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On the Banach–Stone problem

by

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Abstract. Let X and Y be locally compact Hausdorff spaces, let E and F be Banach spaces, and let T be a linear isometry from $C_0(X,E)$ *into* $C_0(Y,F)$. We provide three new answers to the Banach–Stone problem: (1) T can always be written as a generalized weighted composition operator if and only if F is strictly convex; (2) if T is onto then T can be written as a weighted composition operator in a weak sense; and (3) if T is onto and F does not contain a copy of ℓ_2^∞ then T can be written as a weighted composition operator in the classical sense.

1. Introduction. In [18], Jerison got the first vector-valued version of the Banach–Stone Theorem: Suppose X and Y are compact Hausdorff spaces and E is a Banach space. Jerison proved that if E is strictly convex then every linear isometry T from $C(X,E)$ onto $C(Y,E)$ is a *weighted composition operator* $Tf = h \cdot f \circ \varphi$, that is,

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall f \in C(X,E),\ \forall y \in Y,$$

for some continuous map (in fact, homeomorphism) φ from Y onto X and some continuous-operator-valued (in fact, onto-isometry-valued) map h from Y into $L(E,E)$. In [19], Lau gave another version: Suppose the Banach dual space E^* of E is strictly convex instead. Then every linear isometry from $C(X,E)$ onto $C(Y,E)$ is also a weighted composition operator.

Recall that a Banach space E is *strictly convex* if every vector in the unit sphere S_E of E is an extreme point of the closed unit ball U_E of E. We denote by $C_0(X,E)$ the Banach space of continuous vector-valued functions from the locally compact Hausdorff space X into E vanishing at infinity. We write $C(X,E)$ for $C_0(X,E)$ whenever X is compact, as usual. The norm of f in $C_0(X,E)$ is defined to be $\|f\| = \sup\{\|f(x)\| : x \in X\}$. Moreover, the vector space $L(E,F)$ of bounded linear operators from a Banach space E

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into a Banach space F is always equipped with the strong operator topology (SOT) in this paper.

A Banach space E is said to have the *Banach–Stone property* if the existence of a linear isometry T from $C_0(X,E)$ onto $C_0(Y,E)$ ensures X and Y being homeomorphic for all locally compact Hausdorff spaces X and Y. We say that E has the *strong Banach–Stone property* if all such T can be written as a weighted composition operator. It is known that $\ell_2^\infty = \mathbb{R} \oplus_\infty \mathbb{R}$ does not have the Banach–Stone property, while $\mathbb{R} \oplus_\infty (\mathbb{R} \oplus_2 \mathbb{R})$ has the Banach–Stone property but not the strong Banach–Stone property. In fact, every 3-dimensional Banach space has the Banach–Stone property except for $\mathbb{R}\oplus_\infty\mathbb{R}\oplus_\infty\mathbb{R}$ (see e.g. [3, pp. 142–147]). For another example, put $E = C(Q)$, where $Q = [0,1]^\infty$ is the Hilbert cube. Let $X = [0,1]$ and $Y = \{0\}$. Then the spaces $C(X,E)$ and $C(Y,E)$ are isometric while there is no map from Y onto X. In other words, $C(Q)$ does not have the Banach–Stone property.

Definition 1. We say that a Banach space F *solves the Banach–Stone problem* if every linear isometry from $C_0(X,E)$ onto $C_0(Y,F)$ is a weighted composition operator for all locally compact Hausdorff spaces X and Y and Banach spaces E.

Although some authors mainly deal with the case of $E = F$, their arguments can be modified easily to give us solutions of the Banach–Stone problem. In particular, Jerison's result [18] says that strictly convex Banach spaces solve the Banach–Stone problem, while Lau's result [19] says that so do Banach spaces with strictly convex dual. However, not every Banach space solves the Banach–Stone problem. As a basic counterexample, the 2-dimensional Banach space $\ell_2^\infty = \mathbb{R} \oplus_\infty \mathbb{R}$ does not solve the Banach–Stone problem. In fact, the linear isometry T from $C(\{1,2\},\mathbb{R})$ onto $C(\{0\},\ell_2^\infty)$, defined by

$$Tf(0) = (f(1), f(2)),$$

cannot be written as a weighted composition operator. We note that the inverse T^{-1} of T is a weighted composition operation, however. This tells us that the concept of solving the Banach–Stone problem is a non-symmetric generalization of the strong Banach–Stone property. Clearly, every solution of the Banach–Stone problem has the strong Banach–Stone property. We do not know, however, if the converse implication is always true.

In general, every Banach space containing non-trivial M-summands does not solve the Banach–Stone problem (see, e.g., [3, p. 149]). Recall that a non-trivial closed subspace E_1 of a Banach space E is called an *M-summand* of E if $E = E_1 \oplus_\infty E_2$ for some closed proper subspace E_2 of E. In [10], Cambern proved that a *reflexive* Banach space E solves the Banach–Stone problem if and only if E does not have any non-trivial M-summand. However, a reflexive space with a non-trivial M-summand may still have the Banach–

Stone property, for example, $\mathbb{R} \oplus_\infty (\mathbb{R} \oplus_2 \mathbb{R})$. In the non-reflexive case, the Banach–Stone problem is still open. Many counter-examples have been given since then. See, for instance, [8, 9, 3]. Several attempts to attack the Banach–Stone problem have appeared; see [1, 20, 2, 3, 6, 5, 12], to name a few. Among them are the methods of T-sets of Jerison [18] and M-structures of Behrends (see, e.g., [3]). These results proved to be very powerful (cf. [4]).

In this paper, without using any technique of T-sets and M-structures we present three new answers to the Banach–Stone problem. Theorem 3 places the strict convexity in the correct position in solving the Banach–Stone problem. It states that every isometry from $C_0(X,E)$ *into* $C_0(Y,F)$ is a generalized weighted composition operator if and only if F is strictly convex. Theorem 4 says that every Banach space does solve the Banach–Stone problem in a *weak* sense. Finally, Theorem 6 supplements a well known result of Behrends ([3, p. 148]; see also [14]) by showing that Banach spaces containing no copy of ℓ_2^∞ solve the Banach–Stone problem. The proofs of these results are modeled on those employed in the scalar version by Holsztyński [13] and Jarosz [15] (cf. [16]). As applications, we shall derive the classical results of Jerison [18] and Lau [19] (see Corollary 8), and a recent result of Hernandez, Beckenstein and Narici [12] (see Corollary 9) as natural consequences of our Theorems 6 and 4, respectively.

We would like to express our deep thanks to Ka-Sing Lau for sharing with us his conjecture which eventually works out as our Theorem 6, and to K. Jarosz for useful comments on a preliminary version of this paper. We are grateful to the referee for many helpful comments.

2. Three new answers to the Banach–Stone problem. In the following, we always assume X and Y are (non-empty) locally compact Hausdorff spaces and E and F are (non-zero) Banach spaces without any additional structure, unless otherwise stated. We first show that the way to write a linear map from $C_0(X,E)$ into $C_0(Y,F)$ as a weighted composition operator is unique.

PROPOSITION 2. *Let T be a linear map from $C_0(X,E)$ into $C_0(Y,F)$. Suppose there exist a map φ from a non-empty subset Y_0 of Y into X and a non-vanishing map h from Y_0 into $L(E,F)$ such that*

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall y \in Y_0. \tag{1}$$

Then both φ and h are continuous. Moreover, if (Y_0', φ', h') is another triple satisfying all the above conditions then $\varphi(y) = \varphi'(y)$ and $h(y) = h'(y)$ for all y in $Y_0 \cap Y_0'$.

Proof. We divide the proof into the following three claims.

CLAIM 1. *$\varphi : Y_0 \to X$ is continuous.*

Suppose otherwise, and let $\{y_\lambda\}$ be a net convergent to y in Y_0 such that $\{\varphi(y_\lambda)\}$ does not converge to $\varphi(y)$. By passing to a subnet if necessary, we can assume that $\{\varphi(y_\lambda)\}$ converges to some other x in $X_\infty = X \cup \{\infty\}$, the one-point compactification of X. Let U_1 and U_2 be disjoint neighborhoods of x and $\varphi(y)$ in X_∞, respectively. Then $\varphi(y_\lambda) \in U_1$ eventually. Choose an f in $C_0(X, E)$ such that f vanishes outside U_2 and $h(y)(f(\varphi(y))) \neq 0$. We then have $f(\varphi(y_\lambda)) = 0$ and thus $Tf(y_\lambda) = 0$ for all large λ. As a result, $\{Tf(y_\lambda)\}$ cannot converge to $Tf(y) = h(y)(f(\varphi(y))) \neq 0$, a contradiction.

Claim 2. *$h : Y_0 \to (L(E, F), \mathrm{SOT})$ is continuous.*

Let $\{y_\lambda\}$ be a net convergent to y in Y_0. For each e in E, choose an f in $C_0(X, E)$ such that $f(x) = e$ for all x in a neighborhood of $\varphi(y)$. Since φ is continuous, $f(\varphi(y_\lambda)) = e$ for all large enough λ. Consequently, $\|h(y_\lambda)e - h(y)e\| = \|h(y_\lambda)(f(\varphi(y_\lambda))) - h(y)(f(\varphi(y)))\| = \|Tf(y_\lambda) - Tf(y)\|$ eventually. Since $\{Tf(y_\lambda)\}$ converges to $Tf(y)$, the claim is verified.

Claim 3. *$\varphi = \varphi'$ and $h = h'$ on $Y_0 \cap Y_0'$.*

Suppose $\varphi(y) \neq \varphi'(y)$ for some y in $Y_0 \cap Y_0'$. Let $x = \varphi(y)$ and $x' = \varphi'(y)$. Let $f \in C_0(X, E)$ be such that $f(x) = 0$ and $h'(y)(f(x')) \neq 0$. Then $Tf(y) = h(y)(f(\varphi(y))) = 0$ and $Tf(y) = h'(y)(f(\varphi'(y))) \neq 0$, a contradiction. Hence, φ and φ' agree on $Y_0 \cap Y_0'$. It follows that h and h' also agree on $Y_0 \cap Y_0'$. ∎

The family of all triples (Y_0, φ, h) which partially represent a linear isometry T from $C_0(X, E)$ into $C_0(Y, F)$ as a weighted composition operator $Tf|_{Y_0} = h \cdot f \circ \varphi$ is directed in the natural ordering induced by set inclusion. Theorem 3 below ensures that this family is non-trivial if, for example, F is strictly convex. Hence, by taking the set-theoretical union of all such triples, there exists *the* greatest subset Y_0 of Y on which T can be written as a weighted composition operator. By saying that a linear isometry T from $C_0(X, E)$ into $C_0(Y, F)$ is a *generalized weighted composition operator*, we mean there are a subset Y_1 of Y, a continuous map φ from Y_1 *onto* X and a continuous operator-valued map h from Y_1 into $(L(E, F), \mathrm{SOT})$ such that $Tf|_{Y_1} = h \cdot f \circ \varphi$ and $\|Tf\| = \|Tf|_{Y_1}\| = \sup\{\|Tf(y)\| : y \in Y_1\}$.

Our first theorem places the strict convexity in its correct position in the context of the Banach–Stone problem. We remark that we always have the implication (1)⇒(2) of Theorem 3 below, even if the underlying field $\mathbb{K}$ is complex, although the other implication seems open in this case. In fact, Cambern [11] proved the implication (1)⇒(2) when X and Y are compact Hausdorff spaces and $\mathbb{K}$ is either the real or complex field. In [17], we extended this implication to the locally compact case.

Theorem 3. *Let F be a real Banach space. The following two conditions are equivalent*:

(1) *F is strictly convex.*

(2) *For all locally compact Hausdorff spaces X and Y and for all real Banach spaces E, every real linear into isometry T from $C_0(X,E)$ into $C_0(Y,F)$ is a generalized weighted composition operator.*

Proof. Suppose F is strictly convex. For the underlying field $\mathbb{K}$ being either the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$, we have proved in [17] that every linear isometry T from $C_0(X,E)$ into $C_0(Y,F)$ is a generalized weighted composition operator. For the sake of completeness, we present a sketch of the proof below.

The task is to find a subset Y_1 of Y, a map φ from Y_1 onto X and a map h from Y_1 into $L(E,F)$ such that $Tf|_{Y_1} = h \cdot f \circ \varphi$ for all $f \in C_0(X,E)$. Denote by S_{E^*} (resp. S_{F^*}) the unit sphere of the dual space of E (resp. F). Let $x \in X$, $y \in Y$, $\mu \in S_{E^*}$ and $\nu \in S_{F^*}$. Consider the sets

$$\mathcal{S}_{x,\mu} = \{f \in C_0(X,E) : \mu(f(x)) = \|f\| = 1\},$$
$$\mathcal{R}_{y,\nu} = \{g \in C_0(Y,F) : \nu(g(y)) = \|g\| = 1\}.$$

$\mathcal{S}_{x,\mu}$ (resp. $\mathcal{R}_{y,\nu}$) can be considered as the norm attaining set of the norm one linear functional $\mu\circ\delta_x$ (resp. $\nu\circ\delta_y$) of $C_0(X,E)$ (resp. $C_0(Y,F)$), where δ_x (resp. δ_y) is the evaluation map at the point x (resp. y). Set

$$\mathcal{Q}_{x,\mu} = \begin{cases} \{y \in Y : T(\mathcal{S}_{x,\mu}) \subset \mathcal{R}_{y,\nu} \text{ for some } \nu \text{ in } S_{F^*}\} & \text{if } \mathcal{S}_{x,\mu} \neq \emptyset, \\ \emptyset & \text{if } \mathcal{S}_{x,\mu} = \emptyset. \end{cases}$$

By a compactness argument, we can show that

$$\mathcal{Q}_{x,\mu} \neq \emptyset \quad \text{whenever} \quad \mathcal{S}_{x,\mu} \neq \emptyset.$$

Since the norm attaining linear functionals are dense in the unit sphere S_{E^*} of E^* by the Bishop–Phelps Theorem [7], many $\mathcal{S}_{x,\mu}$ are non-empty. Thus

$$\mathcal{Q}_x = \bigcup_{\mu \in S_{E^*}} \mathcal{Q}_{x,\mu} \neq \emptyset$$

for each x in X. Let

$$Y_1 = \bigcup_{x \in X} \mathcal{Q}_x.$$

The strict convexity of F implies that $\mathcal{Q}_{x_1} \cap \mathcal{Q}_{x_2} = \emptyset$ whenever $x_1 \neq x_2$ in X. This partition defines a map φ from Y_1 onto X such that

$$\varphi(y) = x \quad \text{if } y \in \mathcal{Q}_x.$$

Another key step in the proof is to use the strict convexity of F again to assert that

$$\varphi(y) \notin \operatorname{supp} f \ \Rightarrow\ Tf(y) = 0, \qquad \forall f \in C_0(X,E).$$

From this we have the inclusion $\ker \delta_{\varphi(y)} \subseteq \ker \delta_y \circ T$ by Urysohn's Lemma. It follows that there exists a linear map $h(y)$ from E into F such that

$\delta_y \circ T = h(y)\delta_{\varphi(y)}$, or $Tf(y) = h(y)(f(\varphi(y)))$ for all $f \in C_0(X, E)$ and $y \in Y_1$. The continuity of φ and h follows from Proposition 2. It is then easy to see that $\|Tf\| = \|Tf|_{Y_1}\| = \sup\{\|Tf(y)\| : y \in Y_1\}$.

Conversely, we assume that F is not strictly convex. In this case, we also assume that the underlying field is $\mathbb{R}$. We want to find a linear isometry T from $C_0(X, E)$ into $C_0(Y, F)$ which cannot be written as a generalized weighted composition operator. To this end, we set $X = Y = \{1, 2\}$ in the discrete topology. Let $E = \mathbb{R}$. Since F is not strictly convex, there are distinct e_1 and e_2 in the unit sphere S_F of F such that $t_0e_1 + (1-t_0)e_2 \in S_F$ for some $0 < t_0 < 1$. In fact, $te_1 + (1-t)e_2$ belongs to S_F for all t in $[0, 1]$. Consequently,

$$\|\alpha e_1 + \beta e_2\| = \alpha + \beta \quad \text{for all } \alpha, \beta \geq 0. \tag{2}$$

Represent functions f in $C(X)$ as column vectors $\binom{\alpha}{\beta}$ in which $f(1) = \alpha$ and $f(2) = \beta$. Let $f_1 = \binom{1}{1}$ and $f_2 = \binom{1}{-1}$ in $C(X)$. For each $f = \binom{\alpha}{\beta}$ in $C(X)$, we can write

$$f = \frac{\alpha+\beta}{2} f_1 + \frac{\alpha-\beta}{2} f_2.$$

Define a linear map $T : C(X) \to C(Y, F)$ by

$$Tf_1 = \begin{pmatrix} e_1 \\ -e_1 \end{pmatrix} \quad \text{and} \quad Tf_2 = \begin{pmatrix} e_2 \\ e_2 \end{pmatrix}$$

in a similar convention. In other words,

$$T\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha+\beta}{2}\begin{pmatrix} e_1 \\ -e_1 \end{pmatrix} + \frac{\alpha-\beta}{2}\begin{pmatrix} e_2 \\ e_2 \end{pmatrix}.$$

Now we show that T is an isometry. First, assume that $|\alpha| \geq |\beta|$. If $\alpha > 0$, then $(\alpha+\beta)/2 \geq 0$ and $(\alpha-\beta)/2 \geq 0$. By (2),

$$\|Tf(1)\| = \left\|\frac{\alpha+\beta}{2} e_1 + \frac{\alpha-\beta}{2} e_2\right\| = \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2} = \alpha.$$

Moreover,

$$\|Tf(2)\| \leq \frac{\alpha+\beta}{2}\|e_1\| + \frac{\alpha-\beta}{2}\|e_2\| = \alpha.$$

If $\alpha < 0$, then $(\alpha+\beta)/2 \leq 0$ and $(\alpha-\beta)/2 \leq 0$. Again by (2),

$$\begin{aligned}\|Tf(1)\| &= \left\|\frac{\alpha+\beta}{2} e_1 + \frac{\alpha-\beta}{2} e_2\right\| \\ &= \left\|\frac{\alpha+\beta}{-2} e_1 + \frac{\alpha-\beta}{-2} e_2\right\| = \frac{\alpha+\beta}{-2} + \frac{\alpha-\beta}{-2} = -\alpha.\end{aligned}$$

On the other hand,

$$\|Tf(2)\| \leq \frac{\alpha+\beta}{-2}\|-e_1\| + \frac{\alpha-\beta}{-2}\|e_2\| = -\alpha.$$

So $\|Tf\| = \|f\| = |\alpha|$ in both cases. When $|\alpha| < |\beta|$, a similar argument applies and also gives $\|Tf\| = \|f\|$. Hence T is an isometry.

Finally, we show that T is not a generalized weighted composition operator. Suppose it were, and there existed a non-empty subset Y_0 of Y, a continuous map φ from Y_0 into X and a linear map $h(y) : \mathbb{R} \to F$ such that $Tf(y) = h(y)(f(\varphi(y)))$ for all $f \in C(X)$ and all $y \in Y_0$. For the case $1 \in Y_0$ and $\varphi(1) = 1$, we have $e_1 = Tf_1(1) = h(1)(f_1(1)) = h(1)(1) = h(1)(f_2(1)) = Tf_2(1) = e_2$, a contradiction. Similar contradictions can be derived for the other cases. ∎

Our second theorem gives a complete answer to the Banach–Stone problem in a *weak* sense. Subject to *no* constraint on X, Y, E, or F, it says that every linear isometry T from $C_0(X,E)$ onto $C_0(Y,F)$ can be written in a weak form of a weighted composition operator. This version of the Banach–Stone Theorem is good enough for many applications. See, for example, Corollaries 8 and 9 below. Before stating it, recall that if $Tf = h \cdot f \circ \varphi$ is a weighted composition operator from $C_0(X,E)$ into $C_0(Y,F)$ then for each bounded linear functional ν on F, we have

$$\nu(Tf(y)) = \nu \circ h(y)(f(\varphi(y))), \quad \forall f \in C_0(X,E),\ \forall y \in Y.$$

In other words, Tf is again an image of a weighted composition operator when viewed as a function of y *and* ν in $Y \times F^*$. Note that $\nu \circ h(y) \in E^*$.

In the following, U_{F^*} (resp. S_{F^*}) denotes the closed unit ball (resp. unit sphere) of the dual space F^* of F. Since T is a linear isometry, its dual map T^* sends the set of extreme points of the closed dual ball of the range space onto the set of extreme points of $U_{C_0(X,E)^*}$, which contains exactly all functionals of the form $\delta_x \otimes \mu$. Here, δ_x is evaluation at some x in X and μ is an extreme point of U_{E^*}. Note also that every extreme point of the closed dual ball of the range space of T can be extended to an extreme point of $U_{C_0(Y,F)^*}$. Let A_Y be the set of all such extensions. In particular, we can think of A_Y as a subset of $Y \times U_{F^*}$ and T^*A_Y consists of all $\delta_x \otimes \mu$ with x in X and μ being an extreme point of U_{E^*}. Define $\widetilde{\varphi}(y,\nu) = x$ on A_Y if $T^*(\delta_y \otimes \nu) = \delta_x \otimes \mu$ for some μ. In this setting, we have

THEOREM 4. *Let T be a linear isometry from $C_0(X,E)$ into $C_0(Y,F)$. Then there exist a continuous map $\widetilde{\varphi}$ from A_Y onto X, and a weak* continuous map $\widetilde{h}$ from A_Y into E^* such that*

$$\nu(Tf(y)) = \widetilde{h}(y,\nu)(f(\widetilde{\varphi}(y,\nu))), \quad \forall f \in C_0(X,E),\ \forall (y,\nu) \in A_Y.$$

In this case, $\|\widetilde{h}(y,\nu)\| \equiv 1$ for all (y,ν) in A_Y and $\|Tf\| = \sup\{|\nu(Tf(y))| : (y,\nu) \in A_Y\}$. Moreover, if T is onto then the set

$$B_y = \{\nu \in S_{F^*} : (y,\nu) \in A_Y\}$$

contains all extreme points of U_{F^*} for each y in Y.

Theorem 4 can be applied to give some Banach–Stone type theorems in the classical sense. The following lemma is crucial.

LEMMA 5. *Let T be a linear isometry from $C_0(X,E)$ onto $C_0(Y,F)$. Then T is a weighted composition operator $Tf = h\cdot f\circ\varphi$ if and only if $\widetilde{\varphi}(y,\nu_1)=\widetilde{\varphi}(y,\nu_2)$ for all ν_1, ν_2 in B_y and all y in Y. In this case, we have $\widetilde{h}(y,\nu)=\nu\circ h(y)$ and $\widetilde{\varphi}(y,\nu)=\varphi(y)$ for all $\nu\in B_y$ and all $y\in Y$.*

Proof. We verify the sufficiency only. Let $\widetilde{\varphi}(y,\nu_1)=\widetilde{\varphi}(y,\nu_2)$ for all $\nu_1,\nu_2\in B_y$. We can define an onto map $\varphi: Y\to X$ by $\varphi(y)=\widetilde{\varphi}(y,\nu)$ for any ν in B_y. If $f(\varphi(y))=0$, then $\nu(Tf(y))=\widetilde{h}(y,\nu)(f(\varphi(y)))=0$ for all $\nu\in B_y$. Since B_y is total, $Tf(y)=0$. As a result, $\ker\delta_{\varphi(y)}\subseteq\ker\delta_y\circ T$. It follows that there exists a linear map $h(y): E\to F$ such that $Tf(y)=h(y)(f(\varphi(y)))$ for all $f\in C_0(X,E)$ and all $y\in Y$. The continuity of φ and h follows from Proposition 2. ∎

We are now ready to provide an answer to the Banach–Stone problem in the classical sense. Recall that $\ell_2^\infty=\mathbb{R}\oplus_\infty\mathbb{R}$ does not solve the Banach–Stone problem. We say that a (real or complex) Banach space F *does not contain a copy of* ℓ_2^∞ if there is no *real* linear isometric embedding of ℓ_2^∞ into F. It is easy to see that $\ell_2^\infty=\mathbb{R}\oplus_\infty\mathbb{R}$ is real-linear isometrically isomorphic to $\ell_2^1=\mathbb{R}\oplus_1\mathbb{R}$ since their unit balls are both squares. Consequently, F does not contain a copy of ℓ_2^∞ if and only if at least one of the norms $\|e_1\pm e_2\|<2$ whenever $\|e_1\|=\|e_2\|=1$; for else the linear span of $\{e_1,e_2\}$ will be a copy of ℓ_2^1 ($\cong\ell_2^\infty$). For comparison, F is strictly convex if and only if *both* of the norms $\|e_1\pm e_2\|$ are less than 2 whenever $\|e_1\|=\|e_2\|=1$.

THEOREM 6. *Let X and Y be locally compact Hausdorff spaces and let E and F be Banach spaces. Suppose F does not contain a copy of ℓ_2^∞. Then every linear isometry T from $C_0(X,E)$ onto $C_0(Y,F)$ is a weighted composition operator*

$$Tf(y)=h(y)(f(\varphi(y))),\qquad \forall f\in C_0(X,E),\ \forall y\in Y,$$

for some continuous map φ from Y onto X and continuous map h from Y into $(L(E,F),\mathrm{SOT})$.

Proof. We have to verify the condition stated in Lemma 5. Suppose on the contrary that there exist ν_1 and ν_2 in S_{F^*} such that $\widetilde{\varphi}(y,\nu_1)=x_1\neq x_2=\widetilde{\varphi}(y,\nu_2)$. By the definition of $\widetilde{\varphi}$, there exist extreme points μ_1 and μ_2 of U_{E^*} such that $T^*(\delta_y\otimes\nu_1)=\delta_{x_1}\otimes\mu_1$ and $T^*(\delta_y\otimes\nu_2)=\delta_{x_2}\otimes\mu_2$. Let U_1 and U_2 be disjoint neighborhoods of x_1 and x_2, respectively. Choose f_i in $C_0(X,E)$ such that f_i is supported by U_i and $\mu_i(f_i(x_i))=\|f_i\|=1$ for $i=1,2$. Consequently,

$$\|Tf_1(y)\|=\|Tf_2(y)\|=1. \tag{3}$$

Moreover, $\|f_1 \pm f_2\| = 1$ implies $\|T(f_1 \pm f_2)(y)\| \leq 1$. In fact, the inequalities

$$\begin{aligned} 2 = 2\|Tf_1(y)\| &= \|T(f_1 + f_2)(y) + T(f_1 - f_2)(y)\| \\ &\leq \|T(f_1 + f_2)(y)\| + \|T(f_1 - f_2)(y)\| \leq 2 \end{aligned}$$

ensure that $\|T(f_1 \pm f_2)(y)\| = 1$. Since F does not contain a copy of ℓ_2^∞, at least one of the norms $\|T(f_1 + f_2)(y) \pm T(f_1 - f_2)(y)\|$ is less than 2. But this conflicts with (3). ∎

REMARK 7. When neither E nor F contains l_2^∞, Theorem 6 implies that every linear surjective isometry T from $C_0(X, E)$ onto $C_0(Y, F)$ is a weighted composition operator $Tf = h \cdot f \circ \varphi$ such that φ is a homeomorphism from Y onto X. However, a more general statement is known: it is enough to assume that the sets of centralizers of E and F are both trivial (see e.g. [3, pp. 147–148]). In fact, every Banach space with non-trivial multipliers contains ℓ_2^∞. See K. Jarosz [14] for details.

We remark that Theorem 6 is still not optimum for the Banach–Stone problem. For example, the Banach space $F = \mathbb{R} \oplus_1 (\mathbb{R} \oplus_2 \mathbb{R})$ does contain a copy of ℓ_2^1 ($\cong \ell_2^\infty$). Since F is reflexive and contains no non-trivial M-summand, by a theorem of Cambern [10], F solves the Banach–Stone problem. Nevertheless, Theorem 6 does include some famous solutions of the Banach–Stone problem.

COROLLARY 8 (Jerison [18] and Lau [19]). *Let X and Y be locally compact Hausdorff spaces and let E and F be Banach spaces. Suppose F or its Banach dual F^* is strictly convex. Then every linear isometry T from $C_0(X, E)$ onto $C_0(Y, F)$ is a weighted composition operator $Tf = h \cdot f \circ \varphi$. In case E or its Banach dual E^* is also strictly convex, φ is a homeomorphism from Y onto X and $h(y)$ is a linear isometry from E onto F for all y in Y.*

Proof. We claim that a Banach space F does not contain a copy of ℓ_2^∞ whenever F or its dual F^* is not strictly convex. In fact, suppose F contains a copy of ℓ_2^∞. Then it is plain that F cannot be strictly convex. At the same time, the Banach dual F^* of F contains a copy of ℓ_2^1. Thus F^* cannot be strictly convex, either. The desired assertions follow from Theorem 6. ∎

Hernandez, Beckenstein and Narici derived Corollary 8 as a consequence of their results in [12]. Recall that the *cozero* of an f in $C_0(X, E)$ is the set $\{x \in X : f(x) \neq 0\}$. A linear map T from $C_0(X, E)$ into $C_0(Y, F)$ is said to be *separating*, or *disjointness preserving*, if Tf and Tg have disjoint cozeroes whenever f and g have disjoint cozeroes. They showed in [12] that if T is a linear onto isometry such that both T and its inverse T^{-1} are separating then T must be a weighted composition operator. They also verified that a surjective linear isometry T must be separating if E and F are both strictly convex. The same also holds if E^* and F^* are both strictly convex instead.

From these facts, they get Corollary 8. Some parts of their results can also be obtained by our approach. We present a new proof of the following

COROLLARY 9 (Hernandez, Beckenstein and Narici [12]). *Let X and Y be locally compact Hausdorff spaces. Let E and F be Banach spaces. Every separating linear isometry T from $C_0(X,E)$ onto $C_0(Y,F)$ is a weighted composition operator.*

Proof. By Theorem 4, we write

$$\nu(Tf(y)) = \widetilde{h}(y,\nu)(f(\widetilde{\varphi}(y,\nu))), \quad \forall (y,\nu) \in A_Y.$$

It suffices to verify the conditions stated in Lemma 5. Suppose, on the contrary, that $\widetilde{\varphi}(y,\nu_1) \neq \widetilde{\varphi}(y,\nu_2)$ for some y in Y and ν_1, ν_2 in B_y. Let U_1 and U_2 be disjoint open neighborhoods of $x_1 = \widetilde{\varphi}(y,\nu_1)$ and $x_2 = \widetilde{\varphi}(y,\nu_2)$ in X, respectively. Choose f_i in $C_0(X,E)$ such that f_i is supported by U_i and $\widetilde{h}(y,\nu_i)f_i(x_i) \neq 0$, $i = 1,2$. Then f_1 and f_2 have disjoint cozeroes. Since T is assumed to be separating, Tf_1 and Tf_2 also have disjoint cozeroes. However,

$$\nu_1(Tf_1(y)) = \widetilde{h}(y,\nu_1)(f_1(\widetilde{\varphi}(y,\nu_1))) = \widetilde{h}(y,\nu_1)(f_1(x_1)) \neq 0,$$
$$\nu_2(Tf_2(y)) = \widetilde{h}(y,\nu_2)(f_2(\widetilde{\varphi}(y,\nu_2))) = \widetilde{h}(y,\nu_2)(f_2(x_2)) \neq 0,$$

a contradiction. Hence, we have $\widetilde{\varphi}(y,\nu_1) = \widetilde{\varphi}(y,\nu_2)$ for all $\nu_1,\nu_2 \in B_y$, and all $y \in Y$, as asserted. ∎

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