{\boldsymbol\lambda} e_v\|^{2\alpha}|f(v)|^2 < \infty$ (consult the proof of [5, Lemma 2.2.1]). This combined with (7.8) gives $S_{\boldsymbol\lambda} e_u \in \mathcal{D}(|S_{\boldsymbol\lambda}|^\alpha)$ for $u \in V$, which completes the proof. ∎

8. Examples. This section provides examples of weighted shifts on directed trees that illustrate the subject of this paper. We begin by considering the case of quasinormal operators. It follows from Corollary 7.3 that quasinormal weighted shifts on directed trees with nonzero weights are automatically bounded. However, if some of the weights are allowed to be zero, then quasinormal weighted shifts may be unbounded. Below, we construct an example of an injective quasinormal weighted shift on a directed binary tree whose restriction to $\ell^2(\mathsf{Des}(u))$ is unbounded for every $u \in V$.

Example 8.1. Let $\mathscr{T}$ be a directed tree with root such that for every $u \in V$, the set $\mathsf{Chi}(u)$ has exactly two vertices. By [5, Corollary 2.1.5], we have

$$V^\circ = \bigsqcup_{n=1}^{\infty} \mathsf{Chi}^{\langle n\rangle}(\mathsf{root}).$$

We define recursively a sequence $\{\{\lambda_v\}_{v\in\mathsf{Chi}^{\langle n\rangle}(\mathsf{root})}\}_{n=1}^\infty$ of systems of nonnegative real numbers. We begin with $n = 1$. If $v_1, v_2 \in \mathsf{Chi}(\mathsf{root})$ and $v_1 \neq v_2$, then we set $\lambda_{v_1} = 0$ and $\lambda_{v_2} = 1$. Suppose that we have constructed the systems $\{\lambda_v\}_{v\in\mathsf{Chi}^{\langle j\rangle}(\mathsf{root})} \subseteq [0,\infty)$ for $j = 1,\ldots,n$. To construct $\{\lambda_v\}_{v\in\mathsf{Chi}^{\langle n+1\rangle}(\mathsf{root})}$, note that (cf. [5, (6.1.3)])

$$\mathsf{Chi}^{\langle n+1\rangle}(\mathsf{root}) = \bigsqcup_{u\in\mathsf{Chi}^{\langle n\rangle}(\mathsf{root})} \mathsf{Chi}(u). \tag{8.1}$$

Fix $u \in \mathsf{Chi}^{\langle n\rangle}(\mathsf{root})$. By our assumption $\mathsf{Chi}(u) = \{v, w\}$ with $v \neq w$. If $\lambda_u = 0$, then we set $\lambda_v = 0$ and $\lambda_w = n + 1$. If $\lambda_u \neq 0$, then we set $\lambda_v = 0$ and $\lambda_w = \lambda_u$. In view of (8.1), the recursive procedure gives us the system $\boldsymbol\lambda := \{\lambda_v\}_{v\in V^\circ}$. Let $S_{\boldsymbol\lambda}$ be the weighted shift on $\mathscr{T}$ with weights $\boldsymbol\lambda$. By [5, Proposition 3.1.3], $S_{\boldsymbol\lambda}$ is densely defined. It is a routine matter to verify that $S_{\boldsymbol\lambda}$ satisfies the condition (ii) of Corollary 7.3. Hence the operator $S_{\boldsymbol\lambda}$ is quasinormal. It is easily seen, by using [5, Proposition 3.1.8], that for every $u \in V$, the operator $S_{\boldsymbol\lambda}|_{\mathrm{LIN}\{e_v\colon v\in \mathsf{Des}(u)\}}$ (which acts in $\ell^2(\mathsf{Des}(u))$) is unbounded. The injectivity of $S_{\boldsymbol\lambda}$ follows from [5, Proposition 3.1.7].

Now we show how to construct nonquasinormal weighted shifts on certain directed trees that are weakly quasinormal as well as non-weakly quasinormal weighted shifts on the same directed trees that satisfy the condition (i) of Theorem 4.4 (with $A = S_{\boldsymbol\lambda}$). Note that this is not possible for classical weighted shifts (that is, weighted shifts on the directed tress $(\mathbb{Z}_+, \{(n, n+1)\colon n\in\mathbb{Z}_+\})$ and $(\mathbb{Z}, \{(n, n+1)\colon n\in\mathbb{Z}\})$, cf. [5, Remark 3.1.4]), because by Theorem 7.2 and Corollary 7.3 every classical weighted shift which satisfies the condition (i) of Theorem 4.4 is automatically quasinormal. Hence, nonquasinormal classical weighted shifts (many such exist) do not satisfy this condition. We also show that for every $c \in (1, \infty)$, there exists an injective weighted shift $S_{\boldsymbol\lambda}$ on a directed tree such that $\mathfrak{c}_{S_{\boldsymbol\lambda}} = c$, were $\mathfrak{c}_{S_{\boldsymbol\lambda}}$ is understood as in Section 4. In Examples 8.2 and 8.3, we consider the cases of bounded and unbounded nonhyponormal operators with the properties mentioned above. All this can also be achieved in the class of hyponormal operators, as is shown in Example 8.4.

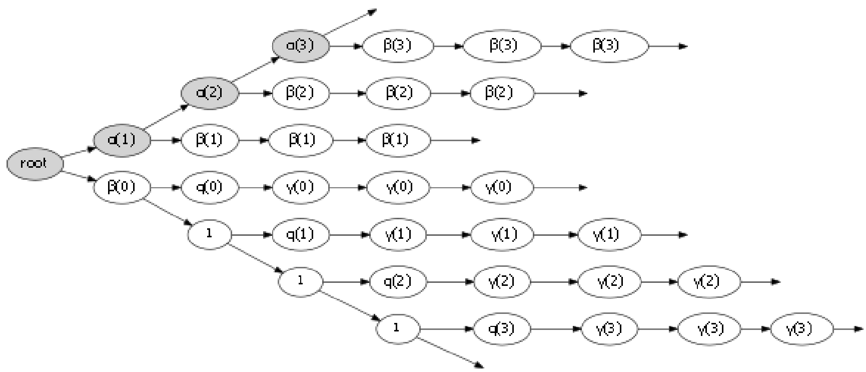

Fig. 1

EXAMPLE 8.2. Let $\mathscr{T}$ be the directed tree as in Figure 1, where $\{\alpha(n)\}_{n=1}^{\infty}$, $\{\beta(n)\}_{n=0}^{\infty}$, $\{q(n)\}_{n=0}^{\infty}$ and $\{\gamma(n)\}_{n=0}^{\infty}$ are sequences of positive real numbers satisfying the following two conditions:

$$\alpha(n)^2 + \beta(n-1)^2 = 1, \quad n \in \mathbb{N}, \tag{8.2}$$

$$1 + q(n)^2 = \gamma(n)^2, \quad n \in \mathbb{Z}_+. \tag{8.3}$$

Let $S_{\boldsymbol\lambda}$ be the weighted shift on $\mathscr{T}$ with weights given by Figure 1. By [5, Proposition 3.1.3], $S_{\boldsymbol\lambda}$ is densely defined (and closed as a weighted shift on a directed tree). It follows from (8.2), (8.3) and [5, Proposition 3.1.8] that $S_{\boldsymbol\lambda}$ is bounded if and only if the sequence $\{q(n)\}_{n=0}^\infty$ is bounded.

We first note that $S_{\boldsymbol\lambda}$ is not hyponormal. Indeed, since

$$\sum_{v\in\mathsf{Chi}(u_i)} \frac{|\lambda_v|^2}{\|S_{\boldsymbol\lambda} e_v\|^2} = 1+\alpha(i+1)^2 > 1, \quad i\in\mathbb{N},$$

where u_i is the vertex corresponding to $\alpha(i)$, we infer from [5, Theorem 5.1.2 and Remark 5.1.5] that $S_{\boldsymbol\lambda}$ is not hyponormal.

Suppose now that $\inf_{n\in\mathbb{N}}\alpha(n)=0$. Then $S_{\boldsymbol\lambda}$ is not weakly quasinormal. Indeed, otherwise by Theorem 7.2(i) applied to $u=u_i$ with $i\in\mathbb{Z}_+$ ($u_0 := \mathsf{root}$), there exists $c>0$ such that $1\le c\alpha(i+1)^2$ for all $i\in\mathbb{Z}_+$, which is impossible. It follows from (8.2) and (8.3) that (7.3) holds, which in view of Theorem 7.2(ii) implies that

$$\langle E(\cdot)|S_{\boldsymbol\lambda}|f, |S_{\boldsymbol\lambda}|f\rangle \ll \langle E(\cdot)S_{\boldsymbol\lambda}f, S_{\boldsymbol\lambda}f\rangle, \quad f\in\mathcal{D}(S_{\boldsymbol\lambda}).$$

Fix $c\in(1,\infty)$. Suppose now that $\inf_{n\in\mathbb{N}}\alpha(n)=1/\sqrt{c}$ and $q(i)^{-2}+1\le c$ for all $i\in\mathbb{Z}_+$ (we still assume that (8.2) and (8.3) are satisfied). It is easily seen that (7.2) holds. Hence, by Theorem 7.2(i), $S_{\boldsymbol\lambda}$ is weakly quasinormal with $\mathfrak{c}_{S_{\boldsymbol\lambda}}\le c$. We show that $\mathfrak{c}_{S_{\boldsymbol\lambda}}=c$. Indeed, by Theorem 7.2(i), $1\le \mathfrak{c}_{S_{\boldsymbol\lambda}}\alpha(i)^2$ for all $i\in\mathbb{N}$, which implies that $1/\sqrt{\mathfrak{c}_{S_{\boldsymbol\lambda}}}\le 1/\sqrt{c}$, and thus $\mathfrak{c}_{S_{\boldsymbol\lambda}}\ge c$.

Finally, note that the so-constructed operator $S_{\boldsymbol\lambda}$ can be made bounded or unbounded according to our needs, still maintaining its properties discussed above. This can be achieved by considering bounded or unbounded sequences $\{q(n)\}_{n=0}^\infty$.

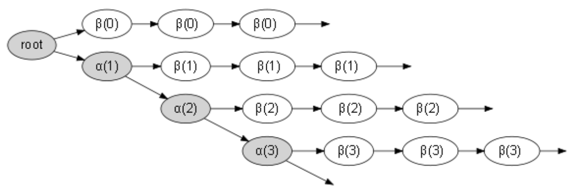

Fig. 2

Example 8.3. Let $\{\alpha(n)\}_{n=1}^\infty$ and $\{\beta(n)\}_{n=0}^\infty$ be sequences of positive real numbers that satisfy (8.2). The reader can easily check that the weighted shifts $S_{\boldsymbol\lambda}$ on the directed tree $\mathscr{T}$ given by Figure 2, which are less complicated than those in Figure 1, have all the properties specified in Example 8.2 (each of which depends on the choice of weights) except for unboundedness, namely $S_{\boldsymbol\lambda}$ are always bounded.

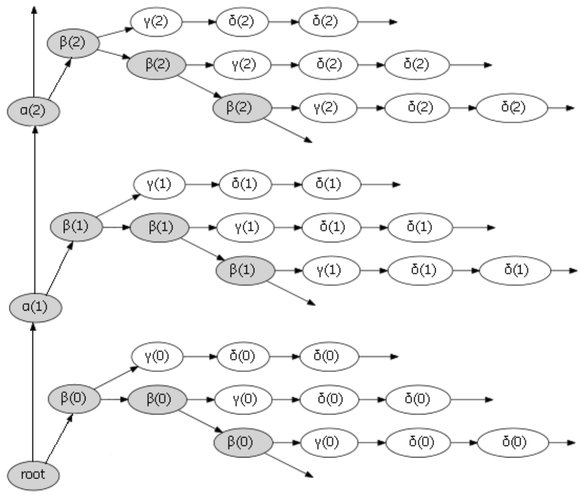

Fig. 3

EXAMPLE 8.4. The previous two constructions can be modified so as to obtain examples of weighted shifts $S_{\boldsymbol{\lambda}}$ on a directed tree with nonzero weights which have all the properties stated in Example 8.2 (each of which depends on the choice of weights) except for nonhyponormality, namely $S_{\boldsymbol{\lambda}}$ are hyponormal. Since the main idea remains the same, we skip the discussion of the present example. For the reader's convenience we draw a figure that contains the necessary data (cf. Figure 3). The sequences $\{\alpha(n)\}_{n=1}^{\infty}$, $\{\beta(n)\}_{n=0}^{\infty}$, $\{\gamma(n)\}_{n=0}^{\infty}$ and $\{\delta(n)\}_{n=0}^{\infty}$ consist of positive real numbers that satisfy (8.2) and the following three conditions:

$$\delta(n)^2 = \beta(n)^2 + \gamma(n)^2, \quad n \in \mathbb{Z}_+,$$

$$\frac{\beta(n)^2}{\delta(n)^2} + \alpha(n+1)^2 < 1, \quad n \in \mathbb{Z}_+, \tag{8.4}$$

$$\delta(n) > 1, \quad n \in \mathbb{Z}_+. \tag{8.5}$$

It is worth pointing out that under the assumption (8.2), the conditions (8.4) and (8.5) are equivalent.

9. Remarks and further results. The absolute continuity approach developed in this paper in the context of quasinormal operators can be generalized to other classes of operators. The class of q-quasinormal operators, a particular case of q-deformed operators introduced by Ôta in [13] (see also [14, 16, 17]) in connection with the theory of quantum groups (see [8]), is well-suited for our purposes.

Let q be a positive real number. Following [13], we say that a closed densely defined operator A in a complex Hilbert space $\mathcal{H}$ is *q-quasinormal* if $U|A| \subseteq \sqrt{q}\,|A|U$, where $A = U|A|$ is the polar decomposition of A (or equivalently $U|A| = \sqrt{q}\,|A|U$; cf. [13, Lemma 2.2]). In view of [13, Theorem 2.5], a closed densely defined operator A in $\mathcal{H}$ is q-quasinormal if and only if

$$UE(\sigma) = E(\psi_q^{-1}(\sigma))U, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

where E is the spectral measure of $|A|$ and $\psi_q\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel function given by $\psi_q(x) = \sqrt{q}\,x$. The above suggests the following generalization.

Proposition 9.1. *Let A be a closed densely defined operator in $\mathcal{H}$, $A = U|A|$ be its polar decomposition and E be the spectral measure of $|A|$. Suppose ϕ and ψ are Borel functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then the following conditions are equivalent* (3)*:*

(i) $UE(\varphi^{-1}(\cdot)) = E(\psi^{-1}(\cdot))U$,
(ii) $U\varphi(|A|) \subseteq \psi(|A|)U$,
(iii) $E(\psi^{-1}(\cdot))A \subseteq AE(\varphi^{-1}(\cdot))$.

Proof. (i)⇔(ii). Use the measure transport theorem (cf. [1, Theorem 5.4.10]) and the "intertwining" version of [1, Theorem 6.3.2].

(i)⇔(iii). Adapt the proof of [21, Proposition 1]. ∎

Below we assume that $\phi, \psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ are fixed Borel functions. Arguing exactly as in the proofs of Theorems 3.1, 4.3, 4.4 and 5.1, and using Proposition 9.1 together with its proof, we obtain the following more general results. It is also worth pointing out that the "moreover" parts of Theorems 4.3 and 4.4 can be easily adapted to this new context as well. We leave the details to the reader.

Theorem 9.2. *Let A be a closed densely defined operator in $\mathcal{H}$, and E be the spectral measure of $|A|$. Then the following conditions are equivalent:*

(i) $E(\psi^{-1}(\cdot))A \subseteq AE(\varphi^{-1}(\cdot))$,
(ii) $\langle E(\psi^{-1}(\cdot))Af, Af\rangle = \langle E(\varphi^{-1}(\cdot))|A|f, |A|f\rangle$ *for all* $f \in \mathcal{D}(A)$,
(iii) $\langle E(\psi^{-1}(\cdot))Af, Af\rangle \ll \langle E(\varphi^{-1}(\cdot))|A|f, |A|f\rangle$ *for all* $f \in \mathcal{D}(A)$.

Theorem 9.3. *Let A be a closed densely defined operator in $\mathcal{H}$, E be the spectral measure of $|A|$ and $c \in \mathbb{R}_+$. Then the following conditions are equivalent:*

(i) $\langle E(\varphi^{-1}(\cdot))|A|f, |A|f\rangle \le c\langle E(\psi^{-1}(\cdot))Af, Af\rangle$ *for all* $f \in \mathcal{D}(A)$,
(ii) *there exists* $T \in \boldsymbol{B}(\mathcal{H})$ *such that*

$$TA = |A|, \quad \|T\| \le \sqrt{c} \quad \text{and} \quad TE(\psi^{-1}(\cdot)) = E(\varphi^{-1}(\cdot))T,$$

(iii) *there exists* $T \in \boldsymbol{B}(\mathcal{H})$ *such that*

$$TA = |A|, \quad \|T\| \le \sqrt{c} \quad \text{and} \quad T\psi(|A|) \subseteq \varphi(|A|)T.$$

(3) $E(\varphi^{-1}(\cdot))$ stands for the spectral measure $\mathfrak{B}(\mathbb{R}_+) \ni \sigma \mapsto E(\varphi^{-1}(\sigma)) \in \boldsymbol{B}(\mathcal{H})$.

THEOREM 9.4. *Let A be a closed densely defined operator in $\mathcal{H}$, and E be the spectral measure of $|A|$. Then the following conditions are equivalent:*

(i) $\langle E(\varphi^{-1}(\cdot))|A|f, |A|f\rangle \ll \langle E(\psi^{-1}(\cdot))Af, Af\rangle$ *for all* $f \in \mathcal{D}(A)$,

(ii) *there exists a (unique) linear map* $T_0\colon \mathscr{H}_0 \to \mathcal{R}(|A|)$ *such that* $T_0 A = |A|$ *and* $T_0 E(\psi^{-1}(\cdot))|_{\mathscr{H}_0} = E(\varphi^{-1}(\cdot))T_0$, *where*

$$\mathscr{H}_0 = \mathrm{LIN}\,\big\{E(\psi^{-1}(\sigma))Af\colon \sigma \in \mathfrak{B}(\mathbb{R}_+),\ f \in \mathcal{D}(A)\big\}.$$

THEOREM 9.5. *Let A be a closed densely defined operator in $\mathcal{H}$, and E be the spectral measure of $|A|$. Then the following conditions are equivalent:*

(i) $E(\psi^{-1}(\cdot))A \subseteq AE(\varphi^{-1}(\cdot))$,

(ii) $\langle E(\varphi^{-1}(\cdot))|A|f, |A|f\rangle \le \langle E(\psi^{-1}(\cdot))Af, Af\rangle$ *for all* $f \in \mathcal{D}(A)$.

Substituting φ = the identity function on $\mathbb{R}_+$ and $\psi = \psi_q$ into the above theorems, we obtain characterizations of q-quasinormal operators, "q-variants" of weakly quasinormal operators and operators satisfying the "q-version" of the condition (i) of Theorem 4.4.

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