

On monotonic functions from the unit interval into a Banach space with uncountable sets of points of discontinuity

by

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Abstract. We say that a function f from $[0, 1]$ to a Banach space X is increasing with respect to $E \subset X^*$ if $x^* \circ f$ is increasing for every $x^* \in E$. We show that if $f : [0, 1] \rightarrow X$ is an increasing function with respect to a norming subset E of X^* with uncountably many points of discontinuity and Q is a countable dense subset of $[0, 1]$, then (1) $\overline{\text{lin}\{f([0, 1])\}}$ contains an order isomorphic copy of $D(0, 1)$, (2) $\overline{\text{lin}\{f(Q)\}}$ contains an isomorphic copy of $C([0, 1])$, (3) $\overline{\text{lin}\{f([0, 1])\}}/\overline{\text{lin}\{f(Q)\}}$ contains an isomorphic copy of $c_0(\Gamma)$ for some uncountable set Γ , (4) if I is an isomorphic embedding of $\overline{\text{lin}\{f([0, 1])\}}$ into a Banach space Z , then no separable complemented subspace of Z contains $I(\overline{\text{lin}\{f(Q)\}})$.

Throughout the paper, X will be a real or complex Banach space and X^* its topological dual. We say that a function $f : [0, 1] \rightarrow X$ is *increasing with respect to a subset E of X^** if $x^* \circ f$ is increasing (= nondecreasing) for every $x^* \in E$. Throughout, we will assume that E is a norming subset of X^* (i.e. there exist constants $C \geq c > 0$ such that $c\|x\| \leq \sup\{x^*(x) : x^* \in E\} \leq C\|x\|$ for every $x \in X$). Every increasing function from the unit interval into a Banach lattice F is increasing with respect to the positive part of the unit ball of F^* . Consequently, the definition covers all natural examples of increasing functions. On the other hand, we can consider an increasing function with respect to E as an increasing function taking values in $C(\overline{E}^{w^*})$. We study the properties of monotonic functions $f : [0, 1] \rightarrow X$ with uncountable sets of points of discontinuity.

Real monotonic functions on $[0, 1]$ have countable sets of points of discontinuity. However, not all monotonic functions from $[0, 1]$ into Banach lattices have this property (see [7], [3]). A Banach lattice F is said to have *property* (λ) if every increasing function from $[0, 1]$ into F has only countably

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many points of discontinuity. The study of properties of Banach lattices with property (λ) was initiated by B. Lavrič in [7]. He showed that a σ -Dedekind complete Banach lattice has property (λ) if and only if it does not contain any lattice copy of l^∞ . L. Drewnowski [3] observed that the space $D(0, 1)$ of all scalar left continuous functions on $(0, 1]$ with right limits on $[0, 1)$, equipped with the supremum norm, contains no isomorphic copy of l^∞ and the function $f : [0, 1] \rightarrow D(0, 1)$ given by $f(\alpha) = \chi_{[0, \alpha]}$ is discontinuous at every point. This shows that the space l^∞ does not play a crucial role in the theory of monotonic functions on $[0, 1]$ taking values in Banach lattices with uncountable sets of points of discontinuity, and simultaneously directs our attention to the space $D(0, 1)$. We show that property (λ) and the fact that a Banach lattice does not contain any order copy of $D(0, 1)$ are equivalent (see Theorem 4(c)).

Preliminaries. We denote by $C(K)$ the Banach space of all scalar continuous functions on a compact Hausdorff space K endowed with the supremum norm. For a given set Γ the Banach space of all functions $x : \Gamma \rightarrow \mathbb{C}$ such that the set $\{\gamma \in \Gamma : |x(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$ equipped with the supremum norm will be denoted by $c_0(\Gamma)$. Moreover, the unit ball of a Banach space X will be denoted by B_X and the weak* topology of X^* by w^* . For other notations and terminology the reader is referred to [12] and [5].

The *Helly space* \mathbb{H} consists of all nondecreasing functions $x : [0, 1] \rightarrow [0, 1]$ and is equipped with the pointwise convergence topology. It is compact, Hausdorff, separable and first countable (see [6, p. 164]). For $\alpha \in [0, 1]$ we denote by $\pi_\alpha : \mathbb{H} \rightarrow [0, 1]$ the function given by $\pi_\alpha(x) = x(\alpha)$. As observed by Drewnowski, these natural projections lead to an increasing function.

EXAMPLE. Let H be a closed subset of the Helly space. The function $f : [0, 1] \rightarrow C(H)$ given by

$$f(\alpha) = \pi_\alpha$$

is increasing.

A fundamental role in our considerations is played by the next theorem. It shows that every increasing function in our sense generates a closed subset of the Helly space, and the properties of the function are determined by the set. More general results on relationships between increasing functions taking values in general order metric spaces and the topology of Helly-like spaces can be found in [4].

THEOREM 1. *Let $f : [0, 1] \rightarrow X$ be an increasing function with respect to a norming subset E of B_{X^*} such that $f(0) = 0$ and $\|f(1)\| = 1$. Let $H_f = \{x^* \circ f : x^* \in \overline{E}^{w^*}\}$ be equipped with the pointwise convergence topology. Then*

- (1) H_f is a closed subset of the Helly space,
- (2) if Y is the closed linear hull in $C(H_f)$ of the functions π_α for $\alpha \in [0, 1]$, then there exists a positive isomorphic embedding $I : Y \rightarrow X$ such that $f(\alpha) = I(\pi_\alpha)$ for every $\alpha \in [0, 1]$, where the order in X is generated by E ,
- (3) moreover, the following assertions are equivalent:

- (a) the function f has countably many points of discontinuity,
- (b) the space H_f is metrizable,
- (c) the function f has separable range.

Proof. Let $\Phi_f : (\overline{E}^{w^*}, w^*) \rightarrow \mathbb{H}$ be given by $\Phi_f(x^*) = x^* \circ f$. It is clear that Φ_f is continuous. Thus H_f is a closed subset of the Helly space. Since E is norming, the operator $S : X \rightarrow C((\overline{E}^{w^*}, w^*))$ given by $S(x) = x|_{\overline{E}^{w^*}}$ is an isomorphism. Let $R : C(H_f) \rightarrow C((\overline{E}^{w^*}, w^*))$ be the isometry given by $R(h) = h \circ \Phi_f$. Then $R(\pi_\alpha)(x^*) = x^*(f(\alpha)) = S(f(\alpha))(x^*)$ for every $x^* \in \overline{E}^{w^*}$. Hence $R(\pi_\alpha) = S(f(\alpha))$ for every $\alpha \in [0, 1]$. The operators R and S are isomorphic embeddings. Moreover, x is a positive element of X if and only if $S(x)$ is positive in $C((\overline{E}^{w^*}, w^*))$. Therefore the operator $I = S^{-1}|_{S(X)} \circ R|_Y$ is a positive isomorphic embedding.

(a) \Rightarrow (b). Let A_f be the set of all points of discontinuity of f . Then the set of all points of discontinuity of elements of H_f is contained in A_f . Therefore the functions π_α for $\alpha \in A_f \cup \mathbb{Q}$ separate points of H_f . Since H_f is Hausdorff and compact, it is metrizable.

The implication (b) \Rightarrow (c) is a straightforward consequence of (2) and the fact that the space $C(H_f)$ is separable.

(c) \Rightarrow (b). Suppose that $f([0, 1])$ is separable. Let V be its closed linear hull. Then the function $\tilde{f} : [0, 1] \rightarrow V$ given by $\tilde{f}(\alpha) = f(\alpha)$ for every $\alpha \in [0, 1]$ is increasing with respect to the norming weak* compact subset $\{x^*|V : x^* \in \overline{E}^{w^*}\}$ of V^* . The set is metrizable in the weak* topology of V^* and $\Phi_{\tilde{f}}(\{x^*|V : x^* \in \overline{E}^{w^*}\}) = \Phi_f(\overline{E}^{w^*})$. Consequently, H_f is metrizable.

(b) \Rightarrow (a). Let D_f be the set of all points of discontinuity of elements of H_f . A subset of \mathbb{H} is separable and metrizable (not necessarily closed) if and only if the set of all points of discontinuity of all members of the set is countable (see [8, Prop. 2]). Thus D_f is countable. If $\beta \notin D_f$ and (β_n) is an increasing (or decreasing) sequence in $[0, 1]$ converging to β , then the sequence (π_{β_n}) of functions converges to π_β pointwise on H_f . Since the π_{β_n} are continuous on H_f , the convergence is uniform by the Dini theorem. Then the function $g : [0, 1] \rightarrow C(H_f)$ given by $g(\alpha) = \pi_\alpha$ is continuous on $[0, 1] \setminus D_f$. An appeal to (2) completes the proof.

An important role in our considerations is played by the subset $\mathbb{L} = \{x \in \mathbb{H} : x([0, 1]) \subset \{0, 1\}\}$ of \mathbb{H} . The space \mathbb{L} is homeomorphic to the

classical two-arrow space with two isolated points added. Consequently, it is Hausdorff, compact, sequentially compact, hereditarily separable and hereditarily Lindelöf (see [5, p. 270]). Note also that $C(\mathbb{L})$ is order isomorphic to $D(0, 1)$.

FACT 2. *If H is a closed nonmetrizable subset of \mathbb{H} , then there exists a separable subset of H with nonmetrizable closure.*

Proof. Let Q be a countable dense subset of $[0, 1]$. Let Γ_1 be the set of all $\gamma \in [0, 1] \setminus Q$ such that there exist $x_0, x_1 \in H$ with $x_0(\alpha) = x_1(\alpha)$ for every $\alpha \in Q$ and $x_0(\gamma) \neq x_1(\gamma)$. The family of functions π_α for $\alpha \in Q \cup \Gamma_1$ separates points of H . Since H is nonmetrizable, the set Γ_1 is uncountable. Then there exist $0 \leq a < b \leq 1$ and an uncountable subset Γ_2 of Γ_1 such that for every $\gamma \in \Gamma_2$ there exist $x_0^\gamma, x_1^\gamma \in H$ with $x_0^\gamma(\alpha) = x_1^\gamma(\alpha)$ for every $\alpha \in Q$ and $x_0^\gamma(\gamma) \leq a < b \leq x_1^\gamma(\gamma)$. Let $\psi : [0, 1] \rightarrow [0, 1]$ be a continuous nondecreasing function such that $\psi([0, a]) = \{0\}$ and $\psi([b, 1]) = \{1\}$. Let $\Psi : H \rightarrow [0, 1]^{[0, 1]}$ be given by

$$\Psi(x) = \psi \circ x.$$

The set $\Psi(H) \cap \mathbb{L}$ is closed, uncountable (consequently, nonmetrizable) and separable. A standard argument completes the proof.

In what follows we will also need the following well known result (a dual result for L^1 is well known; see [2, p. 211]).

FACT 3. *Let (e_n) be a sequence in a Banach space X such that*

- (1) $e_n = e_{2n} + e_{2n+1}$ for every n ,
- (2) there exist constants $0 < c \leq C$ such that for any scalars $a_{2^n}, \dots, a_{2^{n+1}-1}$,

$$c \max_{2^n \leq k < 2^{n+1}} |a_k| \leq \left\| \sum_{k=2^n}^{2^{n+1}-1} a_k e_k \right\| \leq C \max_{2^n \leq k < 2^{n+1}} |a_k|.$$

Then X has a subspace isomorphic to $C([0, 1])$.

We present the proof for the convenience of the reader.

Proof. Let λ be the Haar measure on $A_1 = \{-1, 1\}^{\mathbb{N}}$. For every $n \in \mathbb{N}$ and $0 \leq k \leq 2^n - 1$ let

$$A_{2^n+k} = \left\{ (j_n) \in \{-1, 1\}^{\mathbb{N}} : j_l = i_l \text{ for } l = 1, \dots, n \right. \\ \left. \text{where } \sum_{l=1}^n 2^{n-l-1} (i_l + 1) = k \right\}.$$

The sets $A_{2^n}, \dots, A_{2^{n+1}-1}$ are pairwise disjoint, clopen in $\{-1, 1\}^{\mathbb{N}}$ and $A_{2^n} \cup A_{2^{n+1}} = A_n$ for every $n \in \mathbb{N}$. It is clear that χ_{A_n} belongs to $C(\{-1, 1\}^{\mathbb{N}})$

and for every $a_{2^n}, \dots, a_{2^{n+1}-1} \in \mathbb{C}$,

$$\left\| \sum_{k=2^n}^{2^{n+1}-1} a_k \chi_{A_k} \right\| = \max_{2^n \leq k < 2^{n+1}-1} |a_k|.$$

Let $T_n : C(\{-1, 1\}^{\mathbb{N}}) \rightarrow X$ and $S_n : C(\{-1, 1\}^{\mathbb{N}}) \rightarrow C(\{-1, 1\}^{\mathbb{N}})$ be the linear operators given by

$$T_n(f) = \sum_{k=2^n}^{2^{n+1}-1} e_k \frac{1}{\lambda(A_k)} \int_{A_k} f d\lambda, \quad S_n(f) = \sum_{k=2^n}^{2^{n+1}-1} \chi_{A_k} \frac{1}{\lambda(A_k)} \int_{A_k} f d\lambda.$$

It is clear that $c\|S_n(f)\| \leq \|T_n(f)\| \leq C\|S_n(f)\|$ and $c\|S_{n+k}(f) - S_n(f)\| \leq \|T_{n+k}(f) - T_n(f)\| \leq C\|S_{n+k}(f) - S_n(f)\|$ for all $n, k \in \mathbb{N}$. Since $f = \lim_n S_n(f)$ for every $f \in C(\{-1, 1\}^{\mathbb{N}})$, the operator $T : C(\{-1, 1\}^{\mathbb{N}}) \rightarrow X$ given by $T(f) = \lim_n T_n(f)$ is an isomorphism. An appeal to the Milyutin theorem completes the proof.

Main results. For $\gamma \in [0, 1]$ and $s \in \{0, 1\}$ we denote by y_s^γ the following element of \mathbb{H} :

$$y_s^\gamma(\alpha) = \begin{cases} 0 & \text{if } \alpha < \gamma, \\ s & \text{if } \alpha = \gamma, \\ 1 & \text{if } \alpha > \gamma. \end{cases}$$

For a closed subset Γ of $[0, 1]$ we put $\mathbb{L}_\Gamma = \{y_s^\gamma : \gamma \in \Gamma, s \in \{0, 1\}\}$. Then $\mathbb{L} = \mathbb{L}_{[0,1]}$. It is easy to see that the map $m : \mathbb{L} \rightarrow [0, 1]$ given by $m(y_s^\gamma) = \gamma$ is continuous. Consequently, \mathbb{L}_Γ is a compact subset of \mathbb{L} for every closed subset Γ of $[0, 1]$. For any $\gamma_1 < \dots < \gamma_n$ in Γ and $g_1, \dots, g_n, g_{n+1} \in \mathbb{C}$ we have

$$\begin{aligned} & \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(\mathbb{L}_\Gamma)} \\ &= \left\| \left(\sum_{i=1}^{n+1} g_i \right) \pi_{\gamma_1} + \sum_{i=2}^n \left(\sum_{j=i}^{n+1} g_j \right) (\pi_{\gamma_i} - \pi_{\gamma_{i-1}}) + g_{n+1} (1 - \pi_{\gamma_n}) \right\|_{C(\mathbb{L}_\Gamma)} \\ &= \max \left\{ \left| \sum_{j=i}^{n+1} g_j \right| : i = 1, \dots, n+1 \right\}. \end{aligned}$$

Moreover, the function $g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i}$ is nonnegative on \mathbb{L}_Γ if and only if $\min \{ \sum_{i=j}^{n+1} g_i : j = 1, \dots, n+1 \} \geq 0$. Applying the fact that on \mathbb{L} ,

$$\pi_{\gamma_1} \pi_{\gamma_2} = \begin{cases} \pi_{\gamma_1} & \text{if } \gamma_1 \leq \gamma_2, \\ \pi_{\gamma_2} & \text{if } \gamma_1 \geq \gamma_2, \end{cases}$$

and the Stone–Weierstrass theorem, we find that the closed linear hull of

$\{\pi_\gamma : \gamma \in \Gamma\} \cup \{1\}$ in $C(\mathbb{L}_\Gamma)$ coincides with $C(\mathbb{L}_{\Gamma_2})$. Therefore, if $\varphi : \Gamma_1 \rightarrow \Gamma_2$ is an increasing function, then the operator $M_\varphi : \mathbb{L}_{\Gamma_1} \rightarrow \mathbb{L}_{\Gamma_2}$ given by $M_\varphi(\pi_\gamma) = \pi_{\varphi(\gamma)}$ has a continuous extension. If additionally φ is surjective, then selecting exactly one element $\psi(\gamma)$ in $\varphi^{-1}(\gamma)$ we see that the subspace $\overline{\text{lin}\{\pi_{\psi(\gamma)} : \gamma \in \Gamma_2\}}$ of $C(\mathbb{L}_{\Gamma_1})$ is order isometric to $C(\mathbb{L}_{\Gamma_2})$. The fundamental properties of monotonic functions $f : [0, 1] \rightarrow X$ are gathered in the theorem below. Part (b) of the theorem is closely related to H. Corson's result that the quotient space $D(0, 1)/C([0, 1])$ is isomorphic to $c_0([0, 1])$ (see [1]).

THEOREM 4. *If there exists an increasing function $f : [0, 1] \rightarrow X$ with respect to a norming subset E of X^* with uncountably many points of discontinuity, then*

(a) *the closed linear hull of $f(Q)$ contains a subspace isomorphic to $C([0, 1])$ for every countable dense subset Q of $[0, 1]$,*

(b) *for every countable dense subset Q of $[0, 1]$ the quotient space $X/\overline{\text{lin } f(Q)}$ has a subspace isomorphic to $c_0(\Gamma)$ where Γ is an uncountable set,*

(c) *there exists a positive isomorphic embedding of $C(\mathbb{L})$ into the closed linear hull of $f([0, 1])$.*

Proof. We can assume that $f(0) = 0, \|f(1)\| = 1$ and $E \subset B_{X^*}$. Let H_f be the closed nonmetrizable subset of \mathbb{H} associated with f as in Theorem 1. Let Q be a countable dense subset of $[0, 1]$. Let Γ_1 be the set of all $\gamma \in [0, 1] \setminus Q$ such that there exist $x_0, x_1 \in H_f$ with $x_0(\alpha) = x_1(\alpha)$ for every $\alpha \in Q$ and $x_0(\gamma) \neq x_1(\gamma)$. The family $\{\pi_\alpha : \alpha \in Q \cup \Gamma_1\}$ separates points of H_f . Since H_f is nonmetrizable, the set Γ_1 is uncountable. Thus there exist $0 \leq a < b \leq 1$ and an uncountable subset Γ_2 of Γ_1 such that for every $\gamma \in \Gamma_2$ there exist $x_0^\gamma, x_1^\gamma \in H$ with $x_0^\gamma(\alpha) = x_1^\gamma(\alpha)$ for every $\alpha \in Q$ and $x_0^\gamma(\gamma) \leq a < b \leq x_1^\gamma(\gamma)$. For every $\gamma \in \Gamma_2$ there exists $\delta_\gamma > 0$ such that

$$x_0^\gamma(\gamma + \delta_\gamma) - \lim_{\alpha \rightarrow \gamma+} x_0^\gamma(\alpha) + \lim_{\alpha \rightarrow \gamma-} x_0^\gamma(\alpha) - x_0^\gamma(\gamma - \delta_\gamma) \leq C,$$

where $C = (b - a)/2$ in the proof of (a) and (b), and $C = (b - a)^2/8$ in the proof of (c). It is clear that there exists an uncountable subset Γ_3 of Γ_2 and $\delta > 0$ such that $\delta_\gamma > \delta$ for every $\gamma \in \Gamma_3$. Since $[0, 1]$ can be covered by finitely many intervals of length δ , there exist $0 \leq c < d \leq 1$ with $d - c < \delta$ such that the set $\Gamma_4 = \Gamma_3 \cap (c, d)$ is uncountable. At this point we divide the proof into three parts.

(a) Let $P = \{k/2^n : n \in \mathbb{N}, k = 0, \dots, 2^n\}$. Since Γ_4 is uncountable, there exists a strictly increasing function $\sigma : P \rightarrow Q \cap (c, d)$ such that the interval $(\sigma(s), \sigma(t))$ contains uncountably many elements of Γ_4 for every $s, t \in P$ with $s < t$. Let

$$e_{2^n+k} = \pi_{\sigma((k+1)/2^n)} - \pi_{\sigma(k/2^n)}$$

and let $\omega(n, k)$ be a member of $\Gamma_4 \cap (\sigma(k/2^n), \sigma((k+1)/2^n))$ for every $n \in \mathbb{N}$ and $k = 0, \dots, 2^n - 1$. Then for $g_{2^n}, \dots, g_{2^{n+1}-1} \in \mathbb{C}$ and $j = 0, \dots, 2^n - 1$,

$$\begin{aligned} \left\| \sum_{k=2^n}^{2^{n+1}-1} g_k e_k \right\| &\geq \left| \sum_{k=0}^{2^n-1} g_{2^n+k} (\pi_{\sigma((k+1)/2^n)} - \pi_{\sigma(k/2^n)})(x_0^{\omega(n,j)}) \right| \\ &\geq (b-a) |g_{2^n+j}| \\ &\quad - \max_{0 \leq k < 2^n} |g_{2^n+k}| \sum_{k \neq j} \left(x_0^{\omega(n,j)} \left(\sigma \left(\frac{k+1}{2^n} \right) \right) - x_0^{\omega(n,j)} \left(\sigma \left(\frac{k}{2^n} \right) \right) \right) \\ &\geq (b-a) |g_{2^n+j}| - \max_{0 \leq k < 2^n} |g_{2^n+k}| (x_0^\gamma(d) - \lim_{\alpha \rightarrow \omega(n,j)^+} x_0^{\omega(n,j)}(\alpha)) \\ &\quad + \lim_{\alpha \rightarrow \omega(n,j)^-} x_0^{\omega(n,j)}(\alpha) - x_0^\gamma(c) \\ &\geq (b-a) \left(|g_{2^n+j}| - \frac{1}{2} \max_{0 \leq k < 2^n} |g_{2^n+k}| \right). \end{aligned}$$

Hence

$$\left\| \sum_{k=2^n}^{2^{n+1}-1} g_k e_k \right\| \geq \frac{b-a}{2} \max_{2^n \leq k < 2^{n+1}} |g_k|.$$

It is clear that

$$\left\| \sum_{k=2^n}^{2^{n+1}-1} g_k e_k \right\| \leq \max_{2^n \leq k < 2^{n+1}} |g_k|.$$

In view of Fact 3 and Theorem 1 the closed linear hull of $f(Q)$ contains a copy of $C([0, 1])$.

(b) Let U be the closed linear hull of $\{\pi_\gamma : \gamma \in Q\}$ in $C(H_f)$. Denote by $L : C(H_f) \rightarrow C(H_f)/U$ the quotient map. We show that the closed linear hull of $L(\pi_\alpha)$ for $\alpha \in \Gamma_4$ is isomorphic to $c_0(\Gamma_4)$. Let $\gamma_1, \dots, \gamma_n \in \Gamma_4$ and $g_1, \dots, g_n \in \mathbb{C}$. We can assume that $0 < \gamma_1 < \dots < \gamma_n$. Applying the fact that $2 \max\{|r+s|, |r+t|\} \geq |s-t|$ for any $r, s, t \in \mathbb{C}$, we get, for all $\alpha_1, \dots, \alpha_k \in Q$, $h_1, \dots, h_k \in \mathbb{C}$ and $j = 1, \dots, n$,

$$\begin{aligned} \max_{s \in \{0,1\}} \left| \left(\sum_{i=1}^n g_i \pi_{\gamma_i} - \sum_{i=1}^k h_i \pi_{\alpha_i} \right) (x_s^{\gamma_j}) \right| &\geq \max_{s \in \{0,1\}} \left| \sum_{i=1}^n g_i x_s^{\gamma_j}(\gamma_i) - \sum_{i=1}^k a_i x_1^{\gamma_j}(\alpha_i) \right| \\ &\geq \frac{1}{2} \left| g_j (x_1^{\gamma_j}(\gamma_j) - x_0^{\gamma_j}(\gamma_j)) + \sum_{i \neq j} g_i (x_1^{\gamma_j}(\gamma_i) - x_0^{\gamma_j}(\gamma_i)) \right| \\ &\geq \frac{1}{2} \left((b-a) |g_j| - \max_{i \neq j} |g_i| (x_1^{\gamma_j}(d) - \lim_{\tau \rightarrow \gamma_j^+} x_1^{\gamma_j}(\tau) - \lim_{\tau \rightarrow \gamma_j^-} x_1^{\gamma_j}(\tau) - x_1^{\gamma_j}(c)) \right) \\ &\geq \frac{b-a}{2} \left(|g_j| - \frac{1}{2} \max_{i \neq j} |g_i| \right). \end{aligned}$$

Hence

$$\left\| L\left(\sum_{i=1}^n g_i \pi_{\gamma_i}\right) \right\| \geq \frac{b-a}{4} \max_{1 \leq i \leq n} |g_i|.$$

On the other hand, if $\beta_i \in Q$ and $\beta_i < \gamma_i < \beta_{i+1}$ for $i = 1, \dots, n-1$, then

$$\begin{aligned} \left\| L\left(\sum_{i=1}^n g_i \pi_{\gamma_i}\right) \right\| &\leq \max_{x \in H} \left\| \left(\sum_{i=1}^n g_i \pi_{\gamma_i} - \sum_{i=1}^n g_i \pi_{\beta_i}\right)(x) \right\| \\ &\leq \sum_{i=1}^n |g_i| (x(\gamma_i) - x(\beta_i)) \leq \max_{1 \leq i \leq n} |g_i| \sum_{i=1}^n (x(\gamma_i) - x(\beta_i)) \\ &\leq \max_{1 \leq i \leq n} |g_i|. \end{aligned}$$

Thus we show that the vectors $L(\pi_\gamma)$ for $\gamma \in \Gamma_4$ span a copy of $c_0(\Gamma_4)$ in $C(H_f)/U$. An appeal to Theorem 1 completes the proof of this part.

(c) Let Γ_5 be the closure of Γ_4 in $[0, 1]$. Let $(0, 1) \setminus \Gamma_5 = \bigcup_{n=1}^\infty (\alpha_n, \beta_n)$, where distinct intervals from the sequence $((\alpha_n, \beta_n))$ are disjoint. For every $\gamma \in \Gamma_5 \setminus (\{\beta_n : n \in \mathbb{N}\})$ there exists an increasing sequence of elements of Γ_5 converging to γ and for every $\gamma \in \Gamma_5 \setminus (\{\alpha_n : n \in \mathbb{N}\})$ there exists a decreasing sequence of elements of Γ_5 converging to γ . Let z_1^γ for $\gamma \in \Gamma \setminus \{\beta_n : n \in \mathbb{N}\}$ be any cluster points of a sequence $(x_0^{\gamma_n})$ where (γ_n) is an increasing sequence converging to γ . The point z_1^γ is an element of H_f , $\lim_{\alpha \rightarrow \gamma^-} z_1^\gamma(\alpha) \leq a$, $z_1^\gamma(\gamma) \geq b$ and

$$z_1^\gamma(d) - \lim_{\alpha \rightarrow \gamma^+} z_1^\gamma(\alpha) + \lim_{\alpha \rightarrow \gamma^-} z_1^\gamma(\alpha) - z_1^\gamma(c) \leq \frac{(b-a)^2}{8} = C.$$

Let z_0^γ for $\gamma \in \Gamma_5 \setminus \{\alpha_n : n \in \mathbb{N}\}$ be any cluster point of a sequence $(x_0^{\gamma_n})$ where (γ_n) is a decreasing sequence converging to γ . The point z_0^γ is an element of H_f , $z_0^\gamma(\gamma) \leq a$, $\lim_{\alpha \rightarrow \gamma^+} z_0^\gamma(\alpha) \geq b$ and

$$z_1^\gamma(d) - \lim_{\alpha \rightarrow \gamma^+} z_1^\gamma(\alpha) + \lim_{\alpha \rightarrow \gamma^-} z_1^\gamma(\alpha) - z_1^\gamma(c) \leq C.$$

There exists a subset Γ_6 of $\Gamma_5 \cap (c, d)$ homeomorphic to the Cantor set and an increasing continuous bijection $b : \{0, 1\}^\mathbb{N} \rightarrow \Gamma_6$. Let $\Gamma = b(\{(\varepsilon_n) \in \{0, 1\}^\mathbb{N} : \varepsilon_{2i-1} = \varepsilon_{2i} \text{ for } i \in \mathbb{N}\})$. Let $(0, 1) \setminus \Gamma = \bigcup_{n=1}^\infty (\varrho_n, \eta_n)$, where distinct intervals from the sequence $((\varrho_n, \eta_n))$ are disjoint. The last step of the construction of Γ implies that for every $n \in \mathbb{N}$ there exists $\tau_n \in \Gamma_5 \cap (\varrho_n, \eta_n)$ such that $z_0^{\tau_n}$ and $z_1^{\tau_n}$ are elements of H_f . Let $\gamma_1, \dots, \gamma_n \in \Gamma$ and $g_1, \dots, g_n, g_{n+1} \in \mathbb{C}$. We can assume that $\gamma_1 < \dots < \gamma_n$. Then

$$\begin{aligned} \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(H_f)} &\leq \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(\mathbb{H})} \\ &= \left\| \left(\sum_{i=1}^{n+1} g_i\right) \pi_{\gamma_1} + \sum_{i=2}^n \left(\sum_{j=i}^{n+1} g_j\right) (\pi_{\gamma_i} - \pi_{\gamma_{i-1}}) + g_{n+1}(1 - \pi_{\gamma_n}) \right\|_{C(\mathbb{H})} \end{aligned}$$

$$= \max \left\{ \left| \sum_{j=i}^{n+1} g_j \right| : i = 1, \dots, n+1 \right\} = \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(\mathbb{L}_\Gamma)}.$$

Since Γ is nowhere dense, there exist $\tau_{k_1}, \dots, \tau_{k_n}$ such that $\tau_{k_i} < \gamma_i < \tau_{k_{i+1}}$ for every $i = 1, \dots, n$. Then for $l = 2, \dots, n$,

$$\begin{aligned} \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(H_f)} &\geq \left| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i}(z_1^{\tau_{k_l}}) \right| \\ &\geq \left| \left(\sum_{i=1}^{n+1} g_i \right) \pi_{\gamma_1}(z_1^{\tau_{k_l}}) + \sum_{i=2}^n \left(\sum_{j=i}^{n+1} g_j \right) (\pi_{\gamma_i} - \pi_{\gamma_{i-1}})(z_1^{\tau_{k_l}}) + g_{n+1} (1 - \pi_{\gamma_n})(z_1^{\tau_{k_l}}) \right| \\ &\geq (b-a) \left| \sum_{j=l}^{n+1} g_j \right| - C \max \left\{ \left| \sum_{j=i}^{n+1} g_j \right| : i \neq 1, l, n+1 \right\} - a \left| \sum_{i=1}^{n+1} g_i \right| - (1-b) |g_{n+1}| \end{aligned}$$

and

$$\begin{aligned} \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(H_f)} &\geq \left| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i}(z_1^{\tau_{k_1}}) \right| \\ &\geq b \left| \sum_{i=1}^{n+1} g_i \right| - C \max \left\{ \left| \sum_{l=i}^{n+1} g_l \right| : i \neq 1, n+1 \right\} - (1-b) |g_{n+1}|, \end{aligned}$$

and

$$\begin{aligned} \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(H_f)} &\geq \left| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i}(z_1^{\tau_{k_{n+1}}}) \right| \\ &\geq (1-a) |g_{n+1}| - a \left| \sum_{i=1}^{n+1} g_i \right| - C \max \left\{ \left| \sum_{l=i}^{n+1} g_l \right| : i \neq 1, n+1 \right\}. \end{aligned}$$

If

$$\begin{aligned} \max \left\{ 4 \left| \sum_{i=1}^{n+1} g_i \right| \left(a + \frac{b-a}{4} \right), 4 |g_{n+1}| \left((1-b) + \frac{b-a}{4} \right) \right\} \\ \leq \max \left\{ \left| \sum_{l=i}^{n+1} g_l \right| : i \neq 1, n+1 \right\} (b-a), \end{aligned}$$

then

$$\begin{aligned} \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(H_f)} &\geq \max \left\{ \left| \sum_{l=i}^{n+1} g_l \right| : i \neq 1, n+1 \right\} \frac{b-a}{4} \\ &\geq \frac{b-a}{4} \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(\mathbb{L}_\Gamma)}. \end{aligned}$$

If

$$\begin{aligned} \max \left\{ 4|g_{n+1}| \left((1-b) + \frac{b-a}{4} \right), \max \left\{ \left| \sum_{l=i}^{n+1} g_l \right| : i \neq 1, n+1 \right\} (b-a) \right\} \\ \leq 4 \left| \sum_{i=1}^{n+1} g_i \right| \left(a + \frac{b-a}{4} \right), \end{aligned}$$

then

$$\left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(H_f)} \geq \left| \sum_{i=1}^{n+1} g_i \right| \frac{b-a}{8} \geq \frac{(b-a)^2}{64} \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(\mathbb{L}_\Gamma)}$$

If

$$\begin{aligned} \max \left\{ 4 \left| \sum_{i=1}^{n+1} g_i \right| \left(a + \frac{b-a}{4} \right), \max \left\{ \left| \sum_{l=i}^{n+1} g_l \right| : i \neq 1, n+1 \right\} (b-a) \right\} \\ \leq 4|g_{n+1}| \left((1-b) + \frac{b-a}{4} \right), \end{aligned}$$

then

$$\left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(H_f)} \geq |g_{n+1}| \frac{b-a}{8} \geq \frac{(b-a)^2}{64} \left\| g_{n+1} + \sum_{i=1}^n g_i \pi_{\gamma_i} \right\|_{C(\mathbb{L}_\Gamma)}.$$

Thus we have shown that the operator $I : C(\mathbb{L}_\Gamma) \rightarrow C(H_f)$ given by $I(\pi_\gamma) = \pi_\gamma$ is an isomorphic embedding. Moreover, the embedding is positive. It is clear that there exists an increasing surjection from Γ onto $[0, 1]$. Consequently, the space $C(\mathbb{L}_\Gamma)$ contains a positive and isometric copy of $C(\mathbb{L})$. An appeal to Theorem 1 completes the proof.

As a straightforward consequence of Theorem 4(c) we get

COROLLARY 5. *A Banach lattice F has property (λ) if and only if it does not contain any order copy of $D(0, 1)$.*

COROLLARY 6. *Let $f : [0, 1] \rightarrow X$ be an increasing function with respect to a norming subset E of X^* . Let Q be a countable dense subset of $[0, 1]$. If f has uncountably many points of discontinuity and I is an isomorphic embedding of $\overline{\text{lin } f([0, 1])}$ into a Banach space Z , then no separable complemented subspace of Z contains $I(\overline{\text{lin } f(Q)})$.*

Proof. We can assume that $f(0) = 0$, $\|f(1)\| = 1$ and $E \subset B_{X^*}$. Let H_f be the closed nonmetrizable subset of \mathbb{H} associated with f as in Theorem 1. Let U and V be the closed linear hulls in $C(H_f)$ of $\{\pi_\gamma : \gamma \in Q\}$ and $\{\pi_\gamma : \gamma \in [0, 1]\}$, respectively. Let $I : V \rightarrow Z$ be an isomorphic embedding. Suppose that a separable complemented subspace Y of Z contains $I(U)$. Let W be the closed hull of $I(V) + Y$ in Z . Let $P : W \rightarrow W$ be a projection such that $P(W) = Y$ and $P(y) = y$ for every $y \in Y$. Let the operator

$T : V/U \rightarrow \ker P$ be given by $T(f + U) = I(f) - P(I(f))$. The subset $I^{-1}(I(V) \cap P(I(V)))$ of V is separable. Thus there exists a countable subset Q_1 of $[0, 1]$ such that $I^{-1}(I(V) \cap P(I(V)))$ is contained in the closed linear hull of the functions π_α for $\alpha \in Q_1$ in $C(H_f)$. Let \tilde{U} be the closed linear hull of $\{\pi_\alpha : \alpha \in Q \cup Q_1\}$ in $C(H)$. Let H_1 be a closed, separable, nonmetrizable subset of H_f (existing by Fact 2). Since H_1 is nonmetrizable, as in the proof of Theorem 4(b) we find an uncountable subset Γ of $[0, 1]$, $\varepsilon > 0$ and $0 \leq p < q \leq 1$ such that $\Gamma \subset (p, q)$ and for every $\gamma \in \Gamma$ there exist $x_0^\gamma, x_1^\gamma \in H_1$ with $x_0^\gamma(\alpha) = x_1^\gamma(\alpha)$ for every $\alpha \in Q \cup Q_1$, $x_1^\gamma(\gamma) - x_0^\gamma(\gamma) \geq \varepsilon$ and

$$x_1^\gamma(q) - \lim_{\alpha \rightarrow \gamma+} x_1^\gamma(\alpha) + \lim_{\alpha \rightarrow \gamma-} x_1^\gamma(\alpha) - x_1^\gamma(p) \leq \varepsilon/2.$$

Let $L : V \rightarrow V/U$ be the quotient map. Let $J : c_0(\Gamma) \rightarrow V/U$ be the isomorphic embedding from the proof of Theorem 4(b), i.e. $J(e_\gamma) = L(\pi_\gamma)$ for every $\gamma \in \Gamma$ where e_γ is the γ th unit vector of $c_0(\Gamma)$. Since Y is separable, there exist $\delta > 0$ and an uncountable subset Γ_1 of Γ such that $\|T(J(e_\gamma))\| > \delta$ for every $\gamma \in \Gamma_1$. According to a result of H. P. Rosenthal (see [11, Thm. 3.4]) there exists an uncountable subset Γ_2 of Γ_1 such that $T \circ J : c_0(\Gamma_2) \rightarrow \ker P$ is an isomorphism. Let $S : W \rightarrow W/I(V)$ be the quotient map. It is clear that $W/I(V)$ is separable. Applying once again Rosenthal's result we find an uncountable subset Γ_3 of Γ_2 such that $S(T(J(c_0(\Gamma_3)))) = 0$. Therefore $I^{-1} \circ T \circ J : c_0(\Gamma_3) \rightarrow V$ is an isomorphic embedding. Moreover, $I^{-1}(T(L(\pi_\gamma))) \in \pi_\gamma + \tilde{U}$ for every $\gamma \in \Gamma_3$. Let $R : C(H_f) \rightarrow C(H_1)$ be given by $R(f) = f|_{H_1}$. Let V_1 and U_1 be the closed linear hulls in $C(H_1)$ of $\{\pi_\alpha : \alpha \in [0, 1]\}$ and $\{\pi_\alpha : \alpha \in Q \cup Q_1\}$, respectively. Let $L_1 : V_1 \rightarrow V_1/U_1$ be the quotient map. According to the proof of Theorem 5 the closed linear hull of $\{L_1(\pi_\gamma) : \gamma \in \Gamma_3\}$ is isomorphic to $c_0(\Gamma_3)$. Since $R(\pi_\alpha) = \pi_\alpha$ for every $\alpha \in [0, 1]$, we have $R(V) \subset V_1$ and $R(\tilde{U}) \subset U_1$. Therefore $L_1(R(I^{-1}(T(L(\pi_\gamma)))) = L_1(\pi_\gamma)$ for every $\gamma \in \Gamma_3$. Thus we have shown that $R \circ I^{-1} \circ T \circ J$ is an isomorphic embedding of $c_0(\Gamma_3)$ into V_1 . This contradicts the fact that H_1 , being separable, satisfies the countable chain condition and the space $C(H_1)$ contains no copy of $c_0(\Gamma_3)$ (see [10, Thm. 1.4]).

A Banach space X has the *separable complementation property* if for every separable subspace X_1 of X there exists a separable and complemented subspace X_2 which contains X_1 (see [9]). As a straightforward consequence of the result above we get

COROLLARY 8. *If a Banach space contains an isomorphic copy of $D(0, 1)$, then it does not have the separable complementation property.*

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