STUDIA MATHEMATICA 155 (2) (2003)

Some theorems of Korovkin type

by

TOMOKO HACHIRO (Sendai) and TAKATERU OKAYASU (Yamagata)

Abstract. We take another approach to the well known theorem of Korovkin, in the following situation: X, Y are compact Hausdorff spaces, M is a unital subspace of the Banach space $C(X)$ (respectively, $C_{\mathbb{R}}(X)$) of all complex-valued (resp., real-valued) continuous functions on X, $S \subset M$ a complex (resp., real) function space on X, $\{\phi_n\}$ a sequence of unital linear contractions from M into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$), and ϕ_∞ a linear isometry from M into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$). We show, under the assumption that $\Pi_N \subset \Pi_T$, where Π_N is the Choquet boundary for $N = \mathrm{Span}(\bigcup_{1\le n\le\infty} N_n)$, $N_n = \phi_n(M)$ $(n = 1, 2, \ldots, \infty)$, and Π_T the Choquet boundary for $T = \phi_\infty(S)$, that $\{\phi_n(f)\}$ converges pointwise to $\phi_\infty(f)$ for any $f \in M$ provided $\{\phi_n(f)\}$ converges pointwise to $\phi_\infty(f)$ for any $f \in S$; that $\{\phi_n(f)\}$ converges uniformly on any compact subset of Π_N to $\phi_\infty(f)$ for any $f \in M$ provided $\{\phi_n(f)\}$ converges uniformly to $\phi_\infty(f)$ for any $f \in S$; and that, in the case where S is a function algebra, $\{\phi_n\}$ norm converges to ϕ_∞ on M provided $\{\phi_n(f)\}$ norm converges to ϕ_∞ on S. The proofs are in the spirit of the original one for the theorem of Korovkin.

1. Introduction. The theorem of Korovkin asserts that if $\{\phi_n\}$ is a sequence of positive linear maps from $C([0,1])$ (resp., $C_{\mathbb{R}}([0,1])$), the Banach space of complex-valued (resp., real-valued) continuous functions on the closed interval $[0,1]$, into itself and the sequences $\{\phi_n(\iota^k)\}$ of functions converge uniformly to ι^k on $[0,1]$ $(k = 0, 1, 2)$, where

$$\iota^0(x) = 1, \quad \iota^1(x) = \iota(x) = x, \quad \iota^2(x) = (\iota(x))^2 = x^2, \quad x \in [0,1],$$

then, for any $f \in C([0,1])$ (resp., $C_{\mathbb{R}}([0,1])$), the sequence $\{\phi_n(f)\}$ of functions converges uniformly to f. This fact is well illustrated by the behavior of the Bernstein polynomials

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \le x \le 1,$$

defined for a continuous function f on the interval $[0,1]$. Indeed, it is well known that the sequence $\{B_n(f)\}$ converges uniformly to f on $[0,1]$, not

2000 *Mathematics Subject Classification*: Primary 41A36; Secondary 46E15.
Key words and phrases: Korovkin type approximation, function space, linear contraction.

merely for $f = \iota^k$ $(k = 0, 1, 2)$ (in fact, $B_n(1) = 1,\ B_n(\iota) = \iota$), but also for any $f \in C([0,1])$ (resp., $C_{\mathbb{R}}([0,1])$.

The phenomenon described by this theorem is a prototype of ones where a sequence $\{\phi_n\}$ of maps and a map ϕ_∞ of some specified kind exhibit the property: If the sequence $\{\phi_n(f)\}$ converges to $\phi_\infty(f)$ in some sense for any f belonging to some restricted class S, then $\{\phi_n(f)\}$ converges to $\phi_\infty(f)$ in that sense for all f in a class M properly larger than S.

Korovkin's theorem aroused great interest and stimulated extensive investigation of such phenomena. We refer, e.g., to the large volume [1] by Altomare and Campiti.

The standing assumptions in the present paper are: X, Y are compact Hausdorff spaces, M is a subspace of the Banach space $C(X)$ (resp., $C_{\mathbb{R}}(X)$) of all complex-valued (resp., real-valued) continuous functions on X, $S \subset M$ a complex (resp., real) function space on X, $\{\phi_n\}$ a sequence of unital linear contractions from M into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$), and ϕ_∞ a linear isometry from M into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$); finally, the Choquet boundary $\mathit{\Pi}_N$ for

$$N = \operatorname{Span}\Big(\bigcup_{1\le n\le\infty} N_n\Big), \quad \text{where} \quad N_n = \phi_n(M)\ (n = 1, 2, \ldots, \infty),$$

is contained in the Choquet boundary $\mathit{\Pi}_T$ for $T = \phi_\infty(S)$. The question we ask in this paper is: Does the sequence $\{\phi_n(f)\}$ converge to $\phi_\infty(f)$ for any $f \in M$ provided $\{\phi_n(f)\}$ converges to $\phi_\infty(f)$ for any $f \in S$? Here, we take up three kinds of convergence, namely, the pointwise, uniform, and norm convergences.

The idea, however, of getting the results in this paper rests on the original proof of Korovkin's theorem.

2. Choquet boundaries and linear isometries. Suppose that M is a subspace of the Banach space $C(X)$ (resp., $C_{\mathbb{R}}(X)$) and $\mathit{\Gamma}$ a subset of X. We say that $\mathit{\Gamma}$ is a *boundary* for M if there exists $x' \in \mathit{\Gamma}$ such that

$$|f(x')| = \|f\| = \sup_{x\in X} |f(x)|$$

for any $f \in M$.

For $x \in X$, the *evaluation* $\tau_M(x)$ at x is defined by

$$\tau_M(x)(f) = f(x) \quad \text{for any } f \in M.$$

$\tau_M(x)$ is in the closed unit ball B_{M^*} of the dual space M^* of M. It is known that

$$\mathit{\Pi}_M = \{x \in X : \tau_M(x) \in \operatorname{ext} B_{M^*}\} = \tau_M^{-1}(\operatorname{ext} B_{M^*})$$

is in fact a boundary for M, where $\operatorname{ext} B_{M^*}$ means the set of all extreme points of B_{M^*}. Following Novinger [10], we call $\mathit{\Pi}_M$ the *Choquet boundary* for M. If M is unital, then this notion coincides with the set of H-extremal points due to Bauer [3], [4], and was called the Choquet boundary by Bishop and de Leeuw [6] (see Notes and references to Section 2.6 of [1] for more historical aspects).

Furthermore, we have the following lemma that generalizes the so-called Arens–Kelley theorem [2]:

LEMMA 2.1. *Let M be a subspace of $C(X)$ (resp., $C_{\mathbb{R}}(X)$). Then*

$$\operatorname{ext} B_{M^*} = \mathbb{T}\tau_M(\varPi_M) = \{\lambda\tau_M(x) : \lambda \in \mathbb{T},\, x \in \varPi_M\},$$

where $\mathbb{T}$ is the torus in the complex plane $\mathbb{C}$ (resp.,

$$\operatorname{ext} B_{M^*} = \tau_M(\varPi_M) \cup -\tau_M(\varPi_M)).$$

The following proof is valid for both the real and complex cases.

Proof. We suppose that the closed convex hull of the set $\mathbb{T}\tau_M(X) = \{\lambda\tau_M(x) : \lambda \in \mathbb{T},\, x \in X\}$ is properly contained in B_{M^*}. Then there exists an $f \in M$ such that

$$\sup_{\lambda\in\mathbb{T},\, x\in X} \operatorname{Re}(\lambda\tau_M(x)(f)) < \sup_{\varphi\in B_{M^*}} \operatorname{Re}(\varphi(f)),$$

which yields the strict inequality $\sup_{x\in\varPi_M} |f(x)| < \|f\|$, a contradiction. So the closed convex hull of $\mathbb{T}\tau_M(X)$ coincides with the ball B_{M^*}. It follows by the Milman theorem that the set $\operatorname{ext} B_{M^*}$ is contained in $\mathbb{T}\tau_M(X)$. Hence we conclude that $\operatorname{ext} B_{M^*} = \mathbb{T}\tau_M(\varPi_M)$. ■

For a subspace M of $C(X)$ (resp., $C_{\mathbb{R}}(X)$), we denote by $\varGamma_M$ the closure $\overline{\varPi}_M$ of $\varPi_M$.

We say that a subspace M of $C(X)$ (resp., $C_{\mathbb{R}}(X)$) is a *complex* (resp., *real*) *function space* on X if M is *unital*, that is, the constant function 1 is in M, and separates points of X, that is, given any pair of distinct points $x, x' \in X$ there is an $f \in M$ such that $f(x) \neq f(x')$. Any function space M has a smallest closed boundary $\varSigma_M$, which in fact is $\varGamma_M$, called the *Shilov boundary* for M.

Essential to our considerations is a characterization of Choquet boundaries for function spaces, given in Theorem 2.2.6 of Browder [7] (see also Theorem 4.2.11 in [13]):

LEMMA 2.2. *Let M be a function space* (*either complex or real*) *on X and $x' \in X$. Then $x' \in \varPi_M$ if and only if*, *for any $\alpha, \beta \in (0,\infty)$ with $\alpha < \beta$ and any open neighborhood U of x'*, *there exists a function $f \in M$ such that*

$$\operatorname{Re} f \le 0, \qquad \operatorname{Re} f(x) < -\beta \ \text{ for } x \in U^{\mathrm{c}}, \qquad \operatorname{Re} f(x') > -\alpha.$$

We say that a subspace M of $C(X)$ is *self-conjugate* if $\overline{f} \in M$ whenever $f \in M$. It is obvious that if M is a subspace of $C(X)$, then

$$\overline{M} + M = \{\overline{f}_1 + f_2 : f_1, f_2 \in M\}$$

is self-conjugate. We prove

Lemma 2.3. *If a subspace M of $C(X)$ is unital*, *then*

$$\varPi_{\overline{M}+M} = \varPi_M.$$

Proof. First we consider the case where M is a function space. Suppose that $x' \in \varPi_{\overline{M}+M}$. Then, for any $\alpha, \beta \in (0, \infty)$ with $\alpha < \beta$ and any open neighborhood U of x', there exists a function $f \in \overline{M} + M$ such that

$$\operatorname{Re} f \le 0, \quad \operatorname{Re} f(x) < -\beta \quad \text{for } x \in U^{\mathrm{c}}, \quad \operatorname{Re} f(x') > -\alpha.$$

So, we have functions $f_1, f_2 \in M$ such that $f = \overline{f}_1 + f_2$. Put $g = f_1 + f_2$. Then $g \in M$. We have $\operatorname{Re} g = \operatorname{Re} f$, and so g satisfies

$$\operatorname{Re} g \le 0, \quad \operatorname{Re} g(x) < -\beta \quad \text{for } x \in U^{\mathrm{c}}, \quad \operatorname{Re} g(x') > -\alpha.$$

This shows that $x' \in \varPi_M$. Thus we get the inclusion $\varPi_{\overline{M}+M} \subset \varPi_M$. The opposite inclusion is obvious.

Next we prove the general case. We introduce an equivalence relation $\sim_M$ on X by

$$\begin{aligned} x \sim_M x' &\Leftrightarrow f(x) = f(x') \text{ for any } f \in M \\ &\Leftrightarrow (\overline{f}_1 + f_2)(x) = (\overline{f}_1 + f_2)(x') \text{ for any } f_1, f_2 \in M. \end{aligned}$$

We denote by $X/{\sim_M}$ the quotient space of X by $\sim_M$, which is a compact Hausdorff space, and by $\widehat{\cdot}$ the canonical map from X onto $X/{\sim_M}$. Furthermore, we define, for any $f \in \overline{M}+M$, a continuous function $\widehat{f}$ on $\widehat{X} = X/{\sim_M}$ by

$$\widehat{f}(\widehat{x}) = f(x) \quad \text{for } x \in X.$$

Then $\widehat{M}$ becomes a function space on $\widehat{X}$. Since $(\overline{M} + M)^{\wedge} = \overline{\widehat{M}} + \widehat{M}$, we have

$$\varPi_{\overline{M}+M} = \bigcup \varPi_{(\overline{M}+M)^{\wedge}} = \bigcup \varPi_{\widehat{M}} = \varPi_M. \ \blacksquare$$

A *complex* function space M on X is called a *function algebra* on X if M is uniformly closed, and, at the same time, is an algebra. A function algebra version of Lemma 2.2 is the following:

Lemma 2.4. *Let M be a function algebra on X and $x' \in X$. Then $x' \in \varPi_M$ if and only if*, *for any $\alpha, \beta \in (0, 1)$ with $\alpha < \beta$ and any open neighborhood U of x'*, *there exists a function $f \in M$ such that*

$$|f| \le 1 \quad \text{on } X, \quad |f(x)| < \alpha \quad \text{for } x \in U^{\mathrm{c}}, \quad |f(x')| > \beta.$$

We refer to Theorem 2.3.4 of Browder [7] (and Theorem 4.3.4 of [13], cf. [5] and [6]).

Lemma 2.5. *Let M be a complex* (*resp.*, *real*) *function space on X*, *ϕ a linear isometry from M into $C(Y)$* (*resp.* $C_{\mathbb{R}}(Y)$), *and N the image of M under ϕ. Then there is a unique continuous map η from $\varGamma_N$ onto $\varSigma_M$ such* *that*

$$\phi(f)(y) = \phi(1)(y) f(\eta(y)) \quad \textit{for } y \in \Gamma_N \textit{ and } f \in M.$$

Furthermore,

$$|\phi(1)(y)| = 1 \quad (\textit{resp. } \phi(1)(y) = 1 \textit{ or } -1) \quad \textit{for } y \in \Gamma_N,$$

and η carries Π_N onto Π_M.

Proof. Let $y \in \Pi_N$. Since ϕ is a linear isometry, $\phi^*(\tau_N(y))$ is in $\operatorname{ext} B_{M^*}$ and must be of the form $\lambda\tau_M(x)$, $\lambda \in \mathbb{T}$, $x \in \Pi_M$. So, $\phi(f)(y) = \lambda f(x)$ for any $f \in M$. In particular, $\phi(1)(y) = \lambda$. Hence, $\phi(f)(y) = \phi(1)(y)f(y)$ for any $f \in M$.

We consider the set P of all $(x, y) \in X \times Y$ such that

$$\phi(f)(y) = \phi(1)(y) f(x) \quad \text{for any } f \in M.$$

What we have shown means that Π_N is contained in the set

$$\{y \in Y : (x, y) \in P \text{ for some } x \in \Pi_M\}.$$

Since P is compact, it follows that Γ_N is contained in the set of all $y \in Y$ such that $(x, y) \in P$ for some x in Σ_M.

Since M separates points of X, the restriction $\pi_Y|_P$ to P of the projection π_Y of $X \times Y$ on Y is one-to-one, and its image contains Γ_N. So we can define the map η from Γ_N into Σ_M by

$$\eta = \pi_X \circ (\pi_Y|_{P'})^{-1},$$

where π_X is the projection of $X \times Y$ onto X, and P' the set of all $(x, y) \in P$ with $y \in \Gamma_N$. It is continuous because both π_X and $(\pi_Y|_{P'})^{-1}$ are continuous.

It is clear that

$$\phi(f)(y) = \phi(1)(y) f(\eta(y)) \quad \text{for any } f \in M \text{ and } y \in \Gamma_N.$$

Since, as above, Π_M, Σ_M are contained in the sets of $x \in X$ such that $(x, y) \in P$ for some $y \in \Pi_N$, Γ_N, respectively, we have $\eta(\Gamma_N) = \Gamma_M$ and $\eta(\Pi_N) = \Pi_M$. It also is clear that $|\phi(1)(y)| = 1$ for any $y \in \Gamma_N$, and that the map η is uniquely determined. ∎

3. Pointwise approximations. We say that a linear map ϕ from a subspace M of $C(X)$ (resp., $C_{\mathbb{R}}(X)$) into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$) is a *contraction* if $\|\phi\| \le 1$; that a linear map ϕ from a unital subspace M of $C(X)$ (resp., $C_{\mathbb{R}}(X)$) into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$) is *unital* if $\phi(1) = 1$; and that a linear map ϕ from a unital self-conjugate subspace M of $C(X)$ (resp., $C_{\mathbb{R}}(X)$) into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$) is *positive* if $\phi(f) \ge 0$ whenever $f \ge 0$.

LEMMA 3.1 ([11]). *Any unital linear contraction from a unital subspace of $C(X)$ (resp., $C_{\mathbb{R}}(X)$) into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$) is necessarily positive.*

Lemma 3.2. *If ϕ is a unital linear contraction from a unital subspace M of $C(X)$ (resp., $C_{\mathbb{R}}(X)$) into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$), then ϕ uniquely extends to a positive linear map $\widetilde{\phi}$ from $\overline{M}+M$ into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$).*

Proof. For any $f, g \in M$, we put

$$\widetilde{\phi}(\overline{f}+g) = \overline{\phi(f)} + \phi(g).$$

It is easy to see that $\widetilde{\phi}$ is well defined and positive. In fact, for any $y \in Y$, by the Hahn–Banach extension theorem, $\phi^*\tau_N(y)$ extends to a linear functional ϱ on $\overline{M}+M$ with $\|\varrho\| = 1$. Since ϱ is unital, ϱ must be positive. So, ϱ is self-adjoint. Thus, for $f, g \in M$ with $\overline{f}+g \geq 0$ and $y \in Y$,

$$\widetilde{\phi}(\overline{f}+g)(y) = \overline{\phi(f)(y)} + \phi(g)(y) = \overline{\varrho(f)} + \varrho(g) = \varrho(\overline{f}) + \varrho(g) = \varrho(\overline{f}+g) \geq 0.$$

Hence $\widetilde{\phi}(\overline{f}+g) \geq 0$. It is clear that $\widetilde{\phi}$ must be unique. ■

Theorem 3.3. *Let M be a subspace of $C(X)$ (resp., $C_{\mathbb{R}}(X)$), S a function space on X contained in M, $\{\phi_n\}$ a sequence of unital linear contractions from M into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$), ϕ_∞ a linear isometry from M into $C(Y)$ (resp., $C_{\mathbb{R}}(Y)$), and assume that $\Pi_N \subset \Pi_T$, where*

$$N = \operatorname{Span}\Big(\bigcup_{1\leq n\leq\infty} N_n\Big), \qquad N_n = \phi_n(M)\ (n = 1, 2, \ldots, \infty), \qquad T = \phi_\infty(S).$$

If $\{\phi_n(f)\}$ converges pointwise to $\phi_\infty(f)$ for any $f \in S$, then $\{\phi_n(f)\}$ converges pointwise to $\phi_\infty(f)$ for any $f \in M$.

We only give the proof for M being a subspace of $C(X)$. The argument for the other case is even simpler.

Proof. Let ϕ_∞ be represented, via Lemma 2.5, by a continuous map η from Γ_T onto Σ_S as

$$\phi_\infty(f)(y) = f(\eta(y)), \qquad y \in \Pi_T,$$

for any $f \in S$. Take $f \in M$ and $\varepsilon > 0$. Put $F = f \otimes 1 - 1 \otimes f$. This is a continuous function on $X \times X$, and assumes the value 0 on the diagonal $\Delta_X = \{(x,x) : x \in X\}$ of $X \times X$. So, there exists an open neighborhood U of Δ_X such that $|F(x,x')| < \varepsilon$ for $(x,x') \in U$.

Let $y' \in \Pi_T$ and $x' = \eta(y')$. Then $x' \in \Pi_S$. Thus we have an open neighborhood $V_{x'}$ of x' such that $V_{x'} \times V_{x'} \subset U$. By Lemma 2.2 we can find $f_{y'} \in S$ such that

$$\operatorname{Re} f_{y'} \geq 0 \quad \text{on } X, \qquad \operatorname{Re} f_{y'} \geq 1 \quad \text{on } V_{x'}^{\mathrm{c}}, \qquad \operatorname{Re} f_{y'}(x') < \varepsilon.$$

Put $F_{y'} = f_{y'} \otimes 1 + 1 \otimes f_{y'}$. Then

$$\operatorname{Re} F_{y'} \geq 0 \quad \text{on } X \times X, \qquad \operatorname{Re} F_{y'} \geq 1 \quad \text{on } (V \times V_{x'})^{\mathrm{c}} \supset U^{\mathrm{c}}.$$

So, $|\operatorname{Re} F| \le \|F\| \le \|F\| \operatorname{Re} F_{y'}$ on U^{c}, and hence, $|\operatorname{Re} F| \le \varepsilon + \|F\| \operatorname{Re} F_{y'}$, that is,

$$-(\varepsilon + \|F\| \operatorname{Re} F_{y'}) \le \operatorname{Re} F \le \varepsilon + \|F\| \operatorname{Re} F_{y'}$$

on $X \times X$. Therefore, since $\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty$ is positive,

$$\begin{aligned} -(\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\varepsilon + \|F\| \operatorname{Re} F_{y'}) &\le (\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\operatorname{Re} F) \\ &\le (\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\varepsilon + \|F\| \operatorname{Re} F_{y'}). \end{aligned}$$

So, we have

$$\begin{aligned} &|\operatorname{Re} \phi_n(f) - \operatorname{Re} \phi_\infty(f)| \\ &\quad = |\operatorname{Re}(\phi_n \otimes \phi_\infty)(F)| = |(\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\operatorname{Re} F)| \\ &\quad \le (\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\varepsilon + \|F\| \operatorname{Re} F_{y'}) = \varepsilon + \|F\| (\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\operatorname{Re} F_{y'}) \\ &\quad = \varepsilon + \|F\| \operatorname{Re}(\phi_n \otimes \phi_\infty)(F_{y'}) = \varepsilon + \|F\| (\operatorname{Re}(\phi_n(f_{y'}) + \phi_\infty(f_{y'}))) \\ &\quad \le \varepsilon + \|F\| (|\phi_n(f_{y'}) - \phi_\infty(f_{y'})| + 2 \operatorname{Re} \phi_\infty(f_{y'})), \end{aligned}$$

which implies that, for any sufficiently large integer n,

$$\begin{aligned} &|(\operatorname{Re} \phi_n(f))(y') - (\operatorname{Re} \phi_\infty(f))(y')| \\ &\qquad \le \varepsilon + \|F\| (|\phi_n(f_{y'})(y') - \phi_\infty(f_{y'})(y')| + 2 \operatorname{Re} f_{y'}(x')) \\ &\qquad < \varepsilon + \|F\| (\varepsilon + 2\varepsilon) = (1 + 3\|F\|)\varepsilon. \end{aligned}$$

Hence, $\{\operatorname{Re} \phi_n(f)\}$ converges pointwise to $\operatorname{Re} \phi_\infty(f)$ on Π_T. By replacing f by $-if$, we may see that $\{\operatorname{Im} \phi_n(f)\}$ also converges pointwise to $\operatorname{Im} \phi_\infty(f)$ on Π_T. Therefore, $\{\phi_n(f)\}$ also converges pointwise to $\phi_\infty(f)$ on Π_T.

Now we denote by $\widehat{\cdot}$ the canonical map from Y onto $Y/{\sim_N}$. Then it is obvious that $\{\phi_n(f)^\wedge\}$ converges pointwise to $\phi_\infty(f)^\wedge$ on $\widehat{\Pi}_T = \{\widehat{y} : y \in \Pi_T\}$.

By the Bishop–de Leeuw theorem ([6, Section V], see also [12, Section 4], for any $y \in Y$ there exists a positive measure μ on the σ-ring of subsets of $B_{\widehat{N}^*}$ generated by $\operatorname{ext} B_{\widehat{N}^*}$ and the Baire subsets of $B_{\widehat{N}^*}$ that represents $\widehat{y}$ and satisfies $\mu(B_{\widehat{N}^*}) = 1$. Since $\operatorname{ext} B_{\widehat{N}^*} = \mathbb{T}\Pi_{\widehat{N}} \subset \mathbb{T}\widehat{\Pi}_T$, we conclude, by the Lebesgue dominated convergence theorem, that for any $f \in M$,

$$\begin{aligned} \phi_n(f)(y) &= \phi_n(f)^\wedge(\widehat{y}) = \tau_{\widehat{N}}(\widehat{y})(\phi_n(f)^\wedge) = \int_{B_{\widehat{N}^*}} \phi_n(f)^\wedge \, d\mu \\ &\to \int_{B_{\widehat{N}^*}} \phi_\infty(f)^\wedge \, d\mu = \tau_{\widehat{N}}(\widehat{y})(\phi_\infty(f)^\wedge) = \tau_N(y)(\phi_\infty(f)) = \phi_\infty(f)(y) \end{aligned}$$

as $n \to \infty$. Thus the proof is complete. ∎

4. Uniform approximations. Next we state the following

THEOREM 4.1. *Let* M, S, $\{\phi_n\}$, ϕ_∞, N_n $(n = 1, 2, \ldots, \infty)$, N_∞, N, T *be as in Theorem* 3.3. *If* $\{\phi_n(f)\}$ *converges uniformly to* $\phi_\infty(f)$ *for any* $f \in S$, *then* $\{\phi_n(f)\}$ *converges to* $\phi_\infty(f)$ *uniformly on any compact subset of* Π_N *for any* $f \in M$. *In particular*, *if in addition* Π_N *is compact*, *then it follows that* $\{\phi_n(f)\}$ *converges uniformly to* $\phi_\infty(f)$ *for any* $f \in M$.

Just as for Theorem 3.3, we only give the proof of the complex case.

Proof. Let again ϕ_∞ be represented by a continuous map η of Γ_T onto $\overline{\Pi}_S$ as

$$\phi_\infty(f)(y) = f(\eta(y)), \quad y \in \overline{\Pi}_T,$$

for any $f \in S$. Take $f \in M$, $K \subset \Pi_N$ compact, $y' \in K$, and $\varepsilon > 0$. Put $F = f \otimes 1 - 1 \otimes f$ and $x' = \eta(y')$. Then, as in the proof of Theorem 3.3, we have an open neighborhood $V_{x'}$ of x' and a function $f_{y'} \in S$ such that

$$\operatorname{Re} f_{y'} \geq 0 \quad \text{on } X, \qquad \operatorname{Re} f_{y'} \geq 1 \quad \text{on } V_{x'}^{\mathrm{c}}, \qquad \operatorname{Re} f_{y'}(x') < \varepsilon.$$

It follows that

$$|\operatorname{Re} \phi_n(f) - \operatorname{Re} \phi_\infty(f)| \leq \varepsilon + \|F\|(|\phi_n(f_{y'}) - \phi_\infty(f_{y'})| + 2 \operatorname{Re} \phi_\infty(f_{y'})).$$

Since K is compact and the family of open neighborhoods

$$W_{y'} = \{y \in Y : \operatorname{Re} \phi_\infty(f_{y'})(y) < \varepsilon\}$$

of $y' \in K$ covers K, we can find a finite number of points $y'_1, \ldots, y'_p$ in K such that $K \subset \bigcup_{1 \leq k \leq p} W_{y'_k}$. So, for any $y \in K$, we can find a k with $y \in W_{y'_k}$ and have

$$\begin{aligned}
|(\operatorname{Re} \phi_n(f))(y) &- (\operatorname{Re} \phi_\infty(f))(y)| \\
&\leq \varepsilon + \|F\|(|\phi_n(f_{y'_k})(y) - \phi_\infty(f_{y'_k})(y)| + 2 \operatorname{Re} \phi_\infty(f_{y'_k})(y)) \\
&\leq \varepsilon + \|F\|(\|\phi_n(f_{y'_k}) - \phi_\infty(f_{y'_k})\| + 2\varepsilon) \\
&= \|F\| \max_{1 \leq k \leq p} \|\phi_n(f_{y'_k}) - \phi_\infty(f_{y'_k})\| + (1 + 2\|F\|)\varepsilon.
\end{aligned}$$

Applying the above to $-if = g$ in place of f, we find functions $g_{y''_1}, \ldots, g_{y''_q} \in S$ such that for any $y \in K$,

$$\begin{aligned}
|(\operatorname{Im} \phi_n(f))(y) &- (\operatorname{Im} \phi_\infty(f))(y)| \\
&\leq |(\operatorname{Re} \phi_n(g))(y) - (\operatorname{Re} \phi_\infty(g))(y)| \\
&\leq \|F\| \max_{1 \leq l \leq q} \|\phi_n(g_{y''_l}) - \phi_\infty(g_{y''_l})\| + (1 + 2\|F\|)\varepsilon.
\end{aligned}$$

Therefore we have, for any sufficiently large integer n,

$$\begin{aligned}
&|\phi_n(f)(y)-\phi_\infty(f)(y)| \\
&\quad \le \|F\|(\max_{1\le k\le p}\|\phi_n(f_{y'_k})-\phi_\infty(f_{y'_k})\|+\max_{1\le l\le q}\|\phi_n(g_{y''_l})-\phi_\infty(g_{y''_l})\|) \\
&\qquad + 2(1+2\|F\|)\varepsilon \\
&\quad < (2\|F\|+2(1+2\|F\|))\varepsilon = 2(1+3\|F\|)\varepsilon.
\end{aligned}$$

This shows that, for any $f \in M$, $\{\phi_n(f)\}$ converges uniformly on K to $\phi_\infty(f)$. Thus the theorem is proved. ∎

5. Approximations in norm. In this section, we will state a theorem of Korovkin type on the convergence in norm, parallel to Theorems 3.3 and 4.1, under the assumption, however, that S is a function algebra:

THEOREM 5.1. *Let* M, S, $\{\phi_n\}$, ϕ_∞, N_n $(n = 1, 2, \ldots, \infty)$, N_∞, N, T *be as in Theorem* 3.3, *with the additional assumption that* S *is a function algebra. If*

$$\|\phi_n-\phi_\infty\|_S = \sup_{f\in S,\,\|f\|\le 1}|\phi_n(f)-\phi_\infty(f)|$$

converges to 0 *as* $n\to\infty$, *then*

$$\|\phi_n-\phi_\infty\|_M = \sup_{f\in M,\,\|f\|\le 1}|\phi_n(f)-\phi_\infty(f)|$$

converges to 0 *as* $n\to\infty$.

Proof. Let ϕ_∞ be represented, via Lemma 2.5, by a continuous map η from Γ_T onto Σ_S as

$$\phi_\infty(f)(y) = f(\eta(y)), \qquad y\in\Gamma_T,$$

for any $f\in S$. Take $f\in M$ and $\varepsilon>0$ sufficiently small. Put $F = f\otimes 1 - 1\otimes f$. This is a continuous function on $X\times X$ and assumes the value 0 on the diagonal $\Delta_X = \{(x,x) : x\in X\}$ of $X\times X$. So, there exists an open neighborhood U of Δ_X such that $|F(x,x')|<\varepsilon$ for $(x,x')\in U$.

Let $y'\in\Pi_T$ and $x'=\eta(y')$. Then $x'\in\Pi_S$. We have an open neighborhood $V_{x'}$ of x' such that $V_{x'}\times V_{x'}\subset U$. By Lemma 2.4 we can find $g_{y'}\in S$ such that

$$|g_{y'}|\le 1 \quad\text{on } X,\qquad |g_{y'}(x)|<\varepsilon \quad\text{for } x\in V^{\mathrm{c}}_{x'},\qquad |g_{y'}(x')|>1-\varepsilon.$$

Put $f_{y'} = 1-e^{-i\theta}g_{y'}$ with $\theta=\arg g_{y'}(x')$. Then

$$2\ge \operatorname{Re} f_{y'}\ge 0\quad\text{on } X,\qquad \operatorname{Re} f_{y'}>1-\varepsilon\quad\text{on } V^{\mathrm{c}}_{x'},\qquad \operatorname{Re} f_{y'}(x')<\varepsilon.$$

Put next $F_{y'} = f_{y'}\otimes 1 + 1\otimes f_{y'}$. Then

$$\operatorname{Re} F_{y'}\ge 0\quad\text{on } X\times X,\qquad \operatorname{Re} F_{y'}>1-\varepsilon\quad\text{on } U^{\mathrm{c}}.$$

So,

$$|\operatorname{Re} F| \leq \|F\| \leq \frac{\|F\|}{1-\varepsilon} \operatorname{Re} F_{y'} \quad \text{on } U^{\mathrm{c}},$$

and hence,

$$|\operatorname{Re} F| \leq \varepsilon + \frac{\|F\|}{1-\varepsilon} \operatorname{Re} F_{y'},$$

that is,

$$-\left(\varepsilon + \frac{\|F\|}{1-\varepsilon} \operatorname{Re} F_{y'}\right) \leq \operatorname{Re} F \leq \varepsilon + \frac{\|F\|}{1-\varepsilon} \operatorname{Re} F_{y'}$$

on $X \times X$. So, since $\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty$ is positive,

$$\begin{aligned} -(\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)\left(\varepsilon + \frac{\|F\|}{1-\varepsilon} \operatorname{Re} F_{y'}\right) &\leq (\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\operatorname{Re} F) \\ &\leq (\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)\left(\varepsilon + \frac{\|F\|}{1-\varepsilon} \operatorname{Re} F_{y'}\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} |\operatorname{Re} \phi_n(f) - \operatorname{Re} \phi_\infty(f)| &= |\operatorname{Re}(\phi_n \otimes \phi_\infty)(F)| = |(\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\operatorname{Re} F)| \\ &\leq (\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)\left(\varepsilon + \frac{\|F\|}{1-\varepsilon} \operatorname{Re} F_{y'}\right) = \varepsilon + \frac{\|F\|}{1-\varepsilon} (\widetilde{\phi}_n \otimes \widetilde{\phi}_\infty)(\operatorname{Re} F_{y'}) \\ &= \varepsilon + \frac{\|F\|}{1-\varepsilon} \operatorname{Re}(\phi_n \otimes \phi_\infty)(F_{y'}) = \varepsilon + \frac{\|F\|}{1-\varepsilon} (\operatorname{Re}(\phi_n(f_{y'}) + \phi_\infty(f_{y'}))) \\ &\leq \varepsilon + \frac{\|F\|}{1-\varepsilon} (|\phi_n(f_{y'}) - \phi_\infty(f_{y'})| + 2 \operatorname{Re} \phi_\infty(f_{y'})), \end{aligned}$$

which implies that, for any sufficiently large integer n,

$$\begin{aligned} &|(\operatorname{Re} \phi_n(f))(y') - (\operatorname{Re} \phi_\infty(f))(y')| \\ &\qquad \leq \varepsilon + \frac{\|F\|}{1-\varepsilon} (|\phi_n(f_{y'})(y') - \phi_\infty(f_{y'})(y')| + 2 \operatorname{Re} f_{y'}(x')) \\ &\qquad \leq \varepsilon + \frac{\|F\|}{1-\varepsilon} (\|\phi_n - \phi_\infty\|_S \|f_{y'}\| + 2\varepsilon) < \frac{(1-\varepsilon+4\|F\|)\varepsilon}{1-\varepsilon} \\ &\qquad < \frac{(1+8\|f\|)\varepsilon}{1-\varepsilon}, \end{aligned}$$

since in fact $\|F\| \leq 2\|f\|$.

Replacing f by $-if$ in the above, we have, for any sufficiently large integer n,

$$|(\operatorname{Im} \phi_n(f))(y') - (\operatorname{Im} \phi_\infty(f))(y')| < \frac{(1+8\|f\|)\varepsilon}{1-\varepsilon},$$

and so,

$$\|\phi_n - \phi_\infty\|_M = \sup_{f\in M,\, \|f\|\le 1,\, y'\in \varPi_N} |\phi_n(f)(y') - \phi_\infty(f)(y')|$$
$$\le 2\cdot \frac{9\varepsilon}{1-\varepsilon} = \frac{18\varepsilon}{1-\varepsilon}.$$

Therefore, we conclude that $\|\phi_n - \phi_\infty\|_M \to 0$ as $n\to\infty$, and the proof is complete. ■

6. Applications. Let M be a complex (resp., real) function space on X. Then it may be seen from Lemma 2.2 that, if for any $x' \in X$, there is an $f\in M$ such that

$$\operatorname{Re} f(x') = 0 < \operatorname{Re} f(x) \quad \text{for any } x\in X \text{ distinct from } x',$$

then the Choquet boundary $\varPi_M$ for M coincides with X.

So, from Theorems 3.3 and 4.1, we have the following

Corollary 6.1. *Let S be a complex* (*resp.*, *real*) *function space on X, $\{\phi_n\}$ a sequence of unital linear contractions from $C(X)$* (*resp.*, *$C_{\mathbb{R}}(X)$*) *into $C(Y)$* (*resp.*, *$C_{\mathbb{R}}(Y)$*), *and ϕ_∞ a linear isometry from $C(X)$* (*resp.*, *$C_{\mathbb{R}}(X)$*) *into $C(Y)$* (*resp.*, *$C_{\mathbb{R}}(Y)$*). *Assume that for any $y'\in Y$, there is a $g\in\phi_\infty(S)$ such that*

$$\operatorname{Re} g(y') = 0 < \operatorname{Re} g(y) \quad \textit{for any } y\in Y \textit{ distinct from } y'.$$

If $\{\phi_n(f)\}$ converges pointwise to $\phi_\infty(f)$ for any $f\in S$, then $\{\phi_n(f)\}$ converges pointwise to $\phi_\infty(f)$ for any $f\in C(X)$ (*resp.*, *$C_{\mathbb{R}}(X)$*). *The statement is also true with "converges pointwise" replaced by "converges uniformly".*

Corollary 6.1 gives us some new theorems of Korovkin type. Here we recall theorems due to Volkov [14] (cf. p. 245 of [1]) and to Morozov [9]. The former asserts that if $\{\phi_n\}$ is a sequence of positive linear maps from $C_{\mathbb{R}}(\varOmega)$, $\varOmega$ a compact subset of $\mathbb{R}^p$, into itself, $\{\phi_n(\iota^0)\}$ converges uniformly to ι^0, $\{\phi_n(\iota_k)\}$ converges uniformly to ι_k, where

$$\iota^0(x) = 1,\qquad \iota_k(x) = x_k \qquad \text{for } x = (x_1,\dots,x_p)\in\varOmega \qquad (k=1,\dots,p),$$

and $\{\phi_n(\sum_{k=1}^p \iota_k^2)\}$ converges uniformly to $\sum_{k=1}^p \iota_k^2$, then $\{\phi_n(f)\}$ converges uniformly to f for any $f\in C_{\mathbb{R}}(\varOmega)$; the latter asserts that if $\{\phi_n\}$ is a sequence of positive linear maps from the Banach space $C_{\mathbb{R}}(\mathbb{R}^p)_{2\pi}$ of real-valued continuous functions on $\mathbb{R}^p$ of period 2π in each variable $x_1,\dots,x_n$ into itself, $\{\phi_n(\iota^0)\}$ converges uniformly to ι^0, $\{\phi_n(\cos_k)\}$ converges uniformly to $\cos_k$ and $\{\phi_n(\sin_k)\}$ converges uniformly to $\sin_k$, where

$$\iota^0(x) = 1,\quad \cos_k(x) = \cos x_k,\quad \sin_k(x) = \sin x_k \quad \text{for } x=(x_1,\dots,x_p)\in\mathbb{R}^p$$

$(k=1,\dots,p)$, then $\{\phi_n(f)\}$ converges uniformly to f for any $f\in C_{\mathbb{R}}(\mathbb{R}^p)_{2\pi}$.

Corollary 6.2. *Let $\{\phi_n\}$ be a sequence of unital linear contractions from $C(\Omega)$ (resp., $C_{\mathbb{R}}(\Omega)$) into itself, and ϕ_∞ a linear isometry from $C(\Omega)$ (resp., $C_{\mathbb{R}}(\Omega)$) into itself which satisfies*

$$\phi_\infty(\iota^0) = \iota^0, \quad \phi_\infty(\iota_k) = \iota_k \quad (k = 1, \dots, p), \quad \phi_\infty\Big(\sum_{k=1}^{p} \iota_k^2\Big) = \sum_{k=1}^{p} \iota_k^2.$$

If $\{\phi_n(\iota^0)\}$ converges pointwise to ι^0, $\{\phi_n(\iota_k)\}$ converges pointwise to ι_k ($k = 1, \dots, p$), and $\{\phi_n(\sum_{k=1}^{p} \iota_k^2)\}$ converges pointwise to $\sum_{k=1}^{p} \iota_k^2$, then $\{\phi_n(f)\}$ converges pointwise to f for any $f \in C(\Omega)$ (resp., $C_{\mathbb{R}}(\Omega)$). The statement is also true with "converges pointwise" replaced by "converges uniformly".

Corollary 6.3. *Let $\{\phi_n\}$ be a sequence of unital linear contractions from the Banach space $C(\mathbb{R}^p)_{2\pi}$ (resp., $C_{\mathbb{R}}(\mathbb{R}^p)_{2\pi}$) of complex-valued (resp., real-valued) continuous functions on $\mathbb{R}^p$ of period 2π in each variable $x_1, \dots, x_n$ into itself, and let ϕ_∞ be a linear isometry from $C(\mathbb{R}^p)_{2\pi}$ (resp., $C_{\mathbb{R}}(\mathbb{R}^p)_{2\pi}$) into itself which satisfies*

$$\phi_\infty(\iota^0) = \iota^0, \quad \phi_\infty(\cos_k) = \cos_k, \quad \phi_\infty(\sin_k) = \sin_k \quad (k = 1, \dots, p).$$

If $\{\phi_n(\iota^0)\}$ converges pointwise to ι^0, $\{\phi_n(\cos_k)\}$ converges pointwise to $\cos_k$ and $\{\phi_n(\sin_k)\}$ converges pointwise to $\sin_k$ ($k = 1, \dots, p$), then $\{\phi_n(f)\}$ converges pointwise to f for any $f \in C(\mathbb{R}^p)_{2\pi}$ (resp., $C_{\mathbb{R}}(\mathbb{R}^p)_{2\pi}$). The statement is also true with "converges pointwise" replaced by "converges uniformly".

We denote by $A(\mathbb{T})$ the function algebra on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ consisting of the complex-valued continuous functions on $\mathbb{T}$ which extend to continuous functions on the closed unit disc $D = \{z \in \mathbb{C} : |z| \le 1\}$ and are analytic on its interior. It is known that the Choquet boundary $\Pi_{A(\mathbb{T})}$ of $A(\mathbb{T})$ is the torus. So, from Theorem 5.1, we have the following

Corollary 6.4. *Let $\{\phi_n\}$ be a sequence of unital linear contractions from $C(\mathbb{T})$ into itself, and ϕ_∞ a linear isometry from $C(\mathbb{T})$ into itself. If*

$$\|\phi_n - \phi_\infty\|_{A(\mathbb{T})} = \sup_{f \in A(\mathbb{T}),\, \|f\| \le 1} |\phi_n(f) - \phi_\infty(f)|$$

converges to 0 as $n \to \infty$ (so, ϕ_∞ is an algebra-isomorphism), then

$$\|\phi_n - \phi_\infty\|_{C(\mathbb{T})} = \sup_{f \in C(\mathbb{T}),\, \|f\| \le 1} |\phi_n(f) - \phi_\infty(f)|$$

converges to 0 as $n \to \infty$.

References

[1] F. Altomare and M. Campiti, *Korovkin-type Approximation Theory and its Appri-cations*, de Gruyter Stud. Math. 17, de Gruyter, Berlin, 1994.

[2] R. F. Arens and J. L. Kelley, *Characterizations of the space of continuous functions over compact Hausdorff spaces*, Trans. Amer. Math. Soc. 108 (1963), 265–274.

[3] H. Bauer, *Un problème de Dirichlet pour la frontière de Shilov d'un espace compact*, C. R. Acad. Sci. Paris 247 (1958), 843–846.

[4] —, *Frontière de Shilov et problème de Dirichlet*, Séminaire de Théorie du Potentiel (Institut H. Poincaré) 3 (1958-1959), no. 7.

[5] E. Bishop, *A minimal boundary for function algebras*, Pacific J. Math. 9 (1959), 629–642.

[6] E. Bishop and K. de Leeuw, *The representation of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble) 9 (1959), 305–331.

[7] A. Browder, *Introduction to Function Algebras*, Benjamin, New York, 1969.

[8] W. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous functions*, Studia Math. 26 (1966), 133–136.

[9] E. N. Morozov, *Convergence of a sequence of positive linear operators in the space of continuous* 2π*-periodic functions of two variables*, Kalinin. Gos. Ped. Inst. Uchen. Zap. 26 (1958), 129–142 (in Russian).

[10] W. P. Novinger, *Linear isometries of subspaces of spaces of continuous functions*, Studia Math. 53 (1975), 273–276.

[11] R. R. Phelps, *The range of* Tf *for certain linear operators* T, Proc. Amer. Math. Soc. 16 (1965), 381–382.

[12] —, *Lectures on Choquet's Theorem*, 2nd ed., Lecture Notes in Math. 1757, Springer, Berlin, 2001.

[13] O. Takenouchi, A. Sakai, K. Kishi and T. Jimbo, *Function Algebras*, Math. Sci. Ser. 8, Tokyo, 1977 (in Japanese).

[14] V. I. Volkov, *Conditions for convergence of a sequence of positive linear operators in the space of continuous functions of two variables*, Kalinin. Gos. Ped. Inst. Uchen. Zap. 26 (1958), 27–40 (in Russian).

Mathematics Division
Sendai Shirayuri Gakuen High School
Murasaki-Yama 1-2-1, Izumi-ku
Sendai 981-3205, Japan

Department of Mathematical Sciences
Faculty of Science
Yamagata University
Yamagata 990-8560, Japan
E-mail: okayasu@sci.kj.yamagata-u.ac.jp

Received April 4, 2002
Revised version September 3, 2002 (4924)