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Sets of p-multiplicity in locally compact groups

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Abstract. We initiate the study of sets of p-multiplicity in locally compact groups and their operator versions. We show that a closed subset E of a second countable locally compact group G is a set of p-multiplicity if and only if $E^* = \{(s,t) : ts^{-1} \in E\}$ is a set of operator p-multiplicity. We exhibit examples of sets of p-multiplicity, establish preservation properties for unions and direct products, and prove a p-version of the Stone–von Neumann Theorem.

1. Introduction. The existence of non-zero compact operators acting on a Hilbert space and leaving invariant a given commutative subspace lattice was first examined in [10] (see also [6] and the references therein). That work followed W. B. Arveson's seminal paper [1], and showed that the presence of non-zero compact operators in CSL algebras is closely related to the notion of multiplicity sets in commutative harmonic analysis. This relation was formalised, and generalised to non-commutative locally compact groups, in [17], where the notion of sets of operator multiplicity was introduced, and [18], where it was shown that a closed subset E of a (second countable) locally compact group G is a set of multiplicity if and only if $E^* = \{(s,t) : ts^{-1} \in E\}$ is a set of operator multiplicity.

The study of non-zero operators from Schatten p-classes in CSL algebras was also initiated in [10], where a link between such operators and pseudomeasures on compact abelian groups, whose Fourier transforms belong to the sequence space ℓ^p, was exhibited. If ℓ^p is replaced by c_0, this turns into a special case of the result described in the previous paragraph. It is thus natural to define and study sets of p-multiplicity, their operator analogues, and the relation between these two notions.

This is the aim of the present article. In Section 3, given a locally compact group G, we define a subspace $\mathcal{S}_p(G)$ of the reduced group C^*-algebra $C_r^*(G)$ of G that plays a role analogous to the role of the Schatten p-class within the C^*-algebra of all compact operators on a Hilbert space. If G is compact,

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the space $\mathcal{S}_p(G)$ coincides with the intersection of $C_r^*(G)$ with the Schatten p-class on $L^2(G)$. It should be noted that if G is discrete, then $\mathcal{S}_p(G)$ is equal to $C_r^*(G)$, and thus the interest of our work lies in the case where G is locally compact and non-discrete; for example, when G is connected. After defining sets of p-multiplicity and their operator versions, we show that a closed set $E \subseteq G$ is a set of p-multiplicity if and only if E^* is a set of operator p-multiplicity. We give a number of examples of sets of p-multiplicity, and establish preservation properties for unions and direct products. We include characterisations of the sets of p-multiplicity in the case $p = 1$ and $p = 2$.

In Section 4, we prove a p-version of the Stone–von Neumann Theorem. Recall that this result can be stated by saying that the C^*-algebra of all compact operators on $L^2(G)$ is generated by $C_r^*(G)$ and the multiplication algebra of the space $C_0(G)$ of all continuous functions on G vanishing at infinity. Here, we obtain an analogous result for the Schatten p-class, using the space $\mathcal{S}_p(G)$ in place of $C_r^*(G)$.

In Section 5, using the Fourier theory of compact groups, we give a different proof of the aforementioned transference theorem for sets of p-multiplicity, which we believe is interesting in its own right.

Finally, in Section 2 we collect the necessary background material and set notation.

2. Preliminaries. Let (X,μ) and (Y,ν) be standard (σ-finite) measure spaces. A subset $E \subseteq X \times Y$ is called *marginally null* if $E \subseteq (M \times Y) \cup (X \times N)$, where $M \subseteq X$ and $N \subseteq Y$ are null sets. Let $T(X,Y)$ be the projective tensor product $L^2(X) \hat{\otimes} L^2(Y)$. Every $h \in T(X,Y)$ can be written as a series

$$h = \sum_{i=1}^{\infty} f_i \otimes g_i, \qquad f_i \in L^2(X),\ g_i \in L^2(Y),\ i \in \mathbb{N},$$

where $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$. Such an h may be considered either as a function $h : X \times Y \to \mathbb{C}$, defined up to a marginally null set and given by

$$h(x,y) = \sum_{i=1}^{\infty} f_i(x) g_i(y),$$

or as an element of the predual of the space $\mathcal{B}(L^2(X), L^2(Y))$ of all bounded linear operators from $L^2(X)$ into $L^2(Y)$ via the pairing

$$\langle T, h\rangle := \sum_{i=1}^{\infty} (T f_i, \bar{g}_i).$$

We denote by $\|h\|_T$ the norm of $h \in T(X,Y)$.

Let $\mathfrak{S}(X,Y)$ be the multiplier algebra of $T(X,Y)$; by definition, a measurable function $w : X\times Y \to \mathbb{C}$ belongs to $\mathfrak{S}(X,Y)$ if the map $m_w : h \mapsto wh$ leaves $T(X,Y)$ invariant, that is, wh coincides almost everywhere with a function from $T(X,Y)$, for every $h \in T(X,Y)$. The elements of $\mathfrak{S}(X,Y)$ are called (*measurable*) *Schur multipliers*; we refer the reader to [15] for relevant details. If $w \in \mathfrak{S}(X,Y)$, the adjoint of m_w, acting on $\mathcal{B}(L^2(X),L^2(Y))$, will be denoted by S_w.

Throughout the paper, G is a locally compact group. The Lebesgue spaces $L^p(G)$, $p = 1,2,\infty$, are with respect to left Haar measure m; $dm(x)$ is shortened to dx and the modular function of G is denoted by $\varDelta$. Let $\lambda : G \to \mathcal{B}(L^2(G))$, $s \mapsto \lambda_s$, be the left regular representation. The symbol λ is also used for the corresponding representation of $L^1(G)$ on $L^2(G)$; thus, if $f \in L^1(G)$ then $\lambda(f)$ is the operator on $L^2(G)$ given by $\lambda(f)(g) = f * g$.

The *reduced group C^*-algebra* $C_r^*(G)$ of G is the operator norm closure of $\{\lambda(f) : f \in L^1(G)\}$, while the *group von Neumann algebra* $\mathrm{VN}(G)$ of G is its weak* closure. The *Fourier algebra* $A(G)$ of G is the (commutative, regular, semisimple) Banach algebra consisting of all complex functions u on G of the form

$$x \mapsto u(x) = (\lambda_x\xi, \eta), \tag{2.1}$$

where $\xi, \eta \in L^2(G)$. The norm of an element $u \in A(G)$ is by definition the infimum of the products $\|\xi\|\,\|\eta\|$, where ξ and η are functions from $L^2(G)$ for which (2.1) holds. The Banach space dual of $A(G)$ can be canonically identified with $\mathrm{VN}(G)$: for $T \in \mathrm{VN}(G)$ and u as in (2.1), the pairing is given by

$$\langle T, u\rangle = (T\xi, \eta);$$

we refer the reader to [8] for this and further properties of $A(G)$.

We set $T(G) = T(G,G)$, $\mathfrak{S}(G) = \mathfrak{S}(G,G)$ and $\mathcal{B}(L^2(G)) = \mathcal{B}(L^2(G), L^2(G))$. The map $P : T(G) \to A(G)$ given by

$$P(f\otimes g)(t) = \langle \lambda_t, f\otimes g\rangle = (\lambda_t f, \overline{g}) = \int_G f(t^{-1}s)g(s)\,ds = g * \check{f}(t) \tag{2.2}$$

(where $\check{f}(t) = f(t^{-1})$) is the predual of the inclusion $\mathrm{VN}(G) \to \mathcal{B}(L^2(G))$. Moreover, the following holds (see [18] for a proof):

PROPOSITION 2.1. *For every $h \in T(G)$, we have*

$$P(h)(t) = \int_G h(t^{-1}s, s)\,ds, \qquad t \in G.$$

Define

$$N : L^\infty(G) \to L^\infty(G\times G) \quad \text{by} \quad N(f)(s,t) = f(ts^{-1}).$$

We will often use the fact that if $u \in A(G)$ then $N(u) \in \mathfrak{S}(G)$. More generally, the set of all continuous functions $u : G \to \mathbb{C}$ such that $N(u) \in \mathfrak{S}(G)$ coincides with the algebra $M^{\mathrm{cb}}A(G)$ of all *completely bounded*, or *Herz–Schur, multipliers* of $A(G)$ [3], [19] (see also [11]). For $v \in A(G)$ and $T \in \mathrm{VN}(G)$, let $v \cdot T \in \mathrm{VN}(G)$ be the element of $\mathrm{VN}(G)$ given by

$$\langle v \cdot T, u\rangle = \langle T, vu\rangle, \qquad u \in A(G);$$

we have $v \cdot T = S_{N(v)}(T)$ (see, e.g., [13]).

We denote by $\mathcal{S}_p(H)$ the Schatten p-class on a Hilbert space H (here, $1 \le p < \infty$), and we let $S_\infty(H)$ be the space of all compact operators on H. If H is clear from the context, we simply write $\mathcal{S}_p$. We write $\|T\|_p$ for the Schatten p-norm of an element $T \in \mathcal{S}_p$, $1 \le p < \infty$, and let $\|T\|_\infty = \|T\|$ for the usual operator norm of an element $T \in \mathcal{S}_\infty$.

For a function $h \in L^2(G \times G)$, we let $T_h \in \mathcal{S}_2(L^2(G))$ be the operator given by

$$T_h(\xi)(y) = \int_G h(y,x)\xi(x)\,dx, \qquad \xi \in L^2(G),\ y \in G. \tag{2.3}$$

We call the function h the *integral kernel* of T_h. We note that if $h \in T(G)$ then $T_h \in \mathcal{S}_1(L^2(G))$; conversely, for every operator $T \in \mathcal{S}_1(L^2(G))$ there exists $h \in T(G)$ such that $T = T_h$.

For a measure space (X, μ) and a function $a \in L^\infty(X, \mu)$, we let M_a denote the (bounded) operator on $L^2(X)$ of multiplication by a, and P_K the multiplication by the characteristic function χ_K of a measurable subset $K \subseteq X$.

3. Definitions and properties. Let G be a locally compact group. For each $1 \le p \le \infty$, let

$$\mathcal{S}_p(G) = \{T \in C_r^*(G) : P_K T P_K \in \mathcal{S}_p \text{ for all compact subsets } K \subseteq G\}.$$

Note first that if $f \in C_c(G)$ then $P_K\lambda(f)P_K$ is an integral operator with integral kernel

$$(s,t) \mapsto \chi_{K\times K}(s,t)\Delta(t)^{-1}f(st^{-1}).$$

Thus, $P_K\lambda(f)P_K$ is a Hilbert–Schmidt operator. Since every $T \in C_r^*(G)$ can be approximated in the operator norm by operators of the form $\lambda(f)$ with $f \in C_c(G)$, we conclude that $P_K T P_K \in \mathcal{S}_\infty$ whenever $T \in C_r^*(G)$ and $K \subseteq G$ is compact; thus, $\mathcal{S}_\infty(G) = C_r^*(G)$.

Remarks. (i) Let $v \in M^{\mathrm{cb}}A(G)$ and $T \in \mathcal{S}_p(G)$. Then $v \cdot T \in \mathcal{S}_p(G)$. Indeed, for every compact set $K \subseteq G$, we have

$$P_K(v \cdot T)P_K = P_K S_{N(v)}(T)P_K = S_{N(v)}(P_K T P_K) \in \mathcal{S}_p,$$

since Schur multipliers leave $\mathcal{S}_p$ invariant (the latter fact can be easily seen by using a complex interpolation argument; see [2, 16] and the proof of Theorem 3.3).

(ii) If $p \leq q$ then $\mathcal{S}_p(G) \subseteq \mathcal{S}_q(G)$.

(iii) If G is discrete, then $K \subseteq G$ is compact precisely when it is finite; thus, in this case, $\mathcal{S}_p(G) = C_r^*(G)$ for all values of p.

(iv) If G is compact then $\mathcal{S}_p(G) = C_r^*(G) \cap \mathcal{S}_p$. Indeed, the inclusion $\mathcal{S}_p \cap C_r^*(G) \subseteq \mathcal{S}_p(G)$ holds trivially for any G. If G is compact and $T \in \mathcal{S}_p(G)$ then, taking $K = G$ in the definition of $\mathcal{S}_p(G)$, we see that $T \in \mathcal{S}_p$.

In case G is compact, the previous paragraph shows that $\mathcal{S}_p(G)$ is an ideal of $C_r^*(G)$. Moreover, the inclusion $S_p(G) \subseteq S_q(G)$, $p < q$, is proper if G is infinite (see Remark 5.2). We do not know whether the spaces $\mathcal{S}_p(G)$ are ideals for other classes of locally compact groups G.

(v) The identity from Remark (iv) fails when G is not compact. Indeed, it is known (see, e.g., [20]) that in this case $\mathrm{VN}(G) \cap \mathcal{S}_\infty = \{0\}$.

(vi) Since for any compact subset $K \subseteq G$ and $f \in C_c(G)$ the operator $P_K \lambda(f) P_K$ is Hilbert–Schmidt, we have $\lambda(C_c(G)) \subseteq \mathcal{S}_2(G)$.

(vii) Let G be a compact abelian group and $\widehat{G}$ be its dual group. Then $\mathrm{VN}(G)$ and $C_r^*(G)$ can be identified, via Fourier transform, with the spaces $\ell^\infty(\widehat{G})$ and $c_0(\widehat{G})$. It is easily seen that, under this identification, $\mathcal{S}_p(G)$ is sent onto the sequence space $\ell_p(\widehat{G})$. Thus, in this case we have $\mathcal{S}_p(G) = \mathrm{VN}(G) \cap \mathcal{S}_p = C_r^*(G) \cap \mathcal{S}_p$.

(viii) Let $G = \mathbb{R}$ and $L \subseteq \widehat{\mathbb{R}} = \mathbb{R}$ be a compact interval. Then $\chi_L \in L^\infty(\widehat{\mathbb{R}}) \setminus C_0(\hat{\mathbb{R}})$. Thus, if T is the operator in $\mathrm{VN}(\mathbb{R})$ corresponding to χ_L via Fourier transform, then $T \notin C_r^*(\mathbb{R})$. However, if $K \subseteq \mathbb{R}$ is compact then $P_K T P_K$ is easily seen to be an integral operator with integral kernel

$$(s,t) \mapsto \chi_{K\times K}(s,t) \int_L e^{-ix(s-t)}\, dx.$$

Since $|\int_L e^{-ix(s-t)}\, dx| \leq m(L)$ for all $s,t \in \mathbb{R}$, the operator $P_K T P_K$ belongs to $\mathcal{S}_2$. This example shows that, in contrast to the compact case, replacing $C_r^*(G)$ by $\mathrm{VN}(G)$ in the definition of $\mathcal{S}_p(G)$ will in general yield different spaces.

Recall that, given a closed subset $E \subseteq G$, $I(E)$ (resp. $J(E)$) is the largest (resp. the smallest) ideal of $A(G)$ with null set E:

$$I(E) = \{u \in A(G) : u(s) = 0,\ s \in E\}$$

and

$$J(E) = \overline{\{u \in A(G) : u \text{ has compact support disjoint from } E\}}.$$

For $J \subseteq A(G)$ we denote by $J^\perp$ the annihilator of J in $\mathrm{VN}(G)$.

Sets of multiplicity (or M-sets) in (general) locally compact groups were introduced in [4] (see also [5]), while in [18], the notion of M_1-set was defined. We next formulate p-versions of these concepts.

DEFINITION 3.1. A closed subset $E \subseteq G$ will be called:

(i) an *M^p-set* (or a *set of p-multiplicity*) if

$$J(E)^\perp \cap \mathcal{S}_p(G) \neq \{0\};$$

(ii) an *M_1^p-set* if

$$I(E)^\perp \cap \mathcal{S}_p(G) \neq \{0\}.$$

It is clear that every M_1^p-set is an M^p-set, and that M_1^∞-sets (resp. M^∞-sets) coincide precisely with M_1-sets (resp. M-sets) studied in [18].

Recall that the *support* $\operatorname{supp}(T)$ of an operator $T \in \mathrm{VN}(G)$ is defined by letting

$$\operatorname{supp}(T) = \{t \in G : u \cdot T \neq 0 \text{ whenever } u \in A(G) \text{ and } u(t) \neq 0\}.$$

Note that $J(E)^\perp$ coincides with the space of all operators $T \in \mathrm{VN}(G)$ for which $\operatorname{supp}(T) \subseteq E$. Hence a subset E is an M^p-set if and only if there exists a non-zero operator T in $S_p(G)$ with $\operatorname{supp}(T) \subseteq E$.

Our next aim is to define operator versions of sets of p-multiplicity. We first recall some concepts from [1] and [7]. Given standard measure spaces (X, μ) and (Y, ν), a subset E of $X \times Y$ is called *ω-open* if it is marginally equivalent to the union of a countable set of Borel rectangles. The complements of ω-open sets are called *ω-closed*. A function $w : X \times Y \to \mathbb{C}$ is called *ω-continuous* if $w^{-1}(U)$ is an ω-open set for every open set $U \subseteq \mathbb{C}$. If $F \subseteq X \times Y$ is an ω-closed set, an operator $T \in \mathcal{B}(L^2(X), L^2(Y))$ is said to be *supported on* F if

$$(A \times B) \cap F \simeq \emptyset \ \Rightarrow \ P_B T P_A = 0$$

for all measurable rectangles $A \times B \subseteq X \times Y$. A *masa-bimodule* is a subspace $\mathcal{U} \subseteq \mathcal{B}(L^2(X), L^2(Y))$ such that $\mathcal{D}_Y \mathcal{U} \mathcal{D}_X \subseteq \mathcal{U}$ (where $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$) is the masa multiplication of $L^\infty(X)$ (resp. $L^\infty(Y)$). Given a masa-bimodule $\mathcal{U}$, there exists a smallest, up to marginal equivalence, ω-closed subset $\kappa \subseteq X \times Y$ such that every operator in $\mathcal{U}$ is supported by F; we call F the *support* of $\mathcal{U}$. Given an ω-closed set $\kappa \subseteq X \times Y$, there exist [1], [7] a largest weak* closed masa-bimodule $\mathfrak{M}_{\max}(\kappa)$ and a smallest weak* closed masa-bimodule $\mathfrak{M}_{\min}(\kappa)$ with support κ. The masa-bimodule $\mathfrak{M}_{\max}(\kappa)$ is the space of all $T \in \mathcal{B}(L^2(X), L^2(Y))$ supported on κ.

DEFINITION 3.2. An ω-closed subset $\kappa \subseteq X \times Y$ will be called

(i) an *operator M^p-set* (or a *set of operator p-multiplicity*) if

$$\mathfrak{M}_{\max}(\kappa) \cap \mathcal{S}_p \neq \{0\};$$

(ii) an *operator M_1^p-set* if

$$\mathfrak{M}_{\min}(\kappa) \cap \mathcal{S}_p \neq \{0\}.$$

REMARKS. (i) Note that if $1 \le p \le 2$ then the two notions introduced in Definition 3.2 agree; this follows from the fact that every Hilbert–Schmidt operator is *pseudo-integral*, while the pseudo-integral operators supported on a subset κ are contained (in fact, weak* dense) in $\mathfrak{M}_{\min}(\kappa)$. We refer the reader to [1] for the definition of and more details about the class of pseudo-integral operators.

(ii) A subset $\kappa \subseteq X \times Y$ is an operator M^2-set if and only if $(\mu \times \nu)(\kappa) > 0$; indeed, the latter condition is equivalent to the existence of non-zero functions $h \in L^2(X \times Y)$ supported on κ.

In [18] we established a connection between sets of multiplicity and sets of operator multiplicity. The next theorem is a generalisation of this result to sets of p-multiplicity. For $\varphi \in T(G)$, let $E_\varphi : \mathcal{B}(L^2(G)) \to \mathrm{VN}(G)$ be the map given by

$$\langle E_\varphi(T), u\rangle = \langle T, \varphi N(u)\rangle, \qquad T \in \mathcal{B}(L^2(G)),\ u \in A(G),$$

where the pairing on the left hand side is the one between $\mathrm{VN}(G)$ and $A(G)$, and on the right hand side, the one between $\mathcal{B}(L^2(G))$ and $T(G)$. It was proved in [18, Theorem 3.8] that $E_\varphi(T) \in C_r^*(G)$ for any $\varphi \in T(G)$ whenever T is compact.

For $E \subseteq G$, we let

$$E^* = \{(s,t) : ts^{-1} \in E\} \subseteq G \times G.$$

We will assume, for the rest of the paper, that G is *second countable.*

THEOREM 3.3. *Let G be a locally compact group, $E \subseteq G$ be a closed subset and $p \ge 1$. The following are equivalent:*

(i) *E is an M^p-set (resp. an M_1^p-set);*
(ii) *E^* is an operator M^p-set (resp. an operator M_1^p-set).*

Proof. (i)⇒(ii). Suppose that $E \subseteq G$ is an M^p-set and let T be a non-zero operator in $J(E)^\perp \cap \mathcal{S}_p(G)$. Then there exists a compact set $K \subseteq G$ such that $P_K T P_K$ is non-zero; by [18, Lemma 3.11], $P_K T P_K \in \mathfrak{M}_{\max}(E^*) \cap \mathcal{S}_p$.

Let $E \subseteq G$ be an M_1^p-set. The proof of [18, Theorem 3.12(b)] shows that $I(E)^\perp \subseteq \mathfrak{M}_{\min}(E^*)$. As in the previous paragraph, one can find a non-zero operator in $\mathfrak{M}_{\min}(E^*) \cap \mathcal{S}_p$.

(ii)⇒(i). Assume that E^* is an operator M^p-set; we will show that E is an M^p-set. If $p = \infty$, this follows from [18, Theorem 3.11]. Let $p = 1$ and T be a non-zero trace class operator in $\mathfrak{M}_{\max}(E^*)$; by virtue of (2.3) and the remark following it, write $T = T_h$, where $h = \sum_{i=1}^\infty f_i \otimes g_i$, with $\sum_{i=1}^\infty \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^\infty \|g_i\|_2^2 < \infty$.

Fix $\varphi \in T(G) \cap \mathfrak{S}(G)$ such that the function $\psi = \varphi(1 \otimes \Delta)$ belongs to $\mathfrak{S}(G)$. We will show that $E_\varphi(T) \in \mathcal{S}_1(G)$. For every $u \in A(G)$, we have

$$\begin{aligned}\langle E_\varphi(T), u\rangle = \langle T, \varphi N(u)\rangle &= \iint_{G\times G} h(s,t)\varphi(s,t)u(ts^{-1})\,ds\,dt\\ &= \iint_{G\times G} h(r^{-1}t,t)\varphi(r^{-1}t,t)u(r)\Delta(tr^{-1})\,dr\,dt\\ &= \int_G\Big(\int_G h(r^{-1}t,t)\varphi(r^{-1}t,t)\Delta(t)\,dt\Big)u(r)\Delta(r^{-1})\,dr.\end{aligned}$$

By assumption, $\psi \in \mathfrak{S}(G)$ and hence $\psi h \in T(G)$. By Proposition 2.1,

$$P(\psi h)(r) = \int_G h(r^{-1}t,t)\varphi(r^{-1}t,t)\Delta(t)\,dt$$

and hence

$$\langle E_\varphi(T), u\rangle = \int_G P(\psi h)(r)u(r)\Delta(r^{-1})\,dr. \tag{3.1}$$

Let $\xi, \eta \in L^2(G)$ be such that $u(r) = (\lambda_r(\xi), \eta)$ for all $r \in G$. Then, by (3.1),

$$\begin{aligned}(E_\varphi(T)\xi,\eta) &= \langle E_\varphi(T), u\rangle\\ &= \iint_{G\times G} P(\psi h)(r)\Delta(r^{-1})\xi(r^{-1}x)\overline{\eta(x)}\,dr\,dx\\ &= \iint_{G\times G} P(\psi h)(xy^{-1})\Delta(yx^{-1})\xi(y)\overline{\eta(x)}\Delta(y^{-1})\,dy\,dx\\ &= \iint_{G\times G} P(\psi h)(xy^{-1})\Delta(x^{-1})\xi(y)\overline{\eta(x)}\,dy\,dx.\end{aligned} \tag{3.2}$$

Let

$$w(x,y) = P(\psi h)(xy^{-1})\Delta(x^{-1}), \qquad x, y \in G. \tag{3.3}$$

Identity (3.2) shows that w is an integral kernel and $E_\varphi(T) = T_w$. If $K \subseteq G$ is compact then $P_K T_w P_K = T_{w\chi_{K\times K}}$ and

$$w\chi_{K\times K} = \hat{N}(P(\psi h))((\Delta^{-1}\chi_K)\otimes\chi_K), \tag{3.4}$$

where $\hat{N}(v)(s,t) = v(st^{-1})$ for $s, t \in G$. We have $P(\psi h) \in A(G)$; thus $N(P(\psi h)) \in \mathfrak{S}(G)$ and hence $\hat{N}(P(\psi h)) \in \mathfrak{S}(G)$. Since $(\Delta^{-1}\chi_K)\otimes\chi_K \in T(G)$, identity (3.4) shows that $w\chi_{K\times K} \in T(G)$ and hence $P_K E_\varphi(T) P_K$ is in $\mathcal{S}_1$. Thus, $E_\varphi(T) \in \mathcal{S}_1(G)$.

By [18, Lemma 3.10], there exist $c, d \in L^2(G)$ such that $E_{c\otimes d}(T) \neq 0$. Since the space $\mathcal{F}$ of all compactly supported functions in $L^\infty(G)$ is dense in $L^2(G)$, the continuity of the map $\varphi \mapsto E_\varphi(T)$ and [18, Proposition 3.8] imply that we may choose c and d from $\mathcal{F}$. However, in this case $d\Delta \in L^\infty(G)$ and hence $(c\otimes d)(1\otimes\Delta) \in \mathfrak{S}(G)$. Letting $\varphi = c\otimes d$, we then see by the previous paragraphs that $E_\varphi(T) \neq 0$, and by the proof of [18, Theorem 3.11] that $E_\varphi(T) \in J(E)^\perp$. It follows that E is an M^1-set.

To prove the statement for an arbitrary p, we use complex interpolation. Recall [16] that $(\mathcal{S}_1, \mathcal{S}_\infty)$ is a compatible couple, and $\mathcal{S}_p$ coincides with the interpolation space between $\mathcal{S}_1$ and $\mathcal{S}_\infty$ with parameter $\theta = p^{-1}$. Let, as above, $\varphi \in T(G) \cap \mathfrak{S}(G)$ be such that the function $\psi = \varphi(1 \otimes \Delta)$ is an element of $\mathfrak{S}(G)$. For a fixed compact set $K \subseteq G$ and $p = 1, \infty$, let $\Phi_p : \mathcal{S}_p \to \mathcal{B}(L^2(G))$ be the operator given by

$$\Phi_p(T) = P_K E_\varphi(T) P_K, \quad T \in \mathcal{S}_p.$$

By the previous paragraphs, the image of Φ_1 is in $\mathcal{S}_1$. Moreover,

$$\begin{aligned} \|\Phi_1(T)\|_{\mathcal{S}_1} &= \|\chi_{K\times K} w\|_{T(G)} = \|\hat{N}(P(\psi h))((\Delta^{-1}\chi_K) \otimes \chi_K)\|_{T(G)} \\ &\le \|\hat{N}(P(\psi h))\|_{\mathfrak{S}(G)} \|((\Delta^{-1}\chi_K) \otimes \chi_K)\|_{T(G)} \\ &\le \|P(\psi h)\|_{A(G)} \|((\Delta^{-1}\chi_K) \otimes \chi_K)\|_{T(G)} \\ &\le \|\psi h\|_{T(G)} \|((\Delta^{-1}\chi_K) \otimes \chi_K)\|_{T(G)} \\ &\le m(K) \|\psi\|_{\mathfrak{S}(G)} \|h\|_{T(G)} \|\Delta^{-1}\chi_K\|_\infty \\ &= m(K) \|\psi\|_{\mathfrak{S}(G)} \|\Delta^{-1}\chi_K\|_\infty \|T\|_{\mathcal{S}_1}, \end{aligned}$$

which shows that the operator $\Phi_1 : \mathcal{S}_1 \to \mathcal{S}_1$ is bounded. On the other hand, the image of Φ_∞ is in $\mathcal{S}_\infty$ and, by [18, Theorem 3.8],

$$\|\Phi_\infty(T)\| \le \|\varphi\|_{T(G)} \|T\|, \quad T \in \mathcal{S}_\infty.$$

By complex interpolation, the image of the operator Φ_p is in $\mathcal{S}_p$. The proof is now completed by choosing φ for which $E_\varphi(T)$ is non-zero.

We have thus shown that if E^* is an operator M^p-set then E is an M^p-set. The proof of the case where E^* is an operator M_1^p-set follows similar arguments and uses the fact that $E_\varphi(T)$ belongs to $I(E)^\perp$ if $T \in \mathfrak{M}_{\min}(E^*)$ (see [18, Theorem 3.11]). ■

Corollary 3.4. *A closed subset E of a locally compact group G is an M^1-set if and only if it has a non-empty interior.*

Proof. Suppose that E is an M^1-set. By Theorem 3.3, $\mathfrak{M}_{\max}(E^*)$ contains a non-zero trace class operator; by [7, Theorem 6.7], E^* contains a non-trivial measurable rectangle, say, $\alpha \times \beta$. Thus $\beta\alpha^{-1} \subseteq E$. By Steinhaus' Theorem, $\beta\alpha^{-1}$, and hence E, has a non-empty interior.

Conversely, assume that U is an open subset of E; we may further assume that U has a compact closure contained in E. Let $u \in A(G)$ be a function supported in U; then $u \in L^1(G)$ and thus $\lambda(u) \in C_r^*(G)$. It is easy to see that $\lambda(u) \in J(E)^\perp$. Let $K \subseteq G$ be a compact set. Then $P_K \lambda(u) P_K$ is an integral operator with integral kernel

$$(t, s) \mapsto u(ts^{-1})\chi_K(t)\chi_K(s)\Delta(s)^{-1}.$$

The function $(t, s) \mapsto \chi_K(t)\chi_K(s)\Delta(s)^{-1}$ belongs to $T(G)$ since χ_K is compactly supported and Δ is continuous. Since $N(u) \in \mathfrak{S}(G)$, we conclude that the function $(t,s) \mapsto u(ts^{-1})\chi_K(t)\chi_K(s)\Delta(s)^{-1}$ belongs to $T(G)$; since this holds for all compact sets K, we conclude that $\lambda(u) \in \mathcal{S}_1(G)$. ∎

COROLLARY 3.5. *A closed subset E of a locally compact group G is an M^2-set if and only if it has positive Haar measure.*

Proof. Note that $m(E) > 0$ if and only if $m \times m(E^*) > 0$. The claim follows from Theorem 3.3 and Remark (ii) after Definition 3.2. ∎

We next include some examples.

EXAMPLES. (i) J. Froelich [10, pp. 13–14] has shown that there exists a closed set $E \subseteq \mathbb{T}$ of Lebesgue measure zero which supports a non-zero measure μ whose Fourier transform vanishes at infinity but which does not support a non-zero pseudomeasure with Fourier transform in ℓ^p. The set E is an M-set that is not an M^p-set for any $p \geq 1$.

(ii) Let $\mathbb{T}$ be the group of the unit circle, realised additively as $\mathbb{R}/2\pi\mathbb{Z}$. We identify $\mathbb{T}^n$ with $[-\pi,\pi)^n$, and view the sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ as a subset of $\mathbb{T}^n$. Let μ be the normalised surface area measure of S^{n-1}. A direct calculation (see, e.g., [21, p. 154]) shows that

$$\hat{\mu}(k) = C|k|^{-(n-2)/2}J_{(n-2)/2}(|k|)$$

for some constant C, where $\hat{\mu}$ is the Fourier transform of μ and $J_{(n-2)/2}$ is a Bessel function. As $|J_\nu(r)| \leq C_\nu r^{-1/2}$ (see [9, Theorem 5.1]), for large enough $r > 0$, we obtain

$$\hat{\mu}(k) = O(1/|k|^{(n-1)/2}) \quad \text{as } |k| \to \infty.$$

We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n} |\hat{\mu}(k)|^p &\leq C \sum_{k\in\mathbb{Z}^n,\,|k|\geq 1} \frac{1}{|k|^{p(n-1)/2}} \leq C \int_{x\in\mathbb{R}^n,\,|x|\geq 1} \frac{1}{|x|^{(n-1)p/2}}\,dx \\ &= \int_1^\infty \frac{r^{n-1}}{r^{(n-1)p/2}}\,dr.\end{aligned}$$

Therefore for $p > 2$ and $n > 1 + 2/(p-2)$, the sequence $\{\hat{\mu}(k)\}_{k\in\mathbb{Z}^n}$ belongs to $\ell^p(\mathbb{Z}^n)$ and hence $\lambda(\mu) \in \mathcal{S}_p(\mathbb{T}^n)$. As $\lambda(\mu) \in I(S^{n-1})^\perp$, we conclude that $S^{n-1} \subseteq \mathbb{T}^n$ is an M_1^p-set for $p > 2$ whenever $n > 1 + 2/(p-2)$. Note that, by Corollary 3.5, S^{n-1}, $n > 1$, is not an M_1^2-set since S^{n-1} has zero Lebesgue measure (see Corollary 3.5).

(iii) There exists a closed set $E \subseteq \mathbb{T}$ of Lebesgue measure zero and a non-zero measure μ with $\operatorname{supp}\mu = E$ such that $\widehat{\mu} \in \ell^p$ for any $p > 2$ (see [23, Theorem 10.12]). Hence E is an M_1^p-set for all $p > 2$ but not an M^2-set (see Corollary 3.5).

(iv) In [7, p. 579] an example is given of a set $E \subseteq \mathbb{T}$ and a function $f \in L^2(\mathbb{T})$ such that $\lambda(f)$ is supported in E and $\lambda(f) \in \mathcal{S}_p$ for any $p > 1$

but $\lambda(f) \notin \mathcal{S}_1$. Remark (iv) from the start of Section 3 implies that E is an M^p-set for all $p > 1$ but not an M^1-set.

Following the established terminology in the classical case, let us call a closed subset $E \subseteq G$ a set of *p-uniqueness* if it is not a set of p-multiplicity. In the remainder of this section, we apply Theorem 3.3 to establish some preservation results for sets of p-uniqueness and sets of p-multiplicity.

PROPOSITION 3.6. *Let G be a locally compact group, $1 \le p \le \infty$ and $E_i \subseteq G$ be a closed subset, $i = 1, 2$. Then $E_1 \cup E_2$ is a set of p-uniqueness if and only if E_1 and E_2 are sets of p-uniqueness.*

Proof. To see the "if" part, it suffices, by Theorem 3.3, to show that if $\mathfrak{M}_{\max}(E_i^*) \cap \mathcal{S}_p = \{0\}$ for $i = 1, 2$, then $\mathfrak{M}_{\max}((E_1 \cup E_2)^*) \cap \mathcal{S}_p = \{0\}$.

Let $D_i = \mathfrak{M}_{\max}(E_i^*)_\perp$, $i = 1, 2$, and suppose $T \in \mathfrak{M}_{\max}((E_1 \cup E_2)^*) \cap \mathcal{S}_p$. For each $\theta_i \in D_i \cap \mathfrak{S}(G)$, $i = 1, 2$, we have $\theta_1\theta_2 \in D_1 \cap D_2$, and since $D_1 \cap D_2 = \mathfrak{M}_{\max}((E_1 \cup E_2)^*)_\perp$, we conclude that

$$\langle S_{\theta_1}(T), \theta_2\rangle = \langle T, \theta_1\theta_2\rangle = 0.$$

However, Schur multipliers leave $\mathcal{S}_p$ invariant; since $D_2 \cap \mathfrak{S}(G)$ is dense in D_2 and E_2 is a set of p-uniqueness, it follows that $S_{\theta_1}(T) = 0$. Thus, $\langle T, \theta_1\rangle = 0$ for all $\theta_1 \in D_1 \cap \mathfrak{S}(G)$. Now the density of $D_1 \cap \mathfrak{S}(G)$ in D_1 and the fact that E_1 is a set of p-uniqueness imply that $T = 0$.

The "only if" part follows from the fact that any closed subset of a set of p-uniqueness is a set of p-uniqueness. ∎

PROPOSITION 3.7. *Let G_i be a locally compact group, $1 \le p \le \infty$ and $E_i \subseteq G_i$ be a closed set, $i = 1, 2$. If E_i is an M^p-set (resp. M_1^p-set), $i = 1, 2$, then $E_1 \times E_2 \subseteq G_1 \times G_2$ is an M^p-set (resp. M_1^p-set).*

Proof. Let

$$\rho : G_1 \times G_1 \times G_2 \times G_2 \to G_1 \times G_2 \times G_1 \times G_2$$

be the map given by $\rho(s_1, t_1, s_2, t_2) = (s_1, s_2, t_1, t_2)$. We have $(E_1 \times E_2)^* = \rho(E_1^* \times E_2^*)$.

By Theorem 3.3, it suffices to show that $\rho(E_1^* \times E_2^*)$ is an operator M^p-set (resp. an operator M_1^p-set). Denoting by $\otimes$ the algebraic tensor product, we have (see [18])

$$\mathfrak{M}_{\max}(E_1^*) \otimes \mathfrak{M}_{\max}(E_2^*) \subseteq \mathfrak{M}_{\max}(\rho(E_1^* \times E_2^*))$$

and

$$\mathfrak{M}_{\min}(E_1^*) \otimes \mathfrak{M}_{\min}(E_2^*) \subseteq \mathfrak{M}_{\min}(\rho(E_1^* \times E_2^*)).$$

It follows that if $T_i \in \mathfrak{M}_{\max}(E_1^*) \cap \mathcal{S}_p$ (resp. $S_i \in \mathfrak{M}_{\max}(E_1^*) \cap \mathcal{S}_p$), $i = 1, 2$, then $T_1 \otimes T_2$ (resp. $S_1 \otimes S_2$) is a non-zero operator in $\mathfrak{M}_{\max}(\rho(E_1^* \times E_2^*)) \cap \mathcal{S}_p$ (resp. $\mathfrak{M}_{\min}(\rho(E_1^* \times E_2^*)) \cap \mathcal{S}_p$). ∎

We finish this section with explicit descriptions of the spaces $\mathcal{S}_1(G)$ and $\mathcal{S}_2(G)$. For the next lemma, recall that if G is compact then $\mathfrak{S}(G) \subseteq T(G)$ and hence $N(u) \in T(G)$ for every $u \in A(G)$.

Lemma 3.8. *Let G be a compact group. If $u \in A(G)$ then $T_{N(\check{u})} = \lambda(u)$.*

Proof. If $\xi, \eta \in L^2(G)$ then, using the unimodularity of G, we have

$$\begin{aligned}(T_{N(\check{u})}\xi, \eta) &= \iint\limits_{G\times G} N(\check{u})(t,s)\xi(s)\overline{\eta(t)}\,ds\,dt \\ &= \iint\limits_{G\times G} u(ts^{-1})\xi(s)\overline{\eta(t)}\,ds\,dt \\ &= \iint\limits_{G\times G} u(r)\xi(r^{-1}t)\overline{\eta(t)}\,dr\,dt = (\lambda(u)(\xi), \eta). \ \blacksquare\end{aligned}$$

We say that a function f *belongs to $A(G)$ at the point $t \in G$* if there exists a neighbourhood U of t and a function $u \in A(G)$ such that $f(s) = u(s)$ for all $s \in U$. We let $A(G)^{\mathrm{loc}}$ denote the set of all functions that belong to $A(G)$ at every point $t \in G$.

Lemma 3.9. *Let $w : G \times G \to \mathbb{C}$ be a measurable function (with respect to product measure) such that, for every $r \in G$, $w(xr, yr) = w(x, y)$ for marginally almost all (x, y). Then there exists a measurable function $u : G \to \mathbb{C}$ such that, up to a null set, $w = N(u)$.*

Proof. Let $x \in G$, and $v_x : G \to \mathbb{C}$ be given by $v_x(s) = w(s, xs)$, $s \in G$. For $r \in G$, the set

$$\{(y, z) \in G \times G : w(yr, zr) \neq w(y, z)\}$$

is marginally null. In particular,

$$\Lambda_x := \{(s, xs) \in G \times G : w(sr, xsr) \neq w(s, xs)\}$$

is marginally null. This easily implies that the set $\{s \in G : (s, xs) \in \Lambda_x\}$ is null and hence $v_x(sr) = v_x(s)$ for almost all s. Using arguments similar to [14, Lemma 3.2], one can prove that the function $f_x(s, r) = v_x(sr)$ is $m \times m$-measurable. For every $r \in G$, the set $\{s \in G : v_x(sr) = v_x(s)\}$ is null. By the Fubini Theorem,

$$\iint |v_x(sr) - v_x(s)|\,ds\,dr = \int\Big(\int |v_x(sr) - v_x(s)|\,ds\Big)\,dr = 0,$$

giving $v_x(sr) = v_x(s)$ for almost all pairs (s, r). Thus there exists $s_0 \in G$ such that $v_x(s_0 r) = v_x(s_0)$ for almost all $r \in G$. Hence, there exists $u(x) \in \mathbb{C}$ such that $v_x(s) = u(x)$ for almost all $s \in G$.

The function $u : G \to \mathbb{C}$ is measurable as the composition of w and the measurable functions $x \mapsto (s, xs)$. Since the functions w and $N(u)$ are equal almost everywhere on each set of the form $\{x\}^*$, applying the above arguments we conclude that $w = N(u)$ almost everywhere. $\blacksquare$

THEOREM 3.10. *Let G be a locally compact group. Then*

$$\mathcal{S}_1(G) = \{T \in C_r^*(G) : \text{there exists } u \in A(G)^{\mathrm{loc}} \text{ such that } P_K T P_K = T_{N(u)(\Delta^{-1}\otimes 1)\chi_{K\times K}} \text{ for each compact } K \subseteq G\}. \tag{3.5}$$

Moreover, if G is compact then

$$\mathcal{S}_1(G) = \{\lambda(u) : u \in A(G)\}.$$

Proof. We first show that $\mathcal{S}_1(G)$ is contained in the right hand side of (3.5). By assumption, for every compact set $K \subseteq G$ there exists a function $h_K \in T(G)$ such that $P_K T P_K = T_{h_K}$. If $L \subseteq G$ is compact then

$$\begin{aligned} T_{h_K(\chi_{K\cap L}\otimes\chi_{K\cap L})} &= P_{K\cap L} P_K T P_K P_{K\cap L} = P_{K\cap L} T P_{K\cap L} = T_{h_{K\cap L}} \\ &= P_{K\cap L} P_L T P_L P_{K\cap L} = T_{h_L(\chi_{K\cap L}\otimes\chi_{K\cap L})}. \end{aligned}$$

Thus, $h_K(\chi_{K\cap L}\otimes\chi_{K\cap L}) = h_L(\chi_{K\cap L}\otimes\chi_{K\cap L})$ almost everywhere. Since the functions $h_K(\chi_{K\cap L}\otimes\chi_{K\cap L})$ and $h_L(\chi_{K\cap L}\otimes\chi_{K\cap L})$ are ω-continuous, [17, Lemma 2.2] implies that they are equal up to a marginally null set. Let $(K_n)_{n=1}^\infty$ be an increasing sequence of compact sets such that $G = \bigcup_{n=1}^\infty K_n$. Setting $h(s,t) = h_{K_n}(s,t)$ if $s,t \in K_n$, we obtain a function $h : G\times G \to \mathbb{C}$, defined up to a marginally null set, which has the property that $h|_{K\times K}$ is marginally equivalent to $h_{K\times K}$ for every compact set $K \subseteq G$.

Let $L \subseteq G$ be a compact set and $\xi, \eta \in L^2(G)$ be supported on L. Fix $s \in G$ and let $M \subseteq G$ be a compact set containing both L and Ls^{-1}. Let $\rho : G \to \mathcal{B}(L^2(G))$, $s \mapsto \rho_s$, be the right regular representation given by $\rho_s\xi(x) = \sqrt{\Delta(s)}\,\xi(xs)$. Then

$$\begin{aligned} (T\rho_s\xi,\eta) &= (P_M T P_M \rho_s\xi,\eta) = (T_{h_M}\rho_s\xi,\eta) \\ &= \int_{G\times G} h(x,y)\xi(ys)\overline{\eta(x)}\sqrt{\Delta(s)}\,dx\,dy \\ &= \int_{G\times G} h(x,zs^{-1})\xi(z)\overline{\eta(x)}\sqrt{\Delta(s^{-1})}\,dx\,dz. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\rho_s T\xi,\eta) &= (T\xi,\rho_{s^{-1}}\eta) = (P_M T P_M\xi,\rho_{s^{-1}}\eta) = (T_{h_M}\xi,\rho_{s^{-1}}\eta) \\ &= \int_{G\times G} h(x',z)\xi(z)\overline{\eta(x's^{-1})}\sqrt{\Delta(s^{-1})}\,dx'\,dz \\ &= \int_{G\times G} h(xs,z)\xi(z)\overline{\eta(x)}\sqrt{\Delta(s)}\,dx\,dz. \end{aligned}$$

Since $T \in C_r^*(G)$, we have $T\rho_s = \rho_s T$, and hence

$$\sqrt{\Delta(s^{-1})}\,h(x,zs^{-1}) = \sqrt{\Delta(s)}\,h(xs,z)$$

for marginally almost all $(x,z) \in L\times L$. Since this holds for every compact set L, we deduce that $\sqrt{\Delta(s^{-1})}\,h(x,zs^{-1}) = \sqrt{\Delta(s)}\,h(xs,z)$ for marginally almost all $(x,z) \in G \times G$. Thus,

$$h(x,y) = \Delta(s)h(xs,ys)$$

for marginally almost all $(x,y) \in G \times G$.

Let $\tilde{h} : G \times G \to \mathbb{C}$ be given by

$$\tilde{h}(x,y) = \Delta(x)h(x,y).$$

If $s \in G$ then

$$\tilde{h}(xs,ys) = \Delta(xs)h(xs,ys) = \Delta(x)\Delta(s)h(xs,ys) = \Delta(x)h(x,y) = \tilde{h}(x,y)$$

for marginally almost all (x,y). By Lemma 3.9, there exists a measurable function $u : G \to \mathbb{C}$ such that, up to a null set, $\tilde{h} = N(u)$. Thus, up to a null set, $h = (\Delta^{-1} \otimes 1)N(u)$.

Note that, for every compact set $K \subseteq G$, we have $h\chi_{K\times K} \in T(G)$ and $P_K T P_K = T_{h\chi_{K\times K}}$. Thus, it remains to show that $u \in A(G)^{\mathrm{loc}}$. We see that $\tilde{h}\chi_{K\times K} \in T(G)$ for every compact set $K \subseteq G$. Since $T(G)$ consists of local Schur multipliers (see [17]), the σ-compactness of G implies that $\tilde{h}$ is a local Schur multiplier. By [18, Theorem 8.2], $u \in A(G)^{\mathrm{loc}}$.

To see that the right hand side of (3.5) is contained in $\mathcal{S}_1(G)$, note first that for any compact set L and $u \in A(G)^{\mathrm{loc}}$ there exists $v \in A(G)$ such that $u(t) = v(t)$ for any $t \in L$ (see the discussion before [18, Lemma 6.1]). Let K be a compact set and let $v \in A(G)$ be such that $u = v$ on $L = KK^{-1}$. Then $N(u)\chi_{K\times K} = N(v)\chi_{K\times K}$. Since $N(v)$ is a Schur multiplier and $(\Delta^{-1} \otimes 1)\chi_{K\times K} \in T(G)$, we find that

$$h(x,y) := N(u)(x,y)\Delta^{-1}(x)\chi_{K\times K}(x,y) \in T(G).$$

If G is compact then G is unimodular and $A(G) = A(G)^{\mathrm{loc}}$. By Lemma 3.8, $T_{N(\check{u})} = \lambda(u)$, and the proof is complete. ∎

The proof of the next proposition is similar to that of Theorem 3.10 and is omitted.

Proposition 3.11. *Let G be a locally compact group. The following are equivalent, for an operator $T \in C_r^*(G)$:*

(i) $T \in \mathcal{S}_2(G)$;
(ii) *there exists a measurable function $u : G \to \mathbb{C}$ such that, for every compact set $K \subseteq G$, we have*

$$N(u)\chi_{K\times K} \in L^2(G\times G) \quad \textit{and} \quad P_K T P_K = T_{N(u)(\Delta^{-1}\otimes 1)\chi_{K\times K}}.$$

4. A p-version of the Stone–von Neumann Theorem. The aim of this section is to establish the following p-version of the Stone–von Neumann Theorem [22, Theorem 4.23]. We let $\mathcal{D} = \{M_a : a \in C_0(G)\}$.

THEOREM 4.1. *Let G be a locally compact group and $1 \leq p \leq \infty$. Then the $\|\cdot\|_p$-closed $\mathcal{D}$-bimodule generated by $\mathcal{S}_p(G)$ coincides with $\mathcal{S}_p$.*

Proof. For $p = \infty$, the statement reduces to the Stone–von Neumann Theorem. Fix p with $1 \leq p < \infty$. Let

$$\mathcal{S}'_p(G) = \{T \in C^*_r(G) : P_K T, TP_K \in \mathcal{S}_p \text{ for every compact set } K \subseteq G\}$$

and

$$\mathcal{U}_p = \overline{\operatorname{span}\{M_a T M_b : a, b \in C_c(G),\, T \in \mathcal{S}'_p(G)\}}^{\|\cdot\|_p}.$$

We note first that $\mathcal{S}'_p(G) \subseteq \mathcal{S}_p(G)$ and, by the definition of $\mathcal{S}_p(G)$, we have

$$\mathcal{U}_p \subseteq \overline{\operatorname{span}\{M_a T M_b : a, b \in C_c(G),\, T \in \mathcal{S}_p(G)\}}^{\|\cdot\|_p} \subseteq \mathcal{S}_p.$$

It follows from the proof of Theorem 3.10, that for $u \in A(G) \cap C_c(G)$, $P_K\lambda(u), \lambda(u)P_K \in \mathcal{S}_1 \subseteq \mathcal{S}_p$, $p \geq 1$, which implies that both $\mathcal{S}'_p(G)$ and $\mathcal{U}_p$ are non-zero.

We claim that $\mathcal{U}_p$ is an ideal of $\mathcal{S}_\infty$. First note that if $b \in C_c(G)$ and $f \in L^1(G)$ then by the Stone–von Neumann Theorem $M_b\lambda(f)$ can be approximated in the operator norm by linear combinations $\sum_i \lambda(g_i^n)M_{c_i^n}$. Suppose that $T \in \mathcal{S}'_p(G)$, $S \in \mathcal{S}_\infty$ and $a, b, c \in C_c(G)$. Since $M_a T \in \mathcal{S}_p$ we have

$$\begin{aligned}\Big\|M_a T M_b \lambda(f) - M_a T \sum_i \lambda(g_i^n) M_{c_i^n}\Big\|_p \\ \leq \|M_a T\|_p \Big\|M_b\lambda(f) - \sum_i \lambda(g_i^n) M_{c_i^n}\Big\| \to 0\end{aligned}$$

and therefore $M_a T M_b \lambda(f) \in \mathcal{U}_p$. Again by the Stone–von Neumann Theorem, S can be approximated by linear combinations of operators of the form $\lambda(f)M_c$, where $f \in L^1(G)$ and $c \in C_c(G)$. Hence the corresponding linear combination of $M_a T M_b \lambda(f) M_c$ converges in $\|\cdot\|_p$-norm to $M_a T M_b S$. Thus, $M_a T M_b S \in \mathcal{U}_p$ and hence $\mathcal{U}_p$ is a right ideal of $\mathcal{S}_\infty$; similarly, $\mathcal{U}_p$ is a left ideal in $\mathcal{S}_\infty$. Thus, $\mathcal{U}_p$ is a non-zero ideal of $\mathcal{S}_\infty$, closed in the Schatten p-norm. This easily implies that

$$\mathcal{U}_p = \overline{\operatorname{span}\{M_a T M_b : a, b \in C_c(G),\, T \in \mathcal{S}_p(G)\}}^{\|\cdot\|_p} = \mathcal{S}_p. \blacksquare$$

5. The case of compact groups. In this section, we include a direct proof of Theorem 3.3, that is, a proof that does not use interpolation, in the case where G is a compact group. We first recall some notions from the Fourier theory for compact groups (see, e.g., [12]).

Let G be a compact group with dual $\widehat{G}$; thus, $\widehat{G}$ is a complete family of pairwise inequivalent continuous unitary representations $\pi : G \to \mathcal{B}(H_\pi)$

of G. We let $d_\pi = \dim(H_\pi)$. For $u \in A(G)$ we set

$$\hat{u}(\pi) = \int_G u(s)\pi(s^{-1})\,ds,$$

understood as a linear operator on the finite-dimensional space H_π. Then

$$\sum_{\pi\in\hat{G}} d_\pi \|\hat{u}(\pi)\|_1 < \infty,$$

where $\|\cdot\|_1$ is the trace norm. Moreover, the Fourier algebra $A(G)$ can be identified with the space of operator fields, indexed over $\widehat{G}$,

$$\Big\{(f(\pi))_{\pi\in\widehat{G}} : f(\pi) \in \mathcal{B}(H_\pi),\ \sum_{\pi\in\hat{G}} d_\pi\|f(\pi)\|_1 < \infty\Big\},$$

the identification being given by the map (which we call the Fourier transform) sending an element $u \in A(G)$ to $(\hat{u}(\pi))_{\pi\in\widehat{G}}$. Its inverse sends $(f(\pi))_{\pi\in\widehat{G}}$ to the function f given by $f(s) = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}(f(\pi)\pi(s))$ (where Tr denotes the trace). We can therefore identify the dual space of $A(G)$ with $\prod_{\pi\in\widehat{G}}\mathcal{B}(H_\pi)$ through the duality

$$\langle (T_\pi)_{\pi\in\widehat{G}}, u\rangle = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}(T_\pi \hat{u}(\pi)). \tag{5.1}$$

In particular, the evaluation functional at $s \in G$ corresponds to $(\pi(s))_{\pi\in\widehat{G}}$. The dual space of $A(G)$ is isomorphic to $\mathrm{VN}(G)$, and the identification of $\mathrm{VN}(G)$ with $\prod_{\pi\in\widehat{G}}\mathcal{B}(H_\pi)$ is given by $(T_\pi)_{\pi\in\widehat{G}} \mapsto \bigoplus_{\pi\in\widehat{G}} T_\pi^{(d_\pi)} \in \mathrm{VN}(G)$, where

$$T^{(k)} = \underbrace{T\oplus\cdots\oplus T}_{k}.$$

It follows from Remark (iv) at the start of Section 3 that

$$\mathcal{S}_p(G) = \Big\{T \in C_r^*(G) : \sum_{\pi\in\hat{G}} d_\pi \|T_\pi\|_p^p < \infty\Big\}. \tag{5.2}$$

Thus, $\mathcal{S}_p(G)$ is an ideal not only in $C_r^*(G)$ but also in $\mathrm{VN}(G)$.

Theorem 5.1. *Let G be a compact group, $p \geq 1$ and $E \subseteq G$ be a closed subset. The following are equivalent:*

(i) *E is an M^p-set (resp. an M_1^p-set);*
(ii) *E^* is an operator M^p-set (resp. an operator M_1^p-set).*

Proof. (i)⇒(ii) follows as in Theorem 3.3.

(ii)⇒(i). Suppose that E^* is an operator M^p-set. If $T \in \mathfrak{M}_{\max}(E^*)\cap\mathcal{S}_p$ is non-zero, by [18, Lemma 3.10] there exist $a, b \in L^\infty(G)$ such that $E_{a\otimes b}(T) \in C_r^*(G)$ is non-zero.

By the definition of the map $E_{a\otimes b}$ and Lemma 3.8, for $u \in A(G)$ we have

$$\begin{aligned}(5.3)\qquad \langle E_{a\otimes b}(T), u\rangle &= \langle T, N(u)(a\otimes b)\rangle = \langle M_b T M_a, N(u)\rangle \\ &= \mathrm{Tr}((M_b T M_a)T_{N(u)}) = \mathrm{Tr}((M_b T M_a)\lambda(\check{u})).\end{aligned}$$

Let $u \in A(G)$ be such that $\sum_{\pi\in\hat{G}} d_\pi \|\hat{u}(\pi)\|_q^q < \infty$, where q is conjugate to p. The space of all such elements is dense in $A(G)$ (indeed, it contains all elements of $A(G)$ whose Fourier transform is finitely supported). By the Peter–Weyl Theorem,

$$\lambda = \bigoplus_{\pi\in\hat{G}} \pi^{(d_\pi)}, \quad \text{where} \quad \pi^{(d_\pi)} = \underbrace{\pi\oplus\cdots\oplus\pi}_{d_\pi},$$

and $\lambda(\check{u})|_{H_\pi} = \hat{u}(\pi)$. Thus,

$$\|\lambda(\check{u})\|_q^q = \sum_{\pi\in\hat{G}} d_\pi \|\hat{u}(\pi)\|_q^q < \infty. \tag{5.4}$$

By (5.3) and (5.4),

$$|\langle E_{a\otimes b}(T), u\rangle| \le \|a\|_\infty \|b\|_\infty \|T\|_p \Big(\sum_{\pi\in\hat{G}} d_\pi \|\hat{u}(\pi)\|_q^q\Big)^{1/q}. \tag{5.5}$$

Let $S = E_{a\otimes b}(T)$. By (5.1), $\langle S, u\rangle = \sum_{\pi\in\hat{G}} d_\pi \,\mathrm{Tr}(S_\pi \hat{u}(\pi))$. We claim that

$$\sum_{\pi\in\hat{G}} d_\pi \|S_\pi\|_p^p < \infty. \tag{5.6}$$

In fact, let $S_\pi = V_\pi |S_\pi|$ be the polar decomposition of S_π. For any finite family $\mathcal{F} \subseteq \widehat{G}$, let $u \in A(G)$ be such that

$$\hat{u}(\pi) = \begin{cases} (|S_\pi|)^{p-1} V_\pi^*)/\Big(\sum_{\pi\in\mathcal{F}} d_\pi \|S_\pi\|_p^p\Big)^{(p-1)/p} & \text{if } \pi \in \mathcal{F}, \\ 0 & \text{if } \pi \notin \mathcal{F}. \end{cases}$$

We have

$$\sum_{\pi\in\widehat{G}} d_\pi \|\hat{u}(\pi)\|_q^q = \sum_{\pi\in\mathcal{F}} d_\pi \,\mathrm{Tr}(|S_\pi|^p) / \sum_{\pi\in\mathcal{F}} d_\pi \|S_\pi\|_p^p = 1$$

and

$$\begin{aligned}\sum_{\pi\in\widehat{G}} d_\pi \,\mathrm{Tr}(S_\pi \hat{u}(\pi)) &= \sum_{\pi\in\mathcal{F}} d_\pi \,\mathrm{Tr}(|S_\pi|^p) / \Big(\sum_{\pi\in\mathcal{F}} d_\pi \|S_\pi\|_p^p\Big)^{(p-1)/p} \\ &= \Big(\sum_{\pi\in\mathcal{F}} d_\pi \|S_\pi\|_p^p\Big)^{1/p}.\end{aligned}$$

Inequality (5.5) now implies

$$\Big(\sum_{\pi\in\mathcal{F}} d_\pi \|S_\pi\|_p^p\Big)^{1/p} \le \|a\|_\infty \|b\|_\infty \|T\|_p$$

for any finite $\mathcal{F}$. Inequality (5.6) follows; by (5.2), $E_{a\otimes b}(T) \in \mathcal{S}_p(G)$. By the proof of [18, Theorem 3.11], $E_{a\otimes b}(T) \in J(E)^{\perp}$, and the proof is complete for M^p-sets.

The proof of the statement for M_1^p-sets is similar and uses the fact that $E_{a\otimes b}(T) \in I(E)^{\perp}$ if $T \in \mathfrak{M}_{\min}(E^*)$. ∎

REMARK 5.2. If G is a compact infinite group, then $\mathcal{S}_p(G)$ is a proper ideal of $\mathcal{S}_q(G)$ if $p < q$. In fact, one can easily find $\{\alpha_\pi\}_{\pi\in\hat{G}} \subseteq \mathbb{C}$ such that $\sum_{\pi\in\hat{G}} d_\pi|\alpha_\pi|^p < \infty$ while $\sum_{\pi\in\hat{G}} d_\pi|\alpha_\pi|^q = \infty$. Letting now $T_\pi = \alpha_\pi P_\pi$, where P_π is a projection on a one-dimensional subspace of H_π, and $T = \bigoplus_{\pi\in\hat{G}} T_\pi^{(d_\pi)}$, we have $T \in \mathcal{S}_p(G)$ but $T \notin \mathcal{S}_q(G)$. For general locally compact groups the classes $\mathcal{S}_p(G)$ may coincide, e.g. in the case of discrete groups, where $\mathcal{S}_p(G)$, $p \geq 1$, are all equal to $C_r^*(G)$.

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