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Unicellularity of the multiplication operator on Banach spaces of formal power series

by

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Abstract. Let $\{\beta(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers and $1 \le p < \infty$. We consider the space $\ell^p(\beta)$ of all power series $f(z)=\sum_{n=0}^{\infty} \widehat{f}(n)z^n$ such that $\sum_{n=0}^{\infty} |\widehat{f}(n)|^p|\beta(n)|^p < \infty$. We give some sufficient conditions for the multiplication operator, M_z, to be unicellular on the Banach space $\ell^p(\beta)$. This generalizes the main results obtained by Lu Fang [1].

Introduction. First, we generalize some definitions from [4].

Let $\{\beta(n)\}$ be a sequence of nonzero complex numbers with $\beta(0) = 1$ and $1 \le p < \infty$. We consider the space of sequences $f = \{\widehat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\widehat{f}(n)|^p|\beta(n)|^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$ will be used whether or not the series converges for any value of z. These are called formal power series. Let $\ell^p(\beta)$ denote the space of such formal power series.

For $1 < p < \infty$, $\ell^p(\beta) \cong L^p(\mu)$ where μ is the σ-finite measure defined on the positive integers by $\mu(K) = \sum_{n\in K} \beta(n)^p$, $K \subseteq \mathbb{N} \cup \{0\}$. So $\ell^p(\beta)$ is a reflexive Banach space ([3]) and $(\ell^p(\beta))^* = \ell^q(\beta^{p/q})$ where $\beta^{p/q} = \{\beta(n)^{p/q}\}_n$ ([6]).

Let $\widehat{f_k}(n) = \delta_{nk}$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = |\beta(k)|$. Now consider M_z, the operator of multiplication by z on $\ell^p(\beta)$:

$$(M_z f)(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^{n+1}.$$

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In other words

$$(M_z f)^\wedge(n) = \begin{cases} \widehat{f}(n-1), & n \ge 1, \\ 0, & n = 0. \end{cases}$$

Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case

$$\|M_z^n\| = \sup_k \left| \frac{\beta(n+k)}{\beta(k)} \right|, \qquad n = 0, 1, 2, \ldots$$

Consider the multiplication of formal power series, $fg = h$, given by

$$\Big(\sum_{n=0}^{\infty} \widehat{f}(n) z^n \Big) \cdot \Big(\sum_{n=0}^{\infty} \widehat{g}(n) z^n \Big) = \sum_{n=0}^{\infty} \widehat{h}(n) z^n$$

where

$$\widehat{h}(n) = \sum_{k=0}^{n} \widehat{f}(k) \widehat{g}(n-k), \qquad n = 0, 1, 2, \ldots$$

If $1/p + 1/q = 1$ and

$$\sup_n \sum_{i=1}^{n} \left| \frac{\beta(n)}{\beta(i)\beta(n-i)} \right|^q < \infty$$

then clearly by the Hölder inequality one can see that $\ell^p(\beta)$ is a Banach algebra ([2]).

If $f \in \ell^p(\beta)$ and $P(z)$ is a polynomial, then to the vector $P(M_z)f$ there corresponds the formal power series $P(z)f(z)$.

Let X be a Banach space. We denote by $B(X)$ the set of bounded linear operators on X. Let $A \in B(X)$ and $x \in X$. We say that x is a *cyclic vector* of A if

$$X = \operatorname{span}\{A^n x : n = 0, 1, 2, \ldots\}.$$

Here $\operatorname{span}\{\cdot\}$ is the closed linear span of the set $\{\cdot\}$. A polynomial $p(z) = (z - \lambda_1) \ldots (z - \lambda_k)$ is a cyclic vector of M_z in $\ell^p(\beta)$ iff $\{\lambda_i^n / \beta(n)\}_n \notin \ell^q$ for $i = 1, \ldots, k$, where $1/p + 1/q = 1$ ([6]).

Also an operator A in $B(X)$ is called *unicellular* on X if the set of its invariant subspaces, $\operatorname{Lat}(A)$, is linearly ordered by inclusion.

In the main theorem of this paper we give some sufficient conditions for the multiplication operator, M_z, on $\ell^p(\beta)$ to be unicellular and then we obtain the main results of [1]. Throughout this paper we assume that $M_z \in B(\ell^p(\beta))$.

Unicellularity of M_z. The following theorem is the main result of this paper.

THEOREM. *Let $1 \le p < \infty$. The operator M_z is unicellular on $\ell^p(\beta)$ if $\beta(n)$ is of the form $\beta(n) = \alpha(n)\gamma(n)$ where $\{\alpha(n)\}$ and $\{\gamma(n)\}$ satisfy*:

(i) *There exists a positive number* M *such that*

$$\sup\left\{\left|\frac{\gamma(n+i)}{\gamma(n)\gamma(i)}\right| : i, n = 0, 1, 2, \dots\right\} \leq M.$$

(ii) *There exists a positive integer* m_0 *such that*

$$L_{m_0} = \sup\left\{\left|\frac{\alpha(n+i)\alpha(m_0)}{\alpha(n+m_0)\alpha(i)}\right| : n > 0,\ i \geq m_0\right\} < \infty$$

and

$$\left\{\frac{\alpha(n+m_0)}{\alpha(n)}\right\}_n \in \ell^q,$$

where $1/p + 1/q = 1$.

Proof. Let $\{f_m\}_m$ be the basis for $\ell^p(\beta)$ as defined in the introduction. Put $\ell^p_\infty(\beta) = \{0\}$, $\ell^p_0(\beta) = \ell^p(\beta)$ and

$$\ell^p_n(\beta) = \Big\{\sum_{m\geq n} c_m f_m \in \ell^p(\beta)\Big\} \qquad (n \geq 1).$$

In order to show that M_z is unicellular it suffices to show that the lattice of invariant subspaces of M_z, $\mathrm{Lat}(M_z)$, is a subset of $\{\ell^p_n(\beta) : 0 \leq n \leq \infty\}$. So let $\mathcal{K}$ be a nontrivial element of $\mathrm{Lat}(M_z)$. Then there exists a positive integer n such that $\mathcal{K} \subseteq \ell^p_n(\beta)$ and $\mathcal{K} \not\subseteq \ell^p_{n+1}(\beta)$. Thus we may choose $f = \sum_{m=n}^\infty x_m f_m$ in $\mathcal{K}$ ($x_m = \widehat{f}(m)$) with $x_n \neq 0$. Note that $\{f_{n+k}\}_{k=0}^\infty$ is a basis for $\ell^p_n(\beta)$. We claim that f is a cyclic vector for $M_z|_{\ell^p_n(\beta)}$. If so, then since $M_z\mathcal{K} \subset \mathcal{K}$, we have $M_z^i f \in \mathcal{K}$ for $i \in \mathbb{N}$. Also since

$$\ell^p_n(\beta) = \mathrm{span}\{(M_z^i|_{\ell^p_n(\beta)})f : i = 0, 1, 2, \dots\},$$

we have $\ell^p_n(\beta) \subseteq \mathcal{K}$ and so $\ell^p_n(\beta) = \mathcal{K}$. Now to prove our claim it is sufficient to show that if

$$f = \sum_{m=0}^\infty \widehat{f}(m) f_m \in \ell^p(\beta)$$

is such that $\widehat{f}(0) \neq 0$, then f is a cyclic vector for M_z. Without loss of generality, assume that $\widehat{f}(0) = 1$. Note that $M_z f_k = f_{k+1}$. For the formal power series

$$f(z) = \sum_{m=0}^\infty \widehat{f}(m) z^m,$$

we choose the formal power series

$$g(z) = \sum_{m=0}^\infty \widehat{g}(m) z^m$$

such that $g(z)f(z) = 1$. Indeed $\widehat{g}(0) = 1$ and for $k \geq 1$,

$$\widehat{g}(k) = \sum_{i=1}^{k} \sum_{\substack{m_1+\ldots+m_i=k \\ m_j \geq 1}} (-1)^i \widehat{f}(m_1) \ldots \widehat{f}(m_i)$$

(see [5]). In order to show that f is a cyclic vector of M_z, we show that

$$\text{span}\left\{ \sum_{k=0}^{m_0} \widehat{g}(k) M_z^{k+n} f : n = 0, 1, 2, \ldots \right\} = \ell^p(\beta) \tag{1}$$

(m_0 is the positive integer in condition (ii) of the theorem). Put

$$y_{m_0,n} = \sum_{k=0}^{m_0} \widehat{g}(k) M_z^{k+n} f, \qquad n = 0, 1, 2, \ldots$$

If there exists a positive integer n_0 such that

$$\text{span}\{y_{m_0,n} : n \geq n_0\} = \ell^p_{n_0}(\beta) \tag{2}$$

then clearly one can see that

$$\text{span}\{y_{m_0,n} : n \geq n_0 - 1\} = \ell^p_{n_0-1}(\beta).$$

By continuing this process, we conclude that (1) holds. Now since $g(z)f(z) = 1$, we have

$$\Big(\sum_{k=0}^{m_0} \widehat{g}(k) z^k + \sum_{k>m_0} \widehat{g}(k) z^k \Big) f(z) = 1$$

and so

$$\sum_{k=0}^{m_0} (\widehat{g}(k) M_z^k) f(M_z) f_0 + \sum_{k>m_0} (\widehat{g}(k) M_z^k) f(M_z) f_0 = f_0.$$

Now since for each $n \geq 0$, $M_z^n f_0 = f_n$, by taking the image under M_z^n of both sides of the above equation, we have

$$\sum_{k=0}^{m_0} (\widehat{g}(k) M_z^{k+n}) f(M_z) f_0 - f_n = - \sum_{k>m_0} (\widehat{g}(k) M_z^{k+n}) f(M_z) f_0.$$

Note that $f(M_z) f_0 = f$ and $f = \sum_{m=0}^{\infty} \widehat{f}(m) f_m$. So

$$y_{m_0,n} - f_n = \sum_{k>m_0} \sum_{m=0}^{\infty} \widehat{g}(k) \widehat{f}(m) M_z^{k+n} f_m.$$

Therefore

$$y_{m_0,n} - f_n \in \ell^p_{m_0+n+1}(\beta).$$

Now we show that there exists a positive integer n_0 such that (2) holds. For $i \geq 1$, define the projections $P_i : \ell^p(\beta) \to \ell^p_i(\beta)$ by

$$P_i\Big(\sum_{n=0}^{\infty} \widehat{f}(n)z^n\Big) = \sum_{n=i}^{\infty} \widehat{f}(n)z^n.$$

Note that $\|M_z^n|_{\ell_i^p(\beta)}\| = \sup_m |\beta(i+n+m)/\beta(i+m)|$ and for $i \geq k$, $P_i M_z^k f = M_z^k P_{i-k} f$ for all $f \in \ell^p(\beta)$ ([5]). Thus

$$\begin{aligned}
\frac{1}{|\beta(n)|}\|y_{m_0,n} - f_n\|_p &= \frac{1}{|\beta(n)|}\|P_{m_0+n+1}(y_{m_0,n} - f_n)\|_p \\
&= \frac{1}{|\beta(n)|}\|P_{m_0+n+1}(y_{m_0,n})\|_p \\
&\leq \frac{1}{|\beta(n)|}\sum_{k=0}^{m_0} |\widehat{g}(k)| \cdot \|P_{m_0+n+1} M_z^{k+n} f\|_p \\
&= \frac{1}{|\beta(n)|}\sum_{k=0}^{m_0} |\widehat{g}(k)| \cdot \|M_z^{k+n} P_{m_0-k+1} f\|_p \\
&\leq \frac{\|f\|_p}{|\beta(n)|}\sum_{k=0}^{m_0} |\widehat{g}(k)| \cdot \|M_z^{k+n}|_{\ell^p_{m_0-k+1}(\beta)}\| \\
&= \|f\|_p \sum_{k=0}^{m_0} |\widehat{g}(k)| \sup_i \left|\frac{\beta(m_0+n+i+1)}{\beta(n)\beta(m_0+i+1-k)}\right|.
\end{aligned}$$

Since $\beta(n) = \alpha(n)\gamma(n)$, we have

$$\begin{aligned}
&\sup_i \left|\frac{\beta(m_0+n+i+1)}{\beta(n)\beta(m_0+i+1-k)}\right| \\
&\qquad = \sup_i \left|\frac{\alpha(m_0+n+i+1)}{\alpha(n)\alpha(m_0+i+1-k)}\right|\left|\frac{\gamma(m_0+n+i+1)}{\gamma(n)\gamma(m_0+i+1-k)}\right|.
\end{aligned}$$

But by condition (ii) of the theorem,

$$\begin{aligned}
&\sup_i \left|\frac{\alpha(m_0+n+i+1)}{\alpha(n)\alpha(m_0+i+1-k)}\right| \\
&\quad = \sup_i \left|\frac{\alpha(m_0+n+i+1)\alpha(m_0)}{\alpha(m_0+n)\alpha(m_0+i+1)}\right|\left|\frac{\alpha(m_0+i+1)\alpha(m_0+n)}{\alpha(n)\alpha(m_0)\alpha(m_0+i+1-k)}\right| \\
&\quad \leq L_{m_0}\left|\frac{\alpha(m_0+n)}{\alpha(n)\alpha(m_0)}\right| \sup_i \left|\frac{\alpha(m_0+i+1)}{\alpha(m_0+i+1-k)}\right|,
\end{aligned}$$

and by (i),

$$\begin{aligned}\sup_i &\left|\frac{\gamma(m_0+n+i+1)}{\gamma(n)\gamma(m_0+i+1-k)}\right| \\ &= \sup_i \left|\frac{\gamma(m_0+n+i+1)}{\gamma(n)\gamma(m_0+i+1)}\right|\left|\frac{\gamma(m_0+i+1)}{\gamma(m_0+i+1-k)}\right| \\ &\le M \sup_i \left|\frac{\gamma(m_0+i+1)}{\gamma(m_0+i+1-k)}\right|.\end{aligned}$$

So

$$\begin{aligned}\sup_i &\left|\frac{\beta(m_0+n+i+1)}{\beta(n)\beta(m_0+i+1-k)}\right| \\ &\le ML_{m_0}\left|\frac{\alpha(m_0+n)}{\alpha(n)\alpha(m_0)}\right| \sup_i \left|\frac{\beta(m_0+i+1)}{\beta(m_0+i+1-k)}\right|\end{aligned}$$

and therefore

$$\begin{aligned}&\frac{1}{|\beta(n)|}\|y_{m_0,n}-f_n\|_p \\ &\quad\le ML_{m_0}\left|\frac{\alpha(m_0+n)}{\alpha(n)\alpha(m_0)}\right| \|f\|_p \sum_{k=0}^{m_0} |\widehat{g}(k)| \sup_i \left|\frac{\beta(m_0+i+1)}{\beta(m_0+i+1-k)}\right|.\end{aligned}$$

Since

$$\|M_z^k|_{\ell^p_{m_0+1-k}(\beta)}\| = \sup_i \left|\frac{\beta(m_0+1+i)}{\beta(m_0+1-k+i)}\right| < \infty$$

for $k = 0, 1, 2, \ldots, m_0$, there exists a positive number M' such that

$$\sum_{k=0}^{m_0} |\widehat{g}(k)| \sup_i \left|\frac{\beta(m_0+1+i)}{\beta(m_0+1-k+i)}\right| \le M'.$$

So we have

$$\frac{1}{|\beta(n)|}\|y_{m_0,n}-f_n\|_p \le c_n$$

where

$$c_n = MM'\|f\|_p \frac{L_{m_0}}{|\alpha(m_0)|}\left|\frac{\alpha(m_0+n)}{\alpha(n)}\right|, \qquad n = 1, 2, \ldots$$

Since $\{c_n\} \in \ell^q$, there exists a positive integer $n_0 > m_0$ such that

$$\lambda = \sum_{n>n_0} c_n^q < 1.$$

Therefore for any finite linear combinations

$$\phi = \sum d_k y_{m_0,n_0+k}/\beta(n_0+k), \qquad \psi = \sum d_k f_{n_0+k}/\beta(n_0+k),$$

by the Hölder inequality we have

$$\begin{aligned}\|\phi-\psi\|_p &\le \sum_k |d_k|\cdot\|y_{m_0,n_0+k}-f_{n_0+k}\|_p/|\beta(n_0+k)|\\ &\le \Big(\sum_k |d_k|^p\Big)^{1/p}\Big(\sum_{n=n_0}^{\infty}\|y_{m_0,n}-f_n\|_p^q|\beta(n)|^{-q}\Big)^{1/q}\\ &= \|\psi\|_p\Big(\sum_{n\ge n_0} c_n^q\Big)^{1/q}.\end{aligned}$$

Thus

$$\|\phi-\psi\|_p \le \lambda^{1/q}\|\psi\|_p.$$

Since $0\le\lambda^{1/q}<1$, $\{y_{m_0,n}\}_{n=n_0}^{\infty}$ is in $\ell^p_{n_0}(\beta)$ and $\{f_n\}_{n\ge n_0}$ is a basis for $\ell^p_{n_0}(\beta)$, it follows immediately from Lemma 2.1 of [1] (which is true for Banach spaces) that $\{y_{m_0,n}\}_{n\ge n_0}$ is a complete set, i.e., spanning $\ell^p_{n_0}(\beta)$. So (2) holds and this completes the proof.

From the proof of the theorem, we obtain the following corollary.

Corollary. *Under the hypothesis of the theorem, if* $x=\sum_{m=0}^{\infty}x_m f_m$ *belongs to* $\ell^p(\beta)$ *and* $x_0\neq 0$, *then* x *is a cyclic vector of* M_z.

Now as a consequence of the above theorem, in the following example we prove the main result of [1] which gives sufficient conditions for a Lambert weighted shift operator to be unicellular.

Example. Let H be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. A unilateral weighted shift operator S in $B(H)$ ($Se_n=w_ne_{n+1}$) is called a *Lambert weighted shift operator* if the weights $\{w_n\}$ are given by

$$w_n=a_n\frac{\|A^{n+1}f\|}{\|A^nf\|},\qquad n=0,1,2,\dots,$$

where A is a given injective operator in $B(H)$, f is a nonzero vector in H and $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence of positive numbers. S is unitarily equivalent to the multiplication operator M_z on the space $\ell^2(\beta)$ where the sequence $\beta=\{\beta(n)\}_{n=0}^{\infty}$ satisfies $\beta(0)=1$ and

$$\beta(n)=w_0w_1\dots w_{n-1}\qquad(n\ge 1).$$

The equivalence of these operators is realized by means of the isomorphism U of $\ell^2(\beta)$ onto H defined by the formula $(Uf)_n=\widehat{f}(n)\beta(n)$ ([4]). Now for each nonnegative integer n put

$$\alpha(n)=a_0\dots a_{n-1},\qquad \gamma(n)=\|A^nf\|/\|f\|.$$

If $\{\alpha(n)\}$ and $\{\gamma(n)\}$ satisfy the hypothesis of the theorem, then the Lambert weighted shift operator is unicellular.

Proposition. *Suppose*

$$\sum_{i,n} \left| \frac{\beta(i+n)}{\beta(i)\beta(n)} \right|^q < \infty \qquad \textit{where } 1/p + 1/q = 1.$$

Then M_z is uicellular on $\ell^p(\beta)$.

Proof. Let $\ell^p_n(\beta)$ be defined as in the proof of the previous theorem. As in that proof, it is sufficient to show that if $f = \sum_{n\geq 0} \widehat{f}(n) f_n \in \ell^p(\beta)$ is such that $\widehat{f}(0) \neq 0$, then f is a cyclic vector for M_z. Without loss of generality, assume that $\widehat{f}(0) = 1$. Put $y_n = M_z^n f$ for $n = 0, 1, 2, \ldots$ As before we can see that if

$$\exists n_0 \in \mathbb{N}, \qquad \operatorname{span}\{y_n : n \geq n_0\} = \ell^p_{n_0}(\beta) \tag{1}$$

then

$$\operatorname{span}\{y_n : n \geq n_0 - 1\} = \ell^p_{n_0-1}(\beta).$$

By continuing this process we can conclude that f is a cyclic vector. Note that

$$y_n = f_n + \sum_{i\geq 1} \widehat{f}(i) f_{i+n}, \qquad n \geq 0.$$

Now we have

$$\begin{aligned}
\frac{1}{|\beta(n)|} \|y_n - f_n\|_p &= \frac{1}{\beta(n)|} \Big\| \sum_{i\geq 1} \widehat{f}(i) f_{i+n} \Big\|_p \\
&\leq \sum_{i\geq 1} |\widehat{f}(i)| \cdot |\beta(i)| \left| \frac{\beta(i+n)}{\beta(i)\beta(n)} \right| \\
&\leq \Big(\sum_{i\geq 1} |\widehat{f}(i)|^p |\beta(i)|^p \Big)^{1/p} \Big(\sum_{i\geq 1} \left| \frac{\beta(i+n)}{\beta(i)\beta(n)} \right|^q \Big)^{1/q} \\
&\leq \|f\|_p \Big(\sum_{i\geq 1} \left| \frac{\beta(i+n)}{\beta(i)\beta(n)} \right|^q \Big)^{1/q}.
\end{aligned}$$

Put

$$c_n = \|f\|_p^q \sum_{i\geq 1} \left| \frac{\beta(i+n)}{\beta(i)\beta(n)} \right|^q, \qquad n = 0, 1, 2, \ldots$$

Thus $\sum_{n\geq 0} c_n < \infty$ and so there exists a positive integer n_0 such that $\lambda = \sum_{n\geq n_0} c_n < 1$. Therefore for any finite linear combinations

$$\phi = \sum_k c_k y_{n_0+k}, \qquad \chi = \sum_k c_k f_{n_0+k}$$

we have

$$\begin{aligned}\|\phi - \chi\|_p &= \Big\| \sum_k c_k (y_{n_0+k} - f_{n_0+k}) \Big\|_p \\ &\leq \|\chi\|_p \bigg(\sum_{n \geq n_0} \frac{\|y_n - f_n\|_p^q}{|\beta(n)|^q} \bigg)^{1/q} = \lambda^{1/q} \|\chi\|_p.\end{aligned}$$

Since $\{f_{n_0+k}\}_{k\geq 0}$ is a basis for K_{n_0}, and $0 \leq \lambda^{1/q} < 1$, it follows that (1) holds. This completes the proof.

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