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ℓ^1-Spreading models in subspaces of mixed Tsirelson spaces

by

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Abstract. We investigate the existence of higher order ℓ^1-spreading models in subspaces of mixed Tsirelson spaces. For instance, we show that the following conditions are equivalent for the mixed Tsirelson space $X = T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$:

(1) Every block subspace of X contains an ℓ^1-$\mathcal{S}_\omega$-spreading model,
(2) The Bourgain ℓ^1-index $I_b(Y) = I(Y) > \omega^\omega$ for any block subspace Y of X,
(3) $\lim_m \limsup_n \theta_{m+n}/\theta_n > 0$ and every block subspace Y of X contains a block sequence equivalent to a subsequence of the unit vector basis of X.

Moreover, if one (and hence all) of these conditions holds, then X is arbitrarily distortable.

1. Introduction. The discovery and construction of nontrivial asymptotic ℓ^1 spaces has led to much progress in the structure theory of Banach spaces. The first such space discovered was Tsirelson's space [24]. Subsequently, Schlumprecht constructed what is now called Schlumprecht's space [22]. This space plays a vital role in the solutions of the unconditional basic sequence problem by Gowers and Maurey [12] and the distortion problem by Odell and Schlumprecht [19]. Argyros and Deliyanni [5] introduced the class of mixed Tsirelson spaces which provides a general framework for Tsirelson's space, Schlumprecht's space and related examples such as Tzafriri's space [25]. Mixed Tsirelson spaces have been studied extensively. In particular, results about their finite-dimensional ℓ^1-structure were obtained in [6, 7, 18]. The present authors computed the Bourgain ℓ^1-indices of mixed Tsirelson spaces in [16], and investigated thoroughly the existence of higher order ℓ^1-spreading models in such spaces [17]. (Results in this direction for certain mixed Tsirelson spaces were first proved in [7].)

In the present paper, we investigate when a mixed Tsirelson space contains higher order ℓ^1-spreading models *hereditarily*. Again, the first result of this kind is found in [7]. We prove some general characterizations and obtain the result in [7] as a corollary. Roughly speaking, our results show that

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the complexity of the hereditary finite-dimensional ℓ^1-structure of a mixed Tsirelson space is the same whether it is measured by the existence of higher order ℓ^1-spreading models or Bourgain's ℓ^1-index. These are also related to what may be called "subsequential minimality" of the mixed Tsirelson space in question and imply that it is arbitrarily distortable.

Denote by $\mathbb{N}$ the set of natural numbers. For any infinite subset M of $\mathbb{N}$, let $[M]$, respectively $[M]^{<\infty}$, be the set of all infinite and finite subsets of M respectively. These are subspaces of the power set of $\mathbb{N}$, which is identified with $2^{\mathbb{N}}$ and endowed with the topology of pointwise convergence. A subset $\mathcal{F}$ of $[\mathbb{N}]^{<\infty}$ is said to be *hereditary* if $G \in \mathcal{F}$ whenever $G \subseteq F$ and $F \in \mathcal{F}$. It is *spreading* if for all strictly increasing sequences $(m_i)_{i=1}^k$ and $(n_i)_{i=1}^k$, $(n_i)_{i=1}^k \in \mathcal{F}$ if $(m_i)_{i=1}^k \in \mathcal{F}$ and $m_i \leq n_i$ for all i. We also call $(n_i)_{i=1}^k$ a spreading of $(m_i)_{i=1}^k$. A *regular family* is a subset of $[\mathbb{N}]^{<\infty}$ that is hereditary, spreading and compact (as a subspace of $2^{\mathbb{N}}$). If I and J are nonempty finite subsets of $\mathbb{N}$, we write $I < J$ to mean $\max I < \min J$. We also allow that $\emptyset < I$ and $I < \emptyset$. For a singleton $\{n\}$, $\{n\} < J$ is abbreviated to $n < J$.

Given a regular family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, a sequence of sets $(E_i)_{i=1}^k$ is said to be $\mathcal{F}$-*admissible* if $(\min E_i)_{i=1}^k \in \mathcal{F}$. If $\mathcal{G}$ is another family of sets, let

$$\mathcal{F}[\mathcal{G}] = \Big\{ \bigcup_{i=1}^k G_i : G_i \in \mathcal{G},\ (G_i)_{i=1}^k \text{ is } \mathcal{F}\text{-admissible} \Big\}$$

and

$$(\mathcal{F}, \mathcal{G}) = \{F \cup G : F < G,\ F \in \mathcal{F},\ G \in \mathcal{G}\}.$$

Inductively, set $(\mathcal{F})^1 = \mathcal{F}$ and $(\mathcal{F})^{n+1} = (\mathcal{F}, (\mathcal{F})^n)$ for all $n \in \mathbb{N}$. It is clear that $\mathcal{F}[\mathcal{G}]$ and $(\mathcal{F}, \mathcal{G})$ are regular if both $\mathcal{F}$ and $\mathcal{G}$ are. A class of regular families that has played a central role is the class of generalized Schreier families [1]. The reason for their usefulness as a measure of the complexity of subsets of $[\mathbb{N}]^{<\infty}$ is by now well explained [11, 13]. Let $\mathcal{S}_0$ consist of all singleton subsets of $\mathbb{N}$ together with the empty set. Then define $\mathcal{S}_1$ to be the collection of all $A \in [\mathbb{N}]^{<\infty}$ such that $|A| \leq \min A$ together with the empty set, where $|A|$ denotes the cardinality of the set A. If $\mathcal{S}_\alpha$ has been defined for some countable ordinal α, set $\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha]$. For a countable limit ordinal α, specify a sequence (α_n) that strictly increases to α. Then define

$$\mathcal{S}_\alpha = \{F : F \in \mathcal{S}_{\alpha_n} \text{ for some } n \leq \min F\} \cup \{\emptyset\}.$$

Given a nonempty compact family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, let $\mathcal{F}^{(0)} = \mathcal{F}$ and $\mathcal{F}^{(1)}$ be the set of all limit points of $\mathcal{F}$. Continue inductively to derive $\mathcal{F}^{(\alpha+1)} = (\mathcal{F}^{(\alpha)})^{(1)}$ for all ordinals α and $\mathcal{F}^{(\alpha)} = \bigcap_{\beta<\alpha} \mathcal{F}^{(\beta)}$ for all limit ordinals α. The *index* $\iota(\mathcal{F})$ is taken to be the smallest α such that $\mathcal{F}^{(\alpha+1)} = \emptyset$. Since $[\mathbb{N}]^{<\infty}$ is countable, $\iota(\mathcal{F}) < \omega_1$ for any compact family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$. It is well known that $\iota(\mathcal{S}_\alpha) = \omega^\alpha$ for all $\alpha < \omega_1$ [1, Proposition 4.10].

Denote by c_{00} the space of all finitely supported real sequences. For a finite subset E of $\mathbb{N}$ and $x \in c_{00}$, let Ex be the coordinatewise product of x with the characteristic function of E. The sup norm and the ℓ^1-norm on c_{00} are denoted by $\|\cdot\|_{c_0}$ and $\|\cdot\|_{\ell^1}$ respectively. Given a sequence $(\mathcal{F}_n)$ of regular families and a nonincreasing null sequence $(\theta_n)_{n=1}^\infty$ in $(0,1)$, define a sequence of norms $\|\cdot\|_m$ on c_{00} as follows. Let $\|x\|_0 = \|x\|_{c_0}$ and

$$\|x\|_{m+1} = \max\Big\{\|x\|_m, \sup_n \theta_n \sup \sum_{i=1}^r \|E_i x\|_m\Big\}, \tag{1}$$

where the last sup is taken over all $\mathcal{F}_n$-admissible sequences $(E_i)_{i=1}^r$. Since these norms are all dominated by the ℓ^1-norm, $\|x\| = \lim_m \|x\|_m$ exists and is a norm on c_{00}. The *mixed Tsirelson space* $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ is the completion of c_{00} with respect to the norm $\|\cdot\|$. From (1) we can deduce that the norm in $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ satisfies the implicit equation

$$\|x\| = \max\Big\{\|x\|_{c_0}, \sup_n \theta_n \sup \sum_{i=1}^r \|E_i x\|\Big\}, \tag{2}$$

with the last sup taken over all $\mathcal{F}_n$-admissible sequences $(E_i)_{i=1}^r$. *For the rest of the paper, we consider a fixed sequence $(\theta_n, \mathcal{F}_n)_{n=1}^\infty$ as above and let* $X = T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$. Set $\alpha_n = \iota(\mathcal{F}_n)$ for all n. Families $\mathcal{F}_n$ with $\iota(\mathcal{F}_n) = 1$ contain singletons and the empty set only and may be removed without effect on the norm $\|\cdot\|$. Also the spaces $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ and $T[(\theta_n, \bigcup_{k=1}^n \mathcal{F}_k)_{n=1}^\infty]$ are identical (since (θ_n) is nonincreasing). *Hence there is no loss of generality in assuming that $\alpha_n > 1$ for all n and that (α_n) is nondecreasing. We will also assume that $\alpha_n < \sup_m \alpha_m = \omega^{\omega^\xi}$, $0 < \xi < \omega_1$.* Otherwise, the relevant result has been obtained in [17, Proposition 2], except for the case when $\xi = 0$. The coordinate unit vectors (e_k) form an unconditional basis of X.

Given a Banach space B with a basis (b_k), the *support* of a vector $x = \sum a_k b_k$ (with respect to (b_k)), denoted $\operatorname{supp} x$, is the set of all k such that $a_k \neq 0$. A *block sequence* in B is a sequence (x_k) so that $\operatorname{supp} x_k < \operatorname{supp} x_{k+1}$ for all k. The closed linear span of a block sequence is called a *block subspace.*

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2. Technical preliminaries. In this section, we present some technical results prior to the main discussion. If (x_k) and (y_k) are sequences of vectors residing in (possibly different) normed spaces, we say that (x_k) *dominates* (y_k) if there is a finite positive constant K so that

$$\Big\|\sum a_k y_k\Big\| \le K \Big\|\sum a_k x_k\Big\|$$

for all $(a_k) \in c_{00}$. Two sequences are *equivalent* if they dominate each other. The first lemma shows that under certain mild assumptions on the families $(\mathcal{F}_n)$, any subsequence of (e_k) is equivalent to its left shift. The proof uses essentially the idea in [10, Lemma 2], dressed up in the present language. The family of all subsets of $\mathbb{N}$ with at most k elements is denoted by $\mathcal{A}_k$.

LEMMA 1. *Assume that for all n, either $\mathcal{F}_n = \mathcal{A}_j$ for some $j \in \mathbb{N}$ or $\mathcal{F}_n[\mathcal{A}_3] \subseteq (\mathcal{F}_n)^2$. Suppose that $(i_k) \in [\mathbb{N}]$. Let $x = \sum a_k e_{i_{k+1}}$ and $y = \sum a_k e_{i_k}$ for some $(a_k) \in c_{00}$. Then for any m, there exist $E_1 < E_2 < E_3$ such that*

$$\|x\|_m \le \sum_{i=1}^{3} \|E_i y\|_m.$$

Consequently, the sequences (e_{i_k}) and $(e_{i_{k+1}})$ are equivalent.

Proof. For any set $E \subseteq \mathbb{N}$, let the *left shift* of E be the set $L_E = \{i_k : i_{k+1} \in E\}$. We will prove by induction that for any $m \ge 0$ and any $E \subseteq \mathbb{N}$, there exist $E_1 < E_2 < E_3$, subsets of L_E, such that $\|Ex\|_m \le \sum_{i=1}^{3} \|E_i y\|_m$. The case $m = 0$ is clear. Assume that the lemma holds for some m. Given $E \subseteq \mathbb{N}$, let $z = Ex$. If $\|z\|_{m+1} = \|z\|_m$, there is nothing to prove. Otherwise, $\|z\|_{m+1} = \theta_n \sum_{i=1}^{r} \|F_i z\|_m$ for some n and some $\mathcal{F}_n$-admissible sequence $(F_i)_{i=1}^{r}$. By the inductive hypothesis, there exist $F_1^i < F_2^i < F_3^i$, subsets of L_{F_i}, such that

$$\|F_i z\|_m \le \sum_{k=1}^{3} \|F_k^i y\|_m, \qquad 1 \le i \le r.$$

We claim that $(F_k^i)_{i=1}^{r}{}_{k=1}^{3}$ is $(\mathcal{A}_1 \cup \mathcal{F}_n, (\mathcal{F}_n)^2)$-admissible. Indeed, if $\mathcal{F}_n = \mathcal{A}_j$ for some j, then

$$(\min F_k^i)_{i=1}^{r}{}_{k=1}^{3} \in \mathcal{A}_j[\mathcal{A}_3] = \mathcal{A}_{3j} = (\mathcal{F}_n)^3 \subseteq (\mathcal{A}_1 \cup \mathcal{F}_n, (\mathcal{F}_n)^2).$$

Otherwise, since $\min F_2^i \ge \min F_i$,

$$\bigcup_{i=1}^{r} \{\min F_2^i, \min F_3^i, \min F_1^{i+1}\} \in \mathcal{F}_n[\mathcal{A}_3] \subseteq (\mathcal{F}_n)^2.$$

Clearly, $\{\min F_1^1\} \in \mathcal{A}_1$. Thus

$$\bigcup_{i=1}^{r} \{\min F_1^i, \min F_2^i, \min F_3^i\} \in (\mathcal{A}_1 \cup \mathcal{F}_n, (\mathcal{F}_n)^2),$$

as claimed.

It follows from the claim that there exist $E_1 < E_2 < E_3$ so that $\bigcup_{p=1}^{3} E_p = \bigcup_{i=1}^{r} \bigcup_{k=1}^{3} F_k^i$, each E_p is a union of finitely many F_k^i and that $\mathcal{E}_p = \{F_k^i : F_k^i \subseteq E_p\}$ is $(\mathcal{A}_1 \cup \mathcal{F}_n)$-admissible if $p = 1$ and $\mathcal{F}_n$-admissible if $p = 2, 3$. Notice that $\theta_n \sum_{F_k^i \in \mathcal{E}_p} \|F_k^i y\|_m \le \|E_p y\|_{m+1}$ since $\mathcal{E}_p$ is either $\mathcal{F}_n$-admissible or $\mathcal{A}_1$-admissible. Hence

$$\|Ex\|_{m+1} = \|z\|_{m+1} = \theta_n \sum_{i=1}^{r} \|F_i z\|_m \le \theta_n \sum_{i=1}^{r}\sum_{k=1}^{3} \|F_k^i y\|_m \le \sum_{p=1}^{3} \|E_p y\|_{m+1}.$$

Upon taking the limit as $m \to \infty$, we see that $(e_{i_{k+1}})$ is dominated by (e_{i_k}). Since the reverse domination is clear, the two sequences are equivalent. ∎

A *tree* in a Banach space B is a subset $\mathcal{T}$ of $\bigcup_{n=1}^{\infty} B^n$ so that $(x_1, \ldots, x_n) \in \mathcal{T}$ whenever $(x_1, \ldots, x_n, x_{n+1}) \in \mathcal{T}$. Elements of the tree are called *nodes*. It is *well-founded* if there is no infinite sequence (x_n) so that $(x_1, \ldots, x_m) \in \mathcal{T}$ for all m. If B has a basis, then a tree $\mathcal{T}$ is said to be a *block tree* (with respect to the basis) if every node is a block sequence. For any well-founded tree $\mathcal{T}$, its *derived tree* is the tree $\mathcal{D}^{(1)}(\mathcal{T})$ consisting of all nodes $(x_1, \ldots, x_n)$ so that $(x_1, \ldots, x_n, x) \in \mathcal{T}$ for some x. Inductively, set $\mathcal{D}^{(\alpha+1)}(\mathcal{T}) = \mathcal{D}^{(1)}(\mathcal{D}^{(\alpha)}(\mathcal{T}))$ for all ordinals α and $\mathcal{D}^{(\alpha)}(\mathcal{T}) = \bigcap_{\beta<\alpha} \mathcal{D}^{(\beta)}(\mathcal{T})$ for all limit ordinals α. The *order* of a tree $\mathcal{T}$ is the smallest ordinal $o(\mathcal{T}) = \alpha$ such that $\mathcal{D}^{(\alpha)}(\mathcal{T}) = \emptyset$.

LEMMA 2. *Let $\mathcal{T}$ be a well-founded block tree in a Banach space B with a basis. Define*

$$\mathcal{H} = \{(\max \operatorname{supp} x_j)_{j=1}^{r} : (x_j)_{j=1}^{r} \in \mathcal{T}\},$$
$$\mathcal{G} = \{G : G \text{ is a spreading of a subset of some } H \in \mathcal{H}\}.$$

Then $\mathcal{G}$ is hereditary and spreading. If $\mathcal{G}$ is compact, then $\iota(\mathcal{G}) \ge o(\mathcal{T})$.

Proof. It is clear that $\mathcal{G}$ is hereditary and spreading. Assume that $\mathcal{G}$ is compact. We show by induction on ξ that for all countable ordinals ξ, $\iota(\mathcal{G}) \ge \xi$ if $o(\mathcal{T}) \ge \xi$. There is nothing to prove if $\xi = 0$. Suppose the proposition holds for some $\xi < \omega_1$. Let $\mathcal{T}$ be a well-founded block tree with $o(\mathcal{T}) \ge \xi + 1$. For each $(x) \in \mathcal{T}$, let

$$\mathcal{T}_x = \bigcup_{n=1}^{\infty} \{(x_1, \ldots, x_n) : (x, x_1, \ldots, x_n) \in \mathcal{T}\}.$$

According to [9, Proposition 4], $o(\mathcal{T}) = \sup_{(x)\in\mathcal{T}} (o(\mathcal{T}_x)+1)$. Therefore, there exists $(x_0) \in \mathcal{T}$ such that $o(\mathcal{T}_{x_0}) \ge \xi$. By the inductive hypothesis, $\iota(\mathcal{G}') \ge \xi$, where $\mathcal{G}'$ is defined analogously to $\mathcal{G}$ for the tree $\mathcal{T}_{x_0}$. Let $k_0 = \max \operatorname{supp} x_0$. Then $\{k_0\} \cup G \in \mathcal{G}$ whenever $G \in \mathcal{G}'$. Thus $\{k_0\} \in \mathcal{G}^{(\xi)}$. Since $\mathcal{G}^{(\xi)}$ is spreading, $\{k\} \in \mathcal{G}^{(\xi)}$ for all $k \ge k_0$. It follows that $\iota(\mathcal{G}) \ge \xi + 1$.

Suppose $o(\mathcal{T}) \ge \xi_0$, where ξ_0 is a countable limit ordinal and the proposition holds for all $\xi < \xi_0$. Since $o(\mathcal{T}) \ge \xi$ for all $\xi < \xi_0$, by the inductive hypothesis, $\iota(\mathcal{G}) \ge \xi$ for all $\xi < \xi_0$. Hence $\iota(\mathcal{G}) \ge \xi_0$. This completes the induction. ∎

3. Main results and proofs. The main results concern two measures of the finite-dimensional ℓ^1-complexity of the space X. These are the Bourgain ℓ^1-index and the existence of ℓ^1-spreading models of higher order. Given a finite constant K greater than 1, an *ℓ^1-K-tree* in a Banach space B is a tree in B so that every node $(x_1,\dots,x_n)$ is a normalized sequence such that $\|\sum a_k x_k\| \geq K^{-1}\sum|a_k|$ for all (a_k). If B has a basis, an *ℓ^1-K-block tree* is a block tree that is also an ℓ^1-K-tree. Suppose that B does not contain ℓ^1, and let $I(B,K)=\sup o(\mathcal{T})$, where the sup is taken over the set of all ℓ^1-K-trees in X. The *Bourgain ℓ^1-index* is defined to be $I(B)=\sup_{K<\infty} I(B,K)$. The *block indices* $I_{\mathrm{b}}(B,K)$ and $I_{\mathrm{b}}(B)$ are defined analogously using ℓ^1-block trees. We refer to [2, 14] for thorough investigations of these indices. In particular, it is shown in [14] that for a Banach space B with a basis, $I_{\mathrm{b}}(B)=I(B)$ if either one is $\geq\omega^\omega$. With the same notation as above, a normalized sequence (x_k) is said to be an *ℓ^1-$\mathcal{S}_\beta$-spreading model* with constant K if $\|\sum_{k\in F} a_k x_k\|\geq K^{-1}\sum_{k\in F}|a_k|$ whenever $F\in\mathcal{S}_\beta$.

We are now ready to work our way towards the main Theorem 9. The major parts of the computations are contained in Proposition 4 and Lemma 7 (tree splitting lemma). Let (y_k) be a normalized block sequence in X and let Y be the block subspace $[(y_k)]$. For any $n\in\mathbb{N}$, we call the space $Y_n=[(y_k)_{k=n}^\infty]$ the *n-tail* of Y. We emphasize that in the next lemma both admissibility and the support of a vector are taken with respect to the basis (e_k). Recall the assumption that $(\alpha_n)=(\iota(\mathcal{F}_n))$ is a nondecreasing sequence which converges to ω^{ω^ξ} nontrivially.

LEMMA 3. *Assume that $I_{\mathrm{b}}(Y)>\omega^{\omega^\xi}$. Then there is a constant $C<\infty$ such that for all $n\in\mathbb{N}$, there exists a normalized vector x in the n-tail of Y such that $\sum\|E_i x\|\leq C$ whenever (E_i) is $\mathcal{F}_k$-admissible for some $k\leq n$.*

Proof. There exists $K<\infty$ such that $I_{\mathrm{b}}(Y,K)>\omega^{\omega^\xi}$. Let $\mathcal{T}$ be an ℓ^1-K-block tree in Y such that $o(\mathcal{T})\geq\omega^{\omega^\xi}$. Given n, consider the tree $\widehat{\mathcal{T}}$ consisting of all nodes of the form $(x_j)_{j=n}^r$ for some $(x_j)_{j=1}^r\in\mathcal{T}$, $r\geq n$. Then $\widehat{\mathcal{T}}$ is an ℓ^1-K-block tree in Y_n such that $o(\widehat{\mathcal{T}})\geq\omega^{\omega^\xi}$. Choose α and β so that $\alpha_n<\omega^\alpha<\omega^\beta<\omega^{\omega^\xi}$. Define

$$\mathcal{H}=\{(\max\operatorname{supp} x_j)_{j=n}^r : (x_j)_{j=n}^r\in\widehat{\mathcal{T}}\},$$
$$\mathcal{G}=\{G : G \text{ is a spreading of a subset of some } H\in\mathcal{H}\}.$$

By Lemma 2, $\mathcal{G}$ is hereditary and spreading, and either $\mathcal{G}$ is noncompact or it is compact with $\iota(\mathcal{G})\geq o(\widehat{\mathcal{T}})\geq\omega^{\omega^\xi}$. By [11, Theorem 1.1], there exists $M\in[\mathbb{N}]$ such that

$$\bigcup_{k=1}^n \mathcal{F}_k\cap[M]^{<\infty}\subseteq\mathcal{S}_\alpha\cap[M]^{<\infty}\subseteq\mathcal{S}_\beta\cap[M]^{<\infty}\subseteq\mathcal{G}.$$

Now [21, Proposition 3.6] gives a finite set $G \in \mathcal{S}_\beta \cap [M]^{<\infty}$ and a sequence $(a_p)_{p\in G}$ of positive numbers such that $\sum a_p = 1$ and $\sum_{p\in F} a_p < \theta_n$ whenever $F \subseteq G$ and $F \in \mathcal{S}_\alpha$. By definition, there exist a node $(x_j)_{j=n}^r \in \widehat{\mathcal{T}}$ and a subset J of the integer interval $[n, r]$ such that G is a spreading of $(\max\operatorname{supp} x_j)_{j\in J}$. Denote the unique order preserving bijection from J onto G by u and consider the vector $y = \sum_{j\in J} a_{u(j)} x_j$. Since $(x_j)_{j=n}^r$ is a normalized ℓ^1-K-block sequence in Y_n and $\sum a_{u(j)} = 1$, $y \in Y_n$ and $\|y\| \geq 1/K$. Let (E_i) be $\mathcal{F}_k$-admissible for some $k \leq n$. For each $j \in J$, let $\mathcal{E}_j$ be the collection of all E_i's that have nonempty intersection with $\operatorname{supp} x_{j'}$ if and only if $j' = j$. Also let $\mathcal{E}'$ be the collection of all E_i such that E_i intersects $\operatorname{supp} x_j$ for at least two $j \in J$. Since (E_i) is $\mathcal{F}_k$-admissible, for each $j \in J$,

$$\sum_{E_i\in\mathcal{E}_j} \|E_i y\| \leq a_{u(j)}\theta_k^{-1}\|x_j\| = a_{u(j)}\theta_k^{-1}.$$

Set $J' = \{j \in J : \mathcal{E}_j \neq \emptyset\}$. The $\mathcal{F}_k$-admissiblity of (E_i) implies that $(\max\operatorname{supp} x_j)_{j\in J'} \in \mathcal{F}_k$. Thus $u(J')$, being a spreading of this set, also belongs to $\mathcal{F}_k$. Since $u(J') \subseteq G \in [M]^{<\infty}$, we conclude that $u(J') \in \mathcal{F}_k \cap [M]^{<\infty} \subseteq \mathcal{S}_\alpha$. Hence $\sum_{j\in J'} a_{u(j)} < \theta_n$. Also, since each $\operatorname{supp} x_j$, $j \in J$, intersects at most two E_i in $\mathcal{E}'$,

$$\sum_{E_i\in\mathcal{E}'} \|E_i y\| \leq \sum_{j\in J} a_{u(j)} \sum_{E_i\in\mathcal{E}'} \|E_i x_j\| \leq 2\sum_{j\in J} a_{u(j)} = 2.$$

Therefore,

$$\sum \|E_i y\| = \sum_{E_i\in\mathcal{E}'} \|E_i y\| + \sum_{j\in J'}\sum_{E_i\in\mathcal{E}_j} \|E_i y\| \leq 2 + \theta_k^{-1}\sum_{j\in J'} a_{u(j)} \leq 3.$$

It is clear that the normalized element $x = y/\|y\|$ satisfies the statement of the lemma with the constant $C = 3K$. ∎

We pause to introduce another method of computing the norm of an element in X using norming trees. This is derived from the implicit description of the norm in X (equation (2)) and has been used in [8, 17, 20]. An $((\mathcal{F}_k)$-$)$*admissible tree* is a finite collection of elements (E_i^m), $0 \leq m \leq r$, $1 \leq i \leq k(m)$, in $[\mathbb{N}]^{<\infty}$ with the following properties:

(1) $k(0) = 1$.
(2) For each m, $E_1^m < E_2^m < \cdots < E_{k(m)}^m$.
(3) Every E_i^{m+1} is a subset of some E_j^m.
(4) For each j and m, the collection $\{E_i^{m+1} : E_i^{m+1} \subseteq E_j^m\}$ is $\mathcal{F}_k$-admissible for some k.

The set E_1^0 is called the *root* of the admissible tree. The elements E_i^m are called *nodes* of the tree. If $E_i^n \subseteq E_j^m$ and $n > m$, we say that E_i^n is a *descendant* of E_j^m, and E_j^m is an *ancestor* of E_i^n. If, in the above notation,

$n = m+1$, then E^n_i is said to be an *immediate successor* of E^m_j, and E^m_j the *immediate predecessor* of E^n_i. Nodes with no descendants are called *terminal nodes* or *leaves* of the tree. Assign *tags* to the individual nodes inductively as follows. Let $t(E^0_1) = 1$. If $t(E^m_i)$ has been defined and the collection (E^{m+1}_j) of all immediate successors of E^m_i forms an $\mathcal{F}_k$-admissible collection, then define $t(E^{m+1}_j) = \theta_k t(E^m_i)$ for all immediate successors E^{m+1}_j of E^m_i. If $x \in c_{00}$ and $\mathcal{T}$ is an admissible tree, let $\mathcal{T}x = \sum t(E)\|Ex\|_{c_0}$ where the sum is taken over all leaves in $\mathcal{T}$. It follows from the implicit description (equation (2)) of the norm in X that $\|x\| = \max \mathcal{T}x$, with the maximum taken over the set of all admissible trees. Let us also point out that if $\mathcal{E}$ is a collection of pairwise disjoint nodes of an admissible tree $\mathcal{T}$ so that $E \subseteq \bigcup \mathcal{E}$ for every leaf E of $\mathcal{T}$ and $x \in c_{00}$, then $\mathcal{T}x = \sum_{F\in\mathcal{E}} t(F)\|Fx\|$.

We make the following definitions for notational convenience.

Definition. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are families of finite subsets of $\mathbb{N}$.

(1) An element $G \in \mathcal{G}$ is *maximal* (in $\mathcal{G}$) if it is not properly contained in any other element in $\mathcal{G}$.
(2) The family $\mathcal{F} \ominus \mathcal{G}$ is the collection of all sets F so that there is a maximal $G \in \mathcal{G}$, $G < F$, with $G \cup F \in \mathcal{F}$.

Definition. A sequence of regular families $(\mathcal{F}_n)$ is *tame* if

(1) for each n, either $\mathcal{F}_n = \mathcal{A}_j$ for some j or $\mathcal{F}_n[\mathcal{A}_3] \subseteq (\mathcal{F}_n)^2$,
(2) there exists $n_0 \in \mathbb{N}$ so that $(\mathcal{F}_n \ominus \mathcal{F}_{n_0})[\mathcal{A}_2] \subseteq \mathcal{F}_n$ whenever $n > n_0$.

Proposition 4. *Assume that $(\mathcal{F}_n)$ is a tame sequence. Let Y be a block subspace of X. Suppose that there exists a constant $C < \infty$ such that for all $n \in \mathbb{N}$, there is a normalized vector x in the n-tail of Y such that $\sum \|E_i x\| \le C$ whenever (E_i) is $\mathcal{F}_k$-admissible for some $k \le n$. Then there exists a normalized block sequence (z_n) in Y that is equivalent to a subsequence of (e_k). Moreover, $Z = [(z_n)]$ is a complemented subspace of X.*

Remark. In [18, Propositions 5.7 and 5.8], similar results were shown for mixed Tsirelson spaces of the form $T[(\theta_n, \mathcal{S}_n)_{n=1}^\infty]$ containing certain semi-normalized special convex combinations. Proposition 4 generalizes these results. In [4, Theorems 2.1 and 2.4], it was proved that in the Schlumprecht space S, every block subspace contains a block sequence (z_n) that is equivalent to a subsequence of the unit vector basis with $Z = [(z_n)]$ complemented in S. This case does not follow from Proposition 4 since the sequence $(\mathcal{A}_n)$ is not tame. However, the following result may be proved by similar methods.

Proposition 5. *Let $(\theta_n)_{n=1}^\infty$ be a sequence in $(0,1)$ decreasing to 0 so that $\lim \theta_{2n}/\theta_n = 1$. Suppose that Y is a block subspace of $X = \mathcal{T}[(\theta_n, \mathcal{A}_n)_{n=1}^\infty]$ such that there exists a constant $C < \infty$ so that for all $n \in \mathbb{N}$, there exists a* *normalized vector x in the n-tail of Y with $\sum \|E_i x\| \le C$ whenever (E_i) is $\mathcal{A}_n$-admissible. Then there exists a normalized block sequence (z_n) in Y that is equivalent to a subsequence of the unit vector basis (e_k) of X. Moreover, $Z = [(z_n)]$ is a complemented subspace of X.*

Proof of Proposition 4. Let n_0 be the integer occurring in the definition of tameness for the sequence $(\mathcal{F}_n)$. Inductively, choose a normalized block sequence (z_n) in Y and a strictly increasing sequence $(m_n)_{n=0}^\infty$ in $\mathbb{N}$ so that $m_0 > n_0$, $\theta_{m_n}\|z_n\|_{\ell^1} \le 2^{-n}$ and $\sum \|E_i z_n\| \le C$ whenever (E_i) is $\bigcup_{r=1}^{m_{n-1}} \mathcal{F}_r$-admissible, $n \in \mathbb{N}$. Consider $z = \sum a_n z_n$ for some $(a_n) \in c_{00}$ and let $y = \sum a_n e_{k_n}$, where $k_n = \max\operatorname{supp} z_n$. Let $\mathcal{T}$ be an admissible tree that norms z. Without loss of generality, we may assume that all nodes in $\mathcal{T}$ are integer intervals and that all leaves in $\mathcal{T}$ are singletons. Say that a node is *short* if it intersects $\operatorname{supp} z_n$ for exactly one n. On the other hand, call a node *long* if it intersects $\operatorname{supp} z_n$ for more than one n. The tree $\mathcal{T}$ is endowed with the natural partial order of reverse inclusion. Let $\mathcal{E}$ be the collection of all minimal short nodes in $\mathcal{T}$. Then $\|z\| = \sum_{E\in\mathcal{E}} t(E)\|Ez\|$. For each n, let $\mathcal{E}_n$ be the collection of all nodes in $\mathcal{E}$ that intersects only $\operatorname{supp} z_n$. In particular, $\mathcal{E} = \bigcup \mathcal{E}_n$. Further subdivide each set $\mathcal{E}_n$ into two subsets $\mathcal{E}_n'$ and $\mathcal{E}_n''$ depending on whether $t(E) \le \theta_{m_n}$ or not. We have

$$(3) \qquad \sum_n \sum_{E\in\mathcal{E}_n'} t(E)\|Ez\| \le \sum_n \theta_{m_n}|a_n|\,\|z_n\|_{\ell^1} \le \sum_n \frac{|a_n|}{2^n} \le \|y\|.$$

For each n, let $\mathcal{D}_n$ be the set of all minimal elements in the set of all nodes in $\mathcal{T}$ that are immediate predecessors of some node in $\mathcal{E}_n''$. Since $\mathcal{D}_n$ consists of pairwise disjoint long nodes that intersect $\operatorname{supp} z_n$, $|\mathcal{D}_n| \le 2$ for all n. For each $D \in \mathcal{D}_n$, let $\mathcal{E}_n''(D) = \{E \in \mathcal{E}_n'' : E \subseteq D\}$ and let $\widetilde{\mathcal{E}}_n''(D)$ be the subset of $\mathcal{E}_n''$ consisting of all $E \in \mathcal{E}_n''$ that are immediate successors of D. Fix $E_{n,D} \in \widetilde{\mathcal{E}}_n''(D)$ and $j_{n,D} \in E_{n,D} \cap \operatorname{supp} z_n$ arbitrarily and set

$$w = \sum_n \sum_{D\in\mathcal{D}_n} a_n e_{j_{n,D}}.$$

Since $|\mathcal{D}_n| \le 2$ for all n, $\|w\| \le 2\|y\|$. Any immediate successor of D that contains some $E \in \mathcal{E}_n''(D) \setminus \widetilde{\mathcal{E}}_n''(D)$ must be a long node. Hence there are at most two immediate successors of D, say G_1 and G_2, that all nodes in $\mathcal{E}_n''(D)\setminus\widetilde{\mathcal{E}}_n''(D)$ are descended from. Note that $t(G_1) = t(G_2) = t(E_{n,D})$ since they are all immediate successors of the same node. Thus

$$\sum_{E\in\mathcal{E}_n''(D)\setminus\widetilde{\mathcal{E}}_n''(D)} t(E)\|Ez_n\| \le \sum_{i=1}^2 t(G_i)\|G_i z_n\| \le 2t(E_{n,D}).$$

Hence

$$\begin{aligned}(4)\qquad \sum_n \sum_{D\in\mathcal{D}_n} \sum_{E\in\mathcal{E}_n''(D)\setminus\widetilde{\mathcal{E}}_n''(D)} t(E)\|Ez\| &= \sum_n \sum_{D\in\mathcal{D}_n} \sum_{E\in\mathcal{E}_n''(D)\setminus\widetilde{\mathcal{E}}_n''(D)} t(E)|a_n|\,\|Ez_n\| \\ &\le \sum_n \sum_{D\in\mathcal{D}_n} 2|a_n|t(E_{n,D}) = 2\sum_n \sum_{D\in\mathcal{D}_n} t(E_{n,D})\|E_{n,D}w\|_{c_0} \\ &= 2\mathcal{T}'w \le 2\|w\| \le 4\|y\|,\end{aligned}$$

where $\mathcal{T}'$ is the subtree of $\mathcal{T}$ consisting of all nodes $E_{n,D}$, $D\in\mathcal{D}_n$, and their ancestors. Now let $\mathcal{D}_n'$ consists of those D in $\mathcal{D}_n$ such that $\widetilde{\mathcal{E}}_n''(D)$ is $\bigcup_{r=1}^{m_{n-1}}\mathcal{F}_r$-admissible. Then

$$\begin{aligned}(5)\qquad \sum_n \sum_{D\in\mathcal{D}_n'} \sum_{E\in\widetilde{\mathcal{E}}_n''(D)} t(E)\|Ez\| &= \sum_n \sum_{D\in\mathcal{D}_n'} \sum_{E\in\widetilde{\mathcal{E}}_n''(D)} t(E)|a_n|\,\|Ez_n\| \le C\sum_n \sum_{D\in\mathcal{D}_n'} t(E_{n,D})|a_n| \\ &= C\sum_n \sum_{D\in\mathcal{D}_n'} t(E_{n,D})\|E_{n,D}w\|_{c_0} \le C\|w\| \le 2C\|y\|.\end{aligned}$$

It remains to consider the nodes that belong to $\mathcal{D}_n\setminus\mathcal{D}_n'$ for some n. We have

$$\begin{aligned}\sum_n \sum_{D\in\mathcal{D}_n\setminus\mathcal{D}_n'} \sum_{E\in\widetilde{\mathcal{E}}_n''(D)} t(E)\|Ez\| &= \sum_n \sum_{D\in\mathcal{D}_n\setminus\mathcal{D}_n'} \sum_{E\in\widetilde{\mathcal{E}}_n''(D)} t(E)|a_n|\|Ez_n\| \\ &\le \sum_n \sum_{D\in\mathcal{D}_n\setminus\mathcal{D}_n'} t(D)|a_n|\,\|Dz_n\| \le \sum_n \sum_{D\in\mathcal{D}_n\setminus\mathcal{D}_n'} t(D)|a_n|.\end{aligned}$$

But by Lemma 7 below,

$$\sum_n \sum_{D\in\mathcal{D}_n\setminus\mathcal{D}_n'} t(D)|a_n| \le 4\|y\|.$$

Thus

$$(6)\qquad \sum_n \sum_{D\in\mathcal{D}_n\setminus\mathcal{D}_n'} \sum_{E\in\widetilde{\mathcal{E}}_n''(D)} t(E)\|Ez\| \le 4\|y\|.$$

Combining inequalities (3) to (6), we see that

$$(7)\qquad \|z\| = \sum_{E\in\mathcal{E}} t(E)\|Ez\| \le (9+2C)\|y\|.$$

Hence (z_n) is dominated by (e_{k_n}), where $k_n = \max\operatorname{supp} z_n$. On the other hand, (z_n) dominates $(e_{k_{n-1}})$ (take $k_0 = 1$). Therefore, using the tameness of $(\mathcal{F}_n)$, we see that (z_n) is equivalent to (e_{k_n}) by Lemma 1.

Finally, we show that $Z = [(z_n)]$ is a complemented subspace of X. For each $n \in \mathbb{N}$, let z'_n be a normalized vector in X' such that $\operatorname{supp} z'_n \subseteq \operatorname{supp} z_n = E_n$ and $z'_n(z_n) = 1$. Define $P : X \to X$ by $P(x) = \sum_n z'_n(x) z_n$. Let $l_n = \min \operatorname{supp} z_n$. For any $x \in X$,

$$\begin{aligned}\|Px\| &= \Big\|\sum_n z'_n(x) z_n\Big\| \le \Big\|\sum_n \|E_n x\| z_n\Big\| \quad \text{as } \|z'_n\| \le 1,\\ &\le (9+2C)\Big\|\sum_n \|E_n x\| e_{k_n}\Big\| \quad \text{by (7)},\\ &\le 3(9+2C)\Big\|\sum_n \|E_n x\| e_{l_n}\Big\|\end{aligned}$$

by Lemma 1 and the spreading property of $(\mathcal{F}_n)_{n=1}^{\infty}$. Also, note that since $l_n \le \operatorname{supp} E_n x$,

$$\Big\|\sum_n \|E_n x\| e_{l_n}\Big\| \le \Big\|\sum_n \|E_n x\| \frac{E_n x}{\|E_n x\|}\Big\| = \Big\|\sum_n E_n x\Big\| \le \|x\|.$$

Hence P is bounded. Clearly P is a projection onto Z. ∎

LEMMA 6. *Suppose that $n_1 < n_2$ and $D \in \mathcal{D}_{n_2} \backslash \mathcal{D}'_{n_2}$. Then no descendant of D belongs to $\mathcal{E}''_{n_1}$. In particular, $D \notin \mathcal{D}_{n_1}$.*

Proof. If E is a descendant of $D \in \mathcal{D}_{n_2} \setminus \mathcal{D}'_{n_2}$, then $t(E) \le t(F)$ for any immediate successor F of D. In particular, $t(E) \le t(F)$ for all $F \in \widetilde{\mathcal{E}}''_{n_2}(D)$. By definition of $\mathcal{D}'_{n_2}$, $\widetilde{\mathcal{E}}''_{n_2}(D)$ is not $\mathcal{F}_r$-admissible for all $r \le m_{n_1}$. Hence $t(F) < \theta_{m_{n_1}}$ for all $F \in \widetilde{\mathcal{E}}''_{n_2}(D)$. Therefore, $t(E) < \theta_{m_{n_1}}$ if E is a descendant of $D \in \mathcal{D}_{n_2} \setminus \mathcal{D}'_{n_2}$. This shows that $E \notin \mathcal{E}''_{n_1}$ by definition of $\mathcal{E}''_{n_1}$. ∎

Let $\mathcal{T}'$ be the subtree of $\mathcal{T}$ consisting of all nodes in $\widetilde{\mathcal{D}} = \bigcup_n (\mathcal{D}_n \setminus \mathcal{D}'_n)$ and their ancestors. By Lemma 6, for each $D \in \widetilde{\mathcal{D}}$, there is a unique $n = n_D$ such that $D \in \mathcal{D}_n \setminus \mathcal{D}'_n$. If G is a node in $\mathcal{T}'$, let $\widetilde{\mathcal{D}}(G)$ consist of all $D \in \widetilde{\mathcal{D}}$ such that $D \subseteq G$. Recall the vector w defined in the proof of Proposition 4 above. It was observed that $\|w\| \le 2\|y\|$.

LEMMA 7. *For any $G \in \mathcal{T}'$, there exist subsets G_1 and G_2 of G, $G_1 < G_2$, and admissible trees $\mathcal{T}_1$ and $\mathcal{T}_2$ with roots G_1 and G_2 respectively so that*

$$\sum_{D \in \widetilde{\mathcal{D}}(G)} t(D) |a_{n_D}| \le t(G)(\mathcal{T}_1 w + \mathcal{T}_2 w).$$

In particular,

$$\sum_n \sum_{D \in \mathcal{D}_n \backslash \mathcal{D}'_n} t(D) |a_n| \le 4\|y\|.$$

Proof. The second inequality follows from the first by taking G to be the root of $\mathcal{T}'$ (which is also the root of $\mathcal{T}$). To prove the first inequality, we begin at the terminal nodes of $\mathcal{T}'$ and work our way up the tree. Let G be a terminal node of $\mathcal{T}'$. Then $G \in \widetilde{\mathcal{D}}$. In this case, take $G_1 = [1, \max\operatorname{supp} z_{n_G}] \cap G$ and $G_2 = G \setminus G_1$. Clearly, G_1 and G_2 are subsets of G such that $G_1 < G_2$. Set $\mathcal{T}_1$ and $\mathcal{T}_2$ to be the trivial trees $\mathcal{T}_i = \{G_i\}$, $i = 1, 2$. Now

$$\sum_{D \in \widetilde{\mathcal{D}}(G)} t(D)|a_{n_D}| = t(G)|a_{n_G}| \leq t(G)\|G_1 w\|_{c_0} = t(G)\mathcal{T}_1 w.$$

Thus the lemma holds in this case.

Next, take a node $G \in \mathcal{T}'$ and assume that the lemma has been proved for all descendants of G in $\mathcal{T}'$. List the immediate successors of G in $\mathcal{T}'$ from left to right as $\{H_1, \ldots, H_r\}$. By the assumption, for each j, $1 \leq j \leq r$, there are subsets H_j^i of H_j, and admissible trees $\mathcal{T}_j^i$, $i = 1, 2$, such that $H_j^1 < H_j^2$, the root of $\mathcal{T}_j^i$ is H_j^i and

$$\sum_{D \in \widetilde{\mathcal{D}}(H_j)} t(D)|a_{n_D}| \leq t(H_j)(\mathcal{T}_j^1 w + \mathcal{T}_j^2 w).$$

We divide the rest of the proof into two cases.

Case 1: $G \in \widetilde{\mathcal{D}}$. The sets in the collection $\widetilde{\mathcal{E}}''_{n_G}(G) \cup \{H_j\}_{j=1}^r$ are all immediate successors of G in the tree $\mathcal{T}$. We claim that $E < H_1$ for any $E \in \widetilde{\mathcal{E}}''_{n_G}(G)$. Indeed, either H_1 or a descendant of H_1 belongs to $\widetilde{\mathcal{D}}$. Denote this node by I. Thus $G \in \mathcal{D}_{n_G} \setminus \mathcal{D}'_{n_G}$ has a descendant in $\mathcal{E}''_{n_I}$. By Lemma 6, $n_I \geq n_G$. Since $I \subsetneq G$, $n_I \neq n_G$ by the minimality condition in the definition of $\mathcal{D}_n$. Hence $n_I > n_G$. Now any E in $\widetilde{\mathcal{E}}''_{n_G}(G)$ intersects only $\operatorname{supp} z_{n_G}$ while H_1 must intersect $\operatorname{supp} z_{n_I}$. Therefore, $E < H_1$, as claimed. To continue with the proof, set $G_1 = G \cap [1, k]$, where $k = \max \bigcup \widetilde{\mathcal{E}}''_{n_G}(G)$, and $G_2 = G \setminus G_1$. Then take $\mathcal{T}_1$ to be the trivial tree $\{G_1\}$ and $\mathcal{T}_2$ to be the tree $\{G_2\} \cup \bigcup_{i,j} \mathcal{T}_j^i$. The admissibility of $\mathcal{T}_1$ is clear. To verify the admissibility of $\mathcal{T}_2$, it suffices to show the admissibility of the decomposition of G_2 into $\{H_j^i\}_{i,j}$. Since $\widetilde{\mathcal{E}}''_{n_G}(G) \cup \{H_j\}_{j=1}^r$ are all immediate successors of G in the tree $\mathcal{T}$, the collection is $\mathcal{F}_n$-admissible for some n. However, $\widetilde{\mathcal{E}}''_{n_G}(G)$ is not $\mathcal{F}_r$-admissible for any $r \leq m_{n_G - 1}$. Thus $n > m_{n_G - 1} > n_0$ and $(\min H_j) \in \mathcal{F}_n \ominus \mathcal{F}_{n_0}$. By the tameness of $(\mathcal{F}_n)$, $(\min H_j^i) \in (\mathcal{F}_n \ominus \mathcal{F}_{n_0})[\mathcal{A}_2] \subseteq \mathcal{F}_n$. Hence (H_j^i) is $\mathcal{F}_n$-admissible, as required. Now

$$\mathcal{T}_1 w = \|G_1 w\|_{c_0} \geq |a_{n_G}|$$

and

$$\mathcal{T}_2 w = \theta_n \sum_{i,j} \mathcal{T}_j^i w = \sum_{i,j} \frac{t(H_j)}{t(G)}\, \mathcal{T}_j^i w \geq \sum_j \sum_{D \in \widetilde{\mathcal{D}}(H_j)} \frac{t(D)}{t(G)}\, |a_{n_D}|.$$

Therefore,

$$\sum_{D\in\widetilde{\mathcal{D}}(G)} t(D)|a_{n_D}| = t(G)|a_{n_G}| + \sum_j \sum_{D\in\widetilde{\mathcal{D}}(H_j)} t(D)|a_{n_D}|$$
$$\le t(G)(\mathcal{T}_1 w + \mathcal{T}_2 w).$$

CASE 2: $G \notin \widetilde{\mathcal{D}}$. Suppose that in the tree $\mathcal{T}$, the immediate successors of G form an $\mathcal{F}_n$-admissible collection. In particular, $\{H_j\}_{j=1}^r$ is $\mathcal{F}_n$-admissible. We claim that $(\min H_j^i) \in (\mathcal{F}_n)^2$. This is clear if $\mathcal{F}_n = \mathcal{A}_j$ for some j. Otherwise, $(\min H_j^i) \in \mathcal{F}_n[\mathcal{A}_2] \subseteq (\mathcal{F}_n)^2$ by the tameness of $(\mathcal{F}_n)$. Choose index sets I_1 and I_2 such that $I_1 \cup I_2 = \{(i,j) : 1 \le i \le 2, 1 \le j \le r\}$, $\{H_j^i : (i,j) \in I_k\}$ is $\mathcal{F}_n$-admissible, $k = 1,2$, and that $H_j^i < H_{j'}^{i'}$ whenever $(i,j) \in I_1$ and $(i',j') \in I_2$. Set $G_1 = G \cap [1,p]$, where $p = \max \bigcup\{H_j^i : (i,j) \in I_1\}$ and $G_2 = G \setminus G_1$. Define $\mathcal{T}_k$ to be the tree $\{G_k\} \cup \bigcup_{(i,j)\in I_k} \mathcal{T}_j^i$, $k = 1,2$. The admissibility of $\mathcal{T}_1$ and $\mathcal{T}_2$ follows by construction. Finally,

$$t(G)\sum_k \mathcal{T}_k w = t(G)\theta_n \sum_k \sum_{(i,j)\in I_k} \mathcal{T}_j^i w = \sum_{i,j} t(H_j)\mathcal{T}_j^i w$$
$$\ge \sum_j \sum_{D\in\widetilde{\mathcal{D}}(H_j)} t(D)|a_{n_D}| = \sum_{D\in\widetilde{\mathcal{D}}(G)} t(D)|a_{n_D}|. \ \blacksquare$$

Given a nonzero ordinal α with Cantor normal form $\omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_n} \cdot m_n$, let $\ell(\alpha) = \beta_1$. For any $m \in \mathbb{N}$ and $\varepsilon > 0$, define

$$\gamma(\varepsilon, m) = \max\{\ell(\alpha_{n_s} \cdots \alpha_{n_1}) : \varepsilon\theta_{n_1} \cdots \theta_{n_s} > \theta_m\} \quad (\max\emptyset = 0).$$

The sequence $((\theta_n, \mathcal{F}_n))_{n=1}^\infty$ is said to satisfy (†) if there exists $\varepsilon > 0$ such that for all $\beta < \omega^\xi$, there exists $m \in \mathbb{N}$ such that $\gamma(\varepsilon, m) + 2 + \beta < \ell(\alpha_m)$.

THEOREM 8 ([17, Theorems 4 and 12]). *Assume that* (†) *holds. Then for any $M \in [\mathbb{N}]$, $[(e_k)_{k\in M}]$ contains an ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model. On the other hand, if* (†) *fails, then for all $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that $I_{\mathrm{b}}([(e_k)_{k\in N}]) = \omega^{\omega^\xi}$.*

Recall that a Banach space $(B, \|\cdot\|)$ is said to be λ-*distortable* if there is an equivalent norm $|\cdot|$ on B so that for every infinite-dimensional subspace Y of X, there are $\|\cdot\|$-normalized vectors y and z in Y so that $|y|/|z| > \lambda$. A space is *arbitrarily distortable* if it is λ-distortable for all $\lambda > 1$.

THEOREM 9. *Assume that $(\mathcal{F}_n)$ is a tame sequence. The following statements are equivalent for any block subspace Y of X.*

(1) *Property* (†) *holds and every block subspace Z of Y contains a block sequence equivalent to a subsequence of (e_k).*
(2) *Every block subspace Z of Y contains an ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model.*
(3) *The Bourgain ℓ^1-index satisfies $I_{\mathrm{b}}(Z) = I(Z) > \omega^{\omega^\xi}$ for any block subspace Z of Y.*

Moreover, if one (and hence all) of the equivalent conditions holds for a block subspace Y of X, then Y is arbitrarily distortable.

Proof. The implication (1)⇒(2) follows from the first part of Theorem 8. Let Z be a block subspace of Y. If (2) holds, then $I(Z,K) \geq \omega^{\omega^\xi}$ for some $K < \infty$. By [14, Lemma 5.7], $I_{\mathrm{b}}(Z) = I(Z) > \omega^{\omega^\xi}$. Assume that condition (3) holds. By Lemma 3 and Proposition 4, Z contains a normalized block sequence equivalent to a subsequence of (e_k). Say (z_n) is a normalized block sequence in Z equivalent to $(e_k)_{k\in M}$ for some $M \in [\mathbb{N}]$. If (†) fails, by the second part of Theorem 8, there exists $N \in [M]$ such that $I_{\mathrm{b}}([(e_k)_{k\in N}]) = \omega^{\omega^\xi}$. Hence $I_{\mathrm{b}}([(z_{n_j})]) = \omega^{\omega^\xi}$ for some subsequence (z_{n_j}) of (z_n). This contradicts (3) since $[(z_{n_j})]$ is a block subspace of Y. This proves condition (1).

Assume that the conditions hold for a block subspace Y of X. For each n, consider the equivalent norm $|\!|\!|\cdot|\!|\!|_n$ on X defined by

$$|\!|\!|x|\!|\!|_n = \sup\Big\{\sum \|E_i x\| : (E_i) \text{ is } \mathcal{F}_n\text{-admissible}\Big\}.$$

Let Z be a block subspace of Y. By condition (3) and Lemma 3, there exists $C_1 < \infty$ such that for all n, there exists $z \in Z$ such that $\|z\| = 1$ and $|\!|\!|z|\!|\!|_n \leq C_1$. On the other hand, by (1), Z contains a normalized block sequence $(z_j)_{j=1}^\infty$ that is C_2-equivalent to a subsequence $(e_{k_j})_{j=1}^\infty$ of (e_k). By taking a subsequence if necessary, we may assume that $e_{k_j} < z_{j+1}$ for all $j \in \mathbb{N}$. Let ε be the constant given by property (†). It follows from (†) that there are infinitely many m such that $\gamma(\varepsilon, m) + 2 < \ell(\alpha_m)$. Fix such an m and let $\gamma = \gamma(\varepsilon, m)$. By [11, Theorem 1.1], there exists $N \in [(k_j)]$ such that $\mathcal{S}_{\gamma+2} \cap [N]^{<\infty} \subseteq \mathcal{F}_m$. By [16, Lemma 19], there exists $x \in c_{00}$ such that $\|x\| \leq 1 + \varepsilon^{-1}$, $\|x\|_{\ell^1} = \theta_m^{-1}$ and $\operatorname{supp} x \in \mathcal{S}_{\gamma+2} \cap [N]^{<\infty}$. Say $x = \sum_{j\in I} a_j e_{k_j}$ for some I such that $(k_j)_{j\in I} \in [N]^{<\infty}$. Consider the corresponding element $y = \sum_{j\in I} a_j z_j / \|\sum_{j\in I} a_j z_j\|$. Since $(z_j)_{j\in I}$ is C_2-equivalent to $(e_{k_j})_{j\in I}$,

$$\Big\|\sum_{j\in I} a_j z_j\Big\| \leq C_2 \|x\| \leq C_2(1 + \varepsilon^{-1}).$$

For each j, let $E_j = \operatorname{supp} z_j$. If $j_0 = \min I$, then $(\min E_j)_{j\in I\setminus\{j_0\}}$ is a spreading of a subset of $(k_j)_{j\in I} = \operatorname{supp} x$. Hence $(E_j)_{j\in I\setminus\{j_0\}}$ is $\mathcal{F}_m$-admissible since $\operatorname{supp} x \in \mathcal{F}_m$. Therefore,

$$\begin{aligned}\Big|\!\Big|\!\Big|\sum_{j\in I} a_j z_j\Big|\!\Big|\!\Big|_m &\geq \sum_{i\in I\setminus\{j_0\}} \Big\|E_i \sum_{j\in I} a_j z_j\Big\| = \sum_{i\in I\setminus\{j_0\}} |a_i| = \|x\|_{\ell^1} - |a_{j_0}| \\ &\geq \|x\|_{\ell^1} - \|x\| \geq \theta_m^{-1} - 1 - \varepsilon^{-1}.\end{aligned}$$

Hence $|\!|\!|y|\!|\!|_m \geq C_2^{-1}(1+\varepsilon^{-1})^{-1}(\theta_m^{-1} - 1 - \varepsilon^{-1})$. The existence of z and y shows that Y is $C_1^{-1}C_2^{-1}(1+\varepsilon^{-1})^{-1}(\theta_m^{-1} - 1 - \varepsilon^{-1})$-distortable. Since this holds for infinitely many m, Y is arbitrarily distortable. ∎

COROLLARY 10. *Assume that* $(\mathcal{F}_n)$ *is a tame sequence. If* ξ *is a limit ordinal, the following statements hold.*

(1) *Every block subspace of* X *contains an* ℓ^1-$\mathcal{S}_{\omega^\xi}$*-spreading model.*
(2) *Every block subspace of* X *contains a block sequence equivalent to a subsequence of* (e_k).
(3) X *is arbitrarily distortable.*

Proof. If (z_n) is a normalized block sequence in X, and F is a set such that $\{\min \operatorname{supp} z_n\}_{n\in F} \in \mathcal{F}_m$, then $\|\sum a_n z_n\| \geq \theta_m \sum_F |a_n|$. In particular, $I_{\mathrm{b}}(Y, \theta_m^{-1}) \geq \alpha_m$ for all block subspaces Y of X and all m. By the proof of Theorem 1.1 in [14], if $I_{\mathrm{b}}(Y, K) \geq \alpha^2$, then $I_{\mathrm{b}}(Y, \sqrt{K}) \geq \alpha$. Now for any $\beta < \omega^\xi$, there exists m such that $\omega^{\beta\cdot\omega} < \alpha_m$. Thus $(\omega^\beta)^{2^k} < \alpha_m$ for all k. It follows that $I_{\mathrm{b}}(Y, \theta_m^{-1/2^k}) \geq \omega^\beta$. Hence $I_{\mathrm{b}}(Y, 1+\varepsilon) \geq \omega^\beta$ for any $\varepsilon > 0$ and any $\beta < \omega^\xi$. Therefore, $I_{\mathrm{b}}(Y, 1+\varepsilon) \geq \omega^{\omega^\xi}$ for any $\varepsilon > 0$. By [14, Lemma 5.7], $I_{\mathrm{b}}(Y) > \omega^{\omega^\xi}$. The conclusions of the corollary now follow from Theorem 9. ∎

PROPOSITION 11. *The sequence* $(\mathcal{S}_{\beta_n})$ *is tame for any sequence of nonzero countable ordinals* (β_n).

Proof. Let α be a nonzero countable ordinal. The fact that $\mathcal{S}_\alpha[\mathcal{A}_3] \subseteq (\mathcal{S}_\alpha)^2$ was shown in the Remark following Proposition 9 in [16]. We show that $(\mathcal{S}_\alpha \ominus \mathcal{S}_1)[\mathcal{A}_2] \subseteq \mathcal{S}_\alpha$ by induction on α. If $\alpha = 1$, this is clear. Assume that the inclusion holds for some α. Suppose $E \in (\mathcal{S}_{\alpha+1} \ominus \mathcal{S}_1)[\mathcal{A}_2]$. Then $E = \bigcup_{i=1}^k E_i$, $E_1 < \cdots < E_k$, $E_i \in \mathcal{A}_2$, and $F = \{\min E_i\}_{i=1}^k \in \mathcal{S}_{\alpha+1} \ominus \mathcal{S}_1$. There is a maximal $\mathcal{S}_1$ set G such that $G < F$ and $G \cup F \in \mathcal{S}_{\alpha+1}$. Let $\min G = n$. Then $|G| = n$ and hence $\min F \geq 2n$. Note that $F \subseteq G \cup F \in \mathcal{S}_{\alpha+1}$. Thus we may write F as $\bigcup_{j=1}^r H_j$, where $H_1 < \cdots < H_r$, $H_j \in \mathcal{S}_\alpha$, and $r \leq n$. Since $\mathcal{S}_\alpha[\mathcal{A}_2] \subseteq (\mathcal{S}_\alpha)^2$, $\bigcup\{E_i : \min E_i \in H_j\} \in (\mathcal{S}_\alpha)^2$ for all j. Therefore,

$$E \subseteq \bigcup_{j=1}^r \bigcup \{E_i : \min E_i \in H_j\} \in (\mathcal{S}_\alpha)^{2r}$$

and $2r \leq 2n \leq \min F = \min E$. Hence $E \in \mathcal{S}_{\alpha+1}$, as required.

Finally, suppose the inclusion holds for all $\alpha' < \alpha$, where α is a countable limit ordinal. Let (α_n) be the sequence of ordinals used to define $\mathcal{S}_\alpha$. If $E \in (\mathcal{S}_\alpha \ominus \mathcal{S}_1)[\mathcal{A}_2]$, then $E \in (\mathcal{S}_{\alpha_n} \ominus \mathcal{S}_1)[\mathcal{A}_2]$ for some $n \leq \min E$. Thus $E \in \mathcal{S}_{\alpha_n}$ for some $n \leq \min E$. Hence $E \in \mathcal{S}_\alpha$.

Observe that $\mathcal{S}_1 \subseteq \mathcal{S}_\alpha$ for any nonzero countable ordinal α. Therefore, if $n > 1$,

$$(\mathcal{S}_{\beta_n} \ominus \mathcal{S}_{\beta_1})[\mathcal{A}_2] \subseteq (\mathcal{S}_{\beta_n} \ominus \mathcal{S}_1)[\mathcal{A}_2] \subseteq \mathcal{S}_{\beta_n}. \ \blacksquare$$

THEOREM 12. *Let* (θ_n) *be a nonincreasing null sequence in* $(0,1)$ *and suppose that* (β_n) *is a sequence of ordinals such that* $\sup \beta_m = \omega^\xi > \beta_n > 0$

for all n, where $0 < \xi < \omega_1$. Let

$$\gamma(\varepsilon, m) = \max\{\beta_{n_s} + \cdots + \beta_{n_1} : \varepsilon\theta_{n_s} \cdots \theta_{n_1} > \theta_m\} \quad (\max\emptyset = 0).$$

The following are equivalent for any block subspace Y of $T[(\theta_n, \mathcal{S}_{\beta_n})_{n=1}^{\infty}]$.

(1) *There exists $\varepsilon > 0$ such that for all $\beta < \omega^\xi$, there exists $m \in \mathbb{N}$ such that $\gamma(\varepsilon, m) + 2 + \beta < \beta_m$ and every block subspace Z of Y contains a block sequence equivalent to a subsequence of (e_k).*
(2) *Every block subspace Z of Y contains an ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model.*
(3) *The Bourgain ℓ^1-index satisfies $I_{\mathrm{b}}(Z) = I(Z) > \omega^{\omega^\xi}$ for any block subspace Z of Y.*

If one (and hence all) of these conditions holds for a block subspace Y of $T[(\theta_n, \mathcal{S}_{\beta_n})_{n=1}^{\infty}]$, then Y is arbitrarily distortable. Moreover, all these equivalent conditions hold for the space $T[(\theta_n, \mathcal{S}_{\beta_n})_{n=1}^{\infty}]$ if ξ is a limit ordinal.

When considering the mixed Tsirelson space $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$, it is customary to assume without loss of generality that $\theta_{m+n} \geq \theta_m\theta_n$ for all m, n. In this case, it was shown in the proof of Corollary 28 in [16] that condition (†) is equivalent to $\lim_m \limsup_n \theta_{m+n}/\theta_n > 0$.

Corollary 13. *Let (θ_n) be a nonincreasing null sequence in $(0, 1)$ such that $\theta_{m+n} \geq \theta_m\theta_n$ for all m, n. The following are equivalent for any block subspace Y of $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$.*

(1) $\lim_m \limsup_n \theta_{m+n}/\theta_n > 0$ *and every block subspace Z of Y contains a block sequence equivalent to a subsequence of (e_k).*
(2) *Every block subspace Z of Y contains an ℓ^1-$\mathcal{S}_\omega$-spreading model.*
(3) *The Bourgain ℓ^1-index satisfies $I_{\mathrm{b}}(Z) = I(Z) > \omega^\omega$ for any block subspace Z of Y.*

If one (and hence all) of these conditions holds for a block subspace Y of $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$, then Y is arbitrarily distortable. Moreover, all these equivalent conditions hold for the space $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ if $\lim \theta_n^{1/n} = 1$.

Remark. The fact that every block subspace of $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ contains an ℓ^1-$\mathcal{S}_\omega$-spreading model if $\lim \theta_n^{1/n} = 1$ is due to Argyros, Deliyanni and Manoussakis [7, Proposition 3.1]. Androulakis and Odell [3] showed that if $\lim \theta_n/\theta^n = 0$, where $\theta = \lim \theta_n^{1/n}$, then $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ is arbitrarily distortable.

Proof of Corollary 13. It suffices to prove the "moreover" statement. Clearly every normalized block sequence in $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ is an ℓ^1-$\mathcal{S}_n$-spreading model with constant θ_n^{-1} for any n. Thus for any block subspace Y, $I_{\mathrm{b}}(Y, \theta_{m2^n}^{-1}) \geq \omega^{m2^n}$ for all m, n. By the proof of Theorem 1.1 in [14], it follows that $I_{\mathrm{b}}(Y, \theta_{m2^n}^{-1/2^n}) \geq \omega^m$. Using the hypothesis, we see that $I_{\mathrm{b}}(Y, 1+\varepsilon) \geq$

ω^m for all $\varepsilon > 0$ and all m. Hence $I_{\rm b}(Y, 1+\varepsilon) \geq \omega^\omega$. By [14, Lemma 5.7], $I_{\rm b}(Y) > \omega^\omega$. This proves condition (3). ∎

For an ordinal β with Cantor normal form $\beta = \omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_n} \cdot m_n$, call m_1 the *leading coefficient* of β. The preceding proof shows that if $(\mathcal{F}_n)$ is an increasing sequence of regular families so that $(\iota(\mathcal{F}_n))$ increases nontrivially to ω^{ω^ξ}, where ξ is a countable successor ordinal, and $\sup_n \theta_n^{1/k_n} = 1$, where k_n is the leading coefficient of $\ell(\iota(\mathcal{F}_n))$, then every block subspace of $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ contains an ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model.

LEMMA 14. *Let $\mathcal{F}$ be a regular family and let $M \in [\mathbb{N}]$.*

(1) *If $0 < \iota(\mathcal{F}) < \omega$, then there exists $N \in [M]$ such that $\mathcal{F} \cap [N]^{<\infty} = \mathcal{A}_j \cap [N]^{<\infty}$, where $j = \iota(\mathcal{F})$.*
(2) *If $\iota(\mathcal{F}) \geq \omega$, then there exists $N \in [M]$ with $\mathcal{F}[\mathcal{A}_3] \cap [N]^{<\infty} \subseteq (\mathcal{F})^2$.*
(3) *If $\iota(\mathcal{F}) \geq \omega$, then there exist $N \in [M]$ and $j \in \mathbb{N}$ with the property that $(\mathcal{F} \ominus \mathcal{A}_j)[\mathcal{A}_2] \cap [N]^{<\infty} \subseteq \mathcal{F}$.*

Proof. (1) Since $\iota(\mathcal{F}) = j$, $\mathcal{F} \subseteq \mathcal{A}_j$. On the other hand, choose $n_0 \in M$ such that $\{n_0\} \in \mathcal{F}^{(j-1)}$. If $j = 1$, then set $N = \{n_0, n_0+1, \ldots\} \cap M$. Clearly $\mathcal{A}_1 \cap [N]^{<\infty} \subseteq \mathcal{F}$. If $j > 1$, consider $\mathcal{G} = \{G : n_0 < G,\ \{n_0\} \cup G \in \mathcal{F}\}$. Then $\iota(\mathcal{G}) = j - 1$. Using induction, we obtain $N_1 \in [M]$ with $n_0 < N_1$ such that $\mathcal{G} \cap [N_1]^{<\infty} = \mathcal{A}_{j-1} \cap [N_1]^{<\infty}$. Let $N = \{n_0\} \cup N_1$. If $F \in \mathcal{A}_j \cap [N]^{<\infty}$, then F is a spreading of $\{n_0\} \cup G$ for some $G \in \mathcal{A}_{j-1} \cap [N_1]^{<\infty} = \mathcal{G} \cap [N_1]^{<\infty}$. Hence $F \in \mathcal{F}$.

(2) This follows from [11, Theorem 1.1] since $\iota(\mathcal{F}[\mathcal{A}_3]) \leq \iota(\mathcal{A}_3) \cdot \iota(\mathcal{F}) < \iota(\mathcal{F}) \cdot 2 = \iota((\mathcal{F})^2)$.

(3) Write $\iota(\mathcal{F}) = \alpha + (j-1)$ for some limit ordinal α and some $j \in \mathbb{N}$. It is readily verified that $\mathcal{F} \ominus \mathcal{A}_j$ is a regular family and that $(\mathcal{F} \ominus \mathcal{A}_j)^{(\beta)} \subseteq \mathcal{F}^{(\beta)} \ominus \mathcal{A}_j$ for any β. If $\iota(\mathcal{F} \ominus \mathcal{A}_j) \geq \alpha$, then $\emptyset \in \mathcal{F}^{(\alpha)} \ominus \mathcal{A}_j$. Hence there exists A with $|A| = j$ such that $A \in \mathcal{F}^{(\alpha)}$. But then $\iota(\mathcal{F}) \geq \alpha + j$, a contradiction. Thus $\iota(\mathcal{F} \ominus \mathcal{A}_j) < \alpha$. It follows that $\iota((\mathcal{F} \ominus \mathcal{A}_j)[\mathcal{A}_2]) < 2 \cdot \alpha = \alpha \leq \iota(\mathcal{F})$. By [11, Theorem 1.1], there exists $N \in [M]$ such that $(\mathcal{F} \ominus \mathcal{A}_j)[\mathcal{A}_2] \cap [N]^{<\infty} \subseteq \mathcal{F}$. ∎

Given a regular family $\mathcal{F}$ and $M = (p_k) \in [\mathbb{N}]$, define the family ${}^M\mathcal{F}$ by ${}^M\mathcal{F} = \{F : (p_k)_{k \in F} \in \mathcal{F}\}$. It is clear that ${}^M\mathcal{F}$ is a regular family. Furthermore, the subspace $[(e_k)_{k \in M}]$ of $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ is easily seen to coincide with the mixed Tsirelson space $T[(\theta_n, {}^M\mathcal{F}_n)_{n=1}^\infty]$ under a natural identification. The next proposition shows that the tameness of the sequence of regular families is not a restriction if one is allowed to pass to a subsequence of the unit vector basis.

PROPOSITION 15. *There exists $M \in [\mathbb{N}]$ and a tame sequence of regular families $(\mathcal{G}_n)$ such that $T[(\theta_n, {}^M\mathcal{F}_n)_{n=1}^\infty]$ is isomorphic to $T[(\theta_n, \mathcal{G}_n)_{n=1}^\infty]$ via the formal identity.*

Proof. Recall the assumption that (α_n) increases nontrivially to ω^{ω^ξ}, $\xi > 0$. Let m_0 be the largest number such that $\alpha_{m_0} \le \omega$. (Take m_0 to be 0 if $\alpha_n > \omega$ for all n.) Choose a strictly increasing sequence $(m_k)_{k=1}^\infty$ such that $m_1 > m_0$ and $\theta_{m_{k+1}} \le \theta_{m_k}/2$ for all $k \in \mathbb{N}$. By (1) and (2) of Lemma 14, there exists $M_0 \in [\mathbb{N}]$ such that for each $n \le m_0$, either $\mathcal{F}_n \cap [M_0]^{<\infty} = \mathcal{A}_j \cap [M_0]^{<\infty}$ for some j, or $\mathcal{F}_n[\mathcal{A}_3] \cap [M_0]^{<\infty} \subseteq (\mathcal{F}_n)^2$. It is possible to choose a decreasing sequence $(M_k)_{k=1}^\infty$ of infinite subsets of M_0 and a sequence $(r_k)_{k=1}^\infty$ in $\mathbb{N}$ so that whenever $m_{k-1} < n \le m_k$, $k \in \mathbb{N}$,

(1) $\mathcal{S}_1 \cap [M_k]^{<\infty} \subseteq \mathcal{F}_n$, by [11, Theorem 1.1] since $\iota(\mathcal{S}_1) = \omega < \iota(\mathcal{F}_n)$,
(2) $\mathcal{F}_n[\mathcal{A}_3] \cap [M_k]^{<\infty} \subseteq (\mathcal{F}_n)^2$, by (2) of Lemma 14,
(3) $(\mathcal{F}_n \ominus \mathcal{A}_{r_k})[\mathcal{A}_2] \cap [M_k]^{<\infty} \subseteq \mathcal{F}_n$, by (3) of Lemma 14.

Choose a strictly increasing sequence $(p_k)_{k=1}^\infty$ so that $r_k \le p_k \in M_k$ for all $k \in \mathbb{N}$. Define $M = (p_k)$ and set $\mathcal{G}_n = {}^M\mathcal{F}_n$ if $n \le m_0$ and $\mathcal{G}_n = \{G \in {}^M\mathcal{F}_n : G \ge k\}$ if $m_{k-1} < n \le m_k$, $k \in \mathbb{N}$. By [16, Proposition 1], $T[(\theta_n, {}^M\mathcal{F}_n)_{n=1}^\infty]$ is isomorphic to $T[(\theta_n, \mathcal{G}_n)_{n=1}^\infty]$ via the formal identity. It remains to show that the sequence $(\mathcal{G}_n)$ is tame.

First suppose that $n \le m_0$. If $\mathcal{F}_n \cap [M_0]^{<\infty} = \mathcal{A}_j \cap [M_0]^{<\infty}$ for some j, then clearly $\mathcal{G}_n = {}^M\mathcal{F}_n = \mathcal{A}_j$. Otherwise, $\mathcal{F}_n[\mathcal{A}_3] \cap [M]^{<\infty} \subseteq \mathcal{F}_n[\mathcal{A}_3] \cap [M_0]^{<\infty} \subseteq (\mathcal{F}_n)^2$. If $G \in \mathcal{G}_n[\mathcal{A}_3] = {}^M\mathcal{F}_n[\mathcal{A}_3]$, then $(p_k)_{k\in G} \in \mathcal{F}_n[\mathcal{A}_3] \cap [M]^{<\infty} \subseteq (\mathcal{F}_n)^2$. Hence $G \in ({}^M\mathcal{F}_n)^2$.

Now assume that $n > m_0$. Choose k such that $m_{k-1} < n \le m_k$. If $G \in \mathcal{G}_n[\mathcal{A}_3]$, then $G \in {}^M\mathcal{F}_n[\mathcal{A}_3]$ and $G \ge k$. Hence $p_k \le (p_i)_{i\in G} \in \mathcal{F}_n[\mathcal{A}_3]$. Thus $(p_i)_{i\in G} \in \mathcal{F}_n[\mathcal{A}_3] \cap [M_k]^{<\infty} \subseteq (\mathcal{F}_n)^2$. Therefore $G \in ({}^M\mathcal{F}_n)^2$ and $G \ge k$. It follows that $G \in (\mathcal{G}_n)^2$.

Finally, we show that $(\mathcal{G}_n \ominus \mathcal{G}_m)[\mathcal{A}_2] \subseteq \mathcal{G}_n$ whenever $n > m > m_0$. Choose k and l such that and $m_{k-1} < n \le m_k$ and $m_{l-1} < m \le m_l$. Suppose that $G \in \mathcal{G}_n \ominus \mathcal{G}_m$. There is a maximal $H \in \mathcal{G}_m$ such that $H < G$ and $H \cup G \in \mathcal{G}_n$.

We claim that $|H| \ge r_k$. Indeed, by definition of $\mathcal{G}_n$, $H \ge k$. Thus $r_k \le p_k \le (p_i)_{i\in H}$. If $|H| < r_k$, there exists a nonempty set $I > H$ such that $(p_i)_{i\in H\cup I} \in \mathcal{S}_1$. Clearly, $(p_i)_{i\in H\cup I} \in [M_k]^{<\infty} \subseteq [M_l]^{<\infty}$ as well. Therefore, $(p_i)_{i\in H\cup I} \in \mathcal{F}_m$ by condition (1) above. By definition, $H \cup I \in {}^M\mathcal{F}_m$. Since $H \cup I \ge k \ge l$, $H \cup I \in \mathcal{G}_m$, contrary to the maximality of H. This proves the claim.

It follows from the claim that $(p_i)_{i\in G} \in \mathcal{F}_n \ominus \mathcal{A}_{r_k}$. Therefore, $(p_i)_{i\in J} \in (\mathcal{F}_n \ominus \mathcal{A}_{r_k})[\mathcal{A}_2]$ for all $J \in (\mathcal{G}_n \ominus \mathcal{G}_m)[\mathcal{A}_2]$. Clearly, for such J, $J \ge k$ and hence $(p_i)_{i\in J}$ is in $[M_k]^{<\infty}$. Therefore,

$$(p_i)_{i\in J} \in (\mathcal{F}_n \ominus \mathcal{A}_{r_k})[\mathcal{A}_2] \cap [M_k]^{<\infty} \subseteq \mathcal{F}_n$$

by condition (3) above. This shows that $J \in {}^M\mathcal{F}_n$. As $J \ge k$, we have $J \in \mathcal{G}_n$, as desired. ∎

COROLLARY 16. *Suppose that either* (a) ξ *is a countable limit ordinal or* (b) ξ *is a countable successor ordinal and* $\sup_n \theta_n^{1/k_n} = 1$, *where* k_n *is the leading coefficient of* $\ell(\alpha_n)$. *Then there exists* $M \in [\mathbb{N}]$ *such that the subspace* $Y = [(e_k)_{k\in M}]$ *of* X *has the following properties.*

(1) *Every block subspace of* Y *has an* ℓ^1-$\mathcal{S}_{\omega^\xi}$*-spreading model.*
(2) *Every block subspace of* Y *contains a block sequence equivalent to a subsequence of* $(e_k)_{k\in M}$.
(3) Y *is arbitrarily distortable.*

Schlumprecht proposed a classification of Banach spaces as follows [23]. A Banach space with a normalized basis (u_k) is said to be *Class* 1 if every normalized block sequence has a subsequence equivalent to a subsequence of (u_k). It is *Class* 2 if every block subspace contains two block sequences (y_k) and (z_k) so that the map $y_k \mapsto z_k$ extends to a bounded linear strictly singular operator. Recall that an operator is *strictly singular* if its restriction to any infinite-dimensional subspace is not an isomorphism. Schlumprecht asks whether every infinite-dimensional Banach space contains a subspace with a basis that is either Class 1 or Class 2. He also proved a criterion for a Banach space to be Class 2 [23, Theorem 1.4 and Corollary 1.5]. We conclude with a note showing that his proof applies to mixed Tsirelson spaces satisfying the conditions of Theorem 9. A Banach space is c_0-*saturated* if every closed infinite-dimensional subspace contains an isomorphic copy of c_0.

PROPOSITION 17. *Let* Y *be a block subspace of a mixed Tsirelson space* X *and suppose that* Y *satisfies all the conditions of Theorem* 9. *Then* Y *is a Class* 2 *space.*

Proof. Denote by (e_k) the unit vector basis of X. We will show below that there are a regular family $\mathcal{G}$ with $\iota(\mathcal{G}) \le \omega^{\omega^\xi}$ and a finite constant C so that
$$\Big\|\sum a_k e_k\Big\| \le C \sup_{G\in\mathcal{G}} \sum_{k\in G} |a_k| \quad \text{for all } (a_k) \in c_{00}.$$
Denote the unit vector basis in c_{00} by (u_k) and let U be the completion of c_{00} with respect to the norm $\|\sum a_k u_k\| = \sup_{G\in\mathcal{G}} \sum_{k\in G} |a_k|$ for all $(a_k) \in c_{00}$. The map that sends $\sum a_k u_k$ to the function on $\mathcal{G}$ given by $G \mapsto \sum_{k\in G} a_k$ is an embedding of U into $C(\mathcal{G})$, the space of continuous functions on the countable compact metric space $\mathcal{G}$. Hence U is c_0-saturated. Let Z be a block subspace of Y. By the hypothesis, there is a block sequence (z_k) in Z that is equivalent to a subsequence (e_{m_k}) of (e_k). Also, there is a sequence (y_k) in Z that generates an ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model. We may replace (y_k) with an appropriate subsequence of $(y_{2k} - y_{2k+1})$ if necessary to assume that (y_k) is equivalent to a block sequence. By definition of the norm in X, there is a positive constant K so that $\|\sum a_k y_k\| \ge K^{-1} \sum_{k\in F} |a_k|$ for all

$F \in \mathcal{F}_1[\mathcal{S}_{\omega^\xi}]$. Since $\iota(\mathcal{F}_1) > 1$ by assumption, $\iota(\mathcal{F}_1[\mathcal{S}_{\omega^\xi}]) > \omega^{\omega^\xi} \geq \iota(\mathcal{G})$. Using [11, Theorem 1.1] and replacing $M = (m_k)$ with a subsequence if necessary, we may assume that $\mathcal{G} \cap [M]^{<\infty} \subseteq \mathcal{F}_1[\mathcal{S}_{\omega^\xi}]$. Because (z_k) is equivalent to (e_{m_k}) and (y_k) is equivalent to a block sequence, it follows that the map $y_{m_k} \mapsto z_k$ extends to a bounded linear map $T : [(y_{m_k})] \to [(z_k)]$. Now, for all $(a_k) \in c_{00}$,

$$\Big\| \sum a_k u_{m_k} \Big\| = \sup_{G \in \mathcal{G}} \sum_{m_k \in G} |a_k| \leq \sup_{G \in \mathcal{F}_1[\mathcal{S}_{\omega^\xi}]} \sum_{m_k \in G} |a_k| \leq K \Big\| \sum a_k y_{m_k} \Big\|.$$

Hence $y_{m_k} \mapsto u_{m_k}$ extends to a bounded linear map $S : [(y_{m_k})] \to [(u_{m_k})]$. However, (z_k) is equivalent to (e_{m_k}) and

$$\Big\| \sum a_k e_{m_k} \Big\| \leq C \sup_{G \in \mathcal{G}} \sum_{m_k \in G} |a_k| = C \Big\| \sum a_k u_{m_k} \Big\|.$$

Thus $u_{m_k} \mapsto z_k$ extends to a bounded linear map $R : [(u_{m_k})] \to [(z_k)]$. Therefore, $T = RS$ is a factorization of T through the c_0-saturated space $[(u_{m_k})]$. Since $[(y_{m_k})]$ does not contain a copy of c_0, T is strictly singular.

It remains to show the existence of the family $\mathcal{G}$. Choose a strictly increasing sequence (n_i) such that $\pi_i < 2^{-i}$ for all i, where

$$\pi_i = \max\{\theta_{m_1} \cdots \theta_{m_r} : m_1 + \cdots + m_r > n_i\}.$$

For each i, let $\mathcal{G}_i = \bigcup\{[\mathcal{F}_{m_r}, \ldots, \mathcal{F}_{m_1}] : m_1 + \cdots + m_r \leq n_i\}$. Here $[\mathcal{F}_{m_r}, \ldots, \mathcal{F}_{m_1}]$ is defined inductively as $\mathcal{F}_{m_r}[\mathcal{F}_{m_{r-1}}, \ldots, \mathcal{F}_{m_1}]$. It follows from [15, Proposition 12] that $\iota(\mathcal{G}_i) < \omega^{\omega^\xi}$ since $\iota(\mathcal{F}_n) < \omega^{\omega^\xi}$ for each n. Let $\mathcal{G}$ consist of all sets G such that $G \in \mathcal{G}_i$ for some $i \leq G$ together with all singletons. Then $\iota(\mathcal{G}) \leq \omega^{\omega^\xi}$. Indeed, let $\widetilde{\mathcal{G}}_i = \{G \in \mathcal{G}_i : i \leq G\}$. Then $\mathcal{G} = \mathcal{S}_0 \cup \bigcup \widetilde{\mathcal{G}}_i$. If $G \in \mathcal{G}^{(1)}$, then either $G \in \mathcal{S}_0^{(1)} = \{\emptyset\}$ or there exists a sequence (G_n) converging pointwise to G such that $G_n \neq G$ and $G_n \in \widetilde{\mathcal{G}}_{i_n}$ for some i_n. In particular, $i_n \leq \min G_n = \min G$ for all sufficiently large n. It follows that (i_n) must be bounded. Therefore $G \in \widetilde{\mathcal{G}}_{i_0}^{(1)}$ for some i_0. This shows that $\mathcal{G}^{(1)} \subseteq \bigcup \widetilde{\mathcal{G}}_i^{(1)}$. By induction, $\mathcal{G}^{(\alpha)} \subseteq \bigcup \widetilde{\mathcal{G}}_i^{(\alpha)}$ for all $\alpha < \omega_1$. Hence $\iota(\mathcal{G}) \leq \omega^{\omega^\xi}$.

For any $x = \sum a_k e_k$ with $(a_k) \in c_{00}$, let $\mathcal{T}$ be an admissible tree that norms x. Denote by $\mathcal{E}$ the set of all leaves of $\mathcal{T}$. Also, if $t(E) = \theta_{m_1} \cdots \theta_{m_r}$, $E \in \mathcal{E}$, set $r(E) = m_1 + \cdots + m_r$. Note that $\{E \in \mathcal{E} : r(E) \leq n_i\}$ is $\mathcal{G}_i$-admissible. Thus

$$\begin{aligned} \mathcal{T}x = \sum_{E \in \mathcal{E}} t(E) \|Ex\|_{c_0} &= \sum_{i=1}^{\infty} \sum_{n_{i-1} < r(E) \leq n_i} t(E) \|Ex\|_{c_0} \\ &\leq \sum_{i=1}^{\infty} \pi_{i-1} \varrho_i(x), \end{aligned}$$

where $\varrho_i(x) = \sup_{G\in\mathcal{G}_i} \sum_{k\in G} |a_k|$. However,

$$\varrho_i(x) \le \sum_{k=1}^{i} |a_k| + \sup_{G\in\mathcal{G}_i} \sum_{k\in G, k>i} |a_k| \le i\|x\|_{c_0} + \sup_{G\in\mathcal{G}} \sum_{k\in G} |a_k|.$$

Therefore,

$$\begin{aligned}\|x\| &\le \sum_{i=1}^{\infty} \pi_{i-1}\varrho_i(x) \le \sum_{i=1}^{\infty} \frac{\varrho_i(x)}{2^{i-1}} \\ &\le \|x\|_{c_0} \sum_{i=1}^{\infty} \frac{i}{2^{i-1}} + \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \sup_{G\in\mathcal{G}} \sum_{k\in G} |a_k| \\ &\le 6 \sup_{G\in\mathcal{G}} \sum_{k\in G} |a_k|.\end{aligned}$$

This completes the proof. ■

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