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Interpolation of Cesàro sequence and function spaces

by

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Abstract. The interpolation properties of Cesàro sequence and function spaces are investigated. It is shown that $\mathrm{Ces}_p(I)$ is an interpolation space between $\mathrm{Ces}_{p_0}(I)$ and $\mathrm{Ces}_{p_1}(I)$ for $1 < p_0 < p_1 \le \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$, where $I = [0,\infty)$ or $[0,1]$. The same result is true for Cesàro sequence spaces. On the other hand, $\mathrm{Ces}_p[0,1]$ is not an interpolation space between $\mathrm{Ces}_1[0,1]$ and $\mathrm{Ces}_\infty[0,1]$.

1. Introduction and preliminaries. The structure of Cesàro sequence and function spaces was investigated by several authors (see, for example, [Be], [MPS], [A], [AM] and references therein). Here we are interested in interpolation properties of these spaces. The main purpose is to give interpolation theorems for the Cesàro sequence spaces ces_p and Cesàro function spaces $\mathrm{Ces}_p(I)$ on $I = [0,\infty)$ and $I = [0,1]$. In the case of $I = [0,\infty)$ some interpolation results for Cesàro function spaces are contained implicitly in [MS]. Moreover, using the so-called K^+-method of interpolation it was proved in [CFM] that the Cesàro sequence space ces_p is an interpolation space with respect to the couple $(l_1, l_1(2^{-k}))$.

Our main aim is to give a rather complete description of Cesàro spaces as interpolation spaces with respect to appropriate couples of weighted L_1-spaces as well as Cesàro spaces. For example, if either $I = [0,\infty)$ or $[0,1]$ and $1 < p_0 < p_1 \le \infty$ with $1/p = (1-\theta)/p_0 + \theta/p_1$ for $0 < \theta < 1$, then

$$(1.1)\qquad (\mathrm{Ces}_{p_0}(I), \mathrm{Ces}_{p_1}(I))_{\theta,p} = \mathrm{Ces}_p(I) \quad\text{and}\quad (\mathrm{ces}_{p_0}, \mathrm{ces}_{p_1})_{\theta,p} = \mathrm{ces}_p,$$

where $(\cdot,\cdot)_{\theta,p}$ denotes the K-method of interpolation.

We have a completely different situation in a more interesting and non-trivial case when $I = [0,1]$ and $p_0 = 1$, $p_1 = \infty$. It turns out that $\mathrm{Ces}_p[0,1]$ is not an interpolation space between $\mathrm{Ces}_1[0,1]$ and $\mathrm{Ces}_\infty[0,1]$, whereas $(\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])_{\theta,p}$ for $1 < p < \infty$ is a weighted Cesàro function space.

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Let us collect some necessary definitions and notations related to the interpolation theory of operators as well as Cesàro, Copson and down spaces.

For two normed spaces X and Y the symbol $X \overset{C}{\hookrightarrow} Y$ means that the imbedding $X \subset Y$ is continuous with norm not greater than C, i.e., $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$, and $X \hookrightarrow Y$ means that $X \overset{C}{\hookrightarrow} Y$ for some $C > 0$. Moreover, $X = Y$ means that $X \hookrightarrow Y$ and $Y \hookrightarrow X$, that is, the spaces are the same and the norms are equivalent. If f and g are real functions, then $f \approx g$ means that $c^{-1}g \leq f \leq cg$ for some $c \geq 1$.

For a Banach couple $\bar{X} = (X_0, X_1)$ of two compatible Banach spaces X_0 and X_1 consider the Banach spaces $X_0 \cap X_1$ and $X_0 + X_1$ with their natural norms

$$\|f\|_{X_0 \cap X_1} = \max(\|f\|_{X_0}, \|f\|_{X_1}) \quad \text{for } f \in X_0 \cap X_1,$$

and for $f \in X_0 + X_1$,

$$\|f\|_{X_0 + X_1} = \inf\{\|f_0\|_{X_0}, + \|f_1\|_{X_1} : f = f_0 + f_1,\ f_0 \in X_0,\ f_1 \in X_1\}.$$

For more careful definitions of a Banach couple, intermediate and interpolation spaces with some results introduced briefly below, see [BK, pp. 91–173, 289–314, 338–359] and [BS, pp. 95–116].

A Banach space X is called an *intermediate space* between X_0 and X_1 if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$. Such a space X is called an *interpolation space* between X_0 and X_1 (and we write $X \in \operatorname{Int}(X_0, X_1)$) if, for any bounded linear operator $T : X_0 + X_1 \to X_0 + X_1$ whose restriction $T_{|X_i} : X_i \to X_i$ is bounded for $i = 0, 1$, the restriction $T_{|X} : X \to X$ is also bounded and $\|T\|_{X \to X} \leq C \max\{\|T\|_{X_0 \to X_0}, \|T\|_{X_1 \to X_1}\}$ for some $C \geq 1$. If $C = 1$, then X is called an *exact interpolation space* between X_0 and X_1.

An *interpolation method* or *interpolation functor* $\mathcal{F}$ is a construction (a rule) which assigns to every Banach couple $\bar{X} = (X_0, X_1)$ an interpolation space $\mathcal{F}(\bar{X})$ between X_0 and X_1. The interpolation functor $\mathcal{F}$ is called *exact* if the space $\mathcal{F}(\bar{X})$ is an exact interpolation space for every Banach couple $\bar{X}$. One of the most important interpolation methods is the *K-method*, also known as the *real Lions–Peetre interpolation method.* For a Banach couple $\bar{X} = (X_0, X_1)$ the *Peetre K-functional* of an element $f \in X_0 + X_1$ is defined for $t > 0$ by

$$K(t, f; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1,\ f_0 \in X_0,\ f_1 \in X_1\}.$$

Then the *spaces of the K-method of interpolation* are

$$(X_0, X_1)_{\theta,p} = \Big\{ f \in X_0 + X_1 : \\ \|f\|_{\theta,p} = \Big(\int_0^\infty [t^{-\theta} K(t, f; X_0, X_1)]^p \, \frac{dt}{t} \Big)^{1/p} < \infty \Big\}$$

if $0 < \theta < 1$ and $1 \le p < \infty$, and

$$(X_0, X_1)_{\theta,\infty} = \Big\{ f \in X_0 + X_1 : \|f\|_{\theta,\infty} = \sup_{t>0} K(t, f; X_0, X_1)/t^\theta < \infty \Big\}$$

if $0 \le \theta \le 1$. Very useful in calculations is the so-called *reiteration formula* showing the stability of the K-method of interpolation. If $1 \le p_0, p_1, p \le \infty$, $0 < \theta_0, \theta_1, \theta < 1$ and $\theta_0 \ne \theta_1$, then with equivalent norms

$$((X_0, X_1)_{\theta_0,p_0}, (X_0, X_1)_{\theta_1,p_1})_{\theta,p} = (X_0, X_1)_{\eta,p}, \tag{1.2}$$

where $\eta = (1-\theta)\theta_0 + \theta\theta_1$ (see [BS, Theorem 2.4, p. 311], [BL, Theorems 3.5.3], [BK, Theorem 3.8.10]) and [Tr, Theorem 1.10.2]).

The space $(X_0, X_1)^K_\Phi$ of the *general* K*-method of interpolation*, where Φ is a *parameter of the* K*-method*, i.e., a Banach function space on $((0,\infty), dt/t)$ containing the function $t \mapsto \min\{1, t\}$, is the Banach space of all $f \in X_0+X_1$ such that $K(\cdot, f; X_0, X_1) \in \Phi$ with the norm $\|f\|_{K_\Phi} = \|K(\cdot, f; X_0, X_1)\|_\Phi$. The space $(X_0, X_1)^K_\Phi$ is an exact interpolation space between X_0 and X_1.

In particular, if $L_p = L_p(\Omega, \mu)$, where (Ω, μ) is a complete σ-finite measure space, then for any $f \in L_1 + L_\infty$ we have

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s)\, ds. \tag{1.3}$$

Here and below, f^* denotes the non-increasing rearrangement of $|f|$ defined by $f^*(s) = \inf\{\lambda > 0 : \mu(\{x \in \Omega : |f(x)| > \lambda\}) \le s\}$ (see [BK, Proposition 3.1.1], [KPS, pp. 78–79], [BS, Theorem 6.2, pp. 74–75]). Moreover, for two non-negative weight functions w_0, w_1 and for $f \in L_1(w_0) + L_1(w_1)$ we have

$$K(t, f; L_1(w_0), L_1(w_1)) = \|\min(w_0, tw_1)f\|_{L_1} \tag{1.4}$$

(see [BK, Proposition 3.1.17] and [Ov, p. 391]).

If the inequality $K(t, g; X_0, X_1) \le K(t, f; X_0, X_1)$ $(t > 0)$ with $f \in X$ and $g \in X_0 + X_1$ implies that $g \in X$ and $\|g\|_X \le C\|f\|_X$ for any $X \in \mathrm{Int}(X_0, X_1)$ and some $C \ge 1$ independent of X, f and g, then (X_0, X_1) is called a K*-monotone* or *Calderón–Mityagin couple*. For every K-monotone couple (X_0, X_1) the spaces $(X_0, X_1)^K_\Phi$ of the general K-method are the only interpolation spaces between X_0 and X_1 (see [BK]).

Now, we recall the definitions of Cesàro spaces. The *Cesàro sequence spaces* ces_p are the sets of real sequences $x = \{x_k\}$ such that

$$\|x\|_{\mathrm{ces}(p)} = \Big[\sum_{n=1}^\infty \Big(\frac{1}{n}\sum_{k=1}^n |x_k|\Big)^p\Big]^{1/p} < \infty \quad \text{for } 1 \le p < \infty,$$

and

$$\|x\|_{\mathrm{ces}(\infty)} = \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^n |x_k| < \infty \quad \text{for } p = \infty.$$

The *Cesàro function spaces* $\mathrm{Ces}_p = \mathrm{Ces}_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0,1]$ or $I = [0,\infty)$ such that

$$\|f\|_{\mathrm{Ces}(p)} = \left[\int_I \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right]^{1/p} < \infty \quad \text{for } 1 \le p < \infty,$$

and

$$\|f\|_{\mathrm{Ces}(\infty)} = \sup_{0<x\in I} \frac{1}{x}\int_0^x |f(t)|\,dt < \infty \quad \text{for } p = \infty.$$

Cesàro spaces are Banach lattices which are not symmetric except when they are trivial, namely, $\mathrm{ces}_1 = \{0\}$, $\mathrm{Ces}_1[0,\infty) = \{0\}$. By a *symmetric space* we mean a Banach lattice X on I with the additional property: if $g^*(t) = f^*(t)$ for all $t > 0$, $f \in X$ and $g \in L^0(I)$ (the set of all classes of Lebesgue measurable real functions on I), then $g \in X$ and $\|g\|_X = \|f\|_X$ (cf. [BS], [KPS]). Moreover, $l_p \overset{p'}{\hookrightarrow} \mathrm{ces}_p$, $L_p(I) \overset{p'}{\hookrightarrow} \mathrm{Ces}_p(I)$ for $1 < p \le \infty$ (in what follows $1/p + 1/p' = 1$), and if $1 < p < q < \infty$, then $\mathrm{ces}_p \overset{1}{\hookrightarrow} \mathrm{ces}_q \overset{1}{\hookrightarrow} \mathrm{ces}_\infty$. Also for $I = [0,1]$ and $1 < p < q < \infty$ we have $L_\infty \overset{1}{\hookrightarrow} \mathrm{Ces}_\infty \overset{1}{\hookrightarrow} \mathrm{Ces}_q \overset{1}{\hookrightarrow} \mathrm{Ces}_p \overset{1}{\hookrightarrow} \mathrm{Ces}_1 = L_1(\ln(1/t))$ and $\mathrm{Ces}_\infty \overset{1}{\hookrightarrow} L_1$.

Let $1 \le p < \infty$. The *Copson sequence spaces* cop_p are the sets of real sequences $x = \{x_k\}$ such that

$$\|x\|_{\mathrm{cop}(p)} = \left[\sum_{n=1}^\infty \left(\sum_{k=n}^\infty \frac{|x_k|}{k}\right)^p\right]^{1/p} < \infty,$$

and the *Copson function spaces* $\mathrm{Cop}_p = \mathrm{Cop}_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0,\infty)$ or $I = [0,1]$ such that

$$\|f\|_{\mathrm{Cop}(p)} = \left[\int_0^\infty \left(\int_x^\infty \frac{|f(t)|}{t}\,dt\right)^p dx\right]^{1/p} < \infty \quad \text{for } I = [0,\infty),$$

and

$$\|f\|_{\mathrm{Cop}(p)} = \left[\int_0^1 \left(\int_x^1 \frac{|f(t)|}{t}\,dt\right)^p dx\right]^{1/p} < \infty \quad \text{for } I = [0,1].$$

Sometimes we will use the *Cesàro operators*

$$C_d x(n) = \frac{1}{n}\sum_{k=1}^n |x_k|, \qquad Cf(x) = \frac{1}{x}\int_0^x |f(t)|\,dt$$

and the *Copson operators*

$$C_d^* x(n) = \sum_{k=n}^\infty \frac{|x_k|}{k}, \qquad C^* f(x) = \int_{(x,\infty)\cap I} \frac{|f(t)|}{t}\,dt$$

related to appropriate spaces. Then ces_p (resp. cop_p) consists of all real sequences $x = \{x_k\}$ such that $C_d x \in l_p$ (resp. $C_d^* x \in l_p$), and $\mathrm{Ces}_p(I)$ (resp. $\mathrm{Cop}_p(I)$) consists of all classes of Lebesgue measurable real functions f on I such that $Cf \in L_p(I)$ (resp. $C^* f \in L_p(I)$) with natural norms. By the Copson inequalities (cf. [HLP, Theorems 328 and 331], [Be, p. 25] and [KMP, p. 159]), valid for $1 \le p < \infty$, we have $\|C_d^* x\|_{l_p} \le p\|x\|_{l_p}$ for $x \in l_p$ and $\|C^* f\|_{L_p(I)} \le p\|f\|_{L_p(I)}$ for $f \in L_p(I)$. Therefore, $l_p \overset{p}{\hookrightarrow} \mathrm{cop}_p$ and $L_p \overset{p}{\hookrightarrow} \mathrm{Cop}_p$.

We can define similarly the spaces cop_∞ and Cop_∞ but it is easy to see that $\mathrm{cop}_\infty = l_1(1/k)$ and $\mathrm{Cop}_\infty = L_1(1/t)$. Moreover, for $I = [0, 1]$ we have $L_p \overset{p}{\hookrightarrow} \mathrm{Cop}_p \overset{1}{\hookrightarrow} \mathrm{Cop}_1 = L_1$.

We will also consider more general Cesàro spaces $\mathrm{Ces}_E(I)$, where E is a Banach function space on I with the natural norm $\|f\|_{\mathrm{Ces}(E)} = \|Cf\|_E$.

For a Banach function space E on $I = [0, \infty)$ the *down space* $E^{\downarrow}$ is the collection of all $f \in L^0$ such that

$$\|f\|_{E^{\downarrow}} = \sup \int_I |f(t)| g(t)\, dt < \infty,$$

where the supremum is taken over all non-negative, non-increasing Lebesgue measurable functions g from the Köthe dual E' of E such that $\|g\|_{E'} \le 1$. Let us recall that the *Köthe dual* of a Banach function space E is defined as

$$E' = \Big\{ f \in L^0 : \|f\|_{E'} = \sup_{\|g\|_E \le 1} \int_I |f(t) g(t)|\, dt < \infty \Big\}.$$

It is routine to check that the space $E^{\downarrow}$ has the *Fatou property*, that is, if $0 \le f_n$ increases to f a.e. on I and $\sup_{n \in \mathbb{N}} \|f_n\|_{E^{\downarrow}} < \infty$, then $f \in E^{\downarrow}$ and $\|f_n\|_{E^{\downarrow}}$ increases to $\|f\|_{E^{\downarrow}}$. Moreover, $E'' \overset{1}{\hookrightarrow} E^{\downarrow}$, where E'' is the second Köthe dual of E. Recall also that a Banach function space E has the Fatou property if and only if $E = E''$ with equality of norms.

Sinnamon [Si01, Theorem 3.1] proved that if E is a symmetric space on $I = [0, \infty)$, then $\|f\|_{E^{\downarrow}} \approx \|Cf\|_E$ if and only if the Cesàro operator $C : E \to E$ is bounded. In particular, then $E^{\downarrow} = \mathrm{Ces}_E$. Moreover, $(L_1)^{\downarrow} = L_1$ since

$$\|f\|_{L_1^{\downarrow}} = \sup_{0 \le g} \frac{\int_0^\infty |f(t)| g(t)\, dt}{\|g\|_{L_\infty}} \ge \sup_{0 \le g \downarrow} \frac{\int_0^\infty |f(t)| g(t)\, dt}{\|g\|_{L_\infty}} \ge \frac{\int_0^\infty |f(t)|\, dt}{\|1\|_{L_\infty}} = \|f\|_{L_1}$$

(cf. [MS, p. 194]).

The paper is organized as follows. In Section 2 we prove that the Cesàro and Copson sequence and function spaces on $[0, \infty)$ are interpolation spaces obtained by the K-method from weighted L_1-spaces. At the same time, in the case of $I = [0, 1]$, only the Copson spaces can be described as interpolation spaces with respect to the analogous couple of weighted L_1-spaces (see Theorem 1(iii)). In particular, we obtain a new description of the interpolation spaces $(L_1, L_1(1/t))_{1-1/p,p}$ in the off-diagonal case both for $I = [0,\infty)$ and $I = [0,1]$.

In Section 3 it is shown that the Cesàro function spaces $\mathrm{Ces}_p[0,\infty)$, $1 < p < \infty$, can be obtained by the K-method of interpolation also from the couple $(L_1[0,\infty), \mathrm{Ces}_\infty[0,\infty))$. Hence, applying the reiteration theorem, we conclude that $\mathrm{Ces}_p[0,\infty)$ is an interpolation space with respect to the couple $(\mathrm{Ces}_{p_0}[0,\infty), \mathrm{Ces}_{p_1}[0,\infty))$ for arbitrary $1 < p_0 < p_1 \le \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$.

In Section 4 interpolation of Cesàro function spaces on $[0,1]$ is investigated. We prove that for $1 < p < \infty$,

$$(L_1(1-t)[0,1], \mathrm{Ces}_\infty[0,1])_{\theta,p} = \mathrm{Ces}_p[0,1] \quad \text{with } \theta = 1 - 1/p.$$

As a consequence of this result and the reiteration equality (1.2), we infer

$$(\mathrm{Ces}_{p_0}[0,1], \mathrm{Ces}_{p_1}[0,1])_{\theta,p} = \mathrm{Ces}_p[0,1] \tag{1.5}$$

for all $1 < p_0 < p_1 \le \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$.

We are also interested in description of interpolation spaces between $\mathrm{Ces}_1[0,1]$ and $\mathrm{Ces}_\infty[0,1]$. In Section 5, in Theorem 3, we find an equivalent expression for the K-functional with respect to this couple and then in Section 6 we prove that the real interpolation spaces

$$(\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])_{1-1/p,p}$$

for $1 < p < \infty$ can be identified with the weighted Cesàro function spaces $\mathrm{Ces}_p(\ln(e/t))[0,1]$.

Finally, in Section 7, we show in Theorem 6 that $\mathrm{Ces}_p[0,1]$ for $1 < p < \infty$ are not interpolation spaces between $\mathrm{Ces}_1[0,1]$ and $\mathrm{Ces}_\infty[0,1]$.

2. Cesàro and Copson spaces as interpolation spaces with respect to weighted L_1-spaces. We start with the main result in this part.

THEOREM 2.1.

(i) *If* $1 < p < \infty$, *then*

$$(l_1, l_1(1/k))_{1-1/p,p} = \mathrm{ces}_p = \mathrm{cop}_p.$$

(ii) *If* $I = [0,\infty)$ *and* $1 < p < \infty$, *then*

$$(L_1, L_1(1/t))_{1-1/p,p} = \mathrm{Ces}_p = \mathrm{Cop}_p.$$

(iii) *If* $I = [0,1]$ *and* $1 < p \le \infty$, *then*

$$(L_1, L_1(1/t))_{1-1/p,p} = \mathrm{Cop}_p.$$

Moreover, $\mathrm{Cop}_p \overset{p'}{\hookrightarrow} \mathrm{Ces}_p$ *and the reverse imbedding does not hold.*

Proof. (i) If $f \in l_1 + l_1(1/k)$, then $K(t,x;l_1,l_1(1/k)) = t\sum_{k=1}^{\infty}|x_k|/k$ for $0 < t \le 1$, and

$$K(t,x;l_1,l_1(1/k)) = \sum_{k=1}^{\infty}|x_k|\min(1,t/k) = \sum_{k=1}^{[t]}|x_k| + t\sum_{k=[t]+1}^{\infty}\frac{|x_k|}{k}.$$

for $t \ge 1$. Therefore, for $n \le t < n+1$ $(n \ge 1)$, we have

$$\frac{K(t,x;l_1,l_1(1/k))}{t} \le \frac{1}{n}\sum_{k=1}^{n}|x_k| + \sum_{k=n+1}^{\infty}\frac{|x_k|}{k} = C_d x(n) + C_d^* x(n+1).$$

Since

$$\begin{aligned} C_d C_d^* x(n) &= \frac{1}{n}\sum_{m=1}^{n}\Big(\sum_{k=m}^{\infty}\frac{|x_k|}{k}\Big) \\ &= \frac{1}{n}\Big[\sum_{k=1}^{n}\Big(\sum_{m=1}^{k}\frac{|x_k|}{k}\Big) + \sum_{k=n+1}^{\infty}\Big(\sum_{m=1}^{n}\frac{|x_k|}{k}\Big)\Big] \\ &= \frac{1}{n}\sum_{k=1}^{n}|x_k| + \sum_{k=n+1}^{\infty}\frac{|x_k|}{k} = C_d x(n) + C_d^* x(n+1), \end{aligned}$$

it follows that, for $n \le t < n+1$ $(n \ge 1)$,

$$\frac{K(t,x;l_1,l_1(1/k))}{t} \le C_d C_d^* x(n).$$

Using the classical Hardy inequality (cf. [HLP, Theorem 326] or [KMP, Theorem 1]), we obtain

$$\begin{aligned} \|x\|_{1-1/p,p} &= \Big(\int_0^{\infty}\Big(\frac{K(t,x;l_1,l_1(1/k))}{t}\Big)^p dt\Big)^{1/p} \\ &= \Big[C_d^* x(1)^p + \sum_{n=1}^{\infty}\int_n^{n+1}\Big(\frac{K(t,x)}{t}\Big)^p dt\Big]^{1/p} \\ &\le \Big[C_d^* x(1)^p + \sum_{n=1}^{\infty}(C_d C_d^* x(n))^p\Big]^{1/p} \\ &\le C_d^* x(1) + \|C_d C_d^* x\|_{l_p} \le C_d^* x(1) + p'\|C_d^* x\|_{l_p} \\ &\le (p'+1)\|C_d^* x\|_{l_p} = (p'+1)\|x\|_{\mathrm{cop}(p)}. \end{aligned}$$

This means that $\mathrm{cop}_p \hookrightarrow (l_1, l_1(1/k))_{1-1/p,p}$. On the other hand, for $n \le t < n+1$ $(n \ge 1)$, we have

$$\frac{K(t,x;l_1,l_1(1/k))}{t} \ge \sum_{k=n+1}^{\infty}\frac{|x_k|}{k} = C_d^* x(n+1)$$

and

$$\|x\|_{1-1/p,p} = \Big(\int_0^\infty \Big(\frac{K(t,x;l_1,l_1(1/k))}{t}\Big)^p dt\Big)^{1/p}$$
$$\geq \Big(C_d^*x(1)^p + \sum_{n=1}^\infty C_d^*x(n+1)^p\Big)^{1/p} = \|C_d^*x\|_{l_p} = \|x\|_{\mathrm{cop}(p)},$$

which gives the reverse imbedding $(l_1, l_1(1/k))_{1-1/p,p} \overset{1}{\hookrightarrow} \mathrm{cop}_p$. The equality $\mathrm{ces}_p = \mathrm{cop}_p$, $1 < p < \infty$, was proved by Bennett (cf. [Be, Theorems 4.5 and 6.6]).

(ii) For $f \in L_1 + L_1(1/s) = L_1(\min(1, 1/s))$ we have

$$K(t,f;L_1,L_1(1/s)) = \int_0^\infty |f(s)| \min(1,t/s)\, ds$$
$$= \int_0^t |f(s)|\, ds + t\int_t^\infty \frac{|f(s)|}{s}\, ds.$$

Thus,

$$\frac{K(t,f;L_1,L_1(1/s))}{t} = Cf(t) + C^*f(t), \qquad t > 0,$$

and therefore

$$\|f\|_{1-1/p,p} = \Big(\int_0^\infty \Big(\frac{K(t,f;L_1,L_1(1/s))}{t}\Big)^p dt\Big)^{1/p} = \|Cf + C^*f\|_{L_p(0,\infty)}. \tag{2.1}$$

Since, by the Fubini theorem,

$$C^*Cf(t) = \int_t^\infty \Big(\frac{1}{u^2}\int_0^u |f(s)|\, ds\Big) du$$
$$= \int_0^t \Big(\int_t^\infty \frac{1}{u^2}\, du\Big)|f(s)|\, ds + \int_t^\infty \Big(\int_s^\infty \frac{1}{u^2}\, du\Big)|f(s)|\, ds$$
$$= \frac{1}{t}\int_0^t |f(s)|\, ds + \int_t^\infty \frac{|f(s)|}{s}\, ds = Cf(t) + C^*f(t),$$

from the Copson inequality (cf. [HLP, Theorem 328]) it follows that

$$\|f\|_{\mathrm{Ces}(p)} = \|Cf\|_{L_p(0,\infty)} \leq \|Cf + C^*f\|_{L_p(0,\infty)}$$
$$= \|C^*Cf\|_{L_p(0,\infty)} \leq p\|Cf\|_{L_p(0,\infty)} = p\|f\|_{\mathrm{Ces}(p)}.$$

Combining this with (2.1), we obtain $\|f\|_{1-1/p,p} \approx \|f\|_{\mathrm{Ces}(p)}$.

On the other hand, since

$$CC^*f(t) = \frac{1}{t}\int_0^t \Big(\int_u^\infty \frac{|f(s)|}{s}\,ds\Big)\,du$$
$$= \frac{1}{t}\int_0^t \Big(\int_0^s du\Big)\frac{|f(s)|}{s}\,ds + \frac{1}{t}\int_t^\infty \Big(\int_0^t du\Big)\frac{|f(s)|}{s}\,ds$$
$$= \frac{1}{t}\int_0^t |f(s)|\,ds + \int_t^\infty \frac{|f(s)|}{s}\,ds = Cf(t) + C^*f(t),$$

by the Hardy inequality,

$$\|f\|_{\mathrm{Cop}(p)} = \|C^*f\|_{L_p(0,\infty)} \le \|Cf + C^*f\|_{L_p(0,\infty)}$$
$$= \|CC^*f\|_{L_p(0,\infty)} \le p'\|C^*f\|_{L_p(0,\infty)} = p'\|f\|_{\mathrm{Cop}(p)},$$

and, applying (2.1) once more, we conclude that $\|f\|_{1-1/p,p} \approx \|f\|_{\mathrm{Cop}(p)}$.

(iii) For $I = [0,1]$ and $f \in L_1 + L_1(1/s) = L_1$ we have $K(t, f; L_1, L_1(1/s)) = \|f\|_1$ if $t \ge 1$, and

$$K(t, f; L_1, L_1(1/s)) = \int_0^t |f(s)|\,ds + t\int_t^1 \frac{|f(s)|}{s}\,ds = tCf(t) + tC^*f(t)$$

if $0 < t \le 1$. Therefore, for $1 < p < \infty$,

$$\|f\|_{1-1/p,p} = \Big(\int_0^1 [Cf(t) + C^*f(t)]^p\,dt + \int_1^\infty t^{-p}\|f\|_1^p\,dt\Big)^{1/p}$$
$$= \Big(\|Cf + C^*f\|_p^p + \frac{1}{p-1}\|f\|_1^p\Big)^{1/p}.$$

Firstly, the last expression is not smaller than $\|C^*f\|_p = \|f\|_{\mathrm{Cop}(p)}$. On the other hand, since again $CC^*f(t) = Cf(t) + C^*f(t)$, by the Hardy inequality, it follows that

$$\|f\|_{1-1/p,p} = \Big(\|CC^*f\|_p^p + \frac{1}{p-1}\|f\|_1^p\Big)^{1/p} \le \|CC^*f\|_p + (p-1)^{-1/p}\|f\|_1$$
$$\le p'\|C^*f\|_p + (p-1)^{-1/p}\|f\|_{\mathrm{Cop}(p)} = (p' + (p-1)^{-1/p})\|f\|_{\mathrm{Cop}(p)}.$$

Thus, $(L_1, L_1(1/t))_{1-1/p,p} = \mathrm{Cop}_p$ with equivalent norms for $1 < p < \infty$. For $p = \infty$ we have $(L_1, L_1(1/t))_{1,\infty} = L_1(1/t) = \mathrm{Cop}_\infty[0,1]$.

The imbedding $\mathrm{Cop}_p \overset{p'}{\hookrightarrow} \mathrm{Ces}_p$ for $1 < p \le \infty$ follows from the inequality

$$\|f\|_{\mathrm{Ces}(p)} = \|Cf\|_p \le \|Cf + C^*f\|_p = \|CC^*f\|_p \le p'\|C^*f\|_p = p'\|f\|_{\mathrm{Cop}(p)}.$$

Moreover, $\mathrm{Ces}_p[0,1] \cap L_1[0,1] \overset{p+1}{\hookrightarrow} \mathrm{Cop}_p[0,1]$ for $1 \le p < \infty$. In fact, observe that in the case of $I = [0,1]$ the composition operator C^*C has an additional term. More precisely,

$$C^*Cf(t) = Cf(t) + C^*f(t) - \int_0^1 |f(s)|\,ds.$$

Therefore,

$$\begin{aligned}\|f\|_{\mathrm{Cop}(p)} &= \|C^*f\|_p \le \|Cf + C^*f\|_p \\ &= \|C^*Cf + \int_0^1 |f(s)|\,ds\|_p \le \|C^*Cf\|_p + \|f\|_1 \\ &\le p\|Cf\|_p + \|f\|_1 \le (p+1)\max(\|f\|_{\mathrm{Ces}(p)}, \|f\|_1).\end{aligned}$$

Finally, let us show that $\mathrm{Ces}_p \not\hookrightarrow \mathrm{Cop}_p$ by comparing norms of the functions $f_h(t) = \frac{1}{\sqrt{1-t}}\chi_{[h,1)}(t)$, $0 < h < 1$, in these spaces. We have

$$C^*(f_h)(t) = \begin{cases} \int_h^1 \frac{1}{s\sqrt{1-s}}\,ds & \text{if } 0 < t \le h, \\ \int_t^1 \frac{1}{s\sqrt{1-s}}\,ds & \text{if } h \le t \le 1, \end{cases}$$

and

$$\begin{aligned}\|f_h\|^p_{\mathrm{Cop}(p)} = \|C^*(f_h)\|_p^p &\ge \int_0^h \Big(\int_h^1 \frac{1}{s\sqrt{1-s}}\,ds\Big)^p dt = h\Big(\int_h^1 \frac{1}{s\sqrt{1-s}}\,ds\Big)^p \\ &\ge h\Big(\int_h^1 \frac{1}{\sqrt{1-s}}\,ds\Big)^p = 2^p h(1-h)^{p/2}.\end{aligned}$$

Also,

$$C(f_h)(t) = \begin{cases} 0 & \text{if } 0 < t \le h, \\ (2/t)(\sqrt{1-h} - \sqrt{1-t}) & \text{if } h \le t \le 1, \end{cases}$$

and

$$\begin{aligned}\|f_h\|^p_{\mathrm{Ces}(p)} = \|C(f_h)\|_p^p &= 2^p \int_h^1 \Big(\frac{\sqrt{1-h} - \sqrt{1-t}}{t}\Big)^p dt \\ &\le 2^p \int_h^1 \frac{(1-h)^{p/2}}{t^p}\,dt = 2^p(1-h)^{p/2}\frac{1-h^{p-1}}{(p-1)h^{p-1}}.\end{aligned}$$

Thus,

$$\frac{\|f_h\|^p_{\mathrm{Cop}(p)}}{\|f_h\|^p_{\mathrm{Ces}(p)}} \ge \frac{2^p h(1-h)^{p/2}(p-1)h^{p-1}}{2^p(1-h)^{p/2}(1-h^{p-1})} = (p-1)\frac{h^p}{1-h^{p-1}} \to \infty \ \text{ as } h \to 1^+,$$

and the proof is complete. ∎

REMARK 2.2. Alternatively, the space ces_p for $1 < p < \infty$ can be obtained as an interpolation space with respect to $(l_1, l_1(2^{-n}))$ by the so-called K^+-method, a version of the standard K-method, precisely, $\mathrm{ces}_p = (l_1, l_1(2^{-n}))^{K^+}_{l_p(1/n)}$ (cf. [CFM, proof of Theorem 6.4]).

REMARK 2.3. The results in Theorem 2.1 give a description of the real interpolation spaces $(L_1, L_1(1/t))_{1-1/p,p}$ in the off-diagonal case. Before it was only known that they are intersections of weighted $L_1(w)$-spaces with the weights w from certain sets (cf. [Gi, Theorem 4.1], [MP, Theorem 2]) or some block spaces (cf. [AKMNP, Lemma 3.1]).

The following corollary follows directly from Theorem 2.1, the reiteration formula (1.2) and the equalities $\mathrm{Cop}_\infty[0,1] = L_1(1/t)$ and $\mathrm{Cop}_1[0,1] = L_1$.

COROLLARY 2.4. *If* $1 < p_0 < p_1 < \infty$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$ *with* $0 < \theta < 1$, *then*

$$(\mathrm{ces}_{p_0}, \mathrm{ces}_{p_1})_{\theta,p} = \mathrm{ces}_p, \quad (\mathrm{Ces}_{p_0}[0,\infty), \mathrm{Ces}_{p_1}[0,\infty))_{\theta,p} = \mathrm{Ces}_p[0,\infty). \tag{2.2}$$

If $1 \le p_0 < p_1 \le \infty$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$ *with* $0 < \theta < 1$, *then*

$$(\mathrm{Cop}_{p_0}[0,1], \mathrm{Cop}_{p_1}[0,1])_{\theta,p} = \mathrm{Cop}_p[0,1]. \tag{2.3}$$

REMARK 2.5. A different proof of the second equality in (2.2) was given by Sinnamon [Si91, Corollary 2].

3. Cesàro spaces on $[0,\infty)$ as interpolation spaces with respect to $(L_1, \mathrm{Ces}_\infty)$. All the spaces considered in this part are on the interval $I = [0,\infty)$. By [MS, p. 194] we have $D^\infty := (L_\infty)^\downarrow = \mathrm{Ces}_\infty$ isometrically. On the other hand, for a Banach lattice F with the Fatou property we have $F \in \mathrm{Int}(L_1, D^\infty) = \mathrm{Int}(L_1, \mathrm{Ces}_\infty)$ if and only if $F = E^\downarrow$ with equality of norms for some $E \in \mathrm{Int}(L_1, L_\infty)$ (see [MS, Theorem 6.4]). Then, in particular, $L_p^\downarrow \in \mathrm{Int}(L_1, \mathrm{Ces}_\infty)$. Since the operator C is bounded in L_p for $1 < p \le \infty$, by [Si01, Theorem 3.1] it follows that

$$\|f\|_{L_p^\downarrow} = \big\||f|\big\|_{L_p^\downarrow} \approx \|Cf\|_{L_p} = \|f\|_{\mathrm{Ces}(p)}.$$

Thus, for any $1 < p < \infty$ we have $\mathrm{Ces}_p \in \mathrm{Int}(L_1, \mathrm{Ces}_\infty)$ and $\mathrm{Ces}_p = L_p^\downarrow$.

Moreover, by using Theorem 6.4 from [MS], it is easy to prove the following more precise and general assertion.

PROPOSITION 3.1. *Let* $E, F \in \mathrm{Int}(L_1, L_\infty)$ *and* Φ *be an interpolation Banach lattice with respect to the couple* $(L_\infty, L_\infty(1/u))$ *on* $(0,\infty)$. *Then*

$$(E^\downarrow, F^\downarrow)^K_\Phi = [(E,F)^K_\Phi]^\downarrow. \tag{3.1}$$

In particular, if $1 < p < \infty$, *then*

$$(L_1, \mathrm{Ces}_\infty)_{1-1/p,p} = \mathrm{Ces}_p. \tag{3.2}$$

Proof. Firstly, since the Banach couple (L_1, L_∞) is K-monotone [KPS, Theorem 2.4.3], by the assumption and the Brudnyĭ–Krugljak theorem (cf. [BK, Theorem 4.4.5]), $E = (L_1, L_\infty)^K_{\Phi_0}$ and $F = (L_1, L_\infty)^K_{\Phi_1}$ with some interpolation Banach lattices Φ_0 and Φ_1 with respect to the couple $(L_\infty, L_\infty(1/u))$ on $(0, \infty)$. Applying the reiteration theorem for the general K-method (see [BK, Theorem 3.3.11]), we obtain

$$(E, F)^K_\Phi = ((L_1, L_\infty)^K_{\Phi_0}, (L_1, L_\infty)^K_{\Phi_1})^K_\Phi = (L_1, L_\infty)^K_\Psi,$$

where $\Psi = (\Phi_0, \Phi_1)^K_\Phi$. Moreover, from the proof of Theorem 6.4 in [MS] and the equality $L_1^\downarrow = L_1$ (see Section 1) it follows that

$$E^\downarrow = [(L_1, L_\infty)^K_{\Phi_0}]^\downarrow = (L_1, D^\infty)^K_{\Phi_0}, F^\downarrow = [(L_1, L_\infty)^K_{\Phi_1}]^\downarrow = (L_1, D^\infty)^K_{\Phi_1}$$

and

$$[(E, F)^K_\Phi]^\downarrow = [(L_1, L_\infty)^K_\Psi]^\downarrow = (L_1, D^\infty)^K_\Psi.$$

Therefore, using the reiteration theorem once again, we obtain

$$(E^\downarrow, F^\downarrow)^K_\Phi = ((L_1, D^\infty)^K_{\Phi_0}, (L_1, D^\infty)^K_{\Phi_1})^K_\Phi = (L_1, D^\infty)^K_\Psi = [(E, F)^K_\Phi]^\downarrow.$$

and equality (3.1) is proved. In particular, from (3.1) and the well-known identification formula $(L_1, L_\infty)_{1-1/p,p} = L_p$ [BL, Theorem 5.2.1] it follows that

$$(L_1, \mathrm{Ces}_\infty)_{1-1/p,p} = (L_1^\downarrow, L_\infty^\downarrow)_{1-1/p,p} = L_p^\downarrow = \mathrm{Ces}_p,$$

and equality (3.2) is also proved. ∎

For a given symmetric space E on $I = [0, \infty)$ the Cesàro function space Ces_E is defined by the norm $\|f\|_{\mathrm{Ces}(E)} = \|Cf\|_E$. If the operator C is bounded in E, then, by [Si01, Theorem 3.1], $\mathrm{Ces}_E = E^\downarrow$. Therefore, applying Proposition 3.1, we obtain

Corollary 3.2. *Let the operator C be bounded in symmetric spaces E and F on $[0, \infty)$ and let Φ be an interpolation Banach lattice with respect to the couple $(L_\infty, L_\infty(1/u))$ on $(0, \infty)$. Then*

$$(\mathrm{Ces}_E, \mathrm{Ces}_F)^K_\Phi = \mathrm{Ces}_{(E,F)^K_\Phi}.$$

In particular, for any $1 < p_0 < p_1 \le \infty$,

$$(\mathrm{Ces}_{p_0}, \mathrm{Ces}_{p_1})_{\theta,p} = \mathrm{Ces}_p, \text{ where } 0 < \theta < 1 \text{ and } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{3.3}$$

Remark 3.3. If $1 < p < \infty$, then the restriction of the space $\mathrm{Ces}_p[0, \infty)$ to the interval $[0, 1]$ coincides with $\mathrm{Ces}_p[0, 1] \cap L_1[0, 1]$ (cf. [AM, Remark 5]). Therefore, if we "restrict" (3.3) to $[0, 1]$ we obtain only

$$(\mathrm{Ces}_{p_0}[0, 1] \cap L_1[0, 1], \mathrm{Ces}_{p_1}[0, 1] \cap L_1[0, 1])_{\theta,p} = \mathrm{Ces}_p[0, 1] \cap L_1[0, 1],$$

where $1 < p_0 < p_1 < \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$.

4. Cesàro spaces on $[0,1]$ as interpolation spaces with respect to $(L_1(1-t),\mathrm{Ces}_\infty)$. In contrast to the case of $[0,\infty)$, $\mathrm{Ces}_p[0,1]$ for $1\le p<\infty$ is not even an intermediate space between $L_1[0,1]$ and $\mathrm{Ces}_\infty[0,1]$. In fact, $\mathrm{Ces}_\infty[0,1]\hookrightarrow L_1[0,1]$, but it is easy to show that $\mathrm{Ces}_p[0,1]\not\subset L_1[0,1]$ for every $1\le p<\infty$.

On the other hand, from the inequality $1-u\le \ln(1/u)$ $(0<u\le 1)$ it follows that $\mathrm{Ces}_p[0,1]$, $1\le p<\infty$, is an intermediate space between $L_1(1-t)[0,1]$ and $\mathrm{Ces}_\infty[0,1]$, because

$$\mathrm{Ces}_\infty[0,1]\overset{1}{\hookrightarrow}\mathrm{Ces}_p[0,1]\overset{1}{\hookrightarrow}\mathrm{Ces}_1[0,1]=L_1(\ln(1/t))[0,1]\overset{1}{\hookrightarrow}L_1(1-t)[0,1].$$

THEOREM 4.1. *If $1<p<\infty$, then*

$$(\mathrm{Ces}_1[0,1],\mathrm{Ces}_\infty[0,1])_{1-1/p,p}\overset{1}{\hookrightarrow}\mathrm{Ces}_p[0,1] \tag{4.1}$$

and

$$(L_1(1-t)[0,1],\mathrm{Ces}_\infty[0,1])_{1-1/p,p}=\mathrm{Ces}_p[0,1]. \tag{4.2}$$

Proof. All function spaces in this proof are considered on $I=[0,1]$ unless indicated otherwise.

First, for any $f\in\mathrm{Ces}_1$ and all $0<t\le 1$ we have

$$K(t,f):=K(t,f;\mathrm{Ces}_1,\mathrm{Ces}_\infty)\ge\int_0^t (Cf)^*(s)\,ds. \tag{4.3}$$

In fact, we can assume that $f\ge 0$. If $f=g+h$, $g\ge 0$, $h\ge 0$, $g\in\mathrm{Ces}_1$, $h\in\mathrm{Ces}_\infty$, then $Cf=Cg+Ch$, and therefore, by (1.3),

$$\begin{aligned}\|g\|_{\mathrm{Ces}(1)}+t\|h\|_{\mathrm{Ces}(\infty)}&=\|Cg\|_{L_1}+t\|Ch\|_{L_\infty}\\&\ge\inf\{\|y\|_{L_1}+t\|z\|_{L_\infty}:Cf=y+z,\ y\in L_1,\ z\in L_\infty\}\\&=K(t,Cf;L_1,L_\infty)=\int_0^t(Cf)^*(s)\,ds.\end{aligned}$$

Taking the infimum over all suitable g and h we get (4.3).

Next, by the definition of real interpolation spaces, we obtain

$$\begin{aligned}\|f\|^p_{1-1/p,p}&\ge\int_0^1[t^{1/p-1}K(t,f)]^p\,\frac{dt}{t}=\int_0^1 t^{-p}K(t,f)^p\,dt\\&\ge\int_0^1 t^{-p}\Big[\int_0^t(Cf)^*(s)\,ds\Big]^p\,dt\ge\|Cf\|^p_{L_p[0,1]}=\|f\|^p_{\mathrm{Ces}(p)},\end{aligned}$$

and the proof of (4.1) is complete.

Before proceeding with the proof of (4.2) we introduce the following notation: for a Banach function space E on $I=[0,\infty)$ or $[0,1]$ and any set

$A \subset I$, by $E|_A$ we denote the subspace of E which consists of all functions f such that $\operatorname{supp} f \subset A$. Let also $X_p := (L_1(1-t), \mathrm{Ces}_\infty)_{1-1/p,p}$. Since

$$\|f\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{X_p} + \|f\chi_{[1/2,1]}\|_{X_p},$$

to prove (4.2) it is sufficient to check that

$$\|f\chi_{[0,1/2]}\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{\mathrm{Ces}(p)}, \tag{4.4}$$

$$\|f\chi_{[1/2,1]}\|_{X_p} \approx \|f\chi_{[1/2,1]}\|_{\mathrm{Ces}(p)}. \tag{4.5}$$

Firstly, since $L_1(1-t)|_{[0,1/2]} = L_1[0,\infty)|_{[0,1/2]}$ and

$$\mathrm{Ces}_\infty|_{[0,1/2]} = \mathrm{Ces}_\infty[0,\infty)|_{[0,1/2]},$$

by Proposition 3.1 (see (3.2)) we obtain

$$\|f\chi_{[0,1/2]}\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{(L_1[0,\infty),\mathrm{Ces}_\infty[0,\infty))_{1-1/p,p}} \approx \|f\chi_{[0,1/2]}\|_{\mathrm{Ces}_p[0,\infty)}. \tag{4.6}$$

Note that

$$\mathrm{Ces}_p[0,\infty)|_{[0,1/2]} = \mathrm{Ces}_p[0,1]|_{[0,1/2]} \tag{4.7}$$

with equivalence of norms. In fact, by [AM, Remark 5], $\mathrm{Ces}_p[0,\infty)|_{[0,1]} = \mathrm{Ces}_p \cap L_1$. If $\operatorname{supp} g \subset [0,1/2]$, then

$$\|g\|_{L_1} = \int_0^{1/2} |g(s)|\,ds \le 2^{1/p}\bigg(\int_{1/2}^1 \bigg(\frac{1}{t}\int_0^t |g(s)|\,ds\bigg)^p dt\bigg)^{1/p} \le 2^{1/p}\|g\|_{\mathrm{Ces}(p)}.$$

Combining this with the previous equality, we obtain (4.7). Clearly, (4.4) is an immediate consequence of (4.7) and (4.6).

Now, we prove (4.5). Since $(L_1(1-s)|_{[1/2,1]}, \mathrm{Ces}_\infty|_{[1/2,1]})$ is a complemented subcouple of $(L_1(1-s), \mathrm{Ces}_\infty)$, the well-known result of Baouendi and Goulaouic [BG, Theorem 1], valid for all interpolation methods (see also [Tr, Theorem 1.17.1]), yields

$$\|f\chi_{[1/2,1]}\|_{X_p} \approx \|f\chi_{[1/2,1]}\|_{Y_p},$$

where $Y_p := (L_1(1-s)|_{[1/2,1]}, \mathrm{Ces}_\infty|_{[1/2,1]})_{1-1/p,p}$. To prove (4.5) it is sufficient to show that

$$Y_p = \mathrm{Ces}_p|_{[1/2,1]}. \tag{4.8}$$

On the one hand, since $1-u \le \ln(1/u) \le 2(1-u)$ for all $1/2 \le u \le 1$ and $\mathrm{Ces}_1 = L_1(\ln(1/s))$, we have $\mathrm{Ces}_1|_{[1/2,1]} = L_1(1-s)|_{[1/2,1]}$, and, by the imbedding (4.1), we obtain

$$Y_p = (\mathrm{Ces}_1|_{[1/2,1]}, \mathrm{Ces}_\infty|_{[1/2,1]})_{1-1/p,p} \subset \mathrm{Ces}_p|_{[1/2,1]}.$$

To prove the opposite imbedding we note, firstly, that for any function h

with $\operatorname{supp} h \subset [1/2,1]$ we have

$$\|h\|_{\mathrm{Ces}(\infty)} = \sup_{1/2\le x\le 1} \frac{1}{x}\int_{1/2}^{x} |h(s)|\,ds,$$

whence

$$\|h\|_{L_1} = \int_{1/2}^{1} |h(s)|\,ds \le \|h\|_{\mathrm{Ces}(\infty)} \le 2\int_{1/2}^{1} |h(s)|\,ds = 2\|h\|_{L_1}.$$

Therefore, using the formula for the K-functional with respect to a couple of weighted L_1-spaces (see (1.4)), we obtain

$$G(t,h) \le K(t,h; L_1(1-s)|_{[1/2,1]}, \mathrm{Ces}_\infty|_{[1/2,1]}) \le 2G(t,h), \tag{4.9}$$

where

$$G(t,h) = K(t,h; L_1(1-s)|_{[1/2,1]}, L_1|_{[1/2,1]}) = \int_{1/2}^{1} \min(1-s,t)|h(s)|\,ds.$$

Furthermore, let $h \in L_1|_{[1/2,1]}$. Then

$$Ch(s) = \frac{1}{s}\int_{1/2}^{s} |h(u)|\,du \ge \int_{1/2}^{s} |h(u)|\,du,$$

whence

$$(Ch)^*(s) \ge \int_{1/2}^{1-s} |h(u)|\,du, \qquad 0 < s \le 1.$$

Therefore, for all $0 \le t \le 1$, we obtain

$$\begin{aligned}\int_0^t (Ch)^*(s)\,ds &\ge \int_0^t \Big(\int_{1/2}^{1-s} |h(u)|\,du\Big)\,ds\\ &= \int_{1/2}^{1-t}\Big(\int_0^t |h(u)|\,ds\Big)\,du + \int_{1-t}^{1}\Big(\int_0^{1-u} |h(u)|\,ds\Big)\,du\\ &= t\int_{1/2}^{1-t} |h(u)|\,du + \int_{1-t}^{1} (1-u)|h(u)|\,du = G(t,h).\end{aligned}$$

From this inequality and the definition of $G(t,h)$ it follows that

$$\int_0^{\min(1,t)} (Ch)^*(s)\,ds \ge G(t,h)$$

for all $t > 0$. Hence, by (4.9) and the classical Hardy inequality, for every

$h \in \mathrm{Ces}_p$ with $\operatorname{supp} h \subset [1/2,1]$ we have

$$\begin{aligned}
\|h\|_{Y_p} &= \Big(\int_0^\infty t^{-p} K(t,h; L_1(1-s)|_{[1/2,1]}, \mathrm{Ces}_\infty|_{[1/2,1]})^p \, dt\Big)^{1/p} \\
&\le 2\Big(\int_0^\infty t^{-p} G(t,h)^p \, dt\Big)^{1/p} \\
&\le 2\Big(\int_0^\infty t^{-p} \Big(\int_0^{\min(1,t)} (Ch)^*(s)\, ds\Big)^p \, dt\Big)^{1/p} \\
&\le 2\Big(\int_0^1 t^{-p} \Big(\int_0^t (Ch)^*(s)\, ds\Big)^p \, dt\Big)^{1/p} + 2\Big(\int_1^\infty t^{-p} \Big(\int_0^1 (Ch)^*(s)\, ds\Big)^p \, dt\Big)^{1/p} \\
&\le 2\Big[\frac{p}{p-1} \|Ch\|_{L_p[0,1]} + \frac{1}{(p-1)^{1/p}} \|Ch\|_{L_1[0,1]}\Big] \le \frac{4p}{p-1} \|h\|_{\mathrm{Ces}_p[0,1]}.
\end{aligned}$$

Thus, $\mathrm{Ces}_p|_{[1/2,1]} \subset Y_p$, (4.8) holds, and the proof is complete. ∎

The following result is an immediate consequence of (4.2) and the reiteration equality (1.2).

Corollary 4.2. *If* $1 < p_0 < p_1 \le \infty$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$ *with* $0 < \theta < 1$, *then*

$$(\mathrm{Ces}_{p_0}[0,1], \mathrm{Ces}_{p_1}[0,1])_{\theta,p} = \mathrm{Ces}_p[0,1].$$

Remark 4.3. An inspection of the proof of Theorem 4.1 shows that

$$(\mathrm{Ces}_1|_{[1/2,1]}, \mathrm{Ces}_\infty|_{[1/2,1]})_{1-1/p,p} = \mathrm{Ces}_p|_{[1/2,1]}$$

for every $1 < p < \infty$ with equivalence of norms.

Remark 4.4. Comparison of formulas from Remark 3.3 and Corollary 4.2 shows that the real method $(\cdot,\cdot)_{\theta,p}$ "well" interpolates the intersection of Cesàro spaces on $[0,1]$ with $L_1[0,1]$ or, more precisely,

$$\begin{aligned}
&(\mathrm{Ces}_{p_0}[0,1] \cap L_1[0,1], \mathrm{Ces}_{p_1}[0,1] \cap L_1[0,1])_{\theta,p} \\
&\qquad = (\mathrm{Ces}_{p_0}[0,1], \mathrm{Ces}_{p_1}[0,1])_{\theta,p} \cap L_1[0,1]
\end{aligned}$$

for all $1 < p_0 < p_1 \le \infty, 0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

Remark 4.5. We will see further that the imbedding (4.1) is strict for every $1 < p < \infty$ and, even more, $\mathrm{Ces}_p[0,1]$ is not an interpolation space between $\mathrm{Ces}_1[0,1]$ and $\mathrm{Ces}_\infty[0,1]$. Thus, the weighted L_1-space $L_1(1-t)[0,1]$ is in a sense the "proper" end of the scale of Cesàro spaces $\mathrm{Ces}_p[0,1]$, $1 < p \le \infty$.

5. The K-functional for $(\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])$**.** In this section we will find an equivalent expression for the K-functional

$$K(t,f) = K(t,f;\mathrm{Ces}_1,\mathrm{Ces}_\infty) = K(t,f;\mathrm{Ces}_1[0,1],\mathrm{Ces}_\infty[0,1]).$$

We start with a lemma giving its lower estimate. Let us introduce two functions defined on $(0,1]$ by

$$\tau_1(t) = t/\ln(e/t) \quad \text{and} \quad \tau_2(t) = e^{-t} \quad \text{for } 0 < t \le 1. \tag{5.1}$$

It is easy to see that there exists a unique $t_0 \in (0,1)$ such that $\tau_1(t_0) = \tau_2(t_0)$ and $\tau_1(t) < \tau_2(t)$ if and only if $0 < t < t_0$.

LEMMA 5.1 (lower estimates). *Let $f \in \mathrm{Ces}_1[0,1]$, $f \ge 0$ and $0 < t \le 1$.*

(i) *If $f_0 = f\chi_{[0,\tau_1(t)]\cup[\tau_2(t),1]}$, then*

$$K(t,f) \ge \tfrac{1}{4}\|f_0\|_{\mathrm{Ces}(1)}. \tag{5.2}$$

(ii) *If $f_1 = f\chi_{[\tau_1(t),\tau_2(t)]}$, then*

$$K(t,f) \ge \frac{1}{e^2}t\|f_1\|_{\mathrm{Ces}(\infty)}. \tag{5.3}$$

Proof. (i) Firstly, let us prove that

$$K(t,f) \ge \tfrac{1}{3}\|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} \quad \text{for all } 0 < t \le 1. \tag{5.4}$$

Let $f \in \mathrm{Ces}_1$, $f = g + h$, where $g \in \mathrm{Ces}_1$, $h \in \mathrm{Ces}_\infty$. We may assume that $f \ge 0$ and $0 \le g \le f$, $0 \le h \le f$. Then

$$\begin{aligned}3(\|g\|_{\mathrm{Ces}(1)} + t\|h\|_{\mathrm{Ces}(\infty)}) &\ge \|g\|_{\mathrm{Ces}(1)} + 3t\|h\|_{\mathrm{Ces}(\infty)} \\ &\ge \|(f-h)\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} + 3t\|h\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(\infty)} \\ &= \|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} - \|h\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} + 3t\|h\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(\infty)}.\end{aligned} \tag{5.5}$$

Let us show that for any $v \in \mathrm{Ces}_\infty$, $v \ge 0$, with $\operatorname{supp} v \subset [0,\tau_1(t)]$ we have

$$\|v\|_{\mathrm{Ces}(1)} \le 3t\|v\|_{\mathrm{Ces}(\infty)}. \tag{5.6}$$

In fact, by the assumption on the support of v and the Fubini theorem,

$$\begin{aligned}\|v\|_{\mathrm{Ces}(1)} &= \int_0^{\tau_1(t)} \left(\frac{1}{s}\int_0^s v(u)\,du\right) ds + \int_{\tau_1(t)}^1 \left(\frac{1}{s}\int_0^{\tau_1(t)} v(u)\,du\right) ds \\ &= \int_0^{\tau_1(t)} \left(\frac{1}{s}\int_0^s v(u)\,du\right) ds + \int_0^{\tau_1(t)} \left(\int_{\tau_1(t)}^1 \frac{1}{s}\,ds\right) v(u)\,du \\ &= \int_0^{\tau_1(t)} \left(\frac{1}{s}\int_0^s v(u)\,du\right) ds + \int_0^{\tau_1(t)} v(u)\,du \ln\frac{1}{\tau_1(t)}.\end{aligned}$$

Since $\tau_1(t) \le t$ it follows that

$$\int_0^{\tau_1(t)} \left(\frac{1}{s}\int_0^s v(u)\,du\right) ds \le \tau_1(t) \sup_{0<s\le\tau_1(t)} \frac{1}{s}\int_0^s v(u)\,du \le t\|v\|_{\mathrm{Ces}(\infty)}.$$

Moreover,

$$\int_0^{\tau_1(t)} v(u)\,du \ln\frac{1}{\tau_1(t)} \le \tau_1(t)\ln\frac{1}{\tau_1(t)} \sup_{0<s\le\tau_1(t)} \frac{1}{s}\int_0^s v(u)\,du$$
$$= \frac{\ln\frac{1}{t} + \ln\ln\frac{e}{t}}{\ln\frac{e}{t}} t\|v\|_{\mathrm{Ces}(\infty)} \le 2t\|v\|_{\mathrm{Ces}(\infty)},$$

and (5.6) follows. Combining this estimate for $v = h\chi_{[0,\tau_1(t)]}$ with (5.5) we conclude that

$$3(\|g\|_{\mathrm{Ces}(1)} + t\|h\|_{\mathrm{Ces}(\infty)}) \ge \|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)}.$$

Taking the infimum over all decompositions $f = g + h$, $g \in \mathrm{Ces}_1$, $h \in \mathrm{Ces}_\infty$ with $0 \le g \le f$, $0 \le h \le f$ we obtain (5.4).

Next, since $\mathrm{Ces}_1 = L_1(\ln(1/s))$ and $\mathrm{Ces}_\infty \overset{1}{\hookrightarrow} L_1$, we have

$$K(t, f; L_1(\ln(1/s)), L_1) = K(t, f; \mathrm{Ces}_1, L_1) \le K(t, f).$$

Therefore, applying the well-known equality

$$K(t, f; L_1(\ln(1/s)), L_1) = \int_0^1 \min(\ln(1/s), t)|f(s)|\,ds$$

and the elementary inequality

$$\int_0^1 \min(\ln(1/s), t)|f(s)|\,ds \ge \int_{e^{-t}}^1 \ln(1/s)|f(s)|\,ds = \|f\chi_{[\tau_2(t),1]}\|_{\mathrm{Ces}(1)},$$

we obtain

$$K(t, f) \ge \|f\chi_{[\tau_2(t),1]}\|_{\mathrm{Ces}(1)}.$$

Inequality (5.2) is an immediate consequence of the last inequality and (5.4). The proof of (i) is complete.

(ii) Since (5.3) is obvious for $t \in [t_0, 1]$, it can be assumed that $0 < t < t_0$. Let again $f \in \mathrm{Ces}_1$, $f = g + h$, where $g \in \mathrm{Ces}_1$, $h \in \mathrm{Ces}_\infty$ and $0 \le g \le f$, $0 \le h \le f$. Then for any $c \in (0, 1)$ we have

$$(5.7)\qquad \|g\|_{\mathrm{Ces}(1)} + t\|h\|_{\mathrm{Ces}(\infty)}$$
$$\ge \|g\chi_{[\tau_1(t),\tau_2(t)]}\|_{\mathrm{Ces}(1)} + ct\|(f-g)\chi_{[\tau_1(t),\tau_2(t)]}\|_{\mathrm{Ces}(\infty)}$$
$$\ge \|g\chi_{[\tau_1(t),\tau_2(t)]}\|_{\mathrm{Ces}(1)} - ct\|g\chi_{[\tau_1(t),\tau_2(t)]}\|_{\mathrm{Ces}(\infty)} + ct\|f\chi_{[\tau_1(t),\tau_2(t)]}\|_{\mathrm{Ces}(\infty)}.$$

We want to show that for every positive function $w \in \mathrm{Ces}_1$ with $\operatorname{supp} w \subset [\tau_1(t), \tau_2(t)]$,

$$\frac{1}{e^2}\, t\|w\|_{\mathrm{Ces}(\infty)} \le \|w\|_{\mathrm{Ces}(1)} \quad \text{for any } 0 < t < t_0. \tag{5.8}$$

Since

$$\begin{aligned}
\|w\|_{\mathrm{Ces}(1)} &= \int_0^1 \frac{1}{s}\Big[\int_{\tau_1(t)}^{s} w(u)\,du \cdot \chi_{[\tau_1(t),\tau_2(t)]}(s) + \int_{\tau_1(t)}^{\tau_2(t)} w(u)\,du\, \chi_{[\tau_2(t),1]}(s)\Big]\,ds \\
&= \int_{\tau_1(t)}^{\tau_2(t)} \Big(\frac{1}{s}\int_{\tau_1(t)}^{s} w(u)\,du\Big)\,ds + \int_{\tau_1(t)}^{\tau_2(t)} w(u)\,du \int_{\tau_2(t)}^{1} \frac{ds}{s} \\
&= \int_{\tau_1(t)}^{\tau_2(t)} \Big(\int_{u}^{\tau_2(t)} \frac{ds}{s}\Big) w(u)\,du + \int_{\tau_1(t)}^{\tau_2(t)} w(u)\,du \ln\frac{1}{\tau_2(t)} \\
&= \int_{\tau_1(t)}^{\tau_2(t)} w(u)\ln\frac{\tau_2(t)}{u}\,du + t\int_{\tau_1(t)}^{\tau_2(t)} w(u)\,du,
\end{aligned}$$

to prove (5.8) it suffices to show that for all $t \in (0, t_0)$ and $s \in [\tau_1(t), \tau_2(t)]$ we have

$$\frac{1}{e^2}\, t\int_{\tau_1(t)}^{s} w(u)\,du \le s\Big[\int_{\tau_1(t)}^{\tau_2(t)} w(u)\ln\frac{\tau_2(t)}{u}\,du + t\int_{\tau_1(t)}^{\tau_2(t)} w(u)\,du\Big]. \tag{5.9}$$

We consider the cases when $s \in [\tau_1(t), \tau_2(t)/e]$ and $s \in (\tau_2(t)/e, \tau_2(t)]$ separately. Define a unique $t_1 \in (0, t_0)$ such that $\tau_1(t_1) = \tau_2(t_1)/e$ and note that the interval $[\tau_1(t), \tau_2(t)/e]$ is non-empty only if $0 < t \le t_1$. Let

$$\varphi(s) := s\ln\frac{\tau_2(t)}{s} \quad \text{for } s \in [\tau_1(t), \tau_2(t)/e].$$

Since

$$\varphi'(s) = \ln\frac{\tau_2(t)}{s} - 1 = \ln\frac{\tau_2(t)}{es} \ge 0 \quad \text{for all } s \in [\tau_1(t), \tau_2(t)/e],$$

it follows that φ increases. Therefore, $\varphi(s) \ge \varphi(\tau_1(t))$ for all $s \in [\tau_1(t), \tau_2(t)/e]$ and so

$$\begin{aligned}
s\int_{\tau_1(t)}^{\tau_2(t)} w(u)\ln\frac{\tau_2(t)}{u}\,du &\ge s\ln\frac{\tau_2(t)}{s}\int_{\tau_1(t)}^{s} w(u)\,du \\
&\ge \tau_1(t)\ln\frac{\tau_2(t)}{\tau_1(t)}\int_{\tau_1(t)}^{s} w(u)\,du.
\end{aligned} \tag{5.10}$$

We show that

$$\tau_1(t)\ln\frac{\tau_2(t)}{\tau_1(t)} \geq \frac{1}{e^2}t \quad \text{for all } 0 < t \leq t_1. \tag{5.11}$$

The function

$$\psi(t) = \frac{\tau_1(t)}{t}\ln\frac{\tau_2(t)}{\tau_1(t)} = \frac{\ln\frac{\ln\frac{e}{t}}{t} - t}{\ln\frac{e}{t}} \quad \text{for } t \in (0, t_1]$$

is differentiable and its derivative is

$$\psi'(t) = -\left[(t+1)\left(1+\ln\frac{e}{t}\right) + \ln\tau_1(t)\right] \Big/ \left[t\left(\ln\frac{e}{t}\right)^2\right].$$

It is not hard to check that ψ is increasing on $(0, t_2)$ and decreasing on $(t_2, t_1]$ with $t_2 \in (0, t_1)$. Hence, by the definition of t_1, for all $t \in (0, t_1]$ we have

$$\psi(t) \geq \min[\psi(0^+), \psi(t_1)] = \min\left(1, \ln^{-1}\frac{e}{t_1}\right) = \ln^{-1}\frac{e}{t_1} = t_1^{-1}e^{-1-t_1} \geq e^{-2}.$$

Thus, we obtain (5.11). Combining it with (5.10), we obtain (5.9) in the case when $0 < t \leq t_1$ and $s \in [\tau_1(t), \tau_2(t)/e]$.

In the second case, when $s \in (\tau_2(t)/e, \tau_2(t)]$, we have $s \geq e^{-1-t} \geq e^{-2}$ and so

$$t\int_{\tau_1(t)}^{s} w(u)\,du \leq e^2 ts \int_{\tau_1(t)}^{\tau_2(t)} w(u)\,du.$$

Hence, (5.9) holds again, and so (5.8) is proved. Combining (5.8) and (5.7) with $c = e^{-2}$, we obtain

$$\|g\|_{\mathrm{Ces}(1)} + t\|h\|_{\mathrm{Ces}(\infty)} \geq \frac{1}{e^2}t\|f_1\|_{\mathrm{Ces}(\infty)} \quad \text{for all } 0 < t < t_0.$$

Taking the infimum over all decompositions $f = g + h$, $g \in \mathrm{Ces}_1, h \in \mathrm{Ces}_\infty$ with $0 \leq g \leq f$, $0 \leq h \leq f$ we come to (5.3), and the proof of (ii) is complete. ∎

THEOREM 5.2. *For every $f \in \mathrm{Ces}_1[0,1]$ we have*

$$\frac{1}{2e^2}\left[\|f\chi_{[0,\tau_1(t)]\cup[\tau_2(t),1]}\|_{\mathrm{Ces}(1)} + t\|f\chi_{[\tau_1(t),\tau_2(t)]}\|_{\mathrm{Ces}(\infty)}\right] \leq K(t, f; \mathrm{Ces}_1, \mathrm{Ces}_\infty)$$
$$\leq \|f\chi_{[0,\tau_1(t)]\cup[\tau_2(t),1]}\|_{\mathrm{Ces}(1)} + t\|f\chi_{[\tau_1(t),\tau_2(t)]}\|_{\mathrm{Ces}(\infty)}$$

for all $0 < t < 1$, and $K(t, f; \mathrm{Ces}_1, \mathrm{Ces}_\infty) = \|f\|_{\mathrm{Ces}(1)}$ for all $t \geq 1$.

Proof. The first inequality is a consequence of Lemma 5.1 and the definition of the K-functional. The equality $K(t, f; \mathrm{Ces}_1, \mathrm{Ces}_\infty) = \|f\|_{\mathrm{Ces}(1)}$ $(t \geq 1)$ follows from the imbedding $\mathrm{Ces}_\infty \overset{1}{\hookrightarrow} \mathrm{Ces}_1$. ∎

If a positive function $f \in \mathrm{Ces}_1[0,1]$ is decreasing, then the description of the K-functional can be simplified.

THEOREM 5.3. *If* $f \in \mathrm{Ces}_1[0,1]$, $f \geq 0$ *and* f *is decreasing, then*

$$\tfrac{1}{3}\|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} \leq K(t,f;\mathrm{Ces}_1,\mathrm{Ces}_\infty) \leq \|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} \tag{5.12}$$

for all $0 < t < 1$, *and* $K(t,f;\mathrm{Ces}_1,\mathrm{Ces}_\infty) = \|f\|_{\mathrm{Ces}(1)}$ *for all* $t \geq 1$.

Proof. Taking into account the proof of Lemma 5.1(i) (see (5.4)) it suffices to prove the right-hand inequality in (5.12).

Let $f_0 := [f - f(\tau_1(t))]\chi_{[0,\tau_1(t)]}$ and $f_1 := f - f_0$. Since $f \geq 0$ is decreasing, we have $\|f_1\|_{\mathrm{Ces}(\infty)} = f(\tau_1(t))$. Therefore, by the Fubini theorem,

$$\begin{aligned}
&\|f_0\|_{\mathrm{Ces}(1)} + t\|f_1\|_{\mathrm{Ces}(\infty)} \\
&\quad= \int_0^1 \frac{1}{s}\int_0^s \Big[f(u) - f(\tau_1(t))\Big]\chi_{[0,\tau_1(t)]}(u)\,du\,ds + tf(\tau_1(t)) \\
&\quad= \int_0^1 \frac{1}{s}\int_0^s f(u)\chi_{[0,\tau_1(t)]}(u)\,du\,ds - f(\tau_1(t))\int_0^{\tau_1(t)} \ln\frac{1}{u}\,du + tf(\tau_1(t)) \\
&\quad= \|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} - f(\tau_1(t))\tau_1(t)\left[1 + \ln\frac{1}{\tau_1(t)}\right] + tf(\tau_1(t)) \\
&\quad= \|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} + tf(\tau_1(t))\left[1 - \frac{1+\ln(\ln\frac{e}{t}/t)}{\ln\frac{e}{t}}\right] \\
&\quad= \|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} - \frac{tf(\tau_1(t))\ln(\ln\frac{e}{t})}{\ln\frac{e}{t}} \leq \|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)},
\end{aligned}$$

whence

$$K(t,f;\mathrm{Ces}_1,\mathrm{Ces}_\infty) \leq \|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)},$$

and the desired result is proved. ∎

6. Identification of the real interpolation spaces $(\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])_{1-1/p,p}$ **for** $1 < p < \infty$**.** Let us define the weighted Cesàro function space $\mathrm{Ces}_p(\ln(e/t))[0,1]$ to consist of all Lebesgue measurable functions f on $[0,1]$ such that

$$\|f\|_{\mathrm{Ces}(p,\ln)} := \left(\int_0^1 \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p \ln\frac{e}{x}\,dx\right)^{1/p} < \infty.$$

Clearly, $\mathrm{Ces}_p(\ln(e/t))[0,1] \overset{1}{\hookrightarrow} \mathrm{Ces}_p[0,1]$ for every $1 < p < \infty$, and this imbedding is strict.

THEOREM 6.1. *For* $1 < p < \infty$,

$$(\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])_{1-1/p,p} = \mathrm{Ces}_p\left(\ln\frac{e}{t}\right)[0,1]. \tag{6.1}$$

Proof. Denote $X_p = (\mathrm{Ces}_1, \mathrm{Ces}_\infty)_{1-1/p,p}$, $1 < p < \infty$. Using Theorem 5.2 on the K-functional for the couple $(\mathrm{Ces}_1, \mathrm{Ces}_\infty)$ on $[0,1]$, we have

$$\begin{aligned}\|f\|_{X_p} &\le \Big[\int_0^{t_0} t^{-p}\|f\chi_{[0,\tau_1(t)]}\|^p_{\mathrm{Ces}(1)}\,dt\Big]^{1/p} + \Big[\int_0^{t_0} t^{-p}\|f\chi_{[\tau_2(t),1]}\|^p_{\mathrm{Ces}(1)}\,dt\Big]^{1/p}\\ &\quad + \Big[\int_0^{t_0} t^{-p}(t\|f\chi_{[\tau_1(t),\tau_2(t)]}\|_{\mathrm{Ces}(\infty)})^p\,dt\Big]^{1/p} + \Big[\int_{t_0}^{\infty} t^{-p}\|f\|^p_{\mathrm{Ces}(1)}\,dt\Big]^{1/p}\\ &= I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \Big[\int_0^{t_0} t^{-p}\Big(\int_0^{\tau_1(t)} Cf(s)\,ds + \int_{\tau_1(t)}^1 C(f\chi_{[0,\tau_1(t)]})(s)\,ds\Big)^p\,dt\Big]^{1/p}\\ &\le \Big[\int_0^{t_0} t^{-p}\Big(\int_0^{\tau_1(t)} Cf(s)\,ds\Big)^p\,dt\Big]^{1/p}\\ &\quad + \Big[\int_0^{t_0} t^{-p}\Big(\int_{\tau_1(t)}^1 C(f\chi_{[0,\tau_1(t)]})(s)\,ds\Big)^p\,dt\Big]^{1/p} = I_{11} + I_{12}.\end{aligned}$$

First of all, we estimate all five integrals from above. Since $\tau_1'(t) = (\ln(e/t)+1)/(\ln(e/t))^2$ and so $1/\ln(e/t) \le \tau_1'(t) \le 2/\ln(e/t)$ for all $0 < t \le 1$, we get

$$\begin{aligned}I_{11}^p &\le \int_0^{t_0} t^{-p}\Big(\ln\frac{e}{t}\Big)^{p-1}\Big(\int_0^{\tau_1(t)} Cf(s)\,ds\Big)^p\,dt\\ &\le \int_0^{t_0} \tau_1(t)^{-p}\Big(\int_0^{\tau_1(t)} Cf(s)\,ds\Big)^p\,d\tau_1(t).\end{aligned}$$

Putting $u = \tau_1(t)$ and using the classical Hardy inequality, we obtain

$$\begin{aligned}I_{11} &\le \Big[\int_0^{\tau_1(t_0)}\Big(\frac{1}{u}\int_0^u Cf(s)\,ds\Big)^p\,du\Big]^{1/p} \le \|C^2 f\|_{L_p[0,1]}\\ &\le p'\|Cf\|_{L_p[0,1]} = p'\|f\|_{\mathrm{Ces}(p)} \le p'\|f\|_{\mathrm{Ces}(p,\ln)}.\end{aligned}$$

Next, by the estimate $\ln(1/\tau_1(t)) \le 2\ln(e/t)$, $0 < t \le 1$, we get

$$\begin{aligned}I_{12}^p &= \int_0^{t_0} t^{-p}\Big(\int_{\tau_1(t)}^1\Big(\frac{1}{s}\int_0^{\tau_1(t)} |f(u)|\,du\Big)\,ds\Big)^p\,dt\\ &= \int_0^{t_0} t^{-p}\Big(\int_0^{\tau_1(t)} |f(u)|\,du\Big)^p \ln^p\frac{1}{\tau_1(t)}\,dt\\ &\le 2^p\int_0^{t_0} \tau_1(t)^{-p}\Big(\int_0^{\tau_1(t)} |f(u)|\,du\Big)^p\,dt.\end{aligned}$$

The substitution $t=\tau_1^{-1}(s)$ and the inequalities

$$(\tau_1^{-1})'(s)=\frac{1}{\tau_1'(\tau_1^{-1}(s))}\le \ln\frac{e}{\tau_1^{-1}(s)}\le \ln\frac{e}{s} \tag{6.2}$$

show that

$$\begin{aligned}I_{12}&\le 2\Big[\int_0^{\tau_1(t_0)}\Big(\frac{1}{s}\int_0^s|f(u)|\,du\Big)^p\ln\frac{e}{s}\,ds\Big]^{1/p}\\ &\le 2\Big[\int_0^1(Cf(s))^p\ln\frac{e}{s}\,ds\Big]^{1/p}=2\|f\|_{\mathrm{Ces}(p,\ln)}.\end{aligned}$$

From the equality $\mathrm{Ces}_1[0,1]=L_1(\ln(1/u))$ and the inequalities $\ln(1/u)\le e(1-u)$ $(1/e\le u\le 1)$ and $\tau_2(t)=e^{-t}\ge 1-t$ $(0<t\le 1)$ it follows that

$$\begin{aligned}I_2^p&=\int_0^{t_0}t^{-p}\|f\chi_{[\tau_2(t),1]}\|^p_{\mathrm{Ces}(1)}\,dt=\int_0^{t_0}t^{-p}\Big(\int_{\tau_2(t)}^1|f(u)|\ln\frac{1}{u}\,du\Big)^p\,dt\\ &\le e^p\int_0^{t_0}t^{-p}\Big(\int_{\tau_2(t)}^1|f(u)|(1-u)\,du\Big)^p\,dt\\ &\le e^p\int_0^{t_0}t^{-p}\Big(\int_{1-t}^1|f(u)|(1-u)\,du\Big)^p\,dt.\end{aligned}$$

Arguing in the same way as in the second part of the proof of Theorem 4.1, for $g=f\chi_{[e^{-1},1]}$ and $0<s\le 1$ we have

$$Cg(s)=\frac{1}{s}\int_{e^{-1}}^s|f(u)|\,du\ge\int_{e^{-1}}^s|f(u)|\,du,$$

whence $(Cg)^*(s)\ge\int_{e^{-1}}^{1-s}|f(u)|\,du$ and

$$\begin{aligned}\int_0^t(Cg)^*(s)\,ds&\ge\int_0^t\Big(\int_{e^{-1}}^{1-s}|f(u)|\,du\Big)\,ds\\ &=\int_{e^{-1}}^{1-t}\Big(\int_0^t|f(u)|\,ds\Big)\,du+\int_{1-t}^1\Big(\int_0^{1-u}|f(u)|\,ds\Big)\,du\ge\int_{1-t}^1|f(u)|(1-u)\,du.\end{aligned}$$

Therefore, again by the Hardy inequality,

$$\begin{aligned}I_2&\le e\Big[\int_0^{t_0}t^{-p}\Big(\int_0^t(Cg)^*(s)\,ds\Big)^p\,dt\Big]^{1/p}\le e\|C[(Cg)^*]\|_{L_p[0,1]}\\ &\le ep'\|(Cg)^*\|_{L_p[0,1]}=ep'\|Cg\|_{L_p[0,1]}=ep'\|f\chi_{[e^{-1},1]}\|_{\mathrm{Ces}(p)}\\ &\le ep'\|f\|_{\mathrm{Ces}(p)}\le ep'\|f\|_{\mathrm{Ces}(p,\ln)}.\end{aligned}$$

For the third integral, we have

$$\begin{aligned}
I_3 &= \Big[\int_0^{t_0} \| f \chi_{[\tau_1(t), \tau_2(t)]} \|_{\mathrm{Ces}(\infty)}^p \, dt \Big]^{1/p} \\
&\le \Big[\int_0^{t_0} \sup_{\tau_1(t) < s \le 1/2} \Big(\frac{1}{s} \int_0^s |f(u) \chi_{[\tau_1(t), \tau_2(t)]}(u)| \, du \Big)^p dt \Big]^{1/p} \\
&\quad + \Big[\int_0^{t_0} \sup_{1/2 < s \le \tau_2(t)} \Big(\frac{1}{s} \int_0^s |f(u) \chi_{[\tau_1(t), \tau_2(t)]}(u)| \, du \Big)^p dt \Big]^{1/p} \\
&= \Big[\int_0^{t_0} \sup_{\tau_1(t) < s \le 1/2} \Big(\frac{1}{s} \int_{\tau_1(t)}^s |f(u)| \, du \Big)^p dt \Big]^{1/p} \\
&\quad + \Big[\int_0^{t_0} \sup_{1/2 < s \le \tau_2(t)} \Big(\frac{1}{s} \int_{\tau_1(t)}^s |f(u)| \, du \Big)^p dt \Big]^{1/p} = I_{31} + I_{32}.
\end{aligned}$$

If $\tau_1(t) < s \le 1/2$, then $2s \le 1$ and

$$\begin{aligned}
&\int_{\tau_1(t)}^{2s} \Big(\frac{1}{v} \int_0^v |f(u)| \, du \Big) dv \\
&\qquad = \int_0^{\tau_1(t)} \Big(\int_{\tau_1(t)}^{2s} \frac{1}{v} \, dv \Big) |f(u)| \, du + \int_{\tau_1(t)}^{2s} \Big(\int_u^{2s} \frac{1}{v} \, dv \Big) |f(u)| \, du \\
&\qquad = \int_0^{\tau_1(t)} |f(u)| \, du \ln \frac{2s}{\tau_1(t)} + \int_{\tau_1(t)}^{2s} |f(u)| \ln \frac{2s}{u} \, du \\
&\qquad \ge \frac{2s - \tau_1(t)}{2s} \int_{\tau_1(t)}^{2s} |f(u)| \ln \frac{2s}{u} \, du \ge \ln 2 \, \frac{2s - \tau_1(t)}{2s} \int_{\tau_1(t)}^{s} |f(u)| \, du.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{\tau_1(t) < s \le 1/2} \frac{1}{s} \int_{\tau_1(t)}^s |f(u)| \, du &\le \frac{2}{\ln 2} \sup_{\tau_1(t) < s \le 1/2} \frac{1}{2s - \tau_1(t)} \int_{\tau_1(t)}^{2s} Cf(v) \, dv \\
&\le \frac{2}{\ln 2} \, MCf(\tau_1(t)),
\end{aligned}$$

where M is the maximal Hardy–Littlewood operator on $[0,1]$. The above estimates show that

$$I_{31} \le \frac{2}{\ln 2} \Big(\int_0^{t_0} MCf(\tau_1(t))^p \, dt \Big)^{1/p}.$$

Using once again the substitution $t=\tau_1^{-1}(s)$ and (6.2), we obtain

$$I_{31}\le\frac{2}{\ln 2}\Big[\int_0^{\tau_1(t_0)}[MCf(s)]^p\ln\frac{e}{s}\,ds\Big]^{1/p}\le\frac{2}{\ln 2}\|MCf\|_{L_p(\ln(e/s))}.$$

We will show in the next lemma that the maximal operator M is bounded in $L_p(\ln(e/s))[0,1]$ for $1<p<\infty$, which implies that for some constant $B_p\ge 1$, which depends only on p, we have

$$I_{31}\le\frac{2B_p}{\ln 2}\|Cf\|_{L_p(\ln(e/s))}=\frac{2B_p}{\ln 2}\|f\|_{\mathrm{Ces}(p,\ln)}.$$

The second part of I_3 is estimated in the following way:

$$I_{32}^p=\int_0^{t_0}\sup_{1/2<s\le\tau_2(t)}\Big(\frac{1}{s}\int_{\tau_1(t)}^{s}|f(u)|\,du\Big)^p dt\le 2^p\int_0^{t_0}\Big(\int_{\tau_1(t)}^{\tau_2(t)}|f(u)|\,du\Big)^p dt$$
$$\le 2^p\int_0^{t_0}\Big(\frac{1}{\tau_2(t)}\int_0^{\tau_2(t)}|f(u)|\,du\Big)^p dt,$$

and, changing variable $s=\tau_2(t)=e^{-t}$, we obtain

$$I_{32}\le 2\Big[\int_{e^{-t_0}}^{1}\Big(\frac{1}{s}\int_0^s|f(u)|\,du\Big)^p\frac{ds}{s}\Big]^{1/p}\le 2e^{t_0/p}\Big(\int_0^1 Cf(s)^p\,ds\Big)^{1/p}$$
$$\le 2e\|f\|_{\mathrm{Ces}(p)}\le 2e\|f\|_{\mathrm{Ces}(p,\ln)}.$$

Since $t_0>1/2$, for the last integral we have

$$I_4=\frac{1}{(p-1)^{1/p}t_0^{1-1/p}}\|f\|_{\mathrm{Ces}(1)}\le\frac{2}{p-1}\|f\|_{\mathrm{Ces}(1)}\le\frac{2}{p-1}\|f\|_{\mathrm{Ces}(p,\ln)}.$$

Finally, summing up the above estimates, we get $\|f\|_{X_p}\le C_p\|f\|_{\mathrm{Ces}(p,\ln)}$, where C_p depends only on p. Thus, the imbedding $\mathrm{Ces}(p,\ln)\hookrightarrow X_p$ is proved.

Now, we proceed with estimations from below. Firstly, by (5.4),

$$\|f\|_{X_p}^p\ge 3^{-p}\int_0^{t_0}t^{-p}\|f\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)}^p\,dt=3^{-p}I_1^p\ge 3^{-p}I_{12}^p. \tag{6.3}$$

It is not hard to check that $\ln\frac{1}{\tau_1(t)}=\ln\frac{\ln(e/t)}{t}\ge e^{-1}\ln\frac{e}{t}$ for $t\in(0,t_0]$. Therefore,

$$I_{12}^p=\int_0^{t_0}t^{-p}\Big(\int_0^{\tau_1(t)}|f(u)|\,du\Big)^p\ln^p\frac{1}{\tau_1(t)}\,dt\ge e^{-p}\int_0^{t_0}\tau_1(t)^{-p}\Big(\int_0^{\tau_1(t)}|f(u)|\,du\Big)^p dt.$$

Since $\tau_1'(s)\le 2/\ln(e/s)$, $\tau_1^{-1}(s)\le s\ln(e/s)$ and $\ln\ln(e/s)\le e^{-1}\ln(e/s)$

$(0 < s \le 1)$, we have

$$(\tau_1^{-1})'(s) = \frac{1}{\tau_1'(\tau_1^{-1}(s))} \ge \frac{1}{2}\ln\frac{e}{\tau_1^{-1}(s)} \ge \frac{1}{2}\ln\frac{e}{s\ln(e/s)}$$
$$= \frac{1}{2}\Big(\ln\frac{e}{s} - \ln\ln\frac{e}{s}\Big) \ge \frac{1}{2}\Big(1-\frac{1}{e}\Big)\ln\frac{e}{s}.$$

Hence, after the substitution $t = \tau_1^{-1}(s)$, we obtain

$$I_{12}^p \ge e^{-p}\frac{1}{2}\Big(1-\frac{1}{e}\Big)\int_0^{\tau_1(t_0)}\Big(\frac{1}{s}\int_0^s |f(u)|\,du\Big)^p \ln\frac{e}{s}\,ds$$
$$\ge \frac{1}{4}e^{-p}\int_0^{\tau_1(t_0)} Cf(s)^p \ln\frac{e}{s}\,ds,$$

and so, taking into account (6.3), we get

$$\|f\|_{X_p}^p \ge 4^{-1}(3e)^{-p}\int_0^{\tau_1(t_0)} Cf(s)^p\ln\frac{e}{s}\,ds.$$

On the other hand, by the definition of t_0,

$$\int_{\tau_1(t_0)}^1 Cf(s)^p\ln\frac{e}{s}\,ds \le \ln\frac{e}{\tau_1(t_0)}\int_{\tau_1(t_0)}^1 Cf(s)^p\,ds \le (1+t_0)\|f\|_{\mathrm{Ces}(p)}^p$$
$$\le 2\|f\|_{\mathrm{Ces}(p)}^p \le 2\|f\|_{X_p}^p,$$

where the last inequality follows from (4.1). Hence,

$$\|f\|_{X_p} \ge 8^{-1/p}(3e)^{-1}\Big(\int_0^1 Cf(s)^p\ln\frac{e}{s}\,ds\Big)^{1/p} \ge \frac{1}{72}\|f\|_{\mathrm{Ces}(p,\ln)},$$

and the imbedding $X_p \hookrightarrow \mathrm{Ces}(p,\ln)$ is proved. Thus, the proof of Theorem 6.1 will be finished if we prove the lemma below. ∎

Lemma 6.2. *If $1 < p < \infty$, then the maximal Hardy–Littlewood operator M on $[0,1]$ is bounded in the weighted space $L_p(\ln(e/x))[0,1] = L_p([0,1],\ln(e/x)dx)$.*

Proof. Muckenhoupt [Mu, Theorem 2] proved that the maximal operator M on $[0,1]$ is bounded in $L_p([0,1],w(x)dx)$ if and only if the weight $w(x)$ satisfies the so-called *A_p-condition* on $[0,1]$, that is,

$$\sup_{(a,b)\subset[0,1]}\Big(\frac{1}{b-a}\int_a^b w(x)\,dx\Big)\Big(\frac{1}{b-a}\int_a^b w(x)^{-1/(p-1)}\,dx\Big)^{p-1} < \infty.$$

Therefore, it is enough to show that for all intervals $(a,b)\subset[0,1]$ we have

$$\int_a^b \ln\frac{e}{x}\,dx\Big(\int_a^b\Big(\ln\frac{e}{x}\Big)^{-1/(p-1)}dx\Big)^{p-1} \le 2(b-a)^p. \tag{6.4}$$

Note that for $t \in (0, b)$,

$$\int_t^b \ln \frac{e}{x}\, dx = b \ln \frac{e}{b} - t \ln \frac{e}{t} + b - t$$

and

$$\int_t^b \left(\ln \frac{e}{x}\right)^{-\alpha} dx = b\left(\ln \frac{e}{b}\right)^{-\alpha} - t\left(\ln \frac{e}{t}\right)^{-\alpha} - \alpha \int_t^b \left(\ln \frac{e}{x}\right)^{-\alpha-1} dx$$
$$\le b\left(\ln \frac{e}{b}\right)^{-\alpha} - t\left(\ln \frac{e}{t}\right)^{-\alpha},$$

where $\alpha > 0$. Since the functions

$$\varphi_1(t) = \frac{b \ln(e/b) - t \ln(e/t) + b - t}{b - t}$$

and

$$\varphi_2(t) = \frac{b(\ln(e/b))^{-\alpha} - t(\ln(e/t))^{-\alpha}}{b - t}$$

are both decreasing on $(0, b)$ for every $0 < b \le 1$ it follows that $\max_{0<t<b} \varphi_1(t) = \varphi_1(0^+) = \ln(e^2/b)$ and $\max_{0<t<b} \varphi_2(t) = \varphi_2(0^+) = \ln^{-\alpha}(e/b)$. Therefore, setting $\alpha = 1/(p-1)$, for all $0 \le a < b \le 1$ we have

$$\frac{1}{(b-a)^p} \int_a^b \ln \frac{e}{x}\, dx \left[\int_a^b \left(\ln \frac{e}{x}\right)^{-1/(p-1)} dx\right]^{p-1} \le \ln \frac{e^2}{b} \left[\left(\ln\left(\frac{e}{b}\right)\right)^{-1/(p-1)}\right]^{p-1}$$
$$= \frac{\ln(e^2/b)}{\ln(e/b)} \le 2,$$

and (6.4) is proved. ∎

7. $\mathrm{Ces}_p[0,1]$, $1 < p < \infty$, **is not an interpolation space between** $\mathrm{Ces}_1[0,1]$ **and** $\mathrm{Ces}_\infty[0,1]$**.** We start with two lemmas (it is instructive to compare the first with (4.1)).

Lemma 7.1. *If $1 < p < \infty$, then*

$$\mathrm{Ces}_p[0,1] \not\hookrightarrow (\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])_{1-1/p,\infty}. \tag{7.1}$$

Proof. Let us consider the family of characteristic functions $f_s = \chi_{[0,s]}$, $0 < s < 1$. As we know (cf. Theorem 5.3),

$$K(t, f_s; \mathrm{Ces}_1, \mathrm{Ces}_\infty) \ge \tfrac{1}{3} \|f_s \chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} \quad \text{for all } t > 0.$$

Since

$$\|f_s \chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} = \|\chi_{[0,\min(s,\tau_1(t))]}\|_{\mathrm{Ces}(1)} = \|\chi_{[0,\min(s,\tau_1(t))]}\|_{L_1(\ln(1/s))}$$
$$= \int_0^{\min(s,\tau_1(t))} \ln \frac{1}{s}\, ds = \min(s, \tau_1(t)) \left[\ln \frac{1}{\min(s, \tau_1(t))} + 1\right],$$

it follows that for all t such that $\tau_1(t) \le s$ we have $\|f_s\chi_{[0,\tau_1(t)]}\|_{\mathrm{Ces}(1)} \ge \tau_1(t)\ln(1/\tau_1(t))$. Therefore, using the inequality $\tau_1^{-1}(s) \le s\ln(e/s)$ once again, for $0 < s < e^{-1}$ we obtain

$$\begin{aligned}\|f_s\|_{(\mathrm{Ces}_1,\mathrm{Ces}_\infty)_{1-1/p,\infty}} &= \sup_{t>0} t^{1/p-1}K(t,f_s;\mathrm{Ces}_1,\mathrm{Ces}_\infty)\\ &\ge \frac{1}{3}\sup_{t>0,\,\tau_1(t)\le s} t^{1/p-1}\tau_1(t)\ln\frac{1}{\tau_1(t)}\\ &\ge \frac{1}{3}(\tau_1^{-1}(s))^{1/p-1}s\ln\frac{1}{s} \ge \frac{1}{6}\Big(s\ln\frac{e}{s}\Big)^{1/p-1}s\ln\frac{1}{s}\\ &\ge \frac{1}{6}\,s^{1/p}\Big(\ln\frac{e}{s}\Big)^{1/p}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|f_s\|_{\mathrm{Ces}_p} &= \bigg[\int_0^s\Big(\frac{1}{u}\int_0^u\chi_{[0,s]}(v)\,dv\Big)^p du + \int_s^1\Big(\frac{1}{u}\int_0^u\chi_{[0,s]}(v)\,dv\Big)^p du\bigg]^{1/p}\\ &= \Big(s+s^p\int_s^1 u^{-p}\,du\Big)^{1/p} = \Big(s+\frac{s^p}{p-1}(s^{1-p}-1)\Big)^{1/p}\\ &= \Big(\frac{p}{p-1}\,s-\frac{1}{p-1}\,s^p\Big)^{1/p} \le (p')^{1/p}s^{1/p}.\end{aligned}$$

Therefore, for $0 < s < e^{-1}$,

$$\frac{\|f_s\|_{(\mathrm{Ces}_1,\mathrm{Ces}_\infty)_{1-1/p,\infty}}}{\|f_s\|_{\mathrm{Ces}_p}} \ge \frac{\frac{1}{6}s^{1/p}(\ln\frac{e}{s})^{1/p}}{(p')^{1/p}\,s^{1/p}} \ge \frac{1}{6p'}\Big(\ln\frac{e}{s}\Big)^{1/p},$$

whence

$$\sup_{0<s<1}\frac{\|f_s\|_{(\mathrm{Ces}_1,\mathrm{Ces}_\infty)_{1-1/p,\infty}}}{\|f_s\|_{\mathrm{Ces}_p}} = \infty,$$

which shows that (7.1) holds. ∎

Recall that the *characteristic function* $\varphi(s,t)$ of an exact interpolation functor $\mathcal{F}$ is defined by the equality $\mathcal{F}(s\mathbb{R},t\mathbb{R}) = \varphi(s,t)\mathbb{R}$ for all $s,t>0$. By the Aronszajn–Gagliardo theorem (see [BL, Theorem 2.5.1] or [BK, Theorem 2.3.15]), for every Banach couple (X_0,X_1) and every Banach space $X \in \mathrm{Int}(X_0,X_1)$ there is an exact interpolation functor $\mathcal{F}$ such that $\mathcal{F}(X_0,X_1) = X$.

Lemma 7.2. *Let* $1 < p < \infty$. *Suppose that* $\mathrm{Ces}_p[0,1] \in \mathrm{Int}(\mathrm{Ces}_1[0,1],\mathrm{Ces}_\infty[0,1])$ *and* $\mathcal{F}$ *is an exact interpolation functor such that*

$$\mathcal{F}(\mathrm{Ces}_1[0,1],\mathrm{Ces}_\infty[0,1]) = \mathrm{Ces}_p[0,1]. \tag{7.2}$$

Then the characteristic function $\varphi(1,t)$ *of* $\mathcal{F}$ *is equivalent to* $t^{1/p}$ *for* $0 < t \le 1$.

Proof. To simplify notation set $V_p := \mathrm{Ces}_p|_{[1/2,1]}$ $(1 \le p \le \infty)$, that is, V_p is the subspace of $\mathrm{Ces}_p[0,1]$ which consists of all functions f such that $\operatorname{supp} f \subset [1/2,1]$. Since (V_1, V_∞) is a complemented subcouple of $(\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])$, by (7.2) and the equality in Remark 4.3, we obtain

$$\mathcal{F}(V_1, V_\infty) = V_p = (V_1, V_\infty)_{1-1/p,p}. \tag{7.3}$$

Consider the sequence of functions $g_k(t) = \chi_{[1-2^{-k}, 1-2^{-k-1}]}(t)$, $k = 1, 2, \ldots$, and the linear projection

$$Pf(t) = \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} f(s)\, ds \cdot g_k(t), \qquad f \in V_\infty.$$

We have

$$\begin{aligned} \|Pf\|_{V_\infty} &\le 2\|Pf\|_{L_1|_{[1/2,1]}} \le 2 \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)|\, ds \cdot 2^{-k-1} \\ &= 2\|f\|_{L_1|_{[1/2,1]}} \le 2\|f\|_{V_\infty}, \end{aligned}$$

and, since $1 - u \le \ln(1/u) \le 2(1-u)$ for $1/2 \le u \le 1$,

$$\begin{aligned} \|Pf\|_{V_1} &\le \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)|\, ds \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} \ln\frac{1}{t}\, dt \\ &\le \sum_{k=1}^{\infty} 2^{k+2} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)|\, ds \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} (1-t)\, dt \\ &\le \sum_{k=1}^{\infty} 2^{k+2} \cdot 2^{-2k-1} \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)|\, ds \le 4 \sum_{k=1}^{\infty} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)|(1-s)\, ds \\ &\le 4 \sum_{k=1}^{\infty} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| \ln\frac{1}{s}\, ds = 4\|f\|_{L_1(\ln(1/s))} = 4\|f\|_{V_1}. \end{aligned}$$

Therefore, P is a bounded linear projection from V_∞ onto $\operatorname{Im} P_{|V_\infty}$ and from V_1 onto $\operatorname{Im} P_{|V_1}$. At the same time, it is easy to see that the sequence $\{2^{k+1} g_k\}_{k=1}^{\infty}$ is equivalent in V_∞ (resp. in V_1) to the standard basis in l_1 (resp. in $l_1(2^{-k})$). Hence, $(l_1, l_1(2^{-k}))$ is a complemented subcouple of (V_1, V_∞) and therefore, by (6.3) and by the Baouendi–Goulaouic result [BG, Theorem 1] (see also [Tr, Theorem 1.17.1]),

$$\mathcal{F}(l_1, l_1(2^{-k})) = (l_1, l_1(2^{-k}))_{1-1/p,p}.$$

In particular, from the last relation it follows that

$$\mathcal{F}(\mathbb{R}, 2^{-k}\mathbb{R}) = (\mathbb{R}, 2^{-k}\mathbb{R})_{1-1/p,p} = 2^{-k/p}\mathbb{R}$$

uniformly in $k \in \mathbb{N}$. Since the characteristic function of any exact interpolation functor is quasi-concave [BK, Proposition 2.3.10], this implies the result. ∎

THEOREM 7.3. *For any* $1 < p < \infty$, $\mathrm{Ces}_p[0,1]$ *is not an interpolation space between* $\mathrm{Ces}_1[0,1]$ *and* $\mathrm{Ces}_\infty[0,1]$.

Proof. Assume that $\mathrm{Ces}_p[0,1]$ is an interpolation space between $\mathrm{Ces}_1[0,1]$ and $\mathrm{Ces}_\infty[0,1]$. Then there is an exact interpolation functor $\mathcal{F}$ such that (7.2) holds. By Lemma 7.2, the characteristic function $\varphi(1,t)$ of $\mathcal{F}$ is equivalent to $t^{1/p}$ for $0 < t \leq 1$. Therefore, for any Banach couple (X_0, X_1) we have

$$\mathcal{F}(X_0, X_1) \subset (X_0, X_1)_{\psi,\infty}, \tag{7.4}$$

where $(X_0, X_1)_{\psi,\infty}$ is the real interpolation space consisting of all $x \in X_0 + X_1$ such that $\sup_{t>0} \frac{\psi(t)}{t} K(t, x; X_0, X_1) < \infty$ and $\psi(t) = \min(1, t^{1/p})$ [BK, Proposition 3.8.6]. Since $\mathrm{Ces}_\infty[0,1] \overset{1}{\hookrightarrow} \mathrm{Ces}_1[0,1]$, applying (7.4) to the couple $(\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])$ we obtain

$$\mathcal{F}(\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1]) \subset (\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])_{1-1/p,\infty}, \tag{7.5}$$

whence $\mathrm{Ces}_p[0,1] \subset (\mathrm{Ces}_1[0,1], \mathrm{Ces}_\infty[0,1])_{1-1/p,\infty}$. But in view of Lemma 7.1 the last imbedding does not hold, and the proof is complete. ∎

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References

[AKMNP] I. Asekritova, N. Krugljak, L. Maligranda, L. Nikolova and L.-E. Persson, *Lions–Peetre reiteration formulas for triples and their applications*, Studia Math. 145 (2001), 219–254.

[A] S. V. Astashkin, *On the geometric properties of Cesàro spaces*, Mat. Sb. 203 (2012), 61–80 (in Russian); English transl.: Sb. Math. 203 (2012), 514–533.

[AM] S. V. Astashkin and L. Maligranda, *Structure of Cesàro function spaces*, Indag. Math. (N.S.) 20 (2009), 329–379.

[BG] M. S. Baouendi et C. Goulaouic, *Commutation de l'intersection et des foncteurs d'interpolation*, C. R. Acad. Sci. Paris Sér. A-B 26 (1967), 313–315.

[Be] G. Bennett, *Factorizing the classical inequalities*, Mem. Amer. Math. Soc. 120 (1996), no. 576.

[BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York, 1988.

[BL] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.

[BK] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces*, North-Holland, Amsterdam, 1991.

[CFM] F. Cobos, L. M. Fernández-Cabrera and M. Mastyło, *Abstract limit J-spaces*, J. London Math. Soc. (2) 82 (2010), 501–525.

[Gi] J. E. Gilbert, *Interpolation between weighted L^p-spaces*, Ark. Mat. 10 (1972), 235–249.

[HLP] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1988.

[KPS] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (in Russian); English transl.: Amer. Math. Soc., Providence, 1982.

[KMP] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis, Plzen, 2007.

[MP] L. Maligranda and L.-E. Persson, *Real interpolation between weighted L^p and Lorentz spaces*, Bull. Polish Acad. Sci. Math. 35 (1987), 765–778.

[MPS] L. Maligranda, N. Petrot and S. Suantai, *On the James constant and B-convexity of Cesàro and Cesàro–Orlicz sequences spaces*, J. Math. Anal. Appl. 326 (2007), 312–331.

[MS] M. Mastyło and G. Sinnamon, *A Calderón couple of down spaces*, J. Funct. Anal. 240 (2006), 192–225.

[Mu] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[Ov] V. I. Ovchinnikov, *The Method of Orbits in Interpolation Theory*, Math. Rep. 1 (1984), no. 2, i-x and 349–515.

[Si91] G. Sinnamon, *Interpolation of spaces defined by the level function*, in: Harmonic Analysis (Sendai, 1990), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, 190–193.

[Si94] G. Sinnamon, *Spaces defined by the level function and their duals*, Studia Math. 111 (1994), 19–52.

[Si01] G. Sinnamon, *The level function in rearrangement invariant spaces*, Publ. Mat. 45 (2001), 175–198.

[Si07] G. Sinnamon, *Monotonicity in Banach function spaces*, in: NAFSA 8—Nonlinear Analysis, Function Spaces and Applications, Vol. 8, Czech. Acad. Sci., Praha, 2007, 204–240.

[Tr] H. Triebel, *Interpolation Theory. Function Spaces. Differential Operators*, Deutscher Verlag Wiss., Berlin, 1978.

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