\( \ell^1 \)-Spreading models in subspaces of mixed Tsirelson spaces

by

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Abstract. We investigate the existence of higher order \( \ell^1 \)-spreading models in subspaces of mixed Tsirelson spaces. For instance, we show that the following conditions are equivalent for the mixed Tsirelson space \( X = T[(\theta_n, S_n)_{n=1}^\infty] \):

1. Every block subspace of \( X \) contains an \( \ell^1 \)-\( S_\omega \)-spreading model,
2. The Bourgain \( \ell^1 \)-index \( I_b(Y) = I(Y) > \omega \) for any block subspace \( Y \) of \( X \),
3. \( \lim_m \limsup_n \theta_{m+n}/\theta_n > 0 \) and every block subspace \( Y \) of \( X \) contains a block sequence equivalent to a subsequence of the unit vector basis of \( X \).

Moreover, if one (and hence all) of these conditions holds, then \( X \) is arbitrarily distortable.

1. Introduction. The discovery and construction of nontrivial asymptotic \( \ell^1 \) spaces has led to much progress in the structure theory of Banach spaces. The first such space discovered was Tsirelson’s space [24]. Subsequently, Schlumprecht constructed what is now called Schlumprecht’s space [22]. This space plays a vital role in the solutions of the unconditional basic sequence problem by Gowers and Maurey [12] and the distortion problem by Odell and Schlumprecht [19]. Argyros and Deliyanni [5] introduced the class of mixed Tsirelson spaces which provides a general framework for Tsirelson’s space, Schlumprecht’s space and related examples such as Tzafriri’s space [25]. Mixed Tsirelson spaces have been studied extensively. In particular, results about their finite-dimensional \( \ell^1 \)-structure were obtained in [6, 7, 18]. The present authors computed the Bourgain \( \ell^1 \)-indices of mixed Tsirelson spaces in [16], and investigated thoroughly the existence of higher order \( \ell^1 \)-spreading models in such spaces [17]. (Results in this direction for certain mixed Tsirelson spaces were first proved in [7].)

In the present paper, we investigate when a mixed Tsirelson space contains higher order \( \ell^1 \)-spreading models hereditarily. Again, the first result of this kind is found in [7]. We prove some general characterizations and obtain the result in [7] as a corollary. Roughly speaking, our results show that

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the complexity of the hereditary finite-dimensional $\ell^1$-structure of a mixed Tsirelson space is the same whether it is measured by the existence of higher order $\ell^1$-spreading models or Bourgain’s $\ell^1$-index. These are also related to what may be called “subsequential minimality” of the mixed Tsirelson space in question and imply that it is arbitrarily distortable.

Denote by $\mathbb{N}$ the set of natural numbers. For any infinite subset $M$ of $\mathbb{N}$, let $[M]$, respectively $[M]<\omega$, be the set of all infinite and finite subsets of $M$ respectively. These are subspaces of the power set of $\mathbb{N}$, which is identified with $2^\mathbb{N}$ and endowed with the topology of pointwise convergence. A subset $\mathcal{F}$ of $[\mathbb{N}]<\omega$ is said to be hereditary if $G \in \mathcal{F}$ whenever $G \subseteq F$ and $F \in \mathcal{F}$. It is spreading if for all strictly increasing sequences $(m_i)_{i=1}^k$ and $(n_i)_{i=1}^k$, $(n_i)_{i=1}^k \in \mathcal{F}$ if $(m_i)_{i=1}^k \in \mathcal{F}$ and $m_i \leq n_i$ for all $i$. We also call $(n_i)_{i=1}^k$ a spreading of $(m_i)_{i=1}^k$. A regular family is a subset of $[\mathbb{N}]<\omega$ that is hereditary, spreading and compact (as a subspace of $2^\mathbb{N}$). If $I$ and $J$ are nonempty finite subsets of $\mathbb{N}$, we write $I < J$ to mean $\max I < \min J$. We also allow that $\emptyset < I$ and $I < \emptyset$. For a singleton $\{n\}$, $\{n\} < J$ is abbreviated to $n < J$.

Given a regular family $\mathcal{F} \subseteq [\mathbb{N}]<\omega$, a sequence of sets $(E_i)_{i=1}^k$ is said to be $\mathcal{F}$-admissible if $(\min E_i)_{i=1}^k \in \mathcal{F}$. If $\mathcal{G}$ is another family of sets, let

$$\mathcal{F}[\mathcal{G}] = \left\{ \bigcup_{i=1}^k G_i : G_i \in \mathcal{G}, (G_i)_{i=1}^k \text{ is } \mathcal{F}\text{-admissible} \right\}$$

and

$$(\mathcal{F}, \mathcal{G}) = \{ F \cup G : F < G, F \in \mathcal{F}, G \in \mathcal{G} \}.$$ Inductively, set $(\mathcal{F})^1 = \mathcal{F}$ and $(\mathcal{F})^{n+1} = (\mathcal{F}, (\mathcal{F})^n)$ for all $n \in \mathbb{N}$. It is clear that $\mathcal{F}[\mathcal{G}]$ and $(\mathcal{F}, \mathcal{G})$ are regular if both $\mathcal{F}$ and $\mathcal{G}$ are. A class of regular families that has played a central role is the class of generalized Schreier families [1]. The reason for their usefulness as a measure of the complexity of subsets of $[\mathbb{N}]<\omega$ is by now well explained [11, 13]. Let $S_0$ consist of all singleton subsets of $\mathbb{N}$ together with the empty set. Then define $S_1$ to be the collection of all $A \in [\mathbb{N}]<\omega$ such that $|A| \leq \min A$ together with the empty set, where $|A|$ denotes the cardinality of the set $A$. If $S_\alpha$ has been defined for some countable ordinal $\alpha$, set $S_{\alpha+1} = S_1[S_\alpha]$. For a countable limit ordinal $\alpha$, specify a sequence $(\alpha_n)$ that strictly increases to $\alpha$. Then define

$$S_\alpha = \{ F : F \in S_{\alpha_n} \text{ for some } n \leq \min F \} \cup \{ \emptyset \}.$$ Given a nonempty compact family $\mathcal{F} \subseteq [\mathbb{N}]<\omega$, let $\mathcal{F}^{(0)} = \mathcal{F}$ and $\mathcal{F}^{(1)}$ be the set of all limit points of $\mathcal{F}$. Continue inductively to derive $\mathcal{F}^{(\alpha+1)} = (\mathcal{F}^{(\alpha)})^{(1)}$ for all ordinals $\alpha$ and $\mathcal{F}^{(\alpha)} = \bigcap_{\beta < \alpha} \mathcal{F}^{(\beta)}$ for all limit ordinals $\alpha$. The index $\iota(\mathcal{F})$ is taken to be the smallest $\alpha$ such that $\mathcal{F}^{(\alpha+1)} = \emptyset$. Since $[\mathbb{N}]<\omega$ is countable, $\iota(\mathcal{F}) < \omega_1$ for any compact family $\mathcal{F} \subseteq [\mathbb{N}]<\omega$. It is well known that $\iota(S_\alpha) = \omega^\alpha$ for all $\alpha < \omega_1$ [1, Proposition 4.10].
Denote by $c_{00}$ the space of all finitely supported real sequences. For a finite subset $E$ of $\mathbb{N}$ and $x \in c_{00}$, let $Ex$ be the coordinatewise product of $x$ with the characteristic function of $E$. The sup norm and the $\ell^1$-norm on $c_{00}$ are denoted by $\| \cdot \|_{c_0}$ and $\| \cdot \|_{\ell^1}$ respectively. Given a sequence $(\mathcal{F}_n)$ of regular families and a nonincreasing null sequence $(\theta_n)_{n=1}^\infty$ in $(0,1)$, define a sequence of norms $\| \cdot \|_m$ on $c_{00}$ as follows. Let $\|x\|_0 = \|x\|_{c_0}$ and

$$\|x\|_{m+1} = \max \left\{ \|x\|_m, \sup_n \theta_n \sup_{i=1}^r \|E_i x\|_m \right\},$$

where the last sup is taken over all $\mathcal{F}_n$-admissible sequences $(E_i)_{i=1}^r$. Since these norms are all dominated by the $\ell^1$-norm, $\|x\| = \lim_m \|x\|_m$ exists and is a norm on $c_{00}$. The mixed Tsirelson space $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ is the completion of $c_{00}$ with respect to the norm $\| \cdot \|$. From (1) we can deduce that the norm in $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ satisfies the implicit equation

$$\|x\| = \max \left\{ \|x\|_{c_0}, \sup_n \theta_n \sup_{i=1}^r \|E_i x\|_0 \right\},$$

with the last sup taken over all $\mathcal{F}_n$-admissible sequences $(E_i)_{i=1}^r$. For the rest of the paper, we consider a fixed sequence $(\theta_n, \mathcal{F}_n)_{n=1}^\infty$ as above and let $X = T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$. Set $\alpha_n = \iota(\mathcal{F}_n)$ for all $n$. Families $\mathcal{F}_n$ with $\iota(\mathcal{F}_n) = 1$ contain singletons and the empty set only and may be removed without effect on the norm $\| \cdot \|$. Also the spaces $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ and $T[(\theta_n, \bigcup_{k=1}^n \mathcal{F}_k)_{n=1}^\infty]$ are identical (since $(\theta_n)$ is nonincreasing). Hence there is no loss of generality in assuming that $\alpha_n > 1$ for all $n$ and that $(\alpha_n)$ is nondecreasing. We will also assume that $\alpha_n < \sup_m \alpha_m = \omega^{\omega^\xi}$, $0 < \xi < \omega_1$. Otherwise, the relevant result has been obtained in [17, Proposition 2], except for the case when $\xi = 0$. The coordinate unit vectors $(e_k)$ form an unconditional basis of $X$.

Given a Banach space $B$ with a basis $(b_k)$, the support of a vector $x = \sum a_k b_k$ (with respect to $(b_k)$), denoted supp $x$, is the set of all $k$ such that $a_k \neq 0$. A block sequence in $B$ is a sequence $(x_k)$ so that supp $x_k \subset$ supp $x_{k+1}$ for all $k$. The closed linear span of a block sequence is called a block subspace.

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**2. Technical preliminaries.** In this section, we present some technical results prior to the main discussion. If $(x_k)$ and $(y_k)$ are sequences of vectors residing in (possibly different) normed spaces, we say that $(x_k)$ dominates $(y_k)$ if there is a finite positive constant $K$ so that

$$\left\| \sum a_k y_k \right\| \leq K \left\| \sum a_k x_k \right\|$$
for all \((a_k) \in c_{00}\). Two sequences are *equivalent* if they dominate each other. The first lemma shows that under certain mild assumptions on the families \((\mathcal{F}_n)\), any subsequence of \((e_k)\) is equivalent to its left shift. The proof uses essentially the idea in [10, Lemma 2], dressed up in the present language. The family of all subsets of \(\mathbb{N}\) with at most \(k\) elements is denoted by \(\mathcal{A}_k\).

**Lemma 1.** Assume that for all \(n\), either \(\mathcal{F}_n = \mathcal{A}_j\) for some \(j \in \mathbb{N}\) or \(\mathcal{F}_n[\mathcal{A}_3] \subseteq (\mathcal{F}_n)^2\). Suppose that \((i_k) \in [\mathbb{N}]\). Let \(x = \sum a_k e_{i_k+1}\) and \(y = \sum a_k e_{i_k}\) for some \((a_k) \in c_{00}\). Then for any \(m\), there exist \(E_1 < E_2 < E_3\) such that

\[
\|x\|_m \leq \sum_{i=1}^3 \|E_i y\|_m.
\]

Consequently, the sequences \((e_{i_k})\) and \((e_{i_k+1})\) are equivalent.

**Proof.** For any set \(E \subseteq \mathbb{N}\), let the *left shift* of \(E\) be the set \(L_E = \{i_k : i_{k+1} \in E\}\). We will prove by induction that for any \(m \geq 0\) and any \(E \subseteq \mathbb{N}\), there exist \(E_1 < E_2 < E_3\), subsets of \(L_E\), such that \(\|Ex\|_m \leq \sum_{i=1}^3 \|E_i y\|_m\). The case \(m = 0\) is clear. Assume that the lemma holds for some \(m\). Given \(E \subseteq \mathbb{N}\), let \(z = Ex\). If \(\|z\|_{m+1} = \|z\|_m\), there is nothing to prove. Otherwise, \(\|z\|_{m+1} = \theta_n \sum_{i=1}^r \|F_i z\|_m\) for some \(n\) and some \(\mathcal{F}_n\)-admissible sequence \((F_i)_{i=1}^r\). By the inductive hypothesis, there exist \(F_1^i < F_2^i < F_3^i\), subsets of \(L_{F_i}\), such that

\[
\|F_i z\|_m \leq \sum_{k=1}^3 \|F_k^i y\|_m, \quad 1 \leq i \leq r.
\]

We claim that \((F_k^i)_{i=1}^r\) is \((\mathcal{A}_1 \cup \mathcal{F}_n, (\mathcal{F}_n)^2)\)-admissible. Indeed, if \(\mathcal{F}_n = \mathcal{A}_j\) for some \(j\), then

\[
(min F_k^i)_{i=1}^r 3 = (\mathcal{A}_1 \cup \mathcal{F}_n, (\mathcal{F}_n)^2)\-admissible.
\]

Otherwise, since \(min F_2^i \geq min F_i\),

\[
\bigcup_{i=1}^r \{min F_2^i, min F_3^i, min F_1^i+1\} \in \mathcal{F}_n[\mathcal{A}_3] \subseteq (\mathcal{F}_n)^2.
\]

Clearly, \(\{min F_1^i\} \in \mathcal{A}_1\). Thus

\[
\bigcup_{i=1}^r \{min F_1^i, min F_2^i, min F_3^i\} \in (\mathcal{A}_1 \cup \mathcal{F}_n, (\mathcal{F}_n)^2),
\]

as claimed.

It follows from the claim that there exist \(E_1 < E_2 < E_3\) so that \(\bigcup_{p=1}^3 E_p = \bigcup_{i=1}^r \bigcup_{k=1}^3 F_k^i\), each \(E_p\) is a union of finitely many \(F_k^i\) and that \(\mathcal{E}_p = \{F_k^i : F_k^i \subseteq E_p\}\) is \((\mathcal{A}_1 \cup \mathcal{F}_n)\)-admissible if \(p = 1\) and \(\mathcal{F}_n\)-admissible if \(p = 2, 3\). Notice that \(\theta_n \sum_{F_k^i \in \mathcal{E}_p} \|F_k^i y\|_m \leq \|E_p y\|_{m+1}\) since \(\mathcal{E}_p\) is either \(\mathcal{F}_n\)-admissible.
or $A_1$-admissible. Hence

$$
\|Ex\|_{m+1} = \|z\|_{m+1} = \theta_n \sum_{i=1}^r \|F_iz\|_m \leq \theta_n \sum_{i=1}^r \sum_{k=1}^3 \|F_ky\|_m \leq \sum_{p=1}^3 \|E_py\|_{m+1}.
$$

Upon taking the limit as $m \to \infty$, we see that $(e_{i+1})$ is dominated by $(e_i)$. Since the reverse domination is clear, the two sequences are equivalent.

A tree in a Banach space $B$ is a subset $T$ of $\bigcup_{n=1}^\infty B^n$ so that $(x_1, \ldots, x_n)$ $\in T$ whenever $(x_1, \ldots, x_n, x_{n+1}) \in T$. Elements of the tree are called nodes. It is well-founded if there is no infinite sequence $(x_n)$ so that $(x_1, \ldots, x_m) \in T$ for all $m$. If $T$ has a basis, then a tree $T$ is said to be a block tree (with respect to the basis) if every node is a block sequence. For any well-founded tree $T$, its derived tree is the tree $D^{(1)}(T)$ consisting of all nodes $(x_1, \ldots, x_n)$ so that $(x_1, \ldots, x_n, x) \in T$ for some $x$. Inductively, set $D^{(\alpha+1)}(T) = D^{(1)}(D^{(\alpha)}(T))$ for all ordinals $\alpha$ and $D^{(\alpha)}(T) = \bigcap_{\beta<\alpha} D^{(\beta)}(T)$ for all limit ordinals $\alpha$. The order of a tree $T$ is the smallest ordinal $o(T) = \alpha$ such that $D^{(\alpha)}(T) = \emptyset$.

**Lemma 2.** Let $T$ be a well-founded block tree in a Banach space $B$ with a basis. Define

$$
H = \{ (\max \text{supp } x_j)_{j=1}^r : (x_j)_{j=1}^r \in T \},
$$

$$
G = \{ G : G \text{ is a spreading of a subset of some } H \in H \}.
$$

Then $G$ is hereditary and spreading. If $G$ is compact, then $\iota(G) \geq o(T)$.

**Proof.** It is clear that $G$ is hereditary and spreading. Assume that $G$ is compact. We show by induction on $\xi$ that for all countable ordinals $\xi$, $\iota(G) \geq \xi$ if $o(T) \geq \xi$. There is nothing to prove if $\xi = 0$. Suppose the proposition holds for some $\xi < \omega_1$. Let $T$ be a well-founded block tree with $o(T) \geq \xi + 1$. For each $(x) \in T$, let

$$
T_x = \bigcup_{n=1}^\infty \{ (x_1, \ldots, x_n) : (x, x_1, \ldots, x_n) \in T \}.
$$

According to [9, Proposition 4], $o(T) = \sup_{(x) \in T} (o(T_x) + 1)$. Therefore, there exists $(x_0) \in T$ such that $o(T_{x_0}) \geq \xi$. By the inductive hypothesis, $\iota(G') \geq \xi$, where $G'$ is defined analogously to $G$ for the tree $T_{x_0}$. Let $k_0 = \max \text{supp } x_0$. Then $\{k_0\} \cup G \in G'$ whenever $G \in G'$. Thus $\{k_0\} \in G^{(\xi)}$. Since $G^{(\xi)}$ is spreading, $\{k\} \in G^{(\xi)}$ for all $k \geq k_0$. It follows that $\iota(G) \geq \xi + 1$.

Suppose $o(T) \geq \xi_0$, where $\xi_0$ is a countable limit ordinal and the proposition holds for all $\xi < \xi_0$. Since $o(T) \geq \xi$ for all $\xi < \xi_0$, by the inductive hypothesis, $\iota(G) \geq \xi$ for all $\xi < \xi_0$. Hence $\iota(G) \geq \xi_0$. This completes the induction.
3. Main results and proofs. The main results concern two measures of the finite-dimensional $\ell^1$-complexity of the space $X$. These are the Bourgain $\ell^1$-index and the existence of $\ell^1$-spreading models of higher order. Given a finite constant $K$ greater than 1, an $\ell^1$-$K$-tree in a Banach space $B$ is a tree in $B$ so that every node $(x_1, \ldots, x_n)$ is a normalized sequence such that $\|\sum a_k x_k\| \geq K^{-1} \sum |a_k|$ for all $(a_k)$. If $B$ has a basis, an $\ell^1$-$K$-block tree is a block tree that is also an $\ell^1$-$K$-tree. Suppose that $B$ does not contain $\ell^1$, and let $I(B, K) = \sup o(T)$, where the sup is taken over the set of all $\ell^1$-$K$-trees in $X$. The Bourgain $\ell^1$-index is defined to be $I(B) = \sup_{K<\infty} I(B, K)$. The block indices $I_b(B, K)$ and $I_b(B)$ are defined analogously using $\ell^1$-block trees. We refer to [2, 14] for thorough investigations of these indices.

In particular, it is shown in [14] that for a Banach space $B$ with a basis, $I_b(B) = I(B)$ if either one is $\geq \omega$ with $\omega$. With the same notation as above, a normalized sequence $(x_k)$ is said to be an $\ell^1$-$S_\beta$-spreading model with constant $K$ if $\|\sum_{k \in F} a_k x_k\| \geq K^{-1} \sum_{k \in F} |a_k|$ whenever $F \in S_\beta$.

We are now ready to work our way towards the main Theorem 9. The major parts of the computations are contained in Proposition 4 and Lemma 7 (tree splitting lemma). Let $(y_k)$ be a normalized block sequence in $X$ and let $Y$ be the block subspace $\{(y_k)\}$. For any $n \in \mathbb{N}$, we call the space $Y_n = [(y_k)_{k=n}^\infty]$ the $n$-tail of $Y$. We emphasize that in the next lemma both admissibility and the support of a vector are taken with respect to the basis $(e_k)$. Recall the assumption that $(\alpha_n) = (\epsilon(\mathcal{F}_n))$ is a nondecreasing sequence which converges to $\omega^{\omega^\xi}$ nontrivially.

**Lemma 3.** Assume that $I_b(Y) > \omega^{\omega^\xi}$. Then there is a constant $C < \infty$ such that for all $n \in \mathbb{N}$, there exists a normalized vector $x$ in the $n$-tail of $Y$ such that $\sum \|E_i x\| \leq C$ whenever $(E_i)$ is $\mathcal{F}_k$-admissible for some $k \leq n$.

**Proof.** There exists $K < \infty$ such that $I_b(Y, K) > \omega^{\omega^\xi}$. Let $T$ be an $\ell^1$-$K$-block tree in $Y$ such that $o(T) \geq \omega^{\omega^\xi}$. Given $n$, consider the tree $\hat{T}$ consisting of all nodes of the form $(x_j)_{j=n}^\infty$ for some $(x_j)_{j=1}^r \in T$, $r \geq n$. Then $\hat{T}$ is an $\ell^1$-$K$-block tree in $Y_n$ such that $o(\hat{T}) \geq \omega^{\omega^\xi}$. Choose $\alpha$ and $\beta$ so that $\alpha_n < \omega^\alpha < \omega^\beta < \omega^{\omega^\xi}$. Define

$$\mathcal{H} = \{(\max \supp x_j)_{j=n}^\infty : (x_j)_{j=n}^\infty \in \hat{T}\},$$

$$\mathcal{G} = \{G : G \text{ is a spreading of a subset of some } H \in \mathcal{H}\}.$$

By Lemma 2, $\mathcal{G}$ is hereditary and spreading, and either $\mathcal{G}$ is noncompact or it is compact with $\epsilon(\mathcal{G}) \geq o(\hat{T}) \geq \omega^{\omega^\xi}$. By [11, Theorem 1.1], there exists $M \in [\mathbb{N}]$ such that

$$\bigcup_{k=1}^n \mathcal{F}_k \cap [M]^{<\infty} \subseteq \mathcal{S}_\alpha \cap [M]^{<\infty} \subseteq \mathcal{S}_\beta \cap [M]^{<\infty} \subseteq \mathcal{G}.$$
\[ \text{Now \cite[Proposition 3.6]{21} gives a finite set } G \in S_\beta \cap [M]^{<\infty} \text{ and a sequence } (a_p)_{p \in G} \text{ of positive numbers such that } \sum a_p = 1 \text{ and } \sum_{p \in F} a_p < \theta_n \text{ whenever } F \subseteq G \text{ and } F \in S_n. \text{ By definition, there exist a node } (x_j)_{j=n}^\infty \in \tilde{T} \text{ and a subset } J \text{ of the integer interval } [n,r] \text{ such that } G \text{ is a spreading of } (\max \supp x_j)_{j \in J}. \text{ Denote the unique order preserving bijection from } J \text{ onto } G \text{ by } u \text{ and consider the vector } y = \sum_{j \in J} a_u(j)x_j. \text{ Since } (x_j)_{j=n}^\infty \text{ is a normalized } \ell^1-K\text{-block sequence in } Y_n \text{ and } \sum a_u(j) = 1, y \in Y_n \text{ and } \|y\| \geq 1/K. \]

Let \((E_i)\) be \(F_k\)-admissible for some \(k \leq n\). For each \(j \in J\), let \(E_j\) be the collection of all \(E_i\)'s that have nonempty intersection with \(\supp x_j\) if and only if \(j' = j\). Also let \(E'\) be the collection of all \(E_i\) such that \(E_i\) intersects \(\supp x_j\) for at least two \(j \in J\). Since \((E_i)\) is \(F_k\)-admissible, for each \(j \in J\),

\[
\sum_{E_i \in E_j} \|E_iy\| \leq a_u(j)\theta_k^{-1}\|x_j\| = a_u(j)\theta_k^{-1}.
\]

Set \(J' = \{j \in J : E_j \neq \emptyset\}\). The \(F_k\)-admissibility of \((E_i)\) implies that \((\max \supp x_j)_{j \in J'} \in F_k\). Thus \(u(J')\), being a spreading of this set, also belongs to \(F_k\). Since \(u(J') \subseteq G \in [M]^{<\infty}\), we conclude that \(u(J') \in F_k \cap [M]^{<\infty} \subseteq S_n\). Hence \(\sum_{j \in J'} a_u(j) < \theta_n\). Also, since each \(\supp x_j, j \in J\), intersects at most two \(E_i\) in \(E'\),

\[
\sum_{E_i \in E'} \|E_iy\| \leq \sum_{j \in J} a_u(j) \sum_{E_i \in E'} \|E_i x_j\| \leq 2 \sum_{j \in J} a_u(j) = 2.
\]

Therefore,

\[
\sum \|E_iy\| = \sum_{E_i \in E'} \|E_iy\| + \sum_{j \in J'} \sum_{E_i \in E_j} \|E_iy\| \leq 2 + \theta_k^{-1} \sum_{j \in J'} a_u(j) \leq 3.
\]

It is clear that the normalized element \(x = y/\|y\|\) satisfies the statement of the lemma with the constant \(C = 3K\). \(\blacksquare\)

We pause to introduce another method of computing the norm of an element in \(X\) using norming trees. This is derived from the implicit description of the norm in \(X\) (equation (2)) and has been used in \cite[8, 17, 20]{21}. An \((F_k)\)-admissible tree is a finite collection of elements \(E_i^m\), \(0 \leq m \leq r\), \(1 \leq i \leq k(m)\), in \([N]^{<\infty}\) with the following properties:

1. \(k(0) = 1\).
2. For each \(m\), \(E_i^m < E_2^m < \cdots < E_{k(m)}^m\).
3. Every \(E_i^{m+1}\) is a subset of some \(E_j^m\).
4. For each \(j\) and \(m\), the collection \(\{E_i^{m+1} : E_i^{m+1} \subseteq E_j^m\}\) is \(F_k\)-admissible for some \(k\).

The set \(E_1^0\) is called the root of the admissible tree. The elements \(E_i^m\) are called nodes of the tree. If \(E_i^n \subseteq E_j^m\) and \(n > m\), we say that \(E_i^n\) is a descendant of \(E_j^m\), and \(E_j^m\) is an ancestor of \(E_i^n\). If, in the above notation,
n = m + 1, then $E_i^m$ is said to be an immediate successor of $E_j^m$, and $E_j^m$ the immediate predecessor of $E_i^m$. Nodes with no descendants are called terminal nodes or leaves of the tree. Assign tags to the individual nodes inductively as follows. Let $t(E_0^0) = 1$. If $t(E_i^m)$ has been defined and the collection $(E_j^{m+1})$ of all immediate successors of $E_i^m$ forms an $F_k$-admissible collection, then define $t(E_j^{m+1}) = \theta_k t(E_i^m)$ for all immediate successors $E_j^{m+1}$ of $E_i^m$. If $x \in c_{00}$ and $T$ is an admissible tree, let $T x = \sum t(E)\|Ex\|_0$ where the sum is taken over all leaves in $T$. It follows from the implicit description (equation (2)) of the norm in $X$ that $\|x\| = \max T x$, with the maximum taken over the set of all admissible trees. Let us also point out that if $\mathcal{E}$ is a collection of pairwise disjoint nodes of an admissible tree $T$ so that $E \subseteq \bigcup \mathcal{E}$ for every leaf $E$ of $T$ and $x \in c_{00}$, then $T x = \sum_{F \in \mathcal{E}} t(F)\|Fx\|$.

We make the following definitions for notational convenience.

**Definition.** Suppose that $\mathcal{F}$ and $\mathcal{G}$ are families of finite subsets of $\mathbb{N}$.

1. An element $G \in \mathcal{G}$ is maximal (in $\mathcal{G}$) if it is not properly contained in any other element in $\mathcal{G}$.
2. The family $\mathcal{F} \ominus \mathcal{G}$ is the collection of all sets $F$ so that there is a maximal $G \in \mathcal{G}$, $G < F$, with $G \cup F \in \mathcal{F}$.

**Definition.** A sequence of regular families $(\mathcal{F}_n)$ is tame if

1. for each $n$, either $\mathcal{F}_n = A_j$ for some $j$ or $\mathcal{F}_n[A_3] \subseteq (\mathcal{F}_n)^2$;
2. there exists $n_0 \in \mathbb{N}$ so that $(\mathcal{F}_n \ominus \mathcal{F}_{n_0})[A_2] \subseteq \mathcal{F}_n$ whenever $n > n_0$.

**Proposition 4.** Assume that $(\mathcal{F}_n)$ is a tame sequence. Let $Y$ be a block subspace of $X$. Suppose that there exists a constant $C < \infty$ such that for all $n \in \mathbb{N}$, there is a normalized vector $x$ in the $n$-tail of $Y$ such that $\sum \|E_i x\| \leq C$ whenever $(E_i)$ is $F_k$-admissible for some $k \leq n$. Then there exists a normalized block sequence $(z_n)$ in $Y$ that is equivalent to a subsequence of $(e_k)$. Moreover, $Z = [(z_n)]$ is a complemented subspace of $X$.

**Remark.** In [18, Propositions 5.7 and 5.8], similar results were shown for mixed Tsirelson spaces of the form $T[(\theta_n, S_n)_{n=1}^\infty]$ containing certain semi-normalized special convex combinations. Proposition 4 generalizes these results. In [4, Theorems 2.1 and 2.4], it was proved that in the Schlumprecht space $S$, every block subspace contains a block sequence $(z_n)$ that is equivalent to a subsequence of the unit vector basis with $Z = [(z_n)]$ complemented in $S$. This case does not follow from Proposition 4 since the sequence $(A_n)$ is not tame. However, the following result may be proved by similar methods.

**Proposition 5.** Let $(\theta_n)_{n=1}^\infty$ be a sequence in $(0, 1)$ decreasing to $0$ so that $\lim \theta_{2n}/\theta_n = 1$. Suppose that $Y$ is a block subspace of $X = T[(\theta_n, A_n)_{n=1}^\infty]$ such that there exists a constant $C < \infty$ so that for all $n \in \mathbb{N}$, there exists a
normalized vector $x$ in the $n$-tail of $Y$ with $\sum \|E_ix\| \leq C$ whenever $(E_i)$ is $A_n$-admissible. Then there exists a normalized block sequence $(z_n)$ in $Y$ that is equivalent to a subsequence of the unit vector basis $(e_k)$ of $X$. Moreover, $Z = [(z_n)]$ is a complemented subspace of $X$.

Proof of Proposition 4. Let $n_0$ be the integer occurring in the definition of tameness for the sequence $(\mathcal{F}_n)$. Inductively, choose a normalized block sequence $(z_n)$ in $Y$ and a strictly increasing sequence $(m_n)_{n=0}^\infty$ in $\mathbb{N}$ so that $m_0 > n_0$, $\theta_m \|z_n\|_{\ell^1} \leq 2^{-n}$ and $\sum \|E_i z_n\| \leq C$ whenever $(E_i)$ is $\bigcup_{r=1}^{m_n-1} \mathcal{F}_r$-admissible, $n \in \mathbb{N}$. Consider $z = \sum a_n z_n$ for some $(a_n) \in c_{00}$ and let $y = \sum a_n e_{k_n}$, where $k_n = \max \text{supp } z_n$. Let $T$ be an admissible tree that norms $z$. Without loss of generality, we may assume that all nodes in $T$ are integer intervals and that all leaves in $T$ are singletons. Say that a node is short if it intersects $\text{supp } z_n$ for exactly one $n$. On the other hand, call a node long if it intersects $\text{supp } z_n$ for more than one $n$. The tree $T$ is endowed with the natural partial order of reverse inclusion. Let $\mathcal{E}$ be the collection of all minimal short nodes in $T$. Then $\|z\| = \sum_{E \in \mathcal{E}} t(E) \|Ez\|$. For each $n$, let $\mathcal{E}_n$ be the collection of all nodes in $\mathcal{E}$ that intersects only $\text{supp } z_n$. In particular, $\mathcal{E} = \bigcup \mathcal{E}_n$. Further subdivide each set $\mathcal{E}_n$ into two subsets $\mathcal{E}_n'$ and $\mathcal{E}_n''$ depending on whether $t(E) \leq \theta_m$ or not. We have

$$\sum_n \sum_{E \in \mathcal{E}_n'} t(E) \|Ez\| \leq \sum_n \theta_m \|a_n\| \|z_n\|_{\ell^1} \leq \sum_n \frac{|a_n|}{2^n} \leq \|y\|.$$  

For each $n$, let $D_n$ be the set of all minimal elements in the set of all nodes in $T$ that are immediate predecessors of some node in $\mathcal{E}_n''$. Since $D_n$ consists of pairwise disjoint long nodes that intersect $\text{supp } z_n$, $|D_n| \leq 2$ for all $n$. For each $D \in D_n$, let $\mathcal{E}_n'(D) = \{E \in \mathcal{E}_n'' : E \subseteq D\}$ and let $\mathcal{E}_n''(D)$ be the subset of $\mathcal{E}_n''$ consisting of all $E \in \mathcal{E}_n''$ that are immediate successors of $D$. Fix $E_{n,D} \in \mathcal{E}_n''(D)$ and $j_{n,D} \in E_{n,D} \cap \text{supp } z_n$ arbitrarily and set

$$w = \sum_n \sum_{D \in D_n} a_n e_{j_{n,D}}.$$  

Since $|D_n| \leq 2$ for all $n$, $\|w\| \leq 2\|y\|$. Any immediate successor of $D$ that contains some $E \in \mathcal{E}_n''(D) \setminus \mathcal{E}_n''(D)$ must be a long node. Hence there are at most two immediate successors of $D$, say $G_1$ and $G_2$, that all nodes in $\mathcal{E}_n''(D) \setminus \mathcal{E}_n''(D)$ are descended from. Note that $t(G_1) = t(G_2) = t(E_{n,D})$ since they are all immediate successors of the same node. Thus

$$\sum_{E \in \mathcal{E}_n''(D) \setminus \mathcal{E}_n''(D)} t(E) \|Ez_n\| \leq 2 \sum_{i=1}^2 t(G_i) \|G_i z_n\| \leq 2t(E_{n,D}).$$
Hence
\[ \sum_{n} \sum_{D \in \mathcal{D}_n} \sum_{E \in \mathcal{E}_n(D) \setminus \mathcal{E}'_n(D)} t(E) \| Ez \| \]
\[ = \sum_{n} \sum_{D \in \mathcal{D}_n} \sum_{E \in \mathcal{E}_n(D) \setminus \mathcal{E}'_n(D)} t(E) |a_n| \| Ez_n \| \]
\[ \leq \sum_{n} \sum_{D \in \mathcal{D}_n} 2 |a_n| t(E_n,D) = 2 \sum_{n} \sum_{D \in \mathcal{D}_n} t(E_n,D) \| E_n,Dw \|_{c_0} \]
\[ = 2T'w \leq 2\| w \| \leq 4\| y \|, \]

where $T'$ is the subtree of $T$ consisting of all nodes $E_{n,D}, D \in \mathcal{D}_n$, and their ancestors. Now let $\mathcal{D}'_n$ consists of those $D$ in $\mathcal{D}_n$ such that $\mathcal{E}'_n(D)$ is $\bigcup_{r=1}^{m_{n-1}} \mathcal{F}_r$-admissible. Then
\[ \sum_{n} \sum_{D \in \mathcal{D}'_n} \sum_{E \in \mathcal{E}'_n(D)} t(E) \| Ez \| \]
\[ = \sum_{n} \sum_{D \in \mathcal{D}'_n} \sum_{E \in \mathcal{E}'_n(D)} t(E) |a_n| \| Ez_n \| \leq C \sum_{n} \sum_{D \in \mathcal{D}'_n} t(E_n,D) \| E_n,Dw \|_{c_0} \leq C\| w \| \leq 2C\| y \|. \]

It remains to consider the nodes that belong to $\mathcal{D}_n \setminus \mathcal{D}'_n$ for some $n$. We have
\[ \sum_{n} \sum_{D \in \mathcal{D}_n \setminus \mathcal{D}'_n} \sum_{E \in \mathcal{E}'_n(D)} t(E) \| Ez \| = \sum_{n} \sum_{D \in \mathcal{D}_n \setminus \mathcal{D}'_n} \sum_{E \in \mathcal{E}'_n(D)} t(E) |a_n| \| Ez_n \| \]
\[ \leq \sum_{n} \sum_{D \in \mathcal{D}_n \setminus \mathcal{D}'_n} t(D) |a_n| \| Dz_n \| \leq \sum_{n} \sum_{D \in \mathcal{D}_n \setminus \mathcal{D}'_n} t(D) |a_n|. \]

But by Lemma 7 below,
\[ \sum_{n} \sum_{D \in \mathcal{D}_n \setminus \mathcal{D}'_n} t(D) |a_n| \leq 4\| y \|. \]

Thus
\[ \sum_{n} \sum_{D \in \mathcal{D}_n \setminus \mathcal{D}'_n} \sum_{E \in \mathcal{E}'_n(D)} t(E) \| Ez \| \leq 4\| y \|. \]

Combining inequalities (3) to (6), we see that
\[ \| z \| = \sum_{E \in \mathcal{E}} t(E) \| Ez \| \leq (9 + 2C)\| y \|. \]

Hence $(z_n)$ is dominated by $(e_{k_n})$, where $k_n = \max \text{supp} \ z_n$. On the other hand, $(z_n)$ dominates $(e_{k_{n-1}})$ (take $k_0 = 1$). Therefore, using the tameness of $(\mathcal{F}_n)$, we see that $(z_n)$ is equivalent to $(e_{k_n})$ by Lemma 1.
Finally, we show that \( Z = [(z_n)] \) is a complemented subspace of \( X \). For each \( n \in \mathbb{N} \), let \( z'_n \) be a normalized vector in \( X' \) such that \( \text{supp} z'_n \subseteq \text{supp} z_n = E_n \) and \( z'_n(z_n) = 1 \). Define \( P : X \to X \) by \( P(x) = \sum_n z'_n(x)z_n \). Let \( l_n = \min \text{supp} z_n \). For any \( x \in X \),

\[
\|P x\| = \left\| \sum_n z'_n(x)z_n \right\| \leq \left\| \sum_n \|E_n x\|z_n \right\| \text{ as } \|z'_n\| \leq 1,
\]

\[
\leq (9 + 2C) \| \sum_n \|E_n x\|e_{k_n} \| \text{ by (7),}
\]

\[
\leq 3(9 + 2C) \left\| \sum_n \|E_n x\|e_{l_n} \right\|
\]

by Lemma 1 and the spreading property of \( (F_n)_{n=1}^{\infty} \). Also, note that since \( l_n \leq \text{supp} E_n x \),

\[
\left\| \sum_n \|E_n x\|e_{l_n} \right\| \leq \left\| \sum_n \|E_n x\| \frac{E_n x}{\|E_n x\|} \right\| = \left\| \sum_n E_n x\right\| \leq \|x\|.
\]

Hence \( P \) is bounded. Clearly \( P \) is a projection onto \( Z \). \( \qed \)

**Lemma 6.** Suppose that \( n_1 < n_2 \) and \( D \in \mathcal{D}_{n_2} \setminus \mathcal{D}'_{n_2} \). Then no descendant of \( D \) belongs to \( \mathcal{E}'_{n_1} \). In particular, \( D \notin \mathcal{D}_{n_1} \).

**Proof.** If \( E \) is a descendant of \( D \in \mathcal{D}_{n_2} \setminus \mathcal{D}'_{n_2} \), then \( t(E) \leq t(F) \) for any immediate successor \( F \) of \( D \). In particular, \( t(E) \leq t(F) \) for all \( F \in \mathcal{E}'_{n_2}(D) \). By definition of \( \mathcal{D}'_{n_2} \), \( \mathcal{E}'_{n_2}(D) \) is not \( \mathcal{F}_r \)-admissible for all \( r \leq m_{n_1} \). Hence \( t(F) < \theta_{m_{n_1}} \) for all \( F \in \mathcal{E}'_{n_2}(D) \). Therefore, \( t(E) < \theta_{m_{n_1}} \) if \( E \) is a descendant of \( D \in \mathcal{D}_{n_2} \setminus \mathcal{D}'_{n_2} \). This shows that \( E \notin \mathcal{E}_{n_1}' \) by definition of \( \mathcal{E}_{n_1}' \). \( \qed \)

Let \( T' \) be the subtree of \( T \) consisting of all nodes in \( \tilde{D} = \bigcup_n (\mathcal{D}_n \setminus \mathcal{D}'_n) \) and their ancestors. By Lemma 6, for each \( D \in \tilde{D} \), there is a unique \( n = n_D \) such that \( D \in \mathcal{D}_n \setminus \mathcal{D}'_n \). If \( G \) is a node in \( T' \), let \( \tilde{D}(G) \) consist of all \( D \in \tilde{D} \) such that \( D \subseteq G \). Recall the vector \( w \) defined in the proof of Proposition 4 above. It was observed that \( \|w\| \leq 2\|y\| \).

**Lemma 7.** For any \( G \in T' \), there exist subsets \( G_1 \) and \( G_2 \) of \( G \), \( G_1 < G_2 \), and admissible trees \( T_1 \) and \( T_2 \) with roots \( G_1 \) and \( G_2 \) respectively so that

\[
\sum_{D \in \tilde{D}(G)} t(D)|a_{n_D}| \leq t(G)(T_1w + T_2w).
\]

In particular,

\[
\sum_n \sum_{D \in \mathcal{D}_n \setminus \mathcal{D}'_n} t(D)|a_n| \leq 4\|y\|.
\]

**Proof.** The second inequality follows from the first by taking \( G \) to be the root of \( T' \) (which is also the root of \( T \)). To prove the first inequality, we begin
at the terminal nodes of $T'$ and work our way up the tree. Let $G$ be a terminal node of $T'$. Then $G \in \tilde{D}$. In this case, take $G_1 = [1, \max \supp z_{nG}] \cap G$ and $G_2 = G \setminus G_1$. Clearly, $G_1$ and $G_2$ are subsets of $G$ such that $G_1 \prec G_2$. Set $T_1$ and $T_2$ to be the trivial trees $T_i = \{G_i\}$, $i = 1, 2$. Now

$$\sum_{D \in \tilde{D}(G)} t(D)|a_{nD}| = t(G)|a_{nG}| \leq t(G)||G_1w||_{c_0} = t(G)T_1w.$$ 

Thus the lemma holds in this case.

Next, take a node $G \in T'$ and assume that the lemma has been proved for all descendants of $G$ in $T'$. List the immediate successors of $G$ in $T'$ from left to right as $\{H_1, \ldots, H_r\}$. By the assumption, for each $j$, $1 \leq j \leq r$, there are subsets $H^i_j$ of $H_j$ and admissible trees $T^i_j$, $i = 1, 2$, such that $H^1_j \prec H^2_j$, the root of $T^1_j$ is $H^1_j$ and

$$\sum_{D \in \tilde{D}(H_j)} t(D)|a_{nD}| \leq t(H_j)(T^1_jw + T^2_jw).$$

We divide the rest of the proof into two cases.

**Case 1:** $G \in \tilde{D}$. The sets in the collection $\tilde{E}''_{nG}(G) \cup \{H_j\}_{j=1}^r$ are all immediate successors of $G$ in the tree $T$. We claim that $E < H_1$ for any $E \in \tilde{E}''_{nG}(G)$. Indeed, either $H_1$ or a descendant of $H_1$ belongs to $\tilde{D}$. Denote this node by $I$. Thus $G \in D_{nG} \setminus D'_{nG}$ has a descendant in $E''_{nI}$. By Lemma 6, $n_I \geq n_G$. Since $I \not\subset G$, $n_I \neq n_G$ by the minimality condition in the definition of $D_n$. Hence $n_I > n_G$. Now any $E$ in $\tilde{E}''_{nG}(G)$ intersects only $\supp z_{nG}$ while $H_1$ must intersect $\supp z_{nI}$. Therefore, $E < H_1$, as claimed. To continue with the proof, set $G_1 = G \cap [1, k]$, where $k = \max \tilde{E}''_{nG}(G)$, and $G_2 = G \setminus G_1$. Then take $T_1$ to be the trivial tree $\{G_1\}$ and $T_2$ to be the tree $G_2 \cup \bigcup_{i,j} T^i_j$. The admissibility of $T_1$ is clear. To verify the admissibility of $T_2$, it suffices to show the admissibility of the decomposition of $G_2$ into $\{H^i_j\}_{i,j}$. Since $\tilde{E}''_{nG}(G) \cup \{H_j\}_{j=1}^r$ are all immediate successors of $G$ in the tree $T$, the collection is $F_n$-admissible for some $n$. However, $\tilde{E}''_{nG}(G)$ is not $F_r$-admissible for any $r \leq m_{nG}$. Thus $n > m_{nG} - 1 > n_0$ and $(\min H_j) \in F_n \cap \tilde{F}_{n0}$. By the tameness of $(F_n)$, $(\min H^i_j) \in (F_n \cap F_{n0})[A_2] \subseteq F_n$. Hence $(H^i_j)$ is $F_n$-admissible, as required. Now

$$T_1w = \|G_1w\|_{c_0} \geq |a_{nG}|$$

and

$$T_2w = \theta_n \sum_{i,j} T^i_jw = \sum_{i,j} t(H^i_j) T^i_jw \geq \sum_{j} \sum_{D \in \tilde{D}(H_j)} t(D) \frac{t(G)}{|a_{nD}|}.$$ 

Therefore,
\[
\sum_{D \in \mathcal{D}(G)} t(D)|a_{n_D}| = t(G)|a_{n_G}| + \sum_j \sum_{D \in \mathcal{D}(H_j)} t(D)|a_{n_D}|
\leq t(G)(T_1w + T_2w).
\]

**Case 2:** \( G \notin \mathcal{D} \). Suppose that in the tree \( T \), the immediate successors of \( G \) form an \( \mathcal{F}_n \)-admissible collection. In particular, \( \{H_j\}_{j=1}^r \) is \( \mathcal{F}_n \)-admissible. We claim that \( (\min H_j) \in (\mathcal{F}_n)^2 \). This is clear if \( \mathcal{F}_n = \mathcal{A}_j \) for some \( j \). Otherwise, \( (\min H_j) \in \mathcal{F}_n[\mathcal{A}_2] \subseteq (\mathcal{F}_n)^2 \) by the tameness of \( (\mathcal{F}_n) \). Choose index sets \( I_1 \) and \( I_2 \) such that \( I_1 \cup I_2 = \{(i,j) : 1 \leq i \leq 2, 1 \leq j \leq r\} \), \( \{H_j^i : (i,j) \in I_k\} \) is \( \mathcal{F}_n \)-admissible, \( k = 1,2 \), and that \( H_j^i < H_j^{i'} \) whenever \( (i,j) \in I_1 \) and \( (i',j') \in I_2 \). Set \( G_1 = G \cap [1,p] \), where \( p = \max \bigcup \{H_j^i : (i,j) \in I_1\} \) and \( G_2 = G \setminus G_1 \). Define \( T_k \) to be the tree \( \{G_k\} \cup \bigcup_{(i,j) \in I_k} T_j^i \), \( k = 1,2 \). The admissibility of \( T_1 \) and \( T_2 \) follows by construction. Finally,

\[
t(G) \sum_k T_kw = t(G)\theta_n \sum_{(i,j) \in I_k} T_j^i w = \sum_{i,j} t(H_j^i)T_j^i w \\
\geq \sum_j \sum_{D \in \mathcal{D}(H_j)} t(D)|a_{n_D}| = \sum_{D \in \mathcal{D}(G)} t(D)|a_{n_D}|. \quad \blacksquare
\]

Given a nonzero ordinal \( \alpha \) with Cantor normal form \( \omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_n} \cdot m_n \), let \( \ell(\alpha) = \beta_1 \). For any \( m \in \mathbb{N} \) and \( \varepsilon > 0 \), define

\[
\gamma(\varepsilon, m) = \max\{\ell(\alpha_{n_s} \cdots \alpha_{n_1}) : \varepsilon \theta_{n_1} \cdots \theta_{n_s} > \theta_m\} \quad (\max \emptyset = 0).
\]

The sequence \( (\theta_n, \mathcal{F}_n)_{n=1}^{\infty} \) is said to satisfy (\dag) if there exists \( \varepsilon > 0 \) such that for all \( \beta < \omega^\xi \), there exists \( m \in \mathbb{N} \) such that \( \gamma(\varepsilon, m) + 2 + \beta < \ell(\alpha_m) \).

**Theorem 8** ([17, Theorems 4 and 12]). Assume that (\dag) holds. Then for any \( M \in [\mathbb{N}] \), \( ((e_k)_{k \in M}) \) contains an \( \ell^1 \)-\( \mathcal{S}_{\omega^\xi} \)-spreading model. On the other hand, if (\dag) fails, then for all \( M \in [\mathbb{N}] \), there exists \( N \in [M] \) such that \( I_b((e_k)_{k \in N}) = \omega^{\omega^\xi} \).

Recall that a Banach space \( (B, \| \cdot \|) \) is said to be \( \lambda \)-distortable if there is an equivalent norm \( \| \cdot \|_2 \) on \( B \) so that for every infinite-dimensional subspace \( Y \) of \( X \), there are \( \| \cdot \| \)-normalized vectors \( y \) and \( z \) in \( Y \) so that \( |y|/|z| > \lambda \). A space is arbitrarily distortable if it is \( \lambda \)-distortable for all \( \lambda > 1 \).

**Theorem 9.** Assume that \( (\mathcal{F}_n) \) is a tame sequence. The following statements are equivalent for any block subspace \( Y \) of \( X \).

1. Property (\dag) holds and every block subspace \( Z \) of \( Y \) contains a block sequence equivalent to a subsequence of \( (e_k) \).
2. Every block subspace \( Z \) of \( Y \) contains an \( \ell^1 \)-\( \mathcal{S}_{\omega^\xi} \)-spreading model.
3. The Bourgain \( \ell^1 \)-index satisfies \( I_b(Z) = I(Z) > \omega^{\omega^\xi} \) for any block subspace \( Z \) of \( Y \).
Moreover, if one (and hence all) of the equivalent conditions holds for a block subspace \( Y \) of \( X \), then \( Y \) is arbitrarily distortable.

**Proof.** The implication (1) \( \Rightarrow \) (2) follows from the first part of Theorem 8. Let \( Z \) be a block subspace of \( Y \). If (2) holds, then \( I(Z, K) \geq \omega^\omega \) for some \( K < \infty \). By [14, Lemma 5.7], \( I_b(Z) = I(Z) > \omega^\omega \). Assume that condition (3) holds. By Lemma 3 and Proposition 4, \( Z \) contains a normalized block sequence equivalent to a subsequence of \( (e_k) \). Say \( (z_n) \) is a normalized block sequence in \( Z \) equivalent to \( (e_k)_{k \in M} \) for some \( M \in [\mathbb{N}] \). If (†) fails, by the second part of Theorem 8, there exists \( N \in [M] \) such that \( I_b(\{(e_k)_{k \in N}\}) = \omega^\omega \).

Hence \( I_b(\{(z_{n_j})\}) = \omega^\omega \) for some subsequence \( (z_{n_j}) \) of \( (z_n) \). This contradicts (3) since \( (z_{n_j}) \) is a block subspace of \( Y \). This proves condition (1).

Assume that the conditions hold for a block subspace \( Y \) of \( X \). For each \( n \), consider the equivalent norm \( \| \cdot \|_n \) on \( X \) defined by

\[
\| x \|_n = \sup \left\{ \sum \| E_ix \| : (E_i) \text{ is } F_n\text{-admissible} \right\}.
\]

Let \( Z \) be a block subspace of \( Y \). By condition (3) and Lemma 3, there exists \( C_1 < \infty \) such that for all \( n \), there exists \( z \in Z \) such that \( \| z \| = 1 \) and \( \| z \|_n \leq C_1 \). On the other hand, by (1), \( Z \) contains a normalized block sequence \( (z_j)_{j=1}^\infty \) that is \( C_2 \)-equivalent to a subsequence \( (e_{k_j})_{j=1}^\infty \) of \( (e_k) \). By taking a subsequence if necessary, we may assume that \( e_{k_j} \leq z_{j+1} \) for all \( j \in \mathbb{N} \). Let \( \varepsilon \) be the constant given by property (†). It follows from (†) that there are infinitely many \( m \) such that \( \gamma(\varepsilon, m) + 2 < \ell(\alpha_m) \). Fix such an \( m \) and let \( \gamma = \gamma(\varepsilon, m) \). By [11, Theorem 1.1], there exists \( N \in [(k_j)] \) such that \( S_{\gamma+2} \cap [N]^{< \infty} \subseteq F_m \). By [16, Lemma 19], there exists \( x \in c_{00} \) such that \( \| x \| \leq 1 + \varepsilon^{-1} \), \( \| x \|_I = \theta_m^{-1} \) and \( x \in S_{\gamma+2} \cap [N]^{< \infty} \). Say \( x = \sum_{j \in I} a_j e_{k_j} \) for some \( I \) such that \( (k_j)_{j \in I} \in [N]^{< \infty} \). Consider the corresponding element \( y = \sum_{j \in I} a_j z_j / \sum_{j \in I} a_j z_j \). Since \( (z_j)_{j \in I} \) is \( C_2 \)-equivalent to \( (e_{k_j})_{j \in I} \),

\[
\left\| \sum_{j \in I} a_j z_j \right\| \leq C_2 \| x \| \leq C_2(1 + \varepsilon^{-1}).
\]

For each \( j \), let \( E_j = \text{supp } z_j \). If \( j_0 = \min I \), then \( (\min E_j)_{j \in I \backslash \{j_0\}} \) is a spreading of a subset of \( (k_j)_{j \in I} = \text{supp } x \). Hence \( (E_j)_{j \in I \backslash \{j_0\}} \) is \( F_m \)-admissible since \( \text{supp } x \in F_m \). Therefore,

\[
\left\| \sum_{j \in I} a_j z_j \right\|_m \geq \sum_{i \in I \backslash \{j_0\}} \left\| E_i \sum_{j \in I} a_j z_j \right\|_I = \sum_{i \in I \backslash \{j_0\}} |a_i| = \| x \|_I - |a_{j_0}|
\]

\[
\geq \| x \|_I - \| x \| \geq \theta_m^{-1} - 1 - \varepsilon^{-1}.
\]

Hence \( \| y \|_n \geq C_2^{-1}(1 + \varepsilon^{-1})^{-1}(\theta_m^{-1} - 1 - \varepsilon^{-1}) \). The existence of \( z \) and \( y \) shows that \( Y \) is \( C_1^{-1}C_2^{-1}(1 + \varepsilon^{-1})^{-1}(\theta_m^{-1} - 1 - \varepsilon^{-1}) \)-distortable. Since this holds for infinitely many \( m \), \( Y \) is arbitrarily distortable. \( \blacksquare \)
Corollary 10. Assume that \((F_n)\) is a tame sequence. If \(\xi\) is a limit ordinal, the following statements hold.

1. Every block subspace of \(X\) contains an \(\ell^1\)-\(S_{\omega\xi}\)-spreading model.
2. Every block subspace of \(X\) contains a block sequence equivalent to a subsequence of \((e_k)\).
3. \(X\) is arbitrarily distortable.

Proof. If \((z_n)\) is a normalized block sequence in \(X\), and \(F\) is a set such that \(\{\min \text{supp } z_n\}_{n \in F} \in F_m\), then \(\| \sum a_n z_n \| \geq \theta_m \sum_{F} |a_n|\). In particular, \(I_b(Y, \theta_m^{-1}) \geq \alpha_m\) for all block subspaces \(Y\) of \(X\) and all \(m\). By the proof of Theorem 1.1 in [14], if \(I_b(Y, K) \geq \alpha^2\), then \(I_b(Y, \sqrt{K}) \geq \alpha\). Now for any \(\beta < \omega^\xi\), there exists \(m\) such that \(\omega^\beta \omega < \alpha_m\). Thus \((\omega^\beta)^{2k} < \alpha_m\) for all \(k\). It follows that \(I_b(Y, \theta_m^{-1/2^k}) \geq \omega^\beta\). Hence \(I_b(Y, 1 + \epsilon) \geq \omega^\beta\) for any \(\epsilon > 0\) and any \(\beta < \omega^\xi\). Therefore, \(I_b(Y, 1 + \epsilon) \geq \omega^{\omega^\xi}\) for any \(\epsilon > 0\). By [14, Lemma 5.7], \(I_b(Y) > \omega^{\omega^\xi}\). The conclusions of the corollary now follow from Theorem 9.

 Proposition 11. The sequence \((S_{\beta_n})\) is tame for any sequence of nonzero countable ordinals \((\beta_n)\).

Proof. Let \(\alpha\) be a nonzero countable ordinal. The fact that \(S_\alpha[A_3] \subseteq (S_\alpha)^2\) was shown in the Remark following Proposition 9 in [16]. We show that \((S_\alpha \cap S_1)[A_2] \subseteq S_\alpha\) by induction on \(\alpha\). If \(\alpha = 1\), this is clear. Assume that the inclusion holds for some \(\alpha\). Suppose \(E \in (S_{\alpha+1} \cap S_1)[A_2]\). Then \(E = \bigcup_{i=1}^k E_i, E_1 < \cdots < E_k, E_i \in A_2,\) and \(F = \{ \min E_i \}_{i=1}^k \in S_{\alpha+1} \cap S_1\). There is a maximal \(S_1\) set \(G\) such that \(G < F\) and \(G \cup F \in S_{\alpha+1}\). Let \(\min G = n\). Then \(|G| = n\) and hence \(\min F \geq 2n\). Note that \(F \subseteq G \cup F \in S_{\alpha+1}\). Thus we may write \(F\) as \(\bigcup_{j=1}^r H_j\), where \(H_1 < \cdots < H_r, H_j \in S_\alpha,\) and \(r \leq n\). Since \(S_\alpha[A_2] \subseteq (S_\alpha)^2, \bigcup \{ E_i : \min E_i \in H_j \} \in (S_\alpha)^2\) for all \(j\). Therefore,

\[
E \subseteq \bigcup_{j=1}^r \bigcup \{ E_i : \min E_i \in H_j \} \in (S_\alpha)^{2r}
\]

and \(2r \leq 2n \leq \min F = \min E\). Hence \(E \in S_{\alpha+1}\), as required.

Finally, suppose the inclusion holds for all \(\alpha' < \alpha\), where \(\alpha\) is a countable limit ordinal. Let \((\alpha_n)\) be the sequence of ordinals used to define \(S_\alpha\). If \(E \in (S_\alpha \cap S_1)[A_2]\), then \(E \in (S_{\alpha_n} \cap S_1)[A_2]\) for some \(n \leq \min E\). Thus \(E \in S_{\alpha_n}\) for some \(n \leq \min E\). Hence \(E \in S_\alpha\).

Observe that \(S_1 \subseteq S_\alpha\) for any nonzero countable ordinal \(\alpha\). Therefore, if \(n > 1\),

\[
(S_{\beta_n} \cap S_{\beta_1})[A_2] \subseteq (S_{\beta_n} \cap S_1)[A_2] \subseteq S_{\beta_n}.
\]

Theorem 12. Let \((\theta_n)\) be a nonincreasing null sequence in \((0,1)\) and suppose that \((\beta_n)\) is a sequence of ordinals such that \(\sup \beta_m = \omega^\xi > \beta_n > 0\)
for all $n$, where $0 < \xi < \omega_1$. Let

$$
\gamma(\varepsilon, m) = \max\{\beta_{n_1} + \cdots + \beta_{n_1} : \varepsilon \theta_{n_1} \cdots \theta_{n_1} > \theta_m\} \quad (\max \emptyset = 0).
$$

The following are equivalent for any block subspace $Y$ of $T[(\theta_n, S_{\beta_n})_{n=1}^\infty]$.

1. There exists $\varepsilon > 0$ such that for all $\beta < \omega^\xi$, there exists $m \in \mathbb{N}$ such that $\gamma(\varepsilon, m) + 2 + \beta < \beta_m$ and every block subspace $Z$ of $Y$ contains a block sequence equivalent to a subsequence of $(e_k)$.

2. Every block subspace $Z$ of $Y$ contains an $\ell^1$-$\mathcal{S}_{\omega^\xi}$-spreading model.

3. The Bourgain $\ell^1$-index satisfies $I_b(Z) = I(Z) > \omega^{\omega^\xi}$ for any block subspace $Z$ of $Y$.

If one (and hence all) of these conditions holds for a block subspace $Y$ of $T[(\theta_n, S_{\beta_n})_{n=1}^\infty]$, then $Y$ is arbitrarily distortable. Moreover, all these equivalent conditions hold for the space $T[(\theta_n, S_{\beta_n})_{n=1}^\infty]$ if $\xi$ is a limit ordinal.

When considering the mixed Tsirelson space $T[(\theta_n, S_n)_{n=1}^\infty]$, it is customary to assume without loss of generality that $\theta_{m+n} \geq \theta_m \theta_n$ for all $m, n$. In this case, it was shown in the proof of Corollary 28 in [16] that condition (†) is equivalent to $\lim_m \limsup_n \theta_{m+n}/\theta_n > 0$.

**Corollary 13.** Let $(\theta_n)$ be a nonincreasing null sequence in $(0,1)$ such that $\theta_{m+n} \geq \theta_m \theta_n$ for all $m, n$. The following are equivalent for any block subspace $Y$ of $T[(\theta_n, S_n)_{n=1}^\infty]$.

1. $\lim_m \limsup_n \theta_{m+n}/\theta_n > 0$ and every block subspace $Z$ of $Y$ contains a block sequence equivalent to a subsequence of $(e_k)$.

2. Every block subspace $Z$ of $Y$ contains an $\ell^1$-$\mathcal{S}_{\omega}$-spreading model.

3. The Bourgain $\ell^1$-index satisfies $I_b(Z) = I(Z) > \omega^\omega$ for any block subspace $Z$ of $Y$.

If one (and hence all) of these conditions holds for a block subspace $Y$ of $T[(\theta_n, S_n)_{n=1}^\infty]$, then $Y$ is arbitrarily distortable. Moreover, all these equivalent conditions hold for the space $T[(\theta_n, S_n)_{n=1}^\infty]$ if $\lim \theta_n^{1/n} = 1$.

**Remark.** The fact that every block subspace of $T[(\theta_n, S_n)_{n=1}^\infty]$ contains an $\ell^1$-$\mathcal{S}_{\omega}$-spreading model if $\lim \theta_n^{1/n} = 1$ is due to Argyros, Deliyanni and Manoussakis [7, Proposition 3.1]. Androulakis and Odell [3] showed that if $\lim \theta_n/\theta^n = 0$, where $\theta = \lim \theta_n^{1/n}$, then $T[(\theta_n, S_n)_{n=1}^\infty]$ is arbitrarily distortable.

**Proof of Corollary 13.** It suffices to prove the “moreover” statement. Clearly every normalized block sequence in $T[(\theta_n, S_n)_{n=1}^\infty]$ is an $\ell^1$-$\mathcal{S}_n$-spreading model with constant $\theta_n^{-1}$ for any $n$. Thus for any block subspace $Y$, $I_b(Y, \theta_n^{-1}) \geq \omega^{m2^n}$ for all $m, n$. By the proof of Theorem 1.1 in [14], it follows that $I_b(Y, \theta_n^{-1/2^n}) \geq \omega^m$. Using the hypothesis, we see that $I_b(Y, 1+\varepsilon) \geq$
\( \omega^m \) for all \( \varepsilon > 0 \) and all \( m \). Hence \( I_b(Y, 1 + \varepsilon) \geq \omega^\omega \). By [14, Lemma 5.7], \( I_b(Y) > \omega^\omega \). This proves condition (3).

For an ordinal \( \beta \) with Cantor normal form \( \beta = \omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_n} \cdot m_n \), call \( m_1 \) the leading coefficient of \( \beta \). The preceding proof shows that if \( (\mathcal{F}_n) \) is an increasing sequence of regular families so that \( (\iota(\mathcal{F}_n)) \) increases nontrivially to \( \omega^\xi \), where \( \xi \) is a countable successor ordinal, and \( \sup_n \theta_n^{1/k_n} = 1 \), where \( k_n \) is the leading coefficient of \( \ell(\iota(\mathcal{F}_n)) \), then every block subspace of \( T[(\theta_n, \mathcal{F}_n)_{n=1}^{\infty}] \) contains an \( \ell^1 - \mathcal{S}_\omega^\xi \)-spreading model.

**Lemma 14.** Let \( \mathcal{F} \) be a regular family and let \( M \in [N] \).

1. If \( 0 < \iota(\mathcal{F}) < \omega \), then there exists \( N \in [M] \) such that \( \mathcal{F} \cap [N]^{<\infty} = \mathcal{A}_j \cap [N]^{<\infty} \), where \( j = \iota(\mathcal{F}) \).
2. If \( \iota(\mathcal{F}) \geq \omega \), then there exists \( N \in [M] \) with \( \mathcal{F}[\mathcal{A}_3] \cap [N]^{<\infty} \subseteq (\mathcal{F})^2 \).
3. If \( \iota(\mathcal{F}) \geq \omega \), then there exist \( N \in [M] \) and \( j \in \mathbb{N} \) with the property that \( (\mathcal{F} \ominus \mathcal{A}_j)[\mathcal{A}_2] \cap [N]^{<\infty} \subseteq \mathcal{F} \).

**Proof.** (1) Since \( \iota(\mathcal{F}) = j \), \( \mathcal{F} \subseteq \mathcal{A}_j \). On the other hand, choose \( n_0 \in M \) such that \( \{n_0\} \in \mathcal{F}^{(j-1)} \). If \( j = 1 \), then set \( N = \{n_0, n_0+1, \ldots\} \cap M \). Clearly \( \mathcal{A}_1 \cap [N]^{<\infty} \subseteq \mathcal{F} \). If \( j > 1 \), consider \( \mathcal{G} = \{G : n_0 < G, \{n_0\} \cup G \in \mathcal{F}\} \). Then \( \iota(\mathcal{G}) = j - 1 \). Using induction, we obtain \( N_1 \in [M] \) with \( n_0 < N_1 \) such that \( \mathcal{G} \cap [N_1]^{<\infty} = \mathcal{A}_{j-1} \cap [N_1]^{<\infty} \). Let \( N = \{n_0\} \cup N_1 \). If \( F \in \mathcal{A}_j \cap [N]^{<\infty} \), then \( F \) is a spreading of \( \{n_0\} \cup G \) for some \( G \in \mathcal{A}_{j-1} \cap [N_1]^{<\infty} \). Hence \( F \in \mathcal{F} \).

(2) This follows from [11, Theorem 1.1] since \( \iota(\mathcal{F}[\mathcal{A}_3]) \leq \iota(\mathcal{A}_3) \cdot \iota(\mathcal{F}) < \iota(\mathcal{F}) \cdot 2 = \iota((\mathcal{F})^2) \).

(3) Write \( \iota(\mathcal{F}) = \alpha + (j-1) \) for some limit ordinal \( \alpha \) and some \( j \in \mathbb{N} \). It is readily verified that \( \mathcal{F} \ominus \mathcal{A}_j \) is a regular family and that \( (\mathcal{F} \ominus \mathcal{A}_j)^{\beta} \subseteq (\mathcal{F})^{\beta} \ominus \mathcal{A}_j \) for any \( \beta \). If \( \iota(\mathcal{F} \ominus \mathcal{A}_j) \geq \alpha \), then \( 0 \in (\mathcal{F}^{(\alpha)}) \ominus \mathcal{A}_j \). Hence there exists \( A \) with \( |A| = j \) such that \( A \in (\mathcal{F}^{(\alpha)}) \). But then \( \iota(\mathcal{F}) \geq \alpha + j \), a contradiction. Thus \( \iota(\mathcal{F} \ominus \mathcal{A}_j) < \alpha \). It follows that \( \iota((\mathcal{F} \ominus \mathcal{A}_j)[\mathcal{A}_2]) < 2 \cdot \alpha = \alpha \leq \iota(\mathcal{F}) \). By [11, Theorem 1.1], there exists \( N \in [M] \) such that \( (\mathcal{F} \ominus \mathcal{A}_j)[\mathcal{A}_2] \cap [N]^{<\infty} \subseteq \mathcal{F} \).}

Given a regular family \( \mathcal{F} \) and \( M = (p_k) \in [N] \), define the family \( M \mathcal{F} \) by \( M \mathcal{F} = \{F : (p_k)_{k \in F} \in \mathcal{F} \} \). It is clear that \( M \mathcal{F} \) is a regular family. Furthermore, the subspace \( [e_k]_{k \in M} \) of \( T[(\theta_n, M \mathcal{F}_n)_{n=1}^{\infty}] \) is easily seen to coincide with the mixed Tsirelson space \( T[(\theta_n, M \mathcal{F}_n)_{n=1}^{\infty}] \) under a natural identification. The next proposition shows that the tameness of the sequence of regular families is not a restriction if one is allowed to pass to a subsequence of the unit vector basis.

**Proposition 15.** There exists \( M \in [N] \) and a tame sequence of regular families \( (\mathcal{G}_n) \) such that \( T[(\theta_n, M \mathcal{F}_n)_{n=1}^{\infty}] \) is isomorphic to \( T[(\theta_n, \mathcal{G}_n)_{n=1}^{\infty}] \) via the formal identity.
Proof. Recall the assumption that \((\alpha_n)\) increases nontrivially to \(\omega^\xi\), \(\xi > 0\). Let \(m_0\) be the largest number such that \(\alpha_{m_0} \leq \omega\). (Take \(m_0\) to be 0 if \(\alpha_n > \omega\) for all \(n\).) Choose a strictly increasing sequence \((m_k)_{k=1}^\infty\) such that \(m_1 > m_0\) and \(\theta_{m_k+1} \leq \theta_{m_k}/2\) for all \(k \in \mathbb{N}\). By (1) and (2) of Lemma 14, there exists \(M_0 \in [\mathbb{N}]\) such that for each \(n \leq m_0\), either \(\mathcal{F}_n \cap [M_0]<\infty = A_j \cap [M_0]<\infty\) for some \(j\), or \(\mathcal{F}_n[A_3] \cap [M_0]<\infty \subseteq (\mathcal{F}_n)^2\). It is possible to choose a decreasing sequence \((M_k)_{k=1}^\infty\) of infinite subsets of \(M_0\) and a sequence \((r_k)_{k=1}^\infty\) in \(\mathbb{N}\) so that whenever \(m_{k-1} < n \leq m_k\), \(k \in \mathbb{N}\),

1. \(S_1 \cap [M_k]<\infty \subseteq \mathcal{F}_n\), by [11, Theorem 1.1] since \(\iota(S_1) = \omega < \iota(\mathcal{F}_n)\),
2. \(\mathcal{F}_n[A_3] \cap [M_k]<\infty \subseteq (\mathcal{F}_n)^2\), by (2) of Lemma 14,
3. \((\mathcal{F}_n \cap A_{r_k})[A_2] \cap [M_k]<\infty \subseteq \mathcal{F}_n\), by (3) of Lemma 14.

Choose a strictly increasing sequence \((p_k)_{k=1}^\infty\) so that \(r_k \leq p_k \in M_k\) for all \(k \in \mathbb{N}\). Define \(M = (p_k)\) and set \(\mathcal{G}_n = M \mathcal{F}_n\) if \(n \leq m_0\) and \(\mathcal{G}_n = \{G \in M \mathcal{F}_n : G \geq k\}\) if \(m_{k-1} < n \leq m_k\), \(k \in \mathbb{N}\). By [16, Proposition 1], \(T[(\theta_n, M \mathcal{F}_n)^{\infty}_{n=1}]\) is isomorphic to \(T[(\theta_n, \mathcal{G}_n)^{\infty}_{n=1}]\) via the formal identity. It remains to show that the sequence \((\mathcal{G}_n)\) is tame.

First suppose that \(n \leq m_0\). If \(\mathcal{F}_n \cap [M_0]<\infty = A_j \cap [M_0]<\infty\) for some \(j\), then clearly \(\mathcal{G}_n = M \mathcal{F}_n = A_j\). Otherwise, \(\mathcal{F}_n[A_3] \cap [M]<\infty \subseteq \mathcal{F}_n[A_3] \cap [M_0]<\infty \subseteq (\mathcal{F}_n)^2\). If \(G \in \mathcal{G}_n[A_3] = M \mathcal{F}_n[A_3]\), then \((p_k)_{k \in G} \in \mathcal{F}_n[A_3] \cap [M]<\infty \subseteq (\mathcal{F}_n)^2\). Hence \(G \in (M \mathcal{F}_n)^2\).

Now assume that \(n > m_0\). Choose \(k\) such that \(m_{k-1} < n \leq m_k\). If \(G \in \mathcal{G}_n[A_3]\), then \(G \in M \mathcal{F}_n[A_3]\) and \(G \geq k\). Hence \(p_k \leq (p_i)_{i \in G} \in \mathcal{F}_n[A_3]\). Thus \((p_i)_{i \in G} \in \mathcal{F}_n[A_3] \cap [M_k]<\infty \subseteq (\mathcal{F}_n)^2\). Therefore \(G \in (M \mathcal{F}_n)^2\) and \(G \geq k\). It follows that \(G \in (\mathcal{G}_n)^2\).

Finally, we show that \((\mathcal{G}_n \cap \mathcal{G}_m)[A_2] \subseteq \mathcal{G}_n\) whenever \(n > m > m_0\). Choose \(k\) and \(l\) such that and \(m_{k-1} < n \leq m_k\) and \(m_{l-1} < m \leq m_l\). Suppose that \(G \in \mathcal{G}_n \cap \mathcal{G}_m\). There is a maximal \(H \in \mathcal{G}_m\) such that \(H < G\) and \(H \cup G \in \mathcal{G}_n\).

We claim that \(|H| \geq r_k\). Indeed, by definition of \(\mathcal{G}_n\), \(H \geq k\). Thus \(r_k \leq p_k \leq (p_i)_{i \in H}.\) If \(|H| < r_k\), there exists a nonempty set \(I > H\) such that \((p_i)_{i \in H} \cap I \subseteq S_1\). Clearly, \((p_i)_{i \in H} \subseteq [M_k]<\infty \subseteq [M_i]<\infty\) as well. Therefore, \((p_i)_{i \in H} \subseteq \mathcal{F}_m\) by condition (1) above. By definition, \(H \cup I \in M \mathcal{F}_m\). Since \(H \cup I \geq k \geq l\), \(H \cup I \in \mathcal{G}_m\), contrary to the maximality of \(H\). This proves the claim.

It follows from the claim that \((p_i)_{i \in G} \in \mathcal{F}_n \cap A_{r_k}\). Therefore, \((p_i)_{i \in J} \in (\mathcal{F}_n \cap A_{r_k})[A_2]\) for all \(J \in (\mathcal{G}_n \cap \mathcal{G}_m)[A_2]\). Clearly, for such \(J\), \(J \geq k\) and hence \((p_i)_{i \in J}\) is in \([M_k]<\infty\). Therefore,

\[ (p_i)_{i \in J} \in (\mathcal{F}_n \cap A_{r_k})[A_2] \cap [M_k]<\infty \subseteq \mathcal{F}_n \]

by condition (3) above. This shows that \(J \in M \mathcal{F}_n\). As \(J \geq k\), we have \(J \in \mathcal{G}_n\), as desired.
Corollary 16. Suppose that either (a) $\xi$ is a countable limit ordinal or (b) $\xi$ is a countable successor ordinal and $\sup_n \theta_n^{1/k_n} = 1$, where $k_n$ is the leading coefficient of $\ell(\alpha_n)$. Then there exists $M \in [\mathbb{N}]$ such that the subspace $Y = [(e_k)_{k \in M}]$ of $X$ has the following properties.

1. Every block subspace of $Y$ has an $\ell^1$-$S_{\omega^\xi}$-spreading model.
2. Every block subspace of $Y$ contains a block sequence equivalent to a subsequence of $(e_k)_{k \in M}$.
3. $Y$ is arbitrarily distortable.

Schlumprecht proposed a classification of Banach spaces as follows [23]. A Banach space with a normalized basis $(u_k)$ is said to be Class 1 if every normalized block sequence has a subsequence equivalent to a subsequence of $(u_k)$. It is Class 2 if every block subspace contains two block sequences $(y_k)$ and $(z_k)$ so that the map $y_k \mapsto z_k$ extends to a bounded linear strictly singular operator. Recall that an operator is strictly singular if its restriction to any infinite-dimensional subspace is not an isomorphism. Schlumprecht asks whether every infinite-dimensional Banach space contains a subspace with a basis that is either Class 1 or Class 2. He also proved a criterion for a Banach space to be Class 2 [23, Theorem 1.4 and Corollary 1.5]. We conclude with a note showing that his proof applies to mixed Tsirelson spaces satisfying the conditions of Theorem 9. A Banach space is $c_0$-saturated if every closed infinite-dimensional subspace contains an isomorphic copy of $c_0$.

Proposition 17. Let $Y$ be a block subspace of a mixed Tsirelson space $X$ and suppose that $Y$ satisfies all the conditions of Theorem 9. Then $Y$ is a Class 2 space.

Proof. Denote by $(e_k)$ the unit vector basis of $X$. We will show below that there are a regular family $G$ with $\iota(G) \leq \omega^{\omega^\xi}$ and a finite constant $C$ so that
\[
\left\| \sum a_k e_k \right\| \leq C \sup_{G \in G} \sum_{k \in G} |a_k| \quad \text{for all } (a_k) \in c_{00}.
\]

Denote the unit vector basis in $c_{00}$ by $(u_k)$ and let $U$ be the completion of $c_{00}$ with respect to the norm $\left\| \sum a_k u_k \right\| = \sup_{G \in G} \sum_{k \in G} |a_k|$ for all $(a_k) \in c_{00}$. The map that sends $\sum a_k u_k$ to the function on $G$ given by $G \mapsto \sum_{k \in G} a_k$ is an embedding of $U$ into $C(G)$, the space of continuous functions on the countable compact metric space $G$. Hence $U$ is $c_0$-saturated. Let $Z$ be a block subspace of $Y$. By the hypothesis, there is a block sequence $(z_k)$ in $Z$ that is equivalent to a subsequence $(e_{m_k})$ of $(e_k)$. Also, there is a sequence $(y_k)$ in $Z$ that generates an $\ell^1$-$S_{\omega^\xi}$-spreading model. We may replace $(y_k)$ with an appropriate subsequence of $(y_{2k} - y_{2k+1})$ if necessary to assume that $(y_k)$ is equivalent to a block sequence. By definition of the norm in $X$, there is a positive constant $K$ so that $\left\| \sum a_k y_k \right\| \geq K^{-1} \sum_{k \in F} |a_k|$ for all
\( F \in \mathcal{F}_1[S_\omega] \). Since \( \iota(\mathcal{F}_1) > 1 \) by assumption, \( \iota(\mathcal{F}_1[S_\omega]) > \omega^\omega \geq \iota(\mathcal{G}) \). Using [11, Theorem 1.1] and replacing \( M = (m_k) \) with a subsequence if necessary, we may assume that \( \mathcal{G} \cap [M]^{< \omega} \subseteq \mathcal{F}_1[S_\omega] \). Because \((z_k)\) is equivalent to \((e_{m_k})\) and \((y_k)\) is equivalent to a block sequence, it follows that the map \( y_{m_k} \mapsto z_k \) extends to a bounded linear map \( T : [(y_{m_k})] \to [(z_k)] \). Now, for all \((a_k) \in c_0\),

$$
\left\| \sum_{m_k \in G} a_k u_{m_k} \right\| = \sup_{G \in \mathcal{G}} \sum_{m_k \in G} |a_k| \leq \sup_{G \in \mathcal{F}_1[S_\omega]} \sum_{m_k \in G} |a_k| \leq K \left\| \sum_{m_k \in G} a_k y_{m_k} \right\|
$$

Hence \( y_{m_k} \mapsto u_{m_k} \) extends to a bounded linear map \( S : [(y_{m_k})] \to [(u_{m_k})] \). However, \((z_k)\) is equivalent to \((e_{m_k})\) and

$$
\left\| \sum_{m_k \in G} a_k e_{m_k} \right\| \leq C \sup_{G \in \mathcal{G}} \sum_{m_k \in G} |a_k| = C \left\| \sum_{m_k \in G} a_k u_{m_k} \right\|
$$

Thus \( u_{m_k} \mapsto z_k \) extends to a bounded linear map \( R : [(u_{m_k})] \to [(z_k)] \). Therefore, \( T = RS \) is a factorization of \( T \) through the \( c_0 \)-saturated space \([(u_{m_k})]\). Since \([(y_{m_k})]\) does not contain a copy of \( c_0 \), \( T \) is strictly singular.

It remains to show the existence of the family \( \mathcal{G} \). Choose a strictly increasing sequence \((n_i)\) such that \( \pi_i < 2^{-i} \) for all \( i \), where

$$
\pi_i = \max\{ \theta_{m_1} \cdots \theta_{m_r} : m_1 + \cdots + m_r > n_i \}.
$$

For each \( i \), let \( \mathcal{G}_i = \bigcup\{ [F_{m_1}, \ldots, F_{m_1}] : m_1 + \cdots + m_r \leq n_i \} \). Here \([F_{m_1}, \ldots, F_{m_1}]\) is defined inductively as \( F_{m_r}[F_{m_{r-1}}, \ldots, F_{m_1}] \). It follows from [15, Proposition 12] that \( \iota(\mathcal{G}_1) < \omega^\omega \) since \( \iota(\mathcal{F}_n) < \omega^\omega \) for each \( n \). Let \( \mathcal{G} \) consist of all sets \( G \) such that \( G \in \mathcal{G}_i \) for some \( i \leq G \) together with all singletons. Then \( \iota(\mathcal{G}) \leq \omega^\omega \). Indeed, let \( \tilde{\mathcal{G}}_i = \{ G \in \mathcal{G}_i : i \leq G \} \). Then \( \mathcal{G} = \mathcal{S}_0 \cup \bigcup \tilde{\mathcal{G}}_i \).

If \( G \in \mathcal{G}^{(1)} \), then either \( G \in \mathcal{S}_0^{(1)} = \{ \emptyset \} \) or there exists a sequence \((G_n)\) converging pointwise to \( G \) such that \( G_n \neq G \) and \( G_n \in \tilde{\mathcal{G}}_{i_n} \) for some \( i_n \). In particular, \( i_n \leq \min G_n = \min G \) for all sufficiently large \( n \). It follows that \((i_n)\) must be bounded. Therefore \( G \in \mathcal{G}^{(1)}_{i_0} \) for some \( i_0 \). This shows that \( \mathcal{G}^{(1)} \subseteq \bigcup \mathcal{G}^{(1)}_{i_0} \). By induction, \( \mathcal{G}^{(\alpha)} \subseteq \bigcup \mathcal{G}^{(\alpha)}_{i_0} \) for all \( \alpha < \omega_1 \). Hence \( \iota(\mathcal{G}) \leq \omega^\omega \).

For any \( x = \sum a_k e_k \) with \((a_k) \in c_0\), let \( T \) be an admissible tree that norms \( x \). Denote by \( \mathcal{E} \) the set of all leaves of \( T \). Also, if \( t(E) = \theta_{m_1} \cdots \theta_{m_r} \), \( E \in \mathcal{E} \), set \( r(E) = m_1 + \cdots + m_r \). Note that \( \{ E \in \mathcal{E} : r(E) \leq n_i \} \) is \( \mathcal{G}_i \)-admissible. Thus

\[
T x = \sum_{E \in \mathcal{E}} t(E) \|Ex\|_{c_0} = \sum_{i=1}^\infty \sum_{n_{i-1} < r(E) \leq n_i} t(E) \|Ex\|_{c_0} \\
\leq \sum_{i=1}^\infty \pi_{i-1} g_i(x),
\]
\[ g_i(x) = \sup_{G \in \mathcal{G}} \sum_{k \in G} |a_k| \text{. However,} \\
\]
\[ g_i(x) \leq \sum_{k=1}^{i} |a_k| + \sup_{G \in \mathcal{G}} \sum_{k \in G, k > i} |a_k| \leq i \|x\|_c + \sup_{G \in \mathcal{G}} \sum_{k \in G} |a_k| \text{.} \\
\]

Therefore,
\[ \|x\| \leq \sum_{i=1}^{\infty} \frac{\pi_{i-1} g_i(x)}{2^{i-1}} \leq \|x\|_c \sum_{i=1}^{\infty} \frac{i}{2^{i-1}} + \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \sup_{G \in \mathcal{G}} \sum_{k \in G} |a_k| \leq 6 \sup_{G \in \mathcal{G}} \sum_{k \in G} |a_k| \text{.} \]

This completes the proof. 

**References**


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