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Fréchet algebras, formal power series, and automatic continuity

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Abstract. We describe *all* those commutative Fréchet algebras which may be continuously embedded in the algebra $\mathbb{C}[[X]]$ in such a way that they contain the polynomials. It is shown that these algebras (except $\mathbb{C}[[X]]$ itself) always satisfy a certain equicontinuity condition due to Loy. Using this result, some applications to the theory of automatic continuity are given; in particular, the uniqueness of the Fréchet algebra topology for such algebras is established.

1. Introduction. Throughout the paper, "algebra" will mean a nonzero, complex commutative algebra with identity unless otherwise specified. A *Fréchet algebra* is a complete, metrizable locally convex algebra A whose topology τ may be defined by a sequence $(p_k)_{k\geq 1}$ (assumed increasing without loss of generality) of submultiplicative seminorms. We may refer to τ as "the Fréchet topology of A" in the following. The basic theory of Fréchet algebras was introduced in [4] and [16]. The principal tool for studying Fréchet algebras is the Arens–Michael representation, in which A is given by an inverse limit of Banach algebras A_k. We shall briefly describe this in the next section, in order to establish notation that will be used throughout the paper.

We write $\mathcal{F}$ for the algebra $\mathbb{C}[[X]]$ of all formal power series in an indeterminate X, with complex coefficients. The algebra $\mathcal{F}$ is a Fréchet algebra when endowed with the weak topology defined by the projections $\pi_m : \mathcal{F} \to \mathbb{C}$, $m \in \mathbb{Z}^+$, where $\pi_m(\sum_{n=0}^{\infty} \lambda_n X^n) = \lambda_m$. A defining sequence of seminorms for $\mathcal{F}$ is (p'_k), where $p'_k(\sum_{n=0}^{\infty} \lambda_n X^n) = \sum_{n=0}^{k} |\lambda_n|$ $(k \in \mathbb{N})$. A *Fréchet algebra of power series* is a subalgebra A of $\mathcal{F}$ such that A is a Fréchet algebra containing the indeterminate X and such that the inclusion map $A \hookrightarrow \mathcal{F}$ is continuous. Though Fréchet algebras of power series have been considered earlier by Loy [12] and [14], recently these algebras—and more generally, the power series ideas in general Fréchet algebras—have

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acquired significance in understanding the structure of Fréchet algebras ([2], [5], [6]). Thus, it is of interest to investigate the following:

(1) whether one can describe *all* those commutative Fréchet algebras which may be continuously embedded in $\mathcal{F}$ in such a way that they contain the polynomials,
(2) whether such algebras have a unique Fréchet topology.

In this paper we shall be concerned with the solution to the above problems. We shall show that if A is a commutative, unital Fréchet algebra containing a non-nilpotent, closed maximal ideal M such that: (i) $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$, and (ii) $\dim(M/\overline{M^2}) = 1$, then A is a Fréchet algebra of power series; the converse holds if the polynomials are dense in A. (See Theorem 3.1 below.) The solution to the second problem, presented in Section 4, is broken up into several results of independent interest. Precisely, we shall show that every Fréchet algebra of power series A ($\neq \mathcal{F}$) necessarily satisfies an equicontinuity condition: there is a sequence (γ_n) of positive reals such that $(\gamma_n^{-1}\pi_n)$ is equicontinuous [14]. (See Theorem 3.6 below.)

We remark that the uniqueness of the Fréchet topology of $\mathcal{F}$ was proved in [1]. This was a side result on the way to proving the surprising result that this algebra is normable. In 1971, Loy proved in [14] the uniqueness of the Fréchet space topology of certain topological algebras of formal power series. (The result in the Banach algebra case was also proved by Loy [13].) As a special case of that result, Bhatt and the author proved the uniqueness of the Fréchet topology of the Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^+, \boldsymbol{\omega})$ ([5, Corollary 3.4]). Thus, it is natural to suspect that a Fréchet algebra of power series has a unique Fréchet topology. Here [7] and [8] are good references: Dales proved the uniqueness of the Fréchet topology of $\mathcal{F}_n$ for each $n \in \mathbb{N}$, and posed this problem for a more general case in [7]; but he has not mentioned any result even for the single indeterminate case in [8]. As far as we know, the problem has been unsolved since 1971.

2. Fréchet algebras.

Let A be a Fréchet algebra, with its topology defined by an increasing sequence $(p_k)_{k\geq 1}$ of submultiplicative seminorms. For each k, let $q_k : A \to A/\ker p_k$ be the quotient map. Then $A/\ker p_k$ is naturally a normed algebra, normed by setting $\|x+\ker p_k\|_k = p_k(x)$ $(x \in A)$. We let $(A_k; \|\cdot\|_k)$ be the completion of $A/\ker p_k$. Then $d_k(x+\ker p_{k+1}) = x+\ker p_k$ $(x \in A)$ extends to a norm-decreasing homomorphism $d_k : A_{k+1} \to A_k$ such that

$$A_1 \xleftarrow{d_1} A_2 \xleftarrow{d_2} A_3 \leftarrow \cdots \leftarrow A_k \xleftarrow{d_k} A_{k+1} \leftarrow \cdots$$

is an inverse limit sequence of Banach algebras; and bicontinuously $A = \varprojlim(A_k; d_k)$. This is called an *Arens–Michael representation* of A.

Let M be a closed ideal of a Fréchet algebra A. Then $\overline{M^n}$ for each $n \geq 1$ and $\bigcap_{n\geq 1} \overline{M^n}$ are also closed ideals of A. Here, for each $n \geq 1$, M^n is the ideal generated by products of n elements in M. We have the Arens–Michael representations of M, $\overline{M^n}$ and $\bigcap_{n\geq 1} \overline{M^n}$ as follows:

Lemma 2.1. *Let M be a closed ideal of A. Then the Arens–Michael isomorphism $A \cong \varprojlim(A_k; d_k)$ induces isomorphisms*:

(i) $M \cong \varprojlim(M_k; \overline{d}_k)$;

(ii) $\overline{M^n} \cong \varprojlim(\overline{M_k^n}; \overline{d}_k)$ $(n \geq 1)$;

(iii) $\bigcap_{n\geq 1} \overline{M^n} \cong \varprojlim(\bigcap_{n\geq 1} \overline{M_k^n}; \overline{d}_k)$.

(Here $\overline{d}_k = d_k|_{I_{k+1}} : I_{k+1} \to I_k$, where $I_k = \overline{q_k(I)}$ (closure in A_k), whenever I is a closed ideal in A.)

Proof. (i) is clear from [3, Lemma 1].

(ii) For a fixed $n \geq 1$, we have $\overline{M^n} \cong \varprojlim(\overline{q_k(M^n)}; \overline{d_k})$ by [3, Lemma 1]. Fix $k \in \mathbb{N}$. Since $q_k(M^n) = q_k(M)^n$, we have $\overline{q_k(M^n)} = \overline{(\overline{q_k(M)})^n} = \overline{M_k^n}$.

(iii) For each $m \geq 1$, we have $\bigcap_{n\geq 1} \overline{M_k^n} \subset \overline{M_k^m}$. So $\varprojlim(\bigcap_{n\geq 1} \overline{M_k^n}; \overline{d}_k) \subseteq \varprojlim(\overline{M_k^m}; \overline{d}_k)$, where the latter space is isomorphic to $\overline{M^m}$, by (ii). Thus $\varprojlim(\bigcap_{n\geq 1} \overline{M_k^n}; \overline{d}_k) \subseteq \overline{M^m}$. This is true for each m, so that $\varprojlim(\bigcap_{n\geq 1} \overline{M_k^n}; \overline{d}_k) \subseteq \bigcap_{n\geq 1} \overline{M^n}$.

On the other hand, for each $k \in \mathbb{N}$, $\overline{q_k(\bigcap_{n\geq 1} \overline{M^n})} \subseteq \bigcap_{n\geq 1} \overline{q_k(\overline{M^n})} \subseteq \bigcap_{n\geq 1} \overline{M_k^n}$ since q_k is a continuous algebra homomorphism. Therefore we have $\varprojlim(\overline{q_k(\bigcap_{n\geq 1} \overline{M^n})}; \overline{d}_k) \subseteq \varprojlim(\bigcap_{n\geq 1} \overline{M_k^n}; \overline{d}_k)$, where the former space is isomorphic to $\bigcap_{n\geq 1} \overline{M^n}$. ■

Next, we shall consider the quotient Fréchet algebras $A/\overline{M^n}$ for each $n \geq 1$ and $A/\bigcap_{n\geq 1} \overline{M^n}$. Let $\tilde{d}_k : A_{k+1}/\overline{M_{k+1}^n} \to A_k/\overline{M_k^n}$ be the homomorphism induced by d_k. Then we have the following lemma from [3, Theorem 6].

Lemma 2.2. *With the above notation*, *the Arens–Michael isomorphism $A \cong \varprojlim(A_k; d_k)$ induces isomorphisms*:

(i) $A/\overline{M^n} \cong \varprojlim(A_k/\overline{M_k^n}; \tilde{d}_k)$ $(n \geq 1)$;

(ii) $A/\bigcap_{n\geq 1} \overline{M^n} \cong \varprojlim(A_k/\bigcap_{n\geq 1} \overline{M_k^n}; \tilde{d}_k)$. ■

We conclude this section with the following special case, in order to obtain a complete characterization of Fréchet algebras of power series.

Proposition 2.3. *Let $(A, (p_k))$ be a commutative*, *unital Fréchet algebra with the Arens–Michael isomorphism $A \cong \varprojlim(A_k; d_k)$*, *and let M be a non-nilpotent*, *closed maximal ideal of A such that*: (i) $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$ *and*

(ii) $\dim(M/\overline{M^2}) = 1$. *Then there exists* $t \in M$ *such that* $\overline{M^n} = \overline{M^{n+1}} \oplus \mathbb{C}t^n$ *for each* $n \geq 1$. *Assume further that each* p_k *is a norm. Then, for each sufficiently large* k, M_k *is a non-nilpotent maximal ideal of* A_k *such that*: (a) $\bigcap_{n\geq 1} \overline{M_k^n} = \{0\}$ *and* (b) $\dim(M_k/\overline{M_k^2}) = 1$.

Proof. We clearly have $\overline{M^{n+1}} \neq \overline{M^n} \neq \{0\}$ for each n.

Since $\dim(M/\overline{M^2}) = 1$, there exists $t \in M$ such that $M = \overline{M^2} \oplus \mathbb{C}t$, so that $\overline{M^n} = \overline{M^{n+1}} + \mathbb{C}t^n$ for each $n \in \mathbb{N}$. If it were the case that $t^n \in \overline{M^{n+1}}$ for some $n \in \mathbb{N}$, then $\overline{M^m} = \overline{M^n}$ for all $m \geq n$, and so $\overline{M^n} = \bigcap_{m\geq 1} \overline{M^m} = \{0\}$, a contradiction of the fact that M is non-nilpotent. Thus $\overline{M^n} = \overline{M^{n+1}} \oplus \mathbb{C}t^n$ for each $n \in \mathbb{N}$.

Assume further that each p_k is a norm. Then it is clear that M_k is not nilpotent for each k since M is not nilpotent, and so, by Lemma 2.1(ii), we have $\overline{q_k(M)^n} = \overline{M_k^n} \neq \{0\}$ for all n, k. Also, since $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$, we have $\bigcap_{n\geq 1} \overline{M_k^n} = \{0\}$ for each k, by Lemma 2.1(iii) and [8, Corollary A.1.25]. Since M is a closed maximal ideal of A, we have $A = M + \mathbb{C}$. Thus $q_k(M) + \mathbb{C} = M + \mathbb{C}$ is dense in A_k, and so also is $M_k + \mathbb{C}$. Since M_k is closed in A_k, we have $A_k = M_k + \mathbb{C}$. As it is not true that $M_k = A_k$ for infinitely many $k \in \mathbb{N}$, this proves that M_k is a maximal ideal of A_k for each sufficiently large k. Repeating this argument and using the fact that $\overline{q_k(\overline{M^2})} = \overline{M_k^2}$, we obtain $\dim(M_k/\overline{M_k^2}) = 1$ for each sufficiently large k. ∎

3. Fréchet algebras of power series. We now turn to the problem of describing *all* those commutative Fréchet algebras which may be embedded in $\mathcal{F}$ in such a way that they contain the polynomials in X. This is to generalize Theorem 2 of [15]. The solution of this problem does include the earlier result as a special case. The method of proof of the following theorem will be repeated in the proof of Theorem 3.6.

Theorem 3.1. *Let* A *be a commutative, unital Fréchet algebra. Suppose that* A *contains a non-nilpotent, closed maximal ideal* M *such that*: (i) $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$ *and* (ii) $\dim(M/\overline{M^2}) = 1$. *Then* A *is a Fréchet algebra of power series. The converse holds if the polynomials are dense in* A.

Proof. Let A satisfy the stated conditions. By Proposition 2.3, there exists $t \in M$ such that $\overline{M^n} = \overline{M^{n+1}} \oplus \mathbb{C}t^n$ for each $n \in \mathbb{N}$. Let $x \in A$. Then a simple induction on n shows that for $n \geq 1$,

$$x = \sum_{i=0}^{n} \lambda_i t^i + y_n,$$

where $y_n \in \overline{M^{n+1}}$ and the (λ_i) are uniquely determined. Hence the functionals $\pi_j : x \mapsto \lambda_j$ are uniquely defined, and linear for all $j \in \mathbb{N}$. Let $x \in \ker \pi_j$

for all $j \in \mathbb{N}$. Then $x \in \bigcap_{j\ge 1} \overline{M^j} = \{0\}$. Thus the mapping $x \mapsto \sum_{i=0}^{\infty} \pi_i(x)t^i$ is an isomorphism of A onto an algebra of formal power series. If, further, $z = \sum_{i=0}^{n} \mu_i t^i + y_n'$, where $y_n' \in \overline{M^{n+1}}$, is in A, then $\pi_i(xz) = \sum_{k=0}^{i} \lambda_k \mu_{i-k}$ for $x, z \in A$. It follows that the mapping is multiplicative.

If we now carry over the topology via this isomorphism, the result will follow once we show that the functionals π_j are continuous for each j. Clearly π_0 is continuous since $M = \ker \pi_0$ is a closed maximal ideal of A. Let $k \in \mathbb{N}$, and assume that π_i is continuous for each $i < k$, and take (x_n) in A with $x_n \to 0$ as $n \to \infty$. Then

$$x_n = \sum_{i=0}^{k} \pi_i(x_n)t^i + y_{n,k}$$

for some $y_{n,k} \in \overline{M^{k+1}}$. It follows that $\pi_k(x_n)t^k + y_{n,k} \to 0$, so if $\pi_k(x_n)$ does not converge to 0 we deduce that $t^k \in \overline{M^{k+1}}$, a contradiction. Thus each π_j is continuous.

Conversely, let A be a Fréchet algebra of power series such that the polynomials are dense in A. Clearly A is an integral domain. Set $M = \ker \pi_0$. Then M is a non-nilpotent, closed maximal ideal of A. Further, $\overline{M^n} \subset \ker \pi_{n-1}$ for each $n \ge 1$, so that $\bigcap_{n\ge 1} \overline{M^n} = \{0\}$. Hence $\overline{M^n} \ne \overline{M^{n+1}} \ne \{0\}$ for each $n \ge 1$. We clearly have $\overline{M^2} \oplus \mathbb{C}X \subseteq M$. Let $M_2 = \{a \in A : \pi_0(a) = \pi_1(a) = 0\}$. Then it is clear that $\overline{M^2} \subset M_2$. In fact, $\overline{M^2} = M_2$ because the polynomials are dense in A. Hence $M = \overline{M^2} \oplus \mathbb{C}X$, and so $\dim(M/\overline{M^2}) = 1$. ∎

Remarks 1. (a) The assumption that $\dim(M/\overline{M^2}) = 1$ is essential in Theorem 3.1 since there are Fréchet algebras of power series in several indeterminates (defined analogously to the present case) containing a non-nilpotent, closed maximal ideal M such that $\bigcap_{n\ge 1} \overline{M^n} = \{0\}$. Indeed, without our assumption that Fréchet algebras of power series contain the polynomials any finite value is possible for $\dim(M/\overline{M^2})$ with subalgebras of $A(D)$, D the closed unit disc (similarly, with subalgebras of $\mathrm{Hol}(U)$, U the open unit disc, in the Fréchet case).

(b) The author is indebted to the referee for pointing out an error in the original proof of the converse part of Theorem 3.1; the following counterexamples due to the referee show that the assumption that the polynomials are dense in A cannot be dropped.

Let $\omega = (\omega_n)$ be an increasing sequence of positive reals such that $\omega_0 = 1$ and

$$\omega_n \ge n \sum_{r=1}^{n-1} \omega_r \omega_{n-r} \qquad (n \in \mathbb{N}),$$

and let

$$A := \left\{ a = \sum_{n=0}^{\infty} \alpha_n X^n : \sup \frac{|\alpha_n|}{\omega_n} < \infty \right\}.$$

Clearly $(A, \|\cdot\|)$ is a Banach space, where $\|a\| = \sup |\alpha_n|/\omega_n$ $(a \in A)$. Also the embedding $A \hookrightarrow \mathcal{F}$ is continuous. Let $a, b \in A$. Then

$$\frac{|\pi_n(ab)|}{\omega_n} \le \left(2 + \sum_{r=1}^{n-1} \frac{\omega_r \omega_{n-r}}{\omega_n}\right) \|a\|\,\|b\| \le 3\|a\|\,\|b\|,$$

and so $(A, \|\cdot\|)$ is a Banach algebra of power series.

Let θ be a continuous linear functional on ℓ^∞ such that $\theta|c_0 = 0$ and $\theta(e) = 1$, where $e = (1, 1, 1 \dots)$, and define Θ on A by setting $\Theta(a) = \theta(a/\omega)$ $(a \in A)$. It is clear that Θ is a continuous linear functional on A. Let $a, b \in M$. Then

$$\frac{|\pi_n(ab)|}{\omega_n} \le \left(\sum_{r=1}^{n-1} \frac{\omega_r \omega_{n-r}}{\omega_n}\right) \|a\|\,\|b\| \le \frac{1}{n}\,\|a\|\,\|b\| \to 0$$

as $n \to \infty$, and so $ab/\omega \in c_0$. Thus $\Theta(ab) = 0$, and so $\Theta|M^2 = 0$. However, the element $a = \sum_{n=0}^{\infty} \alpha_n X^n$, where $(\alpha_n) = (0, 0, \omega_2, \omega_3, \omega_4, \dots)$, belongs to M_2, and $\Theta(a) = 1$. This shows that $\overline{M^2}$ is properly contained in M_2, and so $\dim(M/\overline{M^2}) > 1$.

In the Fréchet case, following the same arguments, one can easily check that $A = \bigcap_{p \ge 1} A_p$, where

$$A_p = \left\{ \sum_{n=0}^{\infty} \alpha_n X^n : \sup \frac{|\alpha_n|}{n^{1/(p+2)} \omega_n} < \infty \right\},$$

is a non-Banach Fréchet algebra of power series such that the polynomials are not dense in A.

We now turn to characterizations of $\mathcal{F}$. We first define a class of Fréchet algebras as follows.

Let A be a unital Fréchet algebra, with its Fréchet topology defined by a sequence (p_k). An element x in A is a *power series generator* for A if each $y \in A$ is of the form $y = \sum_{n=0}^{\infty} \lambda_n x^n$, λ_n complex scalars, such that $\sum_{n=0}^{\infty} |\lambda_n| p_k(x^n) < \infty$ for all k [5]. Thus if A is a Fréchet algebra with a power series generator, then A is a commutative, singly generated Fréchet algebra generated by x. Moreover, A is separable.

The following lemma, whose proof we omit, is a Fréchet-algebra-of-power-series analogue of [5, Lemma 2.2]. Notice that

$$A_1 := \left\{ y \in A : \sum_{n=0}^{\infty} |\lambda_n| p_k(X^n) < \infty \text{ for all } k \right\}$$

in the lemma.

Lemma 3.2. *Let A be a Fréchet algebra of power series. Then there exists a Fréchet subalgebra A_1 of A such that*:

(1) *A_1 is continuously embedded in A*;
(2) *A_1 is a Fréchet algebra having a power series generator X*;
(3) *A_1 is a Banach algebra provided that A is a Banach algebra.* ■

Remark 2. The subalgebra A_1 need not be dense in A. For example, if $A = H^\infty(U)$, the algebra of bounded holomorphic functions on the open unit disc U, then the polynomials are not dense in A; indeed, the closure of the polynomials in A is precisely $A(D)$. So A_1, the subalgebra of $A(D)$ consisting of all functions having absolutely convergent Taylor series, is a Banach subalgebra of A which is not dense in A; but it is dense in $A(D)$. Note that A and $A(D)$ are both Banach algebras of power series under the sup-norm and $(A_1, |\cdot|)$ is a Banach algebra with a power series generator Z, where $|f| = \sum_{n=0}^{\infty} |f^n(0)|/n!$.

Then we have the following elementary, but crucial, theorem, generalizing Proposition 4 of [2] and Proposition 7.8 of [9]. By identifying the series expansion in x with the series expansion in X, Fréchet algebras with a cyclic basis are realized as Fréchet algebras of power series, the projections being continuous. Note that by a *proper* seminorm we mean a seminorm that is not a norm.

Theorem 3.3. *Let A be a Fréchet algebra of power series. Then either A is $\mathcal{F}$ or the Fréchet topology of A is defined by a sequence (p_k) of norms.*

Proof. If A is a Banach algebra, then certainly the topology of A is defined by a norm (and $A \neq \mathcal{F}$ because $\mathcal{F}$ is not a Banach algebra for any norm).

Now suppose that A is a non-Banach Fréchet algebra of power series. Let (p_k) be the increasing sequence of seminorms defining the Fréchet topology τ of A, and set

$$G = \{k \in \mathbb{N} : p_k \text{ is a proper seminorm on } A\}.$$

If G is finite the corresponding p_k may be deleted and we have a new sequence of norms, which defines the same Fréchet topology τ of A. If G is infinite the corresponding p_k can be taken to define the Fréchet topology τ of A. Note that these possibilities are mutually exclusive. Then, by Lemma 3.2, there exists a Fréchet subalgebra A_1 of A continuously embedded in A; A_1 is a Fréchet algebra with a power series generator X. By [5, Theorem 2.1], $A_1 = \mathcal{F}$. It follows that $A = \mathcal{F}$ topologically in view of the open mapping theorem. ■

As corollaries, we have the following curious characterizations of $\mathcal{F}$ as a Fréchet algebra.

Corollary 3.4. *Let A be a Fréchet algebra of power series. Then A is equal to $\mathcal{F}$ if and only if the Fréchet topology of A is defined by a sequence (p_k) of proper seminorms.* ∎

The main points of Corollary 3.4 should be emphasized. It has the surprising consequence that for a Fréchet algebra A of power series to have its Fréchet topology τ defined by an increasing sequence (p_k) of proper seminorms is, in fact, an *algebraic* property. Thus the topological structure of A here determines the algebraic structure of A. This is totally in contrast to what we would normally like to examine when and how the algebraic structure determines the topological structure, in particular, the continuity aspect in automatic continuity theory. (We shall encounter this in the next section.) There is a further consequence, which says that $\mathcal{F}$ is the *only* Fréchet algebra of finite type (introduced in [11, p. 218]) among Fréchet algebras of power series since an Arens–Michael representation of $\mathcal{F}$ contains finite-dimensional algebras, and if $A \neq \mathcal{F}$, then A_k, the completion of (A, p_k), cannot be finite-dimensional for each k. In fact, we have the following result on an Arens–Michael representation of A.

Corollary 3.5. *Let A be a Fréchet algebra of power series such that the polynomials are dense in A. Then A is not equal to $\mathcal{F}$ if and only if $A = \varprojlim A_k$, where each A_k is a Banach algebra of power series.*

Proof. Suppose that $A \neq \mathcal{F}$. Evidently, by Corollary 3.4, we may suppose that each p_k is a norm on A. Now, by Theorem 3.1, A contains a non-nilpotent, closed maximal ideal $M = \ker \pi_0$ such that $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$ and $\dim(M/\overline{M^2}) = 1$. By Proposition 2.3, for each sufficiently large k, M_k is a non-nilpotent maximal ideal of A_k such that $\bigcap_{n\geq 1} \overline{M_k^n} = \{0\}$ and $\dim(M_k/\overline{M_k^2}) = 1$. Again, by Theorem 3.1, A_k is a Banach algebra of power series for each sufficiently large k. Hence, by passing to a suitable subsequence of (p_k) defining the same Fréchet topology of A, we conclude that each A_k is a Banach algebra of power series.

The converse has already been discussed above. ∎

The immediate consequence of Corollary 3.5 is: if A $(\neq \mathcal{F})$ is a Fréchet algebra of power series such that the polynomials are dense in A, then, by [5, Corollary 3.2], A satisfies Loy's condition (E). A somewhat more elaborate version of the same idea enables us to drop the condition on the polynomials in order to get a more general result, given below.

Let A be a Fréchet algebra of power series, with its Fréchet topology τ, and let τ_c denote the topology of coordinatewise convergence of $\mathcal{F}$. Let p be a seminorm on A. We say that p is *of type* (E) if for each $m \in \mathbb{Z}^+$ there exists $c_m > 0$ such that $|\pi_m(f)| \leq c_m p(f)$ for all $f \in A$ (see [5]). A seminorm of type (E) is a norm. If the Fréchet topology τ of A is given by a sequence (p_k), then each p_k is of type (E) if and only if A satisfies Loy's condition (E) [5, p. 144]. Also, by [14, Theorem 2], A satisfies Loy's condition (E) if and only if A admits a *growth sequence*, i.e., there is a sequence (σ_n) of positive reals such that $\sigma_n \pi_n(x) \to 0$ for each $x \in A$.

THEOREM 3.6. *Let A be a Fréchet algebra of power series. Then A is not equal to $\mathcal{F}$ if and only if $A = \varprojlim A_k$, where each A_k is a Banach algebra of power series. In particular*, *A satisfies Loy's condition* (E).

Proof. Suppose that $A \neq \mathcal{F}$. Let (p_k) be the increasing sequence of seminorms defining the Fréchet topology τ of A. Then, by Corollary 3.4, we may suppose that each p_k is a norm on A and so, for each k, (A, p_k) is a normed algebra, which is a subalgebra of $\mathcal{F}$ containing the polynomials. We first show that the projections π_m are continuous on (A, p_k) for all k and m, i.e., the inclusion map $(A, p_k) \hookrightarrow \mathcal{F}$ is continuous.

Since $M = \ker \pi_0$ is a non-nilpotent, closed maximal ideal of A such that $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$, M is a non-nilpotent, maximal ideal of (A, p_k) for each k; also, $\overline{M^{n+1}} \neq \overline{M^n} \neq \{0\}$ (closure in (A, τ)) in (A, p_k) for all k and n. In fact, we may suppose that M is also closed in (A, p_k) for each k, and hence that π_0 is p_k-continuous for each k. Also, by Proposition 2.3, we may suppose that M_k is a non-nilpotent maximal ideal of A_k and that $\bigcap_{n\geq 1} \overline{M_k^n} = \{0\}$ for each k. Hence $\overline{M_k^n} \neq \overline{M_k^{n+1}} \neq \{0\}$ for all k and n. Assume inductively that π_i is continuous for $i < m$, and take (x_n) in (A, p_k) with $p_k(x_n) \to 0$. Then, following the argument given in Theorem 3.1, we deduce that $X^m \in A \cap \overline{M_k^{m+1}}$ (which is, in fact, $\overline{M^{m+1}}$ in (A, p_k) for each k), a contradiction of the fact that $\overline{M_k^m} \neq \overline{M_k^{m+1}}$.

Let $k \in \mathbb{N}$. Next we show that if (x_n) is any p_k-Cauchy sequence in A such that $x_n \to 0$ in $(\mathcal{F}, \tau_c)$, then $p_k(x_n) \to 0$. Clearly there exists some $x \in A_k$ such that $p_k(x_n - x) \to 0$. Now, by continuity, the inclusion map $(A, p_k) \hookrightarrow \mathcal{F}$ extends to a continuous homomorphism $\phi : A_k \to \mathcal{F}$. So $x_n \to x$ in $(\mathcal{F}, \tau_c)$. Hence $p_k(x_n) \to 0$.

Further, let $x \in \ker \phi$. Choose a sequence (x_n) in A such that $p_k(x_n - x) \to 0$. By the continuity of ϕ, $\phi(x_n) = x_n \to \phi(x) = 0$ in $\mathcal{F}$. Thus (x_n) has the properties mentioned in the last paragraph. So $p_k(x_n) \to 0$. Hence $x = 0$, and so A_k is a Banach algebra of power series for each k. In particular, A satisfies Loy's condition (E), by [5, p. 144]. ∎

REMARKS 3. (a) We again emphasize the fact that the characterizations of Fréchet algebras of power series and of the algebra $\mathcal{F}$ play an essential role in the proof of Theorem 3.6.

(b) The fact that a Fréchet algebra of power series A satisfies Loy's condition (E) played an essential role in [14], and it is of interest to obtain a simple characterization of this condition. Here it is easy to see that a

Fréchet algebra of power series A satisfies Loy's condition (E) if and only if A admits a continuous norm, i.e., there exists a norm p on A such that for some $K > 0$ and integer $n \geq 1$, $p(x) \leq Kp_n(x)$ $(x \in A)$. In fact, Theorem 3.6 significantly generalizes Proposition 3.1 and Corollary 3.2 of [5].

(c) $\mathcal{F}$ itself does not satisfy Loy's condition (E) [14, Example]. In fact, from Theorem 3.6, it is clear that $\mathcal{F}$ is the *only* example of a Fréchet algebra of power series which does not satisfy Loy's condition (E).

(d) By the method of proofs, it is clear that the characterizations obtained in Corollary 3.5 and Theorem 3.6 are independent of the Arens–Michael representation chosen, in the sense that if (p''_k) is any other sequence of norms defining the Fréchet topology of A, then the proofs are valid with that sequence. Of course, the A''_k, obtained using the sequence (p''_k), may be different Banach algebras of power series in an Arens–Michael representation of A. For example, two different Arens–Michael representations of $\mathrm{Hol}(U)$, U the open unit disc, $\mathrm{Hol}(\mathbb{C})$ and $A^\infty(\Gamma)$, Γ the unit circle, are discussed in [5, Examples 1.4 and 1.5], containing different Banach algebras of power series.

4. Automatic continuity and uniqueness of topology. We now turn to the second problem stated in the introduction: to establish that every Fréchet algebra A of power series has a unique Fréchet topology. By [15, Theorem 10], it is clear that every Fréchet algebra A of power series satisfying condition (E) has a unique Fréchet topology. Since $\mathcal{F}$ does not satisfy condition (E), there may be some other Fréchet algebras of power series not satisfying this condition; but, by Remarks 3(c), we have ruled out that possibility. Hence Theorem 3.6 gives the following result from [15].

THEOREM 4.1. *Let A be a Fréchet algebra of power series such that $A \neq \mathcal{F}$. Then a homomorphism $\phi : B \to A$ from a Fréchet algebra B into A is continuous provided that the range of ϕ is not one-dimensional.* ∎

REMARK 4. From (c) of Remarks 3, it is clear that $\mathcal{F}$ does not admit a growth sequence. So the result of Johnson [10, Theorem 9.1] (which was proved for the Banach algebra case with an indication that some condition such as the existence of a growth sequence is required in the Fréchet case) is here established for Fréchet algebras in a more general form, provided that $A \neq \mathcal{F}$. Thus there can be no special automatic discontinuity result in this case. We do not know whether or not every homomorphism $\phi : B \to \mathcal{U}$ from a Fréchet algebra B into a semisimple Fréchet algebra $\mathcal{U}$ is continuous.

As a corollary, we have the following result in automatic continuity theory. We note that the continuity of automorphisms of $\mathcal{F}$ was proved in [17, §2], and the uniqueness of Fréchet topology of $\mathcal{F}$ was proved in [1, Corollary 2].

Corollary 4.2. *Let A be a Fréchet algebra of power series. Then every automorphism of A is continuous. In particular, A has a unique Fréchet topology.* ∎

In relation to the still unsolved "Michael problem", which is to determine whether every character on a (commutative) Fréchet algebra need be continuous, the following question may have some interest in the theory of automatic continuity.

Question. Is every (surjective) homomorphism $\phi : B \to \mathcal{F}$ from a non-Banach Fréchet algebra B continuous?

We remark that Dales and McClure established the existence of a commutative, unital Banach algebra A which has a totally discontinuous higher point derivation of infinite order at a character, and which is the domain of a discontinuous epimorphism onto $\mathcal{F}$ [8, Theorem 5.5.19]. On the other hand, there is a discontinuous homomorphism between two commutative unital Fréchet algebras having certain properties [2, Theorem 8]; but, in the construction of the discontinuous homomorphism, Allan used a continuous homomorphism from A into $\mathcal{F}$.

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References

[1] G. R. Allan, *Embedding the algebra of formal power series in a Banach algebra*, Proc. London Math. Soc. 25 (1972), 329–340.

[2] —, *Fréchet algebras and formal power series*, Studia Math. 119 (1996), 271–278.

[3] —, *Stable inverse-limit sequences, with application to Fréchet algebras*, ibid. 121 (1996), 277–308.

[4] R. F. Arens, *A generalization of normed rings*, Pacific J. Math. 2 (1952), 455–471.

[5] S. J. Bhatt and S. R. Patel, *On Fréchet algebras of power series*, Bull. Austral. Math. Soc. 66 (2002), 135–148.

[6] G. Carboni and A. Larotonda, *An example of a Fréchet algebra which is a principal ideal domain*, Studia Math. 138 (2000), 265–275.

[7] H. G. Dales, *Automatic continuity: a survey*, Bull. London Math. Soc. 10 (1978), 129–183.

[8] —, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. 24, Clarendon Press, 2000.

[9] T. Husain, *Orthogonal Schauder Bases*, Dekker, New York, 1991.

[10] B. E. Johnson, *Continuity of linear operators commuting with continuous linear operators*, Trans. Amer. Math. Soc. 128 (1967), 88–102.

[11] M. K. Kopp, *Fréchet algebras of finite type*, Arch. Math. (Basel) 83 (2004), 217–228.

[12] R. J. Loy, *Continuity of derivations on topological algebras of power series*, Bull. Austral. Math. Soc. 1 (1969), 419–424.

[13] R. J. Loy, *Uniqueness of the complete norm topology and continuity of derivations on Banach algebras*, Tôhoku Math. J. 22 (1970), 371–378.

[14] —, *Uniqueness of the Fréchet space topology on certain topological algebras*, Bull. Austral. Math. Soc. 4 (1971), 1–7.

[15] —, *Banach algebras of power series*, J. Austral. Math. Soc. 17 (1974), 263–273.

[16] E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).

[17] S. Scheinberg, *Power series in one variable*, J. Math. Anal. Appl. 31 (1970), 321–333.

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