STUDIA MATHEMATICA 147 (3) (2001)

A Kleinecke–Shirokov type condition with Jordan automorphisms

by

MATEJ BREŠAR (Maribor), AJDA FOŠNER (Maribor) and MAJA FOŠNER (Ljubljana)

Abstract. Let φ be a Jordan automorphism of an algebra $\mathcal{A}$. The situation when an element $a \in \mathcal{A}$ satisfies $\frac{1}{2}(\varphi(a) + \varphi^{-1}(a)) = a$ is considered. The result which we obtain implies the Kleinecke–Shirokov theorem and Jacobson's lemma.

Let $\mathcal{A}$ be a finite-dimensional algebra over a field of characteristic 0. Jacobson's lemma [4, Lemma 2] from 1935 asserts, in its original form, that if $a, b \in \mathcal{A}$ are such that

$$[[a, b], b] = 0 \tag{1}$$

then $[a, b]$ is nilpotent (here, $[x, y] = xy - yx$). Actually, from its proof it can be easily extracted that this conclusion holds true if the assumption of finite-dimensionality is replaced by a milder assumption that a is algebraic over the underlying field (see, e.g., Kaplansky's discussion [7] on different proofs and extensions of this lemma). An analytic analogue of Jacobson's lemma states that if elements a, b in a (complex) Banach algebra $\mathcal{A}$ satisfy (1), then $[a, b]$ is quasinilpotent. This theorem was conjectured by Kaplansky (cf. [3]) and proved independently by Kleinecke [9] and Shirokov [13] in the 50's. The usual approach to the proofs of both results, Jacobson's lemma and the Kleinecke–Shirokov theorem, is to interpret the condition (1) as $\delta_b^2(a) = 0$ or as $[\delta_a(b), b] = 0$, where δ_x denotes the inner derivation $\delta_x : y \mapsto [x, y]$, and then consider these two identities for any abstract derivation. In this manner short and elegant proofs of these two results can be obtained. Perhaps the most straightforward way is based on the observation of Kleinecke [9, (2)] that the condition $\delta^2(a) = 0$, where δ is any derivation, implies $\delta^n(a^n) = n!\delta(a)^n$ for any positive integer n; from this identity both results follow almost immediately. On the other hand, the consideration of

2000 *Mathematics Subject Classification*: 46H05, 47B47, 47B48, 16W20.

Partially supported by a grant from the Ministry of Science of Slovenia.

the conditions $\delta^2(a) = 0$ and $[\delta(b), b] = 0$ has led to the study of various interesting problems on derivations (see, e.g., survey articles [10, 11]).

There are many parallels, in both algebra and analysis, between derivations and automorphisms. Therefore it seems natural to seek for an interpretation of the condition (1) in terms of automorphisms. Under the assumption that b is invertible, this is indeed possible. Namely, then (1) can be expressed as $\frac{1}{2}(\varphi_b(a) + \varphi_b^{-1}(a)) = a$ where φ_b is the inner automorphism $\varphi_b : x \mapsto b^{-1}xb$. The question arises how to treat the identity

$$\frac{\varphi(a) + \varphi^{-1}(a)}{2} = a \tag{2}$$

for any automorphism φ. Our aim is to show that, at the first stage, this can be done in a (perhaps surprisingly) similar way to the treatment of an analogous identity $\delta^2(a) = 0$ with derivations, but it yields somewhat more general results.

Moreover, instead of just automorphisms we shall treat Jordan automorphisms. Let us recall the definition. For simplicity we assume (without further mention) that all algebras considered in the present article are unital and the characteristic of their underlying fields is not 2. Let $\mathcal{A}$ be an algebra. We define a new product, the *Jordan product*, in $\mathcal{A}$ by $x \circ y = \frac{1}{2}(xy + yx)$, $x, y \in \mathcal{A}$. A *Jordan automorphism* of $\mathcal{A}$ is a bijective linear map $\varphi : \mathcal{A} \to \mathcal{A}$ that preserves the Jordan product, i.e., $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in \mathcal{A}$. In particular, $\varphi(x^n) = \varphi(x)^n$ for every $x \in \mathcal{A}$, as one easily verifies by induction. Clearly, both automorphisms and antiautomorphisms are also Jordan automorphisms.

The key to everything that follows is the next statement, a perfect analogue of [9, (2)].

LEMMA. *Let φ be a Jordan automorphism of an algebra $\mathcal{A}$. If $a \in \mathcal{A}$ satisfies* (2), *then $(\varphi - 1)^n(a^n) = n!(\varphi(a) - a)^n$ for any positive integer n.*

Proof. Set $\Delta = \varphi - 1$, $q = \Delta(a)$, and denote by $\mathcal{J}_x$, $x \in \mathcal{A}$, the map $\mathcal{J}_x : y \mapsto x \circ y$. Our assumption yields $\varphi(q) = q$, which in turn implies that $\mathcal{J}_q$, φ and Δ mutually commute. We have to show that $\Delta^n(a^n) = n!q^n$. Clearly we may assume that $n \geq 2$ and $\Delta^{n-1}(a^{n-1}) = (n-1)!q^{n-1}$. In particular, this yields $\Delta^n(a^{n-1}) = 0$. Noting that $[\Delta, \mathcal{J}_a] = \varphi\mathcal{J}_q$, and then using $[\Delta^n, \mathcal{J}_a] = \Delta^{n-1}[\Delta, \mathcal{J}_a] + [\Delta^{n-1}, \mathcal{J}_a]\Delta$, one shows by induction that $[\Delta^n, \mathcal{J}_a] = n\varphi\mathcal{J}_q\Delta^{n-1}$. Accordingly, $\Delta^n(a^n) = [\Delta^n, \mathcal{J}_a](a^{n-1}) = n\varphi\mathcal{J}_q\Delta^{n-1}(a^{n-1}) = n!q^n$.

THEOREM 1. *Let $\mathcal{A}$ be an algebra over a field of characteristic* 0, *and let φ be a Jordan automorphism of $\mathcal{A}$. If an algebraic element $a \in \mathcal{A}$ satisfies* (2), *then the element $\varphi(a) - a$ is nilpotent.*

Proof. If $0 \le k < n$, then $(\varphi - 1)^n(a^k) = (\varphi - 1)^{n-k}((\varphi - 1)^k(a^k)) = 0$ by the Lemma. Consequently, $n!(\varphi(a) - a)^n = 0$ where n is the degree of algebraicity of a.

Now let $\mathcal{A}$ be a Banach algebra and $\mathcal{R}$ be its (Jacobson) radical. Recall that $\mathcal{R}$ is a closed ideal of $\mathcal{A}$ consisting of quasinilpotent elements. Moreover, from [8, Proposition 1] it follows immediately that $q \in \mathcal{R}$ if and only if $q \circ x$ is quasinilpotent for every $x \in \mathcal{A}$. Using this fact it is easy to see that $\mathcal{R}$ is invariant under every Jordan automorphism (in fact, even every surjective Jordan endomorphism) φ of $\mathcal{A}$. Indeed, first noting that φ preserves invertibility of elements (see, e.g., [15, Proposition 1.3]), and so, in particular, their quasinilpotency, it follows that for each $q \in \mathcal{R}$, $\varphi(q) \circ \varphi(x) = \varphi(q \circ x)$ is quasinilpotent for all $x \in \mathcal{A}$. But then $\varphi(q) \in \mathcal{R}$.

A Jordan automorphism of a semisimple Banach algebra is automatically continuous [14] (see also [2, Theorem 5.5.2] for a considerably more general result). Incidentally, Ransford's proof [12] of Johnson's theorem [5] on automatic continuity of homomorphisms onto semisimple Banach algebras also works for Jordan homomorphisms. The only modification that one has to make is that at the end of the proof one has to apply the characterization of the radical mentioned above instead of [12, Lemma 1].

Theorem 2. *Let φ be a Jordan automorphism of a complex Banach algebra $\mathcal{A}$. If $a \in \mathcal{A}$ satisfies* (2), *then the element $\varphi(a) - a$ is quasinilpotent.*

Proof. Assume first that φ is continuous. Then the Lemma implies that

$$\|(\varphi(a) - a)^n\| \le \frac{1}{n!}\|\varphi - 1\|^n \|a\|^n$$

and so the quasinilpotency of $\varphi(a) - a$ follows from the spectral radius formula.

In the general case, where φ is not necessarily continuous, we consider the Banach algebra $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{R}$ and the automorphism $\overline{\varphi}$ of $\overline{\mathcal{A}}$ defined by $\overline{\varphi}(x + \mathcal{R}) = \varphi(x) + \mathcal{R}$. We remark that $\overline{\varphi}$ is well defined since φ leaves $\mathcal{R}$ invariant. Since the algebra $\overline{\mathcal{A}}$ is semisimple, $\overline{\varphi}$ is continuous. Therefore, from $\frac{1}{2}(\overline{\varphi}(a+\mathcal{R}) + \overline{\varphi}^{-1}(a+\mathcal{R})) = a+\mathcal{R}$ it follows that $\overline{\varphi}(a+\mathcal{R}) - (a+\mathcal{R}) = \varphi(a) - a + \mathcal{R}$ is a quasinilpotent element in $\overline{\mathcal{A}}$. But then $\varphi(a) - a \in \mathcal{A}$ is also quasinilpotent.

Remarks. 1. Condition (2) is equivalent to the condition $(\varphi - 1)^2(a) = 0$, which makes sense even when φ is not bijective. Indeed, it is clear from the proofs above that when assuming this latter condition, the conclusions of the Lemma and Theorem 1 hold true for any Jordan endomorphism φ of $\mathcal{A}$. Theorem 2 can also be appropriately extended, but we have to assume that φ is either continuous or surjective.

2. The referee informed us that in the case when φ is an automorphism, the Lemma was also observed earlier by Turovskiĭ [17, Section 4]. In fact, Turovskiĭ derived a somewhat more general formula for any endomorphism φ of an algebra $\mathcal{A}$. Specifically, he showed that if $a_i \in \mathcal{A}$, $i = 1, \ldots, n$, are such that $(\varphi - 1)^2(a_i) = 0$, then $(\varphi - 1)^n(a_1 \ldots a_n) = n!(\varphi(a_1) - a_1) \ldots (\varphi(a_n) - a_n)$. He used this to prove some extensions of the Kleinecke–Shirokov theorem in a somewhat different direction than obtained in the present paper. In particular, he showed that the closure of $\mathrm{Ker}(\varphi - 1) \cap \mathrm{Im}(\varphi - 1)$ consists of quasinilpotents (an analogous result was also established for derivations).

3. Condition (2) can be considered for any invertible operator φ on a vector space X. If we assume that X is normed and φ is bounded with $\|\varphi\| \le 1$, then (2) immediately yields $\varphi(a) = a$. Indeed, setting $q = \varphi(a) - a$ we have $\varphi(q) = q$, hence $(\varphi^n - 1)(a) = (1 + \varphi + \ldots + \varphi^{n-1})(\varphi - 1)(a) = (1+\varphi+\ldots+\varphi^{n-1})q = nq$, which in turn implies $n\|q\| \le (\|\varphi\|^n+1)\|a\| \le 2\|a\|$ for any positive integer n. Hence $q = 0$. We remark that, in view of this observation, Theorem 2 is meaningless in the case when φ is an isometry.

4. In view of similarity between the Lemma and [9, (2)] (and their applications) one might wonder whether these two results can be unified. One possible way is to consider linear maps $\varDelta$ of an algebra $\mathcal{A}$ into itself such that, for some Jordan endomorphism φ of $\mathcal{A}$, $\varDelta\varphi = \varphi\varDelta$ and $\varDelta(x \circ y) = \varphi(x) \circ \varDelta(y) + \varDelta(x) \circ y$ for all $x, y \in \mathcal{A}$. The class of these maps includes Jordan derivations (for $\varphi = 1$) as well as maps $\varphi - 1$ where φ is any Jordan endomorphism. An inspection of the proof of the the Lemma shows that the following is true: If such a map $\varDelta$ satisfies $\varDelta^2(a) = 0$ and $\varphi(\varDelta(a)) = \varDelta(a)$ for some $a \in \mathcal{A}$, then $\varDelta^n(a^n) = n!\varDelta(a)^n$ for every n. Thus, the proof of Lemma also indicates an alternative way of proving [9, (2)] (even for Jordan derivations), which avoids the use of Leibniz's rule.

5. In the case when φ is an inner automorphism, Theorem 2 reduces to the Kleinecke–Shirokov theorem. Indeed, since $[b, x] = [b - \lambda 1, x]$ for all $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, when examining condition (1) there is no loss of generality in assuming that b is invertible. But then Theorem 2 tells us that $\varphi_b(a) - a = b^{-1}[a, b]$ is quasinilpotent. Since this element commutes with b by assumption, it follows that $[a, b] = b(b^{-1}[a, b])$ is quasinilpotent.

6. In a similar fashion we see that Jacobson's lemma follows from Theorem 1.

7. Since our results hold, in particular, for antiautomorphisms, they can be applied to algebras with involution. Let us point out a special case of Theorem 2 which could be, in some sense, considered as a $*$-version of the Kleinecke–Shirokov theorem. Let $\mathcal{A}$ be a Banach algebra with involution $*$. Considering the antiautomorphism $\varphi : x \mapsto bx^*b^{-1}$, where $b \in \mathcal{A}$ is any invertible element, we see that the following holds true: If $a \in \mathcal{A}$ is such

that $\frac{1}{2}(ba^*b^{-1} + b^*a^*(b^*)^{-1}) = a$ (equivalently, $a^*(b^{-1}b^*) + (b^{-1}b^*)a^* = 2b^{-1}ab^*$), then $ba^*b^{-1} - a$ is quasinilpotent. Of course, if a is algebraic, then, by Theorem 1, $ba^*b^{-1} - a$ is nilpotent.

8. If δ is a continuous derivation on a Banach algebra $\mathcal{A}$ and $a \in \mathcal{A}$ is such that $\delta^2(a) = 0$, then $\delta(a)$ is quasinilpotent. This generalized version of the Kleinecke–Shirokov theorem, essentially already proved by Kleinecke [9], can also be easily deduced from Theorem 2. Indeed, if $\delta^2(a) = 0$, then the automorphism $\varphi = e^\delta = \sum_{n=0}^{\infty} \delta^n/n!$ satisfies (2), and therefore $\delta(a) = \varphi(a) - a$ is quasinilpotent by Theorem 2.

9. Although derivations are, just as (Jordan) automorphisms, also automatically continuous on semisimple Banach algebras [6], it is usually not so easy to reduce problems concerning the spectrum and derivations to the case when derivations are continuous. The main obstacle is that it is still not known whether the radical of a Banach algebra is invariant under any derivation. Anyway, Thomas [16] proved that the assertion "$\delta^2(a) = 0$ implies $\delta(a)$ is quasinilpotent" holds true even when δ is not continuous. Just recently Villena [18] extended Thomas' result to Jordan–Banach algebras. We remark that Theorem 2 also holds true in the case when $\mathcal{A}$ is a Jordan–Banach algebra and φ is an automorphism of a $\mathcal{A}$. The proof is the same; one just has to apply a result of Aupetit [1, Theorem 2] instead of that of Sinclair [14]. Moreover, the proofs of the Lemma and Theorem 1 work in any power-associative algebra.

Acknowledgements. The authors are indebted to Professors Armando R. Villena and Jaroslav Zemánek for reading the first version of the paper and giving several helpful suggestions and comments. Also, the authors are grateful to the referee for drawing their attention to [8] and [17].

References

[1] B. Aupetit, *The uniqueness of the complete norm topology in Banach algebras and Banach Jordan algebras*, J. Funct. Anal. 47 (1982), 1–6.

[2] —, *A Primer on Spectral Theory*, Springer, 1991.

[3] P. Halmos, *Commutators of operators, II*, Amer. J. Math. 76 (1954), 191–198.

[4] N. Jacobson, *Rational methods in the theory of Lie algebras*, Ann. of Math. 36 (1935), 875–881.

[5] B. E. Johnson, *The uniqueness of the (complete) norm topology*, Bull. Amer. Math. Soc. 73 (1967), 537–539.

[6] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. 90 (1968), 1067–1073.

[7] I. Kaplansky, *Jacobson's lemma revisited*, J. Algebra 62 (1980), 473–476.

[8] A. Katavolos and C. Stamatopoulos, *Commutators of quasinilpotents and invariant subspaces*, Studia Math. 128 (1998), 159–169.

[9] D. C. Kleinecke, *On operator commutators*, Proc. Amer. Math. Soc. 8 (1957), 535–536.
[10] M. Mathieu, *Where to find the image of a derivation?*, in: Banach Center. Publ. 30, Inst. Math., Polish Acad. Sci., 1994, 237–249.
[11] G. J. Murphy, *Aspects of the theory of derivations*, ibid., 267–275.
[12] T. J. Ransford, *A short proof of Johnson's uniqueness-of-norm theorem*, Bull. London Math. Soc. 21 (1989), 487–488.
[13] F. V. Shirokov, *Proof of a conjecture of Kaplansky*, Uspekhi Mat. Nauk 11 (1956), no. 4, 167–168 (in Russian).
[14] A. M. Sinclair, *Jordan automorphisms on a semisimple Banach algebra*, Proc. Amer. Math. Soc. 25 (1970), 526–528.
[15] A. R. Sourour, *Invertibility preserving linear maps on $\mathcal{L}(X)$*, Trans. Amer. Math. Soc. 348 (1996), 13–30.
[16] M. P. Thomas, *Primitive ideals and derivations on non-commutative Banach algebras*, Pacific J. Math. 159 (1993), 139–152.
[17] Yu. V. Turovskiĭ, *Homomorphisms and derivations of rings and algebras*, in: Spectral Theory of Operators and its Applications, No. 7, "Elm", Baku, 1986, 201–207 (in Russian).
[18] A. R. Villena, *A local property of derivations on Jordan–Banach algebras*, Comm. Algebra, to appear.

Department of Mathematics
University of Maribor
PF, Koroška 160
2000 Maribor, Slovenia
E-mail: bresar@uni-mb.si
ajda.fosner@uni-mb.si

Institute of Mathematics, Physics and Mechanics
Jadranska 19
1000 Ljubljana, Slovenia
E-mail: maja.fosner@uni-mb.si

Received October 16, 2000
Revised version March 23, 2001 (4624)