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# Compactness of Sobolev imbeddings involving rearrangement-invariant norms

by

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**Abstract.** We find necessary and sufficient conditions on a pair of rearrangement-invariant norms, $\varrho$ and $\sigma$, in order that the Sobolev space $W^{m,\varrho}(\Omega)$ be compactly imbedded into the rearrangement-invariant space $L_\sigma(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary and $1 \le m \le n-1$. In particular, we establish the equivalence of the compactness of the Sobolev imbedding with the compactness of a certain Hardy operator from $L_\varrho(0,|\Omega|)$ into $L_\sigma(0,|\Omega|)$. The results are illustrated with examples in which $\varrho$ and $\sigma$ are both Orlicz norms or both Lorentz Gamma norms.

**1. Introduction.** Sobolev spaces are one of the key elements of modern functional analysis. In applications their most important property is how they imbed into various function spaces. To be more specific, compactness of Sobolev imbeddings is useful in the theory of PDEs; indeed, it is quite indispensable when the methods of the calculus of variations are used.

Among the function norms defining both the Sobolev and the imbedding spaces those of Lebesgue play a primary role, though a satisfactory description of all cases, especially the limiting ones, requires other, more delicate, norms.

In this paper we characterize precisely when a Sobolev space defined by a rearrangement-invariant (r.i.) norm is compactly imbedded into a function space determined by another such norm. We will use the interpolation methods developed in our previous papers [8], [13] and [14], in which optimal (hence noncompact) imbeddings were studied.

Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \ge 2$, having a Lipschitz boundary, written $\partial\Omega \in \mathrm{Lip}_1$.

Fix $m \in \mathbb{Z}_+$, $1 \le m \le n-1$ and let $N = N(m,n) = \sum_{0\le|\alpha|\le m} 1$ be the number of multiindices $\alpha = (\alpha_1,\ldots,\alpha_k)$ satisfying $0 \le |\alpha| := \alpha_1 + \cdots + \alpha_k \le m$.

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Given a locally integrable function $u : \Omega \to \mathbb{R}$ with weak derivatives of all orders $\le m$, denote by $D^m u$ the $N$-vector $(\partial^\alpha u/\partial x^\alpha)_{0\le|\alpha|\le m}$ of all those derivatives and by $|D^m u|$ the Euclidean length of this vector as an element of $\mathbb{R}^N$.

The Lebesgue norms of $f \in \mathfrak{M}(\Omega)$, the class of real-valued, measurable functions on $\Omega$, are defined by

$$\varrho_p(f) := \Big(\int_\Omega |f(x)|^p\,dx\Big)^{1/p}, \qquad 1 \le p < \infty, \qquad \varrho_\infty(f) := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

Using these norms we define the Lebesgue spaces

$$L_p(\Omega) := \{f \in \mathfrak{M}(\Omega) : \varrho_p(f) < \infty\}$$

and the Sobolev spaces

$$W^{m,p}(\Omega) := \{u : \Omega \to \mathbb{R} : D^m u \text{ is defined and } \varrho_p(|D^m u|) < \infty\}.$$

One form of the Sobolev imbedding theorem states that

$$\varrho_q(u) \le C\varrho_p(|D^m u|), \tag{1.1}$$

where $1 \le p < n/m$, $q \le np/(n-mp)$ and $C > 0$ is independent of $u \in W^{m,p}(\Omega)$. We express (1.1) in the form

$$W^{m,p}(\Omega) \hookrightarrow L_q(\Omega). \tag{1.2}$$

A theorem, which originated in a lemma of Rellich [21] and was proved specifically for Sobolev spaces by Kondrashov [15], asserts that the imbedding (1.2) is compact if $q < np/(n-mp)$, a fact we denote by

$$W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega). \tag{1.3}$$

Standard examples (see [1] or [16]) show it is not compact when $q = np/(n-mp)$.

Our goal is to obtain criteria to determine if one has a compact imbedding, such as (1.3), when the $\varrho_p$ are replaced by more general function norms $\varrho$ having the property of rearrangement-invariance, that is,

$$\varrho(f) = \varrho(g) \quad \text{whenever} \quad f, g \in \mathfrak{M}(\Omega) \text{ and } f^* = g^*,$$

where

$$f^*(t) := \inf\{\lambda > 0 : |\{x \in \Omega : |f(x)| > \lambda\}| \le t\}, \qquad 0 < t < |\Omega|,$$

is called the *nonincreasing rearrangement* of $f$ on $(0, |\Omega|)$; here, $|\Omega|$ denotes the Lebesgue measure of $\Omega$.

One typically starts with an r.i. norm, $\bar\varrho$, defined on $\mathfrak{M}(0, |\Omega|)$; then, for $f : \Omega \to \mathbb{R}$ one sets

$$\varrho(f) := \bar\varrho(f^*). \tag{1.4}$$

We will show, among other things, that, with an obvious extension of notation from the Lebesgue case,

$$W^{m,\varrho}(\Omega) \hookrightarrow\hookrightarrow L_\sigma(\Omega)$$

if and only if the rather simple Hardy operator, $H_{n/m}$, defined at $f \in \mathfrak{M}(0,|\Omega|)$ by

$$(H_{n/m}f)(t) := \int_t^{|\Omega|} f(s)s^{m/n-1}\,ds, \qquad 0<t<|\Omega|,$$

is compact from $L_{\overline{\varrho}}(0,|\Omega|)$ to $L_{\overline{\sigma}}(0,|\Omega|)$, which will be indicated by

$$H_{n/m} : L_{\overline{\varrho}}(0,|\Omega|) \to\to L_{\overline{\sigma}}(0,|\Omega|).$$

We now state our main result.

THEOREM 1.1. *Fix a bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in \mathrm{Lip}_1$. Let $\varrho$ and $\sigma$ be defined on $\mathfrak{M}(0,|\Omega|)$ in terms of the r.i. norms $\overline{\varrho}$ and $\overline{\sigma}$ on $\mathfrak{M}(0,|\Omega|)$, as in* (1.4). *Assume $L_\sigma(\Omega) \neq L_\infty(\Omega)$. Then the following are equivalent*:

$$W^{m,\varrho}(\Omega) \hookrightarrow\hookrightarrow L_\sigma(\Omega); \tag{1.5}$$

$$H_{n/m} : L_{\overline{\varrho}}(0,|\Omega|) \to\to L_{\overline{\sigma}}(0,|\Omega|); \tag{1.6}$$

$$\lim_{a\to 0+} \sup_{\overline{\varrho}(f)\le 1} \overline{\sigma}\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big) = 0. \tag{1.7}$$

The case $L_\sigma(\Omega) = L_\infty(\Omega)$ is different and, as such, is treated in

THEOREM 1.2. *Fix a bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in \mathrm{Lip}_1$. Let $\varrho$ be defined on $\mathfrak{M}(0,|\Omega|)$ in terms of the r.i. norm $\overline{\varrho}$ on $\mathfrak{M}(0,|\Omega|)$, as in* (1.4). *Then the following are equivalent*:

$$W^{m,\varrho}(\Omega) \hookrightarrow\hookrightarrow L_\infty(\Omega); \tag{1.8}$$

$$H_{n/m} : L_{\overline{\varrho}}(0,|\Omega|) \to\to L_\infty(0,|\Omega|); \tag{1.9}$$

$$\lim_{a\to 0+} \sup_{\overline{\varrho}(f)\le 1} \int_0^a f^*(t)t^{m/n-1}\,dt = 0. \tag{1.10}$$

A brief outline of the paper follows. The next section introduces r.i. spaces and Lipschitz domains. In Section 3 we prove, in Theorem 3.1, that one may assume a candidate compact imbedding space has certain interpolation properties. This result seems to be of independent interest; it is crucial in the proof that (1.7) implies (1.5).

The proofs of Theorems 1.1 and 1.2 occupy Section 4. As an application of Theorem 1.1, we relate, in Theorem 5.1, a compact imbedding space of $W^{m,\varrho}(\Omega)$ to its optimal imbedding space, providing thereby a new necessary and sufficient condition for an imbedding to be compact. Also given in Section 5 is a simple sufficient condition for the imbedding (1.2) to be compact, one expressed in terms of the Marcinkiewicz space built on the optimal imbedding space of $W^{m,\varrho}(\Omega)$. Applications of the main theorems to Lorentz Gamma and Orlicz spaces complete the section.

Recently, two papers, [6] and [20], on the compactness of Sobolev imbeddings involving r.i. norms have come to our attention. We discuss the relation of their results to ours in the final section.

**2. Preliminaries.** Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$. Let $\mathfrak{M}(\Omega)$ be the class of real-valued, measurable functions on $\Omega$ and $P(\Omega)$ the class of nonnegative functions in $\mathfrak{M}(\Omega)$.

DEFINITION 2.1. A *Banach function norm* $\varrho$ on $P(\Omega)$ satisfies the following six axioms:

($A_1$) $\varrho(f) \geq 0$, with $\varrho(f) = 0$ if and only if $f = 0$ a.e. on $\Omega$;
($A_2$) $\varrho(cf) = c\varrho(f)$, $c \geq 0$;
($A_3$) $\varrho(f+g) \leq \varrho(f) + \varrho(g)$;
($A_4$) $f_n \uparrow f$ implies $\varrho(f_n) \uparrow \varrho(f)$;
($A_5$) $\varrho(\chi_\Omega) < \infty$;
($A_6$) $\int_\Omega f(x)\,dx \leq C\varrho(f)$, with $C > 0$ independent of $f \in P(\Omega)$.

If, in addition,

($A_7$) $\varrho(f) = \varrho(g)$ whenever $f^* = g^*$,

then $\varrho$ is said to be a *rearrangement-invariant* (r.i.) Banach function norm.

We extend $\varrho$ to $\mathfrak{M}(\Omega)$ by

$$\varrho(f) := \varrho(|f|).$$

Luxemburg has shown (see [2, Chapter 2, Theorem 4.10]) that every r.i. norm $\varrho$ on $\mathfrak{M}(\Omega)$ can be defined in terms of another r.i. norm, $\overline{\varrho}$, on $\mathfrak{M}(0,|\Omega|)$, by

$$\varrho(f) = \overline{\varrho}(f^*).$$

Such a $\overline{\varrho}$ will be introduced without comment in the rest of the paper.

The *Köthe dual* of an r.i. norm $\varrho$ is another such norm, $\varrho'$, with

$$\varrho'(g) := \sup_{\varrho(h)\leq 1} \int_\Omega g(x)h(x)\,dx, \qquad g,h \in P(\Omega).$$

It obeys the *principle of duality*, namely,

$$\varrho'' := (\varrho')' = \varrho.$$

Moreover, the *Hölder inequality*,

$$\int_\Omega f(x)g(x)\,dx \leq \varrho(f)\varrho'(g), \tag{2.1}$$

holds for all $f, g \in P(\Omega)$. Now, if

$$\varrho(f) = \bar{\varrho}(f^*), \qquad f \in \mathfrak{M}(\Omega),$$

then

$$\varrho'(g) = \bar{\varrho}'(g^*), \qquad g \in \mathfrak{M}(\Omega).$$

This leads to the following refinement of (2.1):

$$\int_{\Omega} f(x)g(x)\,dx \le \int_0^{|\Omega|} f^*(t)g^*(t)\,dt \le \bar{\varrho}(f^*)\bar{\varrho}'(g^*), \qquad f, g \in P(\Omega); \tag{2.2}$$

see [2, Chapter 2, Theorem 2.2].

A basic tool for working with a rearrangement-invariant norm $\varrho$ on $\mathfrak{M}(\Omega)$ is the *Hardy–Littlewood–Pólya* (HLP) principle, treated in [2, Chapter 2, Theorem 4.6]). It asserts that

$$f^{**} \le g^{**} \quad \text{implies} \quad \varrho(f) \le \varrho(g), \qquad f, g \in \mathfrak{M}(\Omega),$$

where $f^{**}$ is the *maximal nonincreasing rearrangement* of $f$,

$$f^{**}(t) := t^{-1} \int_0^t f^*(s)\,ds, \qquad 0 < t < |\Omega|.$$

The *fundamental function* of an r.i. norm $\bar{\varrho}$ on $\mathfrak{M}(0, |\Omega|)$ is defined at $t \in (0, |\Omega|)$ by $\bar{\varrho}(\chi_{(0,t)})$. This function is equivalent to one that is concave, and satisfies

$$\bar{\varrho}(\chi_{(0,t)})\bar{\varrho}'(\chi_{(0,t)}) = t, \qquad t \in (0, |\Omega|); \tag{2.3}$$

see [2, Chapter 2, Theorem 5.2 and Corollary 5.3]. Given (2.3), it is not hard to show $\lim_{t\to 0+} \bar{\varrho}(\chi_{(0,t)}) = 0$, unless $\varrho \approx \varrho_\infty$. (An expression of the form $X \approx Y$ means that each of $X$ and $Y$ is dominated by a constant times the other, the constants being independent of all functions involved. More generally, $X \lesssim Y$ means $X$ is dominated by a constant times $Y$, the constant being independent of all functions involved.)

The *dilation operator* $E_s$, $s \in \mathbb{R}_+$, given at $f \in \mathfrak{M}(0, |\Omega|)$ by

$$(E_s f)(t) := \begin{cases} f(t/s), & 0 < t \le |\Omega| s, \\ 0, & t > |\Omega| s, \end{cases}$$

is bounded on any r.i. space $L_\varrho(0, |\Omega|)$, and its operator norm, denoted $h_\varrho(s)$, satisfies $h_\varrho(s) > 1$ for $s > 1$, when $\varrho \not\approx \varrho_\infty$; see [2, Chapter 3, Proposition 5.11].

We define the *lower and upper Boyd indices* of $L_\varrho(0, |\Omega|)$ as

$$i_\varrho := \lim_{s\to\infty} \frac{\log s}{\log h_\varrho(s)} \quad \text{and} \quad I_\varrho := \lim_{s\to 0+} \frac{\log s}{\log h_\varrho(s)},$$

respectively. They satisfy

$$1 \le i_\varrho \le I_\varrho \le \infty.$$

See [17, Vol. II, pp. 131–132].

From [2, Chapter 3, Theorem 5.15], the Hardy operator

$$(Pf)(t) := t^{-1}\int_0^t f(s)\,ds, \qquad f \in \mathfrak{M}(0,|\Omega|),$$

satisfies

$$P : L_\varrho(0,|\Omega|) \to L_\varrho(0,|\Omega|) \quad \text{if and only if} \quad i_\varrho > 1. \tag{2.4}$$

A function $f \in \mathfrak{M}(\Omega)$ is said to be *absolutely continuous with respect to the Banach function norm* $\varrho$ if $\varrho(f\chi_{E_n}) \downarrow 0$ for every sequence $\{E_n\}_{n=1}^\infty$ of measurable subsets of $\Omega$ satisfying $E_n \downarrow \emptyset$. In case $\varrho$ is an r.i. norm, the above condition is readily seen to be equivalent to

$$\lim_{a\to 0+} \bar\varrho(\chi_{(0,a)}f^*) = 0.$$

The first example of what we now call r.i. norms on $\mathfrak{M}(\Omega)$ were the Lebesgue norms $\varrho_p$, defined in the introduction, since

$$\varrho_p(f) = \begin{cases} \Big(\int_0^{|\Omega|} f^*(t)^p\,dt\Big)^{1/p} & \text{when } p < \infty, \\ f^*(0+) & \text{when } p = \infty. \end{cases}$$

It follows from the classical Hölder inequality that $\varrho_p' = \varrho_{p'}$, where $p' = p/(p-1)$ (with the usual modifications when $p = 1$ or $\infty$).

We observe that $(A_4)$, $(A_5)$ and $(A_6)$ ensure

$$L_\infty(\Omega) \subset L_\varrho(\Omega) \subset L_1(\Omega)$$

for any r.i. norm $\varrho$ on $\mathfrak{M}(\Omega)$.

Closely related to the Lebesgue norms are the Lorentz norms

$$\varrho_{p,q}(f) := \Big(\int_0^{|\Omega|} f^{**}(t)^q t^{q/p-1}\,dt\Big)^{1/q} \quad \text{when } 1<p<\infty,\ 1\le q<\infty,\ f\in\mathfrak{M}(\Omega).$$

In addition, one defines

$$\varrho_{p,\infty}(f) := \sup_{0<t<|\Omega|} t^{1/p} f^{**}(t), \qquad 1<p<\infty,\ f\in\mathfrak{M}(\Omega).$$

The corresponding Lorentz spaces are denoted by $L_{p,q}(\Omega)$ and one has, by the Hardy inequality,

$$\|f\|_{L_{p,q}(\Omega)} \approx \begin{cases} \Big(\int_0^{|\Omega|} f^*(t)^q t^{q/p-1}\,dt\Big)^{1/q} & \text{when } 1 \le q < \infty, \\ \sup_{0<t<|\Omega|} t^{1/p} f^*(t) & \text{when } q = \infty. \end{cases}$$

Finally, we recall what it means for a bounded domain to have a Lipschitz boundary.

DEFINITION 2.2. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to have a *Lipschitz boundary*, written $\partial\Omega \in \mathrm{Lip}_1$, if there exist a finite number of open parallelepipeds $V_j$, $j = 1, \ldots, s$, and a constant $d_0 > 0$ satisfying

$$\Omega \subset \bigcup_{j=1}^{s} \{x \in V_j : d(x, \partial V_j) \geq d_0\}, \qquad \{x \in V_j : d(x, \partial V_j) \geq d_0\} \cap \Omega \neq \emptyset$$

and certain additional properties. Namely, one has maps $\lambda_j$ that are compositions of rotations, reflections and translations, for which

$$\lambda_j(V_j) = \{x \in \mathbb{R}^n : a_{ij} < x_i < b_{ij},\, i = 1, \ldots, n\}, \qquad j = 1, \ldots, s;$$

also, functions $\phi_j$ on $\overline{W}_j$, where

$$W_j = \{\overline{x} \in \mathbb{R}^{n-1} : a_{ij} < x_i < b_{ij},\, j = 1, \ldots, n-1\},$$

such that, for $j = 1, \ldots, s$,

$$|\phi_j(\overline{x}) - \phi_j(\overline{y})| \leq M|\overline{x} - \overline{y}|,$$

the constant $M > 0$ being independent of $\overline{x}, \overline{y} \in \overline{W}_j$, with

$$\lambda_j(\Omega \cap V_j) = \{x \in \mathbb{R}^n : a_{nj} < x_n < \phi_j(\overline{x}),\, \overline{x} \in W_j\},$$
$$a_{nj} + d_0 \leq \phi_j(\overline{x}) \leq b_{nj} - d_0, \quad \overline{x} \in W_j, \quad \text{if } V_j \cap \partial\Omega \neq \emptyset,$$

and

$$\phi_j(\overline{x}) = b_{nj} \quad \text{if } V_j \subset \Omega.$$

**3. An auxiliary result of independent interest.** The r.i. imbedding theory in [14] boils down to applying the methods and theorems of interpolation theory to the endpoint imbeddings

$$W^{m,1}(\Omega) \hookrightarrow L_{n/(n-m),1}(\Omega) \quad \text{and} \quad W^{m,\varrho_{n/m,1}}(\Omega) \hookrightarrow L_\infty(\Omega).$$

In particular, given an r.i. norm $\varrho$ on $\mathfrak{M}(\Omega)$, there is another (optimal) r.i. norm, $\sigma_\varrho$, for which

$$W^{m,\varrho}(\Omega) \hookrightarrow L_{\sigma_\varrho}(\Omega).$$

Moreover, $L_{\overline{\sigma}_\varrho}(0, |\Omega|)$ is an interpolation space between $L_{n/(n-m),1}(0, |\Omega|)$ and $L_\infty(0, |\Omega|)$. By this we mean the following. Given r.i. spaces of functions in $\mathfrak{M}(0, |\Omega|)$, namely $X_0$, $X_1$ and $X$, satisfying

$$X_0 \subset X \subset X_1 \quad \text{or} \quad X_0 \supset X \supset X_1.$$

$X$ is said to be an *interpolation space between* $X_0$ *and* $X_1$, denoted $X \in \mathrm{Int}(X_0, X_1)$, if, for any linear operator $T$, $T : X_0 \to X_0$ and $T : X_1 \to X_1$ implies $T : X \to X$.

The main result of this section, which we now state, allows one to assume a candidate *compact* range of $H_{n/m}$ is in $\mathrm{Int}(L_{n/(n-m),1}(0,|\Omega|), L_\infty(0,|\Omega|))$; it is crucial to the proof that (1.7) implies (1.5).

THEOREM 3.1. *Let $\varrho$ and $\sigma$ be r.i. norms on $\mathfrak{M}(0,|\Omega|)$ satisfying* (1.7). *Then there exists another r.i. norm, $\tau$, with*

$$L_\tau(0,|\Omega|) \in \mathrm{Int}(L_{n/(n-m),1}(0,|\Omega|), L_\infty(0,|\Omega|)),$$

*such that $\sigma \lesssim \tau$ and*

$$\lim_{a\to 0+} \sup_{\varrho(f)\le 1} \tau\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big) = 0.$$

The proof of Theorem 3.1 requires extensive preparation. We carry this out in the next two subsections.

**3.1.** *Two assumptions.* One may assume the norm $\sigma$ in Theorem 3.1 satisfies two conditions. The first one,

$$\sigma \ge \varrho_{n/(n-m),1}, \tag{3.1}$$

is justified by

LEMMA 3.2. *Let $\varrho$ be an r.i. norm on $\mathfrak{M}(0,|\Omega|)$ with $L_\varrho(0,|\Omega|) \neq L_1(0,|\Omega|)$. Then*

$$\lim_{a\to 0+} \sup_{\varrho(f)\le 1} \varrho_{n/(n-m),1}\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big) = 0.$$

*Proof.* Given $f \in \mathfrak{M}(0,|\Omega|)$ with $\varrho(f) \le 1$, and $a \in (0,|\Omega|)$, we have

$$\varrho_{n/(n-m),1}\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big) = \int_0^a t^{-m/n} \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\, dt$$

$$= \int_0^a t^{-m/n} \int_t^a f^*(s) s^{m/n-1}\, ds\, dt + \int_0^a t^{-m/n} \int_a^{|\Omega|} f^*(s) s^{m/n-1}\, ds\, dt$$

$$= \int_0^a f^*(s) s^{m/n-1} \int_0^s t^{-m/n}\, dt\, ds + \Big(\int_0^a t^{-m/n}\, dt\Big)\Big(\int_a^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big)$$

$$= \frac{n}{n-m}\Big[\int_0^a f^*(s)\, ds + a^{1-m/n} \int_a^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big]$$

$$\le \frac{n}{n-m}[\varrho'(\chi_{(0,a)}) + \varrho'((a/t)^{1-m/n}\chi_{(a,|\Omega|)}(t))].$$

But, since $L_{\varrho'}(0,|\Omega|) \neq L_\infty(0,|\Omega|)$,

$$\lim_{a\to 0+} \varrho'(\chi_{(0,a)}) = 0.$$

Again, for any $\varepsilon > 0$,

$$\varrho'((a/t)^{1-m/n}\chi_{(a,|\Omega|)}(t)) \le \varrho'(\chi_{(a,a\varepsilon^{-n/(n-m)})}(t)) + \varepsilon\varrho'(\chi_{(0,|\Omega|)}),$$

so we get

$$\lim_{a\to 0+} \varrho'((a/t)^{1-m/n}\chi_{(a,|\Omega|)}(t)) = 0. \ \blacksquare$$

To justify the second assumption, stated in Lemma 3.4 below, we require an interpolation result for positive operators on $\mathfrak{M}(0,|\Omega|)$, that is, for operators $T$ such that $Tf \in P(0,|\Omega|)$ whenever $f \in P(0,|\Omega|)$.

Let $X_0$ and $X_1$ be r.i. spaces of functions on $\mathfrak{M}(0,|\Omega|)$ and fix $\theta$ with $0<\theta<1$. The Calderón space $X_0^{1-\theta}X_1^\theta$ consists of all $f \in \mathfrak{M}(0,|\Omega|)$ such that for some $f_i \in X_i$ with $\|f_i\|_{X_i} \le 1$, $i=0,1$, and for some $\lambda > 0$,

$$|f(t)| \le \lambda |f_0(t)|^{1-\theta}|f_1(t)|^\theta, \quad \text{a.e. } t \in (0,|\Omega|).$$

We define $\|f\|_{X_0^{1-\theta}X_1^\theta}$ to be the infimum of $\lambda$ over all such functions $f_i$, $i=0,1$. One then has

THEOREM 3.3 (Calderón [3]). *A positive operator $T$ on $\mathfrak{M}(0,|\Omega|)$ for which*

$$\|Tf\|_{X_i} \le M_i\|f\|_{X_i}, \quad f \in X_i,\ i=0,1,$$

*is bounded on $X_0^{1-\theta}X_1^\theta$, $0<\theta<1$, and*

$$\|Tf\|_{X_0^{1-\theta}X_1^\theta} \le M_0^{1-\theta}M_1^\theta\|f\|_{X_0^{1-\theta}X_1^\theta}, \quad f \in X_0^{1-\theta}X_1^\theta.$$

LEMMA 3.4. *Let $\varrho$ and $\sigma$ be r.i. norms on $\mathfrak{M}(0,|\Omega|)$ satisfying* (1.7). *Then there exists another r.i. norm, $\tau$, on $\mathfrak{M}(0,|\Omega|)$, with*

$$P : L_\tau(0,|\Omega|) \to L_\tau(0,|\Omega|), \tag{3.2}$$

*such that $\sigma \lesssim \tau$ and*

$$\lim_{a\to 0+} \sup_{\varrho(f)\le 1} \tau\Big(\chi_{(0,a)}(t)\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big) = 0. \tag{3.3}$$

*Proof.* To begin, observe that (1.7) implies

$$H_{n/m} : L_\varrho(0,|\Omega|) \to L_\sigma(0,|\Omega|),$$

which means $L_{\sigma_\varrho}(0,|\Omega|)$, the optimal range of $L_\varrho(0,|\Omega|)$ under $H_{n/m}$, is contained in $L_\sigma(0,|\Omega|)$, so $\sigma \lesssim \sigma_\varrho$. Theorem 3.3 thus ensures

$$H_{n/m} : L_\varrho(0,|\Omega|) \to L_\tau(0,|\Omega|), \quad L_\tau(0,|\Omega|) = L_\sigma(0,|\Omega|)^{1/2}L_{\sigma_\varrho}(0,|\Omega|)^{1/2},$$

where

$$L_{\sigma_\varrho}(0,|\Omega|) \subset L_\tau(0,|\Omega|) \subset L_\sigma(0,|\Omega|),$$

and so $\sigma \lesssim \tau$. Moreover, by Theorem 3.3 again, this time applied to $T = E_s$,

$$h_\tau(s) \le \sqrt{h_\sigma(s) h_{\sigma_\varrho}(s)}.$$

We then have

$$\log h_\tau(s) \le \frac{1}{2}\left[\log h_\sigma(s) + \log h_{\sigma_\varrho}(s)\right]$$

and hence

$$\frac{\log s}{\log h_\tau(s)} \ge \frac{\log s}{\frac{1}{2}[\log h_\sigma(s) + \log h_{\sigma_\varrho}(s)]} = \frac{1}{\frac{1}{2}\left[\frac{\log h_\sigma(s)}{\log s} + \frac{\log h_{\sigma_\varrho}(s)}{\log s}\right]},$$

when $s > 1$. Therefore,

$$i_\tau \ge \frac{1}{\frac{1}{2}\left[\frac{1}{i_\sigma} + \frac{1}{i_{\sigma_\varrho}}\right]} \ge \frac{1}{\frac{1}{2}\left[1 + \frac{1}{i_{\sigma_\varrho}}\right]} \ge \frac{1}{\frac{1}{2}\left[1 + 1 - \frac{m}{n}\right]} \quad \text{(by [13, (5.1)])}$$
$$\ge \frac{1}{1 - \frac{m}{2n}} > 1,$$

so (3.2) holds, in view of (2.4).

Next, we show (3.3). To that end, consider $f \in \mathfrak{M}(0, |\Omega|)$ with $\varrho(f) \le 1$, and $a \in (0, |\Omega|)$. Then

$$\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds$$
$$= \left[\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\right]^{1/2} \left[\int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\right]^{1/2}$$
$$= \sqrt{M_\sigma M_{\sigma_\varrho}} \left[\frac{\chi_{(0,a)}(t)}{M_\sigma} \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\right]^{1/2} \left[\frac{1}{M_{\sigma_\varrho}} \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\right]^{1/2},$$

where

$$M_\sigma := \sigma\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big), \qquad M_{\sigma_\varrho} := \sigma_\varrho\Big(\int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big),$$

with

$$M_\sigma \lesssim M_{\sigma_\varrho} \le \sup_{\varrho(f) \le 1} \sigma_\varrho\Big(\int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big)$$
$$\le \|H_{n/m}\|_{L_\varrho(0,|\Omega|) \to L_{\sigma_\varrho}(0,|\Omega|)} < \infty.$$

We conclude from Theorem 3.3 that

$$\tau\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big) \lesssim \sqrt{M_\sigma M_{\sigma_\varrho}},$$

and thus,

$$\sup_{\varrho(f)\le 1} \tau\Big(\chi_{(0,a)}(t)\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big)$$
$$\lesssim \sqrt{\|H_{n/m}\|_{L_\varrho(0,|\Omega|)\to L_{\sigma_\varrho}(0,|\Omega|)}}\sqrt{\sup_{\varrho(f)\le 1} \sigma\Big(\chi_{(0,a)}(t)\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big)} \to 0$$

as $a\to 0+$, by (1.7). ∎

**3.2.** *Optimal imbedding spaces.* In this subsection we collect results from our previous papers, [13] and [14], which will be needed later on. (At the moment, [14] is not yet published. For this reason we describe the results in some detail. A complete treatment can be found in the paper at the URL address given in the references.)

Our fundamental result in [13] is

THEOREM 3.5. *Fix a bounded domain $\Omega\subset\mathbb{R}^n$ with $\partial\Omega\in \mathrm{Lip}_1$. Let $\varrho$ and $\sigma$ be defined on $\mathfrak{M}(\Omega)$ in terms of the r.i. norms $\overline{\varrho}$ and $\overline{\sigma}$ on $\mathfrak{M}(0,|\Omega|)$, as in* (1.4). *Then*

$$W^{m,\varrho}(\Omega)\hookrightarrow L_\sigma(\Omega) \tag{3.4}$$

*if and only if*

$$H_{n/m}: L_{\overline{\varrho}}(0,|\Omega|)\to L_{\overline{\sigma}}(0,|\Omega|).$$

Given $\varrho$ and $\sigma$, the expressions for their optimal partners involve $\overline{\varrho}'$ and $\overline{\sigma}$, respectively. Thus, for $\varrho$, the $\sigma$ giving the optimal (smallest) imbedding space $L_\sigma(\Omega)$ in (3.4)—call it $\sigma_\varrho$—has

$$\sigma_\varrho'(g) := \overline{\varrho}'(t^{m/n}g^{**}),\qquad g\in\mathfrak{M}(\Omega)$$

(see [13, Theorem 3.2]), while, for $\sigma$, the $\varrho$ defining the optimal (largest) Sobolev space $W^{m,\varrho}(\Omega)$—denote it by $\varrho_\sigma$—satisfies

$$\varrho_\sigma(f) := \sup_{h^*=f^*} \overline{\sigma}\Big(\int_t^{|\Omega|} h(s)s^{m/n-1}\,ds\Big),\qquad f\in\mathfrak{M}(\Omega),\ h\in\mathfrak{M}(0,|\Omega|) \tag{3.5}$$

(see [13, Theorem 3.3]). The optimal partners, $\varrho_\sigma$ and $\sigma_\varrho$, have important interpolation properties:

$$\begin{aligned} &L_{\overline{\varrho}_\sigma}(0,|\Omega|)\in \mathrm{Int}(L_1(0,|\Omega|), L_{n/m,1}(0,|\Omega|)) && [13, \text{Corollary 3.14}],\\ &L_{\overline{\varrho}_\sigma}(0,|\Omega|)\in \mathrm{Int}(L_{n/(n-m),1}(0,|\Omega|), L_\infty(0,|\Omega|)) && [13, \text{Theorem 3.12}]. \end{aligned}\tag{3.6}$$

There are simple tests to guarantee the inclusions (3.6) involving the *supremum operators*

$$(S_{n/m}f)(t) := t^{m/n-1}\sup_{0<s\le t} s^{1-m/n}f^*(s)$$

and

$$(T_{n/m}f)(t) := t^{-m/n} \sup_{t\le s<1} s^{m/n} f^*(s), \qquad f \in \mathfrak{M}(0,|\Omega|),\ 0<t<|\Omega|.$$

Specifically, we have

THEOREM 3.6. *Let $\varrho$ and $\sigma$ be r.i. norms on $\mathfrak{M}(0,|\Omega|)$. Then, given $L_\varrho(0,|\Omega|) \supset L_{n/m,1}(0,|\Omega|)$,*

$$L_\varrho(0,|\Omega|) \in \operatorname{Int}(L_1(0,|\Omega|), L_{n/m,1}(0,|\Omega|))$$

*if and only if*

$$S_{n/m} : L_{\varrho'}(0,|\Omega|) \to L_{\varrho'}(0,|\Omega|).$$

*Again, given $L_\sigma(0,|\Omega|) \subset L_{n/(n-m),1}(0,|\Omega|)$,*

$$L_\sigma(0,|\Omega|) \in \operatorname{Int}(L_{n/(n-m),1}(0,|\Omega|), L_\infty(0,|\Omega|))$$

*if and only if*

$$T_{n/m} : L_{\sigma'}(0,|\Omega|) \to L_{\sigma'}(0,|\Omega|).$$

At certain points in the proof of Theorem 3.1 in Subsection 3.4 we require results, proved in [14] for r.i. norms, to hold for a Banach function norm—namely, those embodied in (4.10), (3.8) and (3.10) of that paper. We now indicate why these generalizations are valid.

Corollary 3.7 and Theorem 3.13 in [13] ultimately depend on the equivalences

$$(S_{n/m}g)^{**} \approx S_{n/m}g \approx S_{n/m}g^{**}, \qquad g \in \mathfrak{M}(0,|\Omega|), \tag{3.7}$$

and the fact that, given an r.i. norm $\lambda$ on $\mathfrak{M}(0,|\Omega|)$ with

$$L_{\lambda'}(0,|\Omega|) \subset L_{n/(n-m),\infty}(0,|\Omega|), \tag{3.8}$$

the functional

$$\nu(g) := \lambda'(S_{n/m}g^{**}), \qquad g \in \mathfrak{M}(0,|\Omega|),$$

is an r.i. norm. As the last assertion is easily shown to be true when $\lambda$ is any Banach function norm, we conclude that the corollary and theorem hold when the functionals $\mu$ in the former and $\lambda'$ in the latter are (only) Banach function norms. This, in turn, yields the following extension of Theorem 3.4 in [14].

THEOREM 3.7. *Let $\lambda$ be a Banach function norm on $\mathfrak{M}(0,|\Omega|)$ satisfying* (3.8). *Then the functional*

$$\mu(g) := \lambda'(S_{n/m}g)$$

*is equivalent to*

$$\nu(g) := \lambda'(S_{n/m}g^{**}), \qquad g \in \mathfrak{M}(0,|\Omega|),$$

*the latter being an r.i. norm on* $\mathfrak{M}(0,|\Omega|)$ *such that* $S_{n/m} : L_\nu(0,|\Omega|) \to L_\nu(0,|\Omega|)$, *and hence*

$$L_{\nu'}(0,|\Omega|) \in \mathrm{Int}(L_{n/(n-m),1}(0,|\Omega|), L_\infty(0,|\Omega|)).$$

*Moreover*,

$$\nu(t^{m/n}g^{**}(t)) \approx \lambda'(t^{m/n}g^{**}(t)), \qquad g \in \mathfrak{M}(0,|\Omega|).$$

With $\lambda$ as in Theorem 3.7, set

$$\mu(g) := \lambda'(t^{m/n}g^{**}(t)) = \lambda'(H'_{n/m}g^*), \qquad g \in \mathfrak{M}(0,|\Omega|),$$

where $H'_{n/m}$ is the operator associated to $H_{n/m}$. Then $L_\mu(0,|\Omega|)$ is the largest r.i. space $H'_{n/m}$ maps into $L_{\lambda'}(0,|\Omega|)$. So, if $\sigma_\lambda := \mu'$, we see that

$$L_{\sigma_\lambda}(0,|\Omega|) = L_{\sigma_{\nu'}}(0,|\Omega|) \tag{3.9}$$

is the smallest r.i. range of $L_\lambda(0,|\Omega|)$ under $H_{n/m}$. Further,

$$\sigma_\lambda(f) \approx \sup_{\lambda'(S_{n/m}g)\le 1} \int_0^1 t^{-m/n}[f^{**}(t) - f^*(t)]g^*(t)\,dt + \varrho_1(f), \qquad f,g \in \mathfrak{M}(0,|\Omega|). \tag{3.10}$$

Indeed, in view of (3.9) and [14, Proposition C],

$$\begin{aligned}\sigma_\lambda(f) &\approx \sigma_{\nu'}(f)\\ &\approx \sup_{\nu(S_{n/m}g)\le 1} \int_0^1 t^{-m/n}[f^{**}(t) - f^*(t)]g^*(t)\,dt + \varrho_1(f), \qquad f,g \in \mathfrak{M}(0,|\Omega|).\end{aligned}$$

But (3.7) ensures

$$\nu(S_{n/m}g) = \lambda'(S_{n/m}(S_{n/m}g)^{**}) \approx \lambda'(S_{n/m}(S_{n/m}g)) = \lambda'(S_{n/m}g),$$

and (3.10) follows.

**3.3.** *A necessary condition for compactness.* The final result needed for the proof of Theorem 3.1 is

THEOREM 3.8. *Fix a bounded domain* $\Omega \subset \mathbb{R}^n$ *with* $\partial\Omega \in \mathrm{Lip}_1$. *Let* $\varrho$ *and* $\sigma$ *be defined on* $\mathfrak{M}(\Omega)$ *in terms of the r.i. norms* $\overline{\varrho}$ *and* $\overline{\sigma}$ *on* $\mathfrak{M}(0,|\Omega|)$, *as in* (1.4). *Assume* $L_\sigma(\Omega) \ne L_\infty(\Omega)$. *Then each of* (1.5), (1.6) *and* (1.7) *implies*

$$\lim_{a\to 0+} \overline{\sigma}(\chi_{(0,a)})\overline{\varrho}'(t^{m/n-1}\chi_{(a,|\Omega|)}(t)) = 0. \tag{3.11}$$

*Proof.* It suffices to show that every sequence $\{a_k\}$ in $(0,|\Omega|)$, with $a_k \downarrow 0$, has a subsequence, $\{a_{k_j}\}$, for which

$$\lim_{j\to\infty} \overline{\sigma}(\chi_{(0,a_{k_j})})\overline{\varrho}'(t^{m/n-1}\chi_{(a_{k_j},|\Omega|)}(t)) = 0.$$

To this end, associate to each $a_k$ a function $0 \le f_k \in L_\infty(0,|\Omega|)$ such that

$$\text{(3.12)} \qquad \overline{\varrho}(f_k) \le 1 \quad \text{and} \quad \int_{a_k}^{|\Omega|} f_k(t) t^{m/n-1}\,dt > \frac{1}{1+a_k}\,\overline{\varrho}'(t^{m/n-1}\chi_{(a_k,|\Omega|)}(t)).$$

For each $k = 1, 2, \ldots$, $H_{n/m} f_k \in L_\infty(0,|\Omega|)$ and, as well,

$$\text{(3.13)} \qquad L_\infty(\Omega) \ni u_k(x) := \int_{K_n|x|^n}^{|\Omega|} f_k(s)\,\frac{(s-K_n|x|^n)^{m-1}}{(m-1)!}\,s^{-m+m/n}\,ds \gtrsim (H_{n/m} f_k)(2K_n|x|^n),$$

where $x \in \Omega$, $2K_n|x|^n \le 1$ and $K_n := \pi^{n/2}/\Gamma(1+n/2)$, the volume of the unit ball in $\mathbb{R}^n$. Here, we have assumed $\{x \in \mathbb{R}^n : |x| < 1\} \subset \Omega$, with no loss of generality.

Given (1.6) and $\overline{\varrho}(f_k) \le 1$, there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ and $g \in L_{\overline{\sigma}}(0,|\Omega|)$ satisfying

$$\lim_{j\to\infty} \overline{\sigma}(H_{n/m} f_{k_j} - g) = 0.$$

Again, [13, proof of Theorem A] implies

$$\overline{\varrho}(|D^m u_k|^*) \lesssim \overline{\varrho}(f_k) \lesssim 1,$$

so, if (1.5) holds, there will be a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and $u \in L_\sigma(\Omega)$ with

$$\text{(3.14)} \qquad \lim_{j\to\infty} \overline{\sigma}((u_{k_j} - u)^*) = 0.$$

The functions $g$ and $u$ of the previous paragraph have absolutely continuous norms in $L_{\overline{\sigma}}(0,|\Omega|)$ and $L_\sigma(\Omega)$, respectively, being norm limits of such functions; see [2, Chapter 1, Theorem 3.8]. Hence, since $L_\sigma(\Omega) \ne L_\infty(\Omega)$, (1.5) yields

$$\begin{aligned}
&\lim_{j\to\infty} \overline{\sigma}(\chi_{(0,a_{k_j})})\overline{\varrho}'(t^{m/n-1}\chi_{(a_{k_j},|\Omega|)}(t)) \\
&\quad \lesssim \limsup_{j\to\infty} \overline{\sigma}(\chi_{(0,a_{k_j})} H_{n/m} f_{k_j}) \quad \text{(by (3.12))} \\
&\quad \lesssim \limsup_{j\to\infty} \overline{\sigma}(\chi_{(0,a_{k_j})} u^*_{k_j}) \quad \text{(by (3.13) and the boundedness of } E_s) \\
&\quad \lesssim \limsup_{j\to\infty} \overline{\sigma}(\chi_{(0,a_{k_j})}(u_{k_j} - u)^*) + \limsup_{j\to\infty} \overline{\sigma}(\chi_{(0,a_{k_j})} u^*) = 0
\end{aligned}$$

by (3.14) and the absolute continuity of $u$ with respect to $\sigma$. By a similar, even simpler, argument, (1.6) gives the same conclusion.

It remains to show that (3.11) follows from (1.7). Suppose (3.11) is *not* satisfied. This means there exist sequences $\{a_k\}$ in $(0,|\Omega|)$ and $\{f_k\}$ in $P(0,|\Omega|)$ such that $a_k \downarrow 0$, $\overline{\varrho}(f_k) \le 1$ and

$$\overline{\sigma}(\chi_{(0,a_k)}) \int_{a_k}^{|\Omega|} f_k(t) t^{m/n-1}\, dt \ge \delta$$

for some $\delta > 0$. Clearly, in view of (2.2), $f_k$ can be assumed to be nonincreasing on $(a_k,|\Omega|)$.

Now, define

$$g_k := \chi_{(0,a_k)} f_k(a_k) + \chi_{(a_k,|\Omega|)} f_k, \qquad k=1,2,\ldots.$$

Then $g_k = g_k^*$ and $\overline{\varrho}(g_k) \le \overline{\varrho}(f_k) \le 1$. Altogether,

$$\overline{\sigma}\Big(\chi_{(0,a_k)}(t) \int_t^{|\Omega|} g_k(s) s^{m/n-1}\, ds\Big) \ge \overline{\sigma}(\chi_{(0,a_k)}) \int_{a_k}^{|\Omega|} f_k(s) s^{m/n-1}\, ds \ge \delta,$$

whence (1.7) does not hold. ∎

**3.4.** *Proof of Theorem 3.1.* In view of Lemmas 3.4 and 3.2 we may assume

$$P : L_\sigma(0,|\Omega|) \to L_\sigma(0,|\Omega|) \tag{3.15}$$

and

$$\sigma \ge \varrho_{n/(n-m),1}. \tag{3.16}$$

Now, the functional

$$\lambda(f) := \sigma\Big(\int_t^{|\Omega|} f(s) s^{m/n-1}\, ds\Big), \qquad f \in \mathfrak{M}(0,|\Omega|), \tag{3.17}$$

satisfies axioms $(A_1)$–$(A_5)$ in Definition 2.1. Given (3.16) as well,

$$\lambda(f) \ge \varrho_{n/(n-m),1}\Big(\int_t^{|\Omega|} f(s) s^{m/n-1}\, ds\Big)$$

$$= \frac{n}{n-m} \int_0^{|\Omega|} f(s)\, ds, \qquad f \in \mathfrak{M}(0,|\Omega|),$$

so $\lambda$ is a Banach function norm.

As observed following Theorem 3.7, the smallest r.i. range of

$$L_\lambda(0,|\Omega|) := \{f \in \mathfrak{M}(0,|\Omega|) : \lambda(|f|) < \infty\}$$

under $H_{n/m}$ is given by $\sigma_\lambda = \mu'$, where

$$\mu(g) := \lambda'(t^{m/n} g^{**}(t)).$$

Setting $\tau=\sigma_\lambda$ we have

$L_\tau(0,|\Omega|)\in \operatorname{Int}(L_{n/(n-m),1}(0,|\Omega|),L_\infty(0,|\Omega|))$ by (3.9) and Theorem 3.7,

and, by (3.10),

$$\tau(h)\approx \sup_{\lambda'(S_{n/m}g)\le 1}\int_0^{|\Omega|} t^{-m/n}[h^{**}(t)-h^*(t)]g^*(t)\,dt+\varrho_1(h),\qquad h\in\mathfrak{M}(0,|\Omega|).$$

Further, the pointwise estimate $S_{n/m}g\ge g^*$ for $g\in\mathfrak{M}(0,|\Omega|)$ gives

$$\begin{aligned}\tau(h)&\lesssim \sup_{\lambda'(g^*)\le 1}\int_0^{|\Omega|} t^{-m/n}[h^{**}(t)-h^*(t)]g^*(t)\,dt+\varrho_1(h)\\ &\le \lambda(t^{-m/n}[h^{**}(t)-h^*(t)])+\varrho_1(h),\qquad h\in\mathfrak{M}(0,|\Omega|).\end{aligned}$$

Fix $a\in(0,|\Omega|)$, $f\in\mathfrak{M}(0,|\Omega|)$ and set

$$h(t):=\chi_{(0,a)}(t)\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds.$$

A simple calculation involving Fubini's theorem shows

$$\begin{aligned}h^{**}(t)-h^*(t)&=\chi_{(0,a)}(t)t^{-1}\int_0^t f^*(s)s^{m/n}\,ds+\chi_{(a,|\Omega|)}(t)t^{-1}\int_0^a f^*(s)s^{m/n}\,ds\\ &\quad+\chi_{(a,|\Omega|)}(t)at^{-1}\int_a^{|\Omega|} f^*(s)s^{m/n-1}\,ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lambda(t^{-m/n}[h^{**}(t)-h^*(t)])\\ &\quad\lesssim \lambda\Big(\chi_{(0,a)}(t)t^{-m/n-1}\int_0^t f^*(s)s^{m/n}\,ds+\chi_{(a,|\Omega|)}(t)t^{-m/n-1}\int_0^a f^*(s)s^{m/n}\,ds\Big)\\ &\qquad+\Big(\int_a^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big)\lambda(at^{-m/n-1}\chi_{(a,|\Omega|)}(t))\\ &\quad=:L+M.\end{aligned}$$

Using the definition of $\lambda$ in (3.17) and applying Fubini's theorem a number of times, we obtain

$$L\lesssim \sigma(Ph)+\Big(\frac{1}{a}\int_0^a f^*(s)s^{m/n}\,ds\Big)\sigma(P\chi_{(0,a)})=:L_1+L_2.$$

From (3.15),

$$L_1 \lesssim \sigma(h) = \sigma\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s)\, ds\Big),$$

and thus,

$$\lim_{a\to 0+} \sup_{\varrho(f)\le 1} L_1 \lesssim \lim_{a\to 0+} \sup_{\varrho(f)\le 1} \sigma\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big) = 0.$$

As for $L_2$, (3.15) implies

$$L_2 \lesssim \Big(\frac{1}{a}\int_0^a f^*(t) t^{m/n}\, dt\Big)\sigma(\chi_{(0,a)}),$$

whence

$$\begin{aligned}
\lim_{a\to 0+} \sup_{\varrho(f)\le 1} L_2 &\lesssim \lim_{a\to 0+} a^{-1}\varrho'(t^{m/n}\chi_{(0,a)}(t))\sigma(\chi_{(0,a)}) \\
&\le \lim_{a\to 0+} a^{m/n}\sigma(\chi_{(0,a)})\,\frac{\varrho'(\chi_{(0,a)})}{a} \\
&\le \lim_{a\to 0+} a^{m/n}\,\frac{\sigma(\chi_{(0,a)})}{\varrho(\chi_{(0,a)})} \qquad \text{(by (2.3))} \\
&\lesssim \lim_{a\to 0+} \sigma\Big(\chi_{(0,a)} \int_a^{2a} \frac{1}{\varrho(\chi_{(0,2a)})}\, s^{m/n-1}\, ds\Big) \\
&\lesssim \lim_{a\to 0+} \sigma\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} \frac{\chi_{(0,2a)}(s)}{\varrho(\chi_{(0,2a)})}\, s^{m/n-1}\, ds\Big) \\
&\lesssim \lim_{a\to 0+} \sup_{\varrho(f)\le 1} \sigma\Big(\chi_{(0,a)}(t) \int_t^{|\Omega|} f^*(s) s^{m/n-1}\, ds\Big) = 0.
\end{aligned}$$

It only remains to show

$$\lim_{a\to 0+} \sup_{\varrho(f)\le 1} M \le \lim_{a\to 0+} a\lambda(t^{-m/n-1}\chi_{(a,|\Omega|)}(t))\varrho'(t^{m/n-1}\chi_{(a,|\Omega|)}(t)) = 0.$$

We first note that

$$a^{m/n+1}\lambda(t^{-m/n-1}\chi_{(a,|\Omega|)}(t)) \lesssim \lambda(\chi_{(0,a)}).$$

This is a consequence of

$$\lambda\Big(t^{-m/n-1}\int_0^t \chi_{(0,a)}(s) s^{m/n}\, ds\Big) \lesssim \lambda(\chi_{(0,a)}),$$

and indeed,

$$\begin{aligned}\lambda\Big(t^{-m/n-1}\int_0^t \chi_{(0,a)}(s)s^{m/n}\,ds\Big) &= \sigma\Big(\int_t^{|\Omega|} s^{-m/n-1}\int_0^s \chi_{(0,a)}(y)y^{m/n}\,dy\,s^{m/n-1}\,ds\Big)\\ &= \sigma\Big(\int_t^{|\Omega|} s^{-1}\int_0^s \chi_{(0,a)}(y)y^{m/n}\,dy\,\frac{ds}{s}\Big)\\ &\le \sigma\Big(t^{-1}\int_0^t\int_s^{|\Omega|} \chi_{(0,a)}(y)y^{m/n-1}\,dy\,ds\Big)\\ &\le \sigma\Big(\int_t^{|\Omega|} \chi_{(0,a)}(s)s^{m/n-1}\,ds\Big) \quad \text{(by (3.15))}\\ &= \lambda(\chi_{(0,a)}).\end{aligned}$$

Next, by the definition of $\lambda$,

$$\lambda(\chi_{(0,a)}) = \sigma\Big(\int_t^{|\Omega|} \chi_{(0,a)}(s)s^{m/n-1}\,ds\Big) \lesssim a^{m/n}\sigma(\chi_{(0,a)}).$$

Putting all these estimates together, we obtain

$$\lim_{a\to 0+}\sup_{\varrho(f)\le 1} M \lesssim \lim_{a\to 0+}\sigma(\chi_{(0,a)})\varrho'(t^{m/n-1}\chi_{(a,|\Omega|)}(t)) = 0,$$

by Theorem 3.8. This completes the proof. ∎

## 4. Proofs of the main results

**4.1.** *Proof of Theorem 1.1.* Assume (1.6) holds, but not (1.7). This means we can find an $\varepsilon > 0$, a sequence $\{\beta_k\}$ in $(0,|\Omega|)$ with $\beta_k \downarrow 0$, and a sequence $\{f_k\}$ in $\mathfrak{M}(0,|\Omega|)$ with $\overline{\varrho}(f_k)\le 1$, for which

$$\overline{\sigma}(\chi_{(0,\beta_k)}H_{n/m}f_k^*) > \varepsilon, \qquad k=1,2,\ldots.$$

Further, by $(A_4)$, there exists a sequence $\{\alpha_k\}$ in $(0,|\Omega|)$ with

$$0<\beta_{k+1}<\alpha_k<\beta_k, \qquad \alpha_k\downarrow 0,$$

(we pass to a subsequence of $\{\beta_k\}$ if necessary) and

$$\overline{\sigma}(\chi_{(\alpha_k,\beta_k)}H_{n/m}f_k^*) > \varepsilon, \qquad k=1,2,\ldots.$$

As

$$\overline{\sigma}\Big(\chi_{(\alpha_k,\beta_k)}\int_{\beta_k}^{|\Omega|} f_k^*(t)t^{m/n-1}\,dt\Big) \le \overline{\sigma}(\chi_{(0,\beta_k)})\overline{\varrho}'(t^{m/n-1}\chi_{(\beta_k,|\Omega|)}(t))$$

and

$$\lim_{a\to 0+}\overline{\sigma}(\chi_{(0,a)})\overline{\varrho}'(t^{m/n-1}\chi_{(a,|\Omega|)}(t)) = 0$$

by Theorem 3.8, we may take $f_k^* = f_k^* \chi_{(0,\beta_k)}$. Hence, for $k > j$,

$$\begin{aligned}\overline{\sigma}(H_{n/m} f_k^* - H_{n/m} f_j^*) &\geq \overline{\sigma}(\chi_{(\alpha_j,\beta_j)}(H_{n/m} f_k^* - H_{n/m} f_j^*)) \\ &\geq \overline{\sigma}(\chi_{(\alpha_j,\beta_j)} H_{n/m} f_j^*) > \varepsilon,\end{aligned}$$

contradicting (1.6).

The argument that (1.5) implies (1.7) is similar to the one above, though a little more complicated. Thus, if (1.5) holds, but not (1.7), there exists an $\varepsilon > 0$ and sequences $\{\alpha_k\}$, $\{\beta_k\}$ in $(0, |\Omega|)$ and $\{f_k\}$ in $\mathfrak{M}(0, |\Omega|)$ such that $0 < \beta_{k+1} < \alpha_k < \beta_k, \alpha_k, \beta_k \downarrow 0$, $f_k^* = f_k^* \chi_{(0,\beta_k)}$, $\overline{\varrho}(f_k) \leq 1$ and

$$\overline{\sigma}(\chi_{(\alpha_k,\beta_k)} H_{n/m} f_k^*) > \varepsilon, \qquad k = 1, 2, \ldots .$$

With no essential loss of generality

$$\Omega \supset B := \{x \in \mathbb{R} : |x| < K_n^{-1/n}\}, \qquad K_n := \pi^{n/2}/\Gamma(1 + n/2),$$

and

$$2\alpha_k/3 > \beta_{k+1}, \qquad k = 1, 2, \ldots .$$

Define $u_k(x) = 0$ when $x \in \Omega \setminus B$ or $K_n^{-1/n} > |x| > K_n^{-1/n} \beta_k^{1/n}$; when $|x| \leq K_n^{-1/n} \beta_k^{1/n}$ set

$$\begin{aligned}u_k(x) &= \int_{K_n|x|^n}^{\beta_k} \int_{t_1}^{\beta_k} \int_{t_2}^{\beta_k} \cdots \int_{t_{m-1}}^{\beta_k} f_k^*(t) t^{-m+m/n}\, dt\, dt_{m-1} \ldots dt_1 \\ &= \int_{K_n|x|^n}^{\beta_k} f_k^*(t) \frac{(t - K_n|x|^n)^{m-1}}{(m-1)!} t^{-m+m/n}\, dt, \qquad k = 1, 2, \ldots .\end{aligned}$$

For

$$x \in A_j := \{x \in \Omega : K_n^{-1/n}(2\alpha_j/3)^{1/n} < |x| < K_n^{-1/n}(2\beta_j/3)^{1/n}\}$$

and $k > j$ we have

$$u_k(x) - u_j(x) = -u_j(x) = -\int_{K_n|x|^n}^{\beta_j} f_j^*(t) \frac{(t - K_n|x|^n)^{m-1}}{(m-1)!} t^{-m+m/n}\, dt.$$

Therefore,

$$\begin{aligned}&\sigma(u_k - u_j) \geq \overline{\sigma}([\chi_{A_j}(u_k - u_j)]^*) \\ &\quad \geq \overline{\sigma}\bigg(\bigg[\chi_{(2\alpha_j/3, 2\beta_j/3)}(K_n|\cdot|^n) \int_{K_n|\cdot|^n}^{\beta_j} f_j^*(t) \frac{(t - K_n|\cdot|^n)^{m-1}}{(m-1)!} t^{-m+m/n}\, dt\bigg]^*\bigg)\end{aligned}$$

$$\geq \overline{\sigma}\Big(\chi_{(2\alpha_j/3,2\beta_j/3)}(t)\int_t^{\beta_j} f_j^*(s)\,\frac{(s-t)^{m-1}}{(m-1)!}\,s^{-m+m/n}\,ds\Big)$$

$$\gtrsim \overline{\sigma}\Big(\chi_{(\alpha_j,\beta_j)}(3t/2)\int_{3t/2}^{\beta_j} f_j^*(s)s^{m/n-1}\,ds\Big)$$

$$\geq \overline{\sigma}(\chi_{(\alpha_j,\beta_j)}H_{n/m}f_j^*)\geq \varepsilon,$$

contradicting (1.5).

Suppose now (1.7) holds. In view of Theorem 3.1 we may assume for the rest of the proof that

$$L_{\overline{\sigma}}(0,|\Omega|)\in \operatorname{Int}(L_{n/(n-m),1}(0,|\Omega|),L_\infty(0,|\Omega|)),$$

which, according to Theorem 3.6, is equivalent to

$$T_{n/m}: L_{\overline{\sigma}'}(0,|\Omega|)\to L_{\overline{\sigma}'}(0,|\Omega|). \tag{4.1}$$

This ensures that (1.7) implies

$$H_{n/m}: L_{\overline{\varrho}}(0,|\Omega|)\to L_{\overline{\sigma}}(0,|\Omega|). \tag{4.2}$$

Indeed, for $f,g\in\mathfrak{M}(0,|\Omega|)$, $f\geq 0$,

$$\begin{aligned}
\overline{\sigma}(H_{n/m}f) &= \sup_{\overline{\sigma}'(g)\leq 1}\int_0^{|\Omega|} g^*(t)\int_t^{|\Omega|} f(s)s^{m/n-1}\,ds\,dt\\
&= \sup_{\overline{\sigma}'(g)\leq 1}\int_0^{|\Omega|} f(s)s^{m/n-1}\int_0^s g^*(t)\,dt\,ds\\
&\leq \sup_{\overline{\sigma}'(g)\leq 1}\int_0^{|\Omega|} f(s)s^{m/n-1}\int_0^s t^{-m/n}\sup_{t\leq y<|\Omega|} y^{m/n}g^*(y)\,dt\,ds\\
&\leq \sup_{\overline{\sigma}'(g)\leq 1}\int_0^{|\Omega|} f^*(s)s^{m/n-1}\int_0^s (T_{n/m}g)(t)\,dt\,ds\\
&= \sup_{\overline{\sigma}'(g)\leq 1}\int_0^{|\Omega|} (T_{n/m}g)(t)\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\,dt\\
&\leq \overline{\sigma}(H_{n/m}f^*)\sup_{\overline{\sigma}'(g)\leq 1}\overline{\sigma}'(T_{n/m}g)\lesssim \overline{\varrho}(f),
\end{aligned}$$

by (1.7) and (4.1).

Arguing as in the last paragraph of [13, proof of Theorem A], we obtain

$$\overline{\sigma}(\chi_{(0,a)}(t)u^*(t))\lesssim \overline{\sigma}(\chi_{(0,a)}(t)H_{n/m}(|D^m u|^*)(t/2)) \tag{4.3}$$

for all $u \in W^{m,\varrho}(\Omega)$ and $0 < a \le |\Omega|$. In particular, taking $a = |\Omega|$ in (4.3), we get

$$\overline{\sigma}(u^*) \lesssim \overline{\sigma}(H_{n/m}(|D^m u|^*)(t/2)) \lesssim \overline{\varrho}(|D^m u|^*),$$

by (4.2); that is, $W^{m,\varrho}(\Omega) \hookrightarrow L_\sigma(\Omega)$. More generally, (4.3) together with (1.7) yields

$$\lim_{a\to 0+} \sup_{\overline{\varrho}(|D^m u|^*)\le 1} \overline{\sigma}(\chi_{(0,a)} u^*) = 0. \tag{4.4}$$

Our intention is to use (4.4) to prove (1.5), or, more precisely, to prove that for each sequence $\{u_k\}$ in $W^{m,\varrho}(\Omega)$ satisfying

$$\sup_k \overline{\varrho}(|D^m u_k|^*) \le 1,$$

there is a subsequence $\{u_{k_j}\}$ and a function $u \in L_\sigma(\Omega)$ with

$$\lim_{j\to\infty} \overline{\sigma}((u_{k_j} - u)^*) = 0.$$

To this end, let $\varepsilon > 0$ be given and choose $\delta > 0$ for which

$$\sup_k \overline{\sigma}(\chi_{(0,a)}(t) u_k^*) < \varepsilon/3,$$

when $0 < a < \delta$. As shown, for example, in [16, p. 367], there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ that converges in measure to a function $u \in L_\varrho(\Omega)$; in particular, we may suppose $(u_{k_j} - u)^*(|\Omega| t) \to 0$ a.e. Thus, for some $j_0 \in \mathbb{Z}_+$,

$$|\Omega_j| < \delta, \qquad j \ge j_0,$$

where

$$\Omega_j := \left\{ x \in \Omega : |u_{k_j}(x) - u(x)| > \frac{\varepsilon}{3\sigma(\chi_{(0,|\Omega|)})} \right\}, \qquad j = 1, 2, \ldots.$$

Consequently, when $j \ge j_0$,

$$\overline{\sigma}((u_{k_j} - u)^*) \le \overline{\sigma}(\chi_{(0,|\Omega_j|)} u_{k_j}^*) + \overline{\sigma}(\chi_{(0,|\Omega_j|)} u^*) + \varepsilon/3 < \varepsilon,$$

whence (1.5) is verified.

It only remains to obtain (1.6) from (1.7). To begin, suppose $L_{\overline{\varrho}}(0,|\Omega|) \ne L_1(0,|\Omega|)$. Given $a \in (0,|\Omega|)$ and $f \in P(0,|\Omega|)$, we write

$$H_{n/m} f = H_{n/m}(\chi_{(0,a)} f) + H_{n/m}(\chi_{(a,|\Omega|)} f) =: R_a f + U_a f.$$

Our strategy will be to prove $U_a$ is a compact operator from $L_{\overline{\varrho}}(0,|\Omega|)$ to $L_{\overline{\sigma}}(0,|\Omega|)$, while

$$\lim_{a\to 0+} \|R_a\|_{L_{\overline{\varrho}}(0,|\Omega|)\to L_{\overline{\sigma}}(0,|\Omega|)} = 0. \tag{4.5}$$

Then, of course, $H_{n/m}$ would be a compact operator from $L_{\overline{\varrho}}(0,|\Omega|)$ to $L_{\overline{\sigma}}(0,|\Omega|)$, being the limit in norm, as $a\to 0+$, of such compact operators.

Fix an $\varepsilon>0$ and let $a\in(0,|\Omega|)$ be small enough to guarantee

$$\sup_{\overline{\varrho}(f)\le 1}\overline{\varrho}(\chi_{(0,a)}H_{n/m}f^*)<\varepsilon,\qquad f\in\mathfrak{M}(\Omega).$$

Then we have, with $f,g\in\mathfrak{M}(0,|\Omega|)$, $f\ge 0$,

$$\begin{aligned}
&\|R_a\|_{L_{\overline{\varrho}}(0,|\Omega|)\to L_{\overline{\sigma}}(0,|\Omega|)}\\
&\quad=\sup_{\overline{\varrho}(f)\le 1}\sup_{\overline{\sigma}'(g)\le 1}\int_0^{|\Omega|} g^*(t)\int_t^{|\Omega|}\chi_{(0,a)}(s)f(s)s^{m/n-1}\,ds\,dt\\
&\quad=\sup_{\overline{\varrho}(f)\le 1}\sup_{\overline{\sigma}'(g)\le 1}\int_0^{a} f(s)s^{m/n-1}\int_0^{s}g^*(t)\,dt\,ds\\
&\quad\le\sup_{\overline{\varrho}(f)\le 1}\sup_{\overline{\sigma}'(g)\le 1}\int_0^{a} f(s)s^{m/n-1}\int_0^{s}t^{-m/n}\sup_{t\le y<|\Omega|}y^{m/n}g^*(y)\,dt\,ds\\
&\quad\le\sup_{\overline{\varrho}(f)\le 1}\sup_{\overline{\sigma}'(g)\le 1}\int_0^{a} f^*(s)s^{m/n-1}\int_0^{s}(T_{n/m}g)(t)\,dt\,ds\\
&\quad=\sup_{\overline{\varrho}(f)\le 1}\sup_{\overline{\sigma}'(g)\le 1}\int_0^{a} (T_{n/m}g)(t)\int_t^{a}f^*(s)s^{m/n-1}\,ds\,dt\\
&\quad\le\sup_{\overline{\sigma}'(g)\le 1}\overline{\sigma}'(T_{n/m}g)\sup_{\overline{\varrho}(f)\le 1}\overline{\sigma}(\chi_{(0,a)}H_{n/m}f^*)\le C\varepsilon,
\end{aligned}$$

where $C$ is the norm of $T_{n/m}$ on $L_{\sigma'}(0,|\Omega|)$. This proves (4.5).

Next, we show that when $L_{\overline{\varrho}}(0,|\Omega|)\ne L_1(0,|\Omega|)$ and $a\in(0,|\Omega|)$ is fixed, $U_a$ is a compact operator from $L_{\overline{\varrho}}(0,|\Omega|)$ to $L_\infty(0,|\Omega|)$ and hence also to $L_{\overline{\sigma}}(0,|\Omega|)$. Indeed, by Hölder's inequality, with $f\in\mathfrak{M}(0,|\Omega|)$,

$$\begin{aligned}
\sup_{\overline{\varrho}(f)\le 1}\varrho_\infty(U_af)\le\sup_{\overline{\varrho}(f)\le 1}\int_a^{|\Omega|}|f(s)|s^{m/n-1}\,ds&=\overline{\varrho}'(t^{m/n-1}\chi_{(a,|\Omega|)}(t))\\
&\le a^{m/n-1}\overline{\varrho}'(\chi_{(a,|\Omega|)})<\infty;
\end{aligned}$$

therefore, the set

$$K_{\overline{\varrho}}:=\{U_af:\overline{\varrho}(f)\le 1\}$$

is equibounded.

Again, $L_{\overline{\varrho}}(0,|\Omega|)\ne L_1(0,|\Omega|)$ means $L_{\overline{\varrho}'}(0,|\Omega|)\ne L_\infty(0,|\Omega|)$, whence, given $\varepsilon>0$ there exists $\delta>0$ with

$$\overline{\varrho}'(t^{m/n-1}\chi_{(a,a+\delta)}(t))\le a^{m/n-1}\overline{\varrho}'(\chi_{(0,\delta)})<\varepsilon. \tag{4.6}$$

Then, for any $t_1$ and $t_2$ with $0 < t_1 < t_2 < |\Omega|$ and $0 < t_2 - t_1 < \delta$,

$$\begin{aligned}\sup_{\overline{\varrho}(f)\le 1} \varrho_\infty((U_a f)(t_2) - (U_a f)(t_1)) &\le \sup_{\overline{\varrho}(f)\le 1} \Bigl| \int_{\max[t_1,a]}^{\max[t_2,a]} |f(s)| s^{m/n-1}\, ds \Bigr| \\ &= \overline{\varrho}'(t^{m/n-1}\chi_{(\max[t_1,a],\max[t_2,a])}(t)) \\ &\le \overline{\varrho}'(t^{m/n-1}\chi_{(a,a+\delta)}(t)) < \varepsilon,\end{aligned}$$

in view of (4.6). The Ascoli–Arzelà theorem now yields the relative compactness of $K_{\overline{\varrho}}$ in $L_\infty(0,|\Omega|)$.

Finally, we suppose $L_{\overline{\varrho}}(0,|\Omega|) = L_1(0,|\Omega|)$. In this case we write

$$H_{n/m} f = \chi_{(0,a)} H_{n/m} f + \chi_{(a,|\Omega|)} H_{n/m} f =: \widetilde{R}_a f + \widetilde{U}_a f, \qquad f \in P(0,|\Omega|).$$

Then

$$\begin{aligned}\|\widetilde{R}_a f\|_{L_1(0,|\Omega|)\to L_{\overline{\sigma}}(0,|\Omega|)} &= \sup_{\varrho_1(f)\le 1} \sup_{\overline{\sigma}'(g)\le 1} \int_0^{|\Omega|} g^*(t)\, \chi_{(0,a)}(t) \int_t^{|\Omega|} f(s) s^{m/n-1}\, ds\, dt \\ &\le \sup_{\varrho_1(f)\le 1} \sup_{\overline{\sigma}'(g)\le 1} \int_0^a g^*(t) \int_t^a f(s) s^{m/n-1}\, ds\, dt \\ &\quad + \sup_{\varrho_1(f)\le 1} \sup_{\overline{\sigma}'(g)\le 1} \int_0^a g^*(t)\, dt \int_a^{|\Omega|} f(s) s^{m/n-1}\, ds.\end{aligned}$$

The second term goes to 0 with $a$ by (1.7). To deal with the first term, we observe that

$$\begin{aligned}&\sup_{\varrho_1(f)\le 1} \sup_{\overline{\sigma}'(g)\le 1} \int_0^a g^*(t) \int_t^a f(s) s^{m/n-1}\, ds\, dt \\ &\qquad \le \sup_{\varrho_1(f)\le 1} \sup_{\overline{\sigma}'(g)\le 1} \int_0^a (T_{n/m} g)(t) \int_t^a f^*(s) s^{m/n-1}\, ds\, dt \\ &\qquad \le \sup_{\overline{\sigma}'(g)\le 1} \overline{\sigma}'(T_{n/m} g) \sup_{\varrho_1(f)\le 1} \overline{\sigma}(\chi_{(0,a)} H_{n/m} f^*),\end{aligned}$$

which, again, goes to 0 with $a$, by (1.7).

To complete the proof we show $\widetilde{U}_a : L_1(0,|\Omega|) \to\to L_{\overline{\sigma}}(0,|\Omega|)$ for every fixed $a \in (0,|\Omega|)$, which is equivalent to $\widetilde{U}'_a : L_{\overline{\sigma}'}(0,|\Omega|) \to\to L_\infty(0,|\Omega|)$, $L_{\overline{\sigma}'}(0,|\Omega|) \ne L_1(0,|\Omega|)$, where

$$(\widetilde{U}'_a g)(t) := \chi_{(a,|\Omega|)}(t) t^{m/n-1} \int_a^t g(s)\, ds.$$

The argument for this is similar to the one for $U_a$ when $L_\varrho(0,|\Omega|) \ne L_1(0,|\Omega|)$. ∎

**4.2.** *Proof of Theorem 1.2.* Assume (1.10) and fix $a \in (0, |\Omega|)$. Then, for $f \in P(0, |\Omega|)$,

$$H_{n/m} f = H_{n/m}(\chi_{(0,a)} f) + H_{n/m}(\chi_{(a,|\Omega|)} f) =: R_a f + U_a f.$$

Now,

$$\|R_a\|_{L_{\overline{\varrho}}(0,|\Omega|)\to L_\infty(0,|\Omega|)} = \sup_{\overline{\varrho}(f)\le 1} \varrho_\infty(R_a f) = \sup_{\overline{\varrho}(f)\le 1} \int_0^a f(s) s^{m/n-1}\, ds = \overline{\varrho}'(t^{m/n-1}\chi_{(0,a)}(t)),$$

so, by (1.10),

$$\lim_{a\to 0+} \|R_a\|_{L_{\overline{\varrho}}(0,|\Omega|)\to L_\infty(0,|\Omega|)} = 0. \tag{4.7}$$

Next, let $K_{\overline{\varrho}} := \{U_a f : \overline{\varrho}(f) \le 1\}$. Then

$$\varrho_\infty(U_a f) = \int_a^{|\Omega|} f(t) t^{m/n-1}\, dt \le \overline{\varrho}'(t^{m/n-1}\chi_{(a,|\Omega|)}(t)) < \infty,$$

hence, $K_{\overline{\varrho}}$ is equibounded. It is also equicontinuous, since, for $0 < s < t < |\Omega|$,

$$\sup_{\overline{\varrho}(f)\le 1} |(U_a f)(t) - (U_a f)(s)| = \sup_{\overline{\varrho}(f)\le 1} \int_{\max[s,a]}^{\max[t,a]} f(y) y^{m/n-1}\, dy \le \overline{\varrho}'(y^{m/n-1}\chi_{(0,t-s)}(y)),$$

which will go to 0 with $t - s$. Thus, $K_{\overline{\varrho}}$ is relatively compact in $L_\infty(0, |\Omega|)$ by virtue of the Ascoli–Arzelà theorem. In other words, for every fixed $a \in (0, |\Omega|)$, $U_a$ is a compact operator from $L_{\overline{\varrho}}(0, |\Omega|)$ to $L_\infty(0, |\Omega|)$. Combining this with (4.7), we see that $H_{n/m}$ is the norm-limit of compact operators, and so it is compact.

Conversely, assume (1.10) is not satisfied. This means that there is some $\delta > 0$ such that

$$\overline{\varrho}'(t^{m/n-1}\chi_{(0,a)}(t)) \ge 3\delta, \qquad a \in (0, |\Omega|).$$

We now define a sequence $\{a_k\}$ as follows: set $a_1 = |\Omega|$ and suppose that, for $k \in \mathbb{Z}_+$, $a_k$ has been determined. Then there is a function $f_k \in P(0, |\Omega|)$ with $\overline{\varrho}(f_k) \le 1$ and $\operatorname{supp} f_k \subset (0, a_k)$ for which

$$\int_0^{a_k} f_k(t) t^{m/n-1}\, dt \ge 2\delta.$$

By the absolute continuity of the Lebesgue integral, there is an $a_{k+1} \in (0, a_k)$ with

$$\int_{a_{k+1}}^{a_k} f_k(t) t^{m/n-1}\, dt \ge \delta.$$

But $\{f_k\}$ is a sequence in the unit ball of $L_{\overline{\varrho}}(0,|\Omega|)$ having the property that, when $k>j$,

$$\varrho_\infty(H_{n/m}f_k - H_{n/m}f_j) \geq \varrho_\infty(\chi_{(a_{j+1},a_j)}(H_{n/m}f_k - H_{n/m}f_j)) = \varrho_\infty(\chi_{(a_{j+1},a_j)}H_{n/m}f_j) = \int\limits_{a_{j+1}}^{a_j} f_j(t)t^{m/n-1}\,dt \geq \delta,$$

so $H_{n/m}$ is not compact from $L_{\overline{\varrho}}(0,|\Omega|)$ to $L_\infty(0,|\Omega|)$, contradicting (1.9).

The equivalence of (1.10) and (1.8) follows by a similar, more technically complicated argument. The key ideas can be found in the proof of Theorem 1.1 and are thus omitted here. ∎

**5. Examples.** Our first result in this section relates a compact imbedding space of $W^{m,\varrho}(\Omega)$ to its optimal imbedding space, $L_{\sigma_\varrho}(\Omega)$, providing thereby a new necessary and sufficient condition for an imbedding to be compact.

THEOREM 5.1. *Fix a bounded domain $\Omega\subset\mathbb{R}^n$, having $\partial\Omega\in \mathrm{Lip}_1$. Suppose $\varrho$ and $\sigma$ are two r.i. norms defined on $\mathfrak{M}(\Omega)$ in terms of the r.i. norms $\overline{\varrho}$ and $\overline{\sigma}$ on $\mathfrak{M}(0,|\Omega|)$, as in* (1.4). *Assume $L_\sigma(\Omega)\neq L_\infty(\Omega)$ and let $L_{\sigma_\varrho}(\Omega)$ be the optimal r.i. imbedding space for $W^{m,\varrho}(\Omega)$. Then*

$$W^{m,\varrho}(\Omega) \hookrightarrow\hookrightarrow L_\sigma(\Omega) \tag{5.1}$$

*if and only if the functions in the unit ball of $L_{\overline{\sigma}_\varrho}(0,|\Omega|)$ have uniformly absolutely continuous norms in $L_{\overline{\sigma}}(0,|\Omega|)$ or, what is the same,*

$$\lim_{a\to 0+}\sup_{\overline{\sigma}_\varrho(f)\leq 1}\overline{\sigma}(\chi_{(0,a)}f^*) = 0, \qquad f\in\mathfrak{M}(0,|\Omega|). \tag{5.2}$$

*Proof.* Since $H_{n/m}$ is bounded from $L_{\overline{\varrho}}(0,|\Omega|)$ to $L_{\overline{\sigma}_\varrho}(0,|\Omega|)$, (5.2) implies (1.7) and thus, in view of Theorem 1.1, also (5.1).

It follows from Theorem 3.1 combined with Theorem 1.1 that, given (5.1), there exists an r.i. norm $\tau$ on $\mathfrak{M}(0,|\Omega|)$ satisfying

$$L_\tau(0,|\Omega|)\in \mathrm{Int}(L_{n/(n-m),1}(0,|\Omega|), L_\infty(0,|\Omega|)), \tag{5.3}$$

$$H_{n/m}: L_{\overline{\varrho}}(0,|\Omega|)\to\to L_\tau(0,|\Omega|)\subset L_{\overline{\sigma}}(0,|\Omega|)$$

(in particular, $\overline{\sigma}\lesssim\tau$) and

$$\lim_{a\to 0+}\sup_{\overline{\varrho}(f)\leq 1}\tau\Big(\chi_{(0,a)}(t)\int\limits_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big) = 0. \tag{5.4}$$

Thus, it suffices to show (5.3) and (5.4) together imply (5.2).

Assuming (5.2) does not hold, there must be a sequence $a_k\downarrow 0$, an $\varepsilon>0$ and functions $f_k\in\mathfrak{M}(0,|\Omega|)$ such that

$$\overline{\sigma}_\varrho(f_k)\leq 1 \quad\text{and}\quad \tau(\chi_{(0,a_k)}f_k^*)\geq\varepsilon, \qquad k\in\mathbb{Z}_+. \tag{5.5}$$

By (3.5),

$$\overline{\sigma}'_{\varrho}(g) = \overline{\varrho}'(t^{m/n} g^{**}(t)), \qquad g \in \mathfrak{M}(0, |\Omega|),$$

so

$$1 \geq \overline{\sigma}_{\varrho}(f_k) = \sup_{g \not\equiv 0} \frac{\int_0^{|\Omega|} f_k^* g^*}{\overline{\sigma}'_{\varrho}(g)} = \sup_{g \not\equiv 0} \frac{\int_0^{|\Omega|} f_k^* g^*}{\overline{\varrho}'(t^{m/n} g^{**}(t))},$$

and hence

$$\int_0^1 f_k^* g^* \leq \overline{\varrho}'(t^{m/n} g^{**}(t)), \qquad g \in \mathfrak{M}(0, |\Omega|). \tag{5.6}$$

To each $f_k$ we may associate $g_k \in \mathfrak{M}(0, |\Omega|)$ having $\operatorname{supp} g_k \subset (0, a_k)$, $\tau'(g_k) \leq 1$ and

$$\int_0^{|\Omega|} f_k^* g_k^* > \frac{1}{2} \tau(\chi_{(0,a_k)} f_k^*), \qquad k \in \mathbb{Z}_+.$$

This, together with (5.6) and (5.5), yields

$$\overline{\varrho}'\Big(t^{m/n-1} \int_0^t g_k^*(s)\, ds\Big) > \frac{\varepsilon}{2}, \qquad k \in \mathbb{Z}_+.$$

Again, by duality, we are guaranteed $h_k \in P(\Omega)$ for which $\overline{\varrho}(h_k) \leq 1$ and

$$\begin{aligned}
\frac{\varepsilon}{2} &< \int_0^{|\Omega|} h_k(t) t^{m/n} g_k^{**}(t)\, dt = \int_0^{|\Omega|} h_k(t) t^{m/n-1} \int_0^t g_k^*(s)\, ds\, dt \\
&= \int_0^{|\Omega|} h_k(t) t^{m/n-1} \int_0^t s^{-m/n} \sup_{s \leq y < a_k} y^{m/n} g_k^*(y)\, ds\, dt \\
&\leq \int_0^{|\Omega|} h_k^*(t) t^{m/n-1} \int_0^t s^{-m/n} \sup_{s \leq y < a_k} y^{m/n} g_k^*(y)\, ds\, dt \\
&= \int_0^{|\Omega|} h_k^*(t) t^{m/n-1} \int_0^t (T_{n/m} g_k)(s)\, ds\, dt \\
&= \int_0^{|\Omega|} (T_{n/m} g_k)(s) \int_s^{|\Omega|} h_k^*(t) t^{m/n-1}\, dt\, ds \\
&\leq \overline{\sigma}'(T_{n/m} g_k) \overline{\sigma}\Big(\chi_{(0,a_k)}(t) \int_t^{|\Omega|} h_k^*(s) s^{m/n-1}\, ds\Big) \\
&\qquad\qquad \text{(since } \operatorname{supp}(T_{n/m} g_k) \subset (0, a_k)) \\
&\lesssim \overline{\sigma}\Big(\chi_{(0,a_k)}(t) \int_t^{|\Omega|} h_k^*(s) s^{m/n-1}\, ds\Big) \qquad \text{(by (5.3) and Theorem 3.6).}
\end{aligned}$$

This contradicts (5.4) and completes the proof. ∎

To be able to state our next theorem we require

LEMMA 5.2. *Let $\varrho$ be an r.i. norm on $\mathfrak{M}(\Omega)$. Then the fundamental function $\overline{\sigma}_\varrho(\chi_{(0,t)})$ satisfies*

$$\overline{\sigma}_\varrho(\chi_{(0,t)}) \approx \frac{1}{\overline{\varrho}'(s^{m/n-1}\chi_{(t,|\Omega|)}(s))}, \qquad t \in (0, |\Omega|/2). \tag{5.7}$$

*Proof.* By (2.3) and (3.5),

$$\begin{aligned}\overline{\sigma}_\varrho(\chi_{(0,t)}) &= \frac{t}{\overline{\sigma}'_\varrho(\chi_{(0,t)})} \\ &= \frac{t}{\overline{\varrho}'(s^{m/n}\chi^{**}_{(0,t)}(s))} \approx \frac{t}{\overline{\varrho}'(s^{m/n}\chi_{(0,t)}(s) + ts^{m/n-1}\chi_{(t,|\Omega|)}(s))}.\end{aligned}$$

However,

$$\begin{aligned}\overline{\varrho}'(ts^{m/n-1}\chi_{(t,|\Omega|)}(t)) \ge \overline{\varrho}'(ts^{m/n-1}\chi_{(t,2t)}(s)) &\gtrsim \overline{\varrho}'(s^{m/n}\chi_{(t,2t)}(s)) \\ &\gtrsim \overline{\varrho}'(s^{m/n}\chi_{(0,t)}(s)),\end{aligned}$$

which yields (5.7). ∎

THEOREM 5.3. *Fix a bounded domain $\Omega \subset \mathbb{R}^n$ having $\partial\Omega \in \mathrm{Lip}_1$. Let $\varrho$ and $\sigma$ be r.i. norms defined on $\mathfrak{M}(\Omega)$, with*

$$\sigma(f) = \overline{\sigma}(f^*), \qquad \varrho(f) = \overline{\varrho}(f^*), \qquad f \in \mathfrak{M}(0, |\Omega|),$$

*as in* (1.4). *Set*

$$\phi_R(t) := \frac{dc}{dt},$$

*where $c(t)$ is the least concave majorant of $t\overline{\varrho}'(s^{m/n-1}\chi_{(t,|\Omega|)}(s))$. Then*

$$\lim_{a\to 0+} \overline{\sigma}(\chi_{(0,a)}\phi_R) = 0 \tag{5.8}$$

*suffices for $W^{m,\varrho}(\Omega) \hookrightarrow\hookrightarrow L_\sigma(\Omega)$.*

*Proof.* Denote by $M_{\phi_R}$ the Marcinkiewicz space in which $\phi_R$ is essentially the largest element in the HLP sense, so that

$$\|f\|_{M_{\phi_R}} := \sup_{0<t<|\Omega|} \frac{\int_0^t f^*(s)\,ds}{\int_0^t \phi_R(s)\,ds} \approx \sup_{0<t<|\Omega|/2} \frac{f^{**}(t)}{\overline{\varrho}'(s^{m/n-1}\chi_{(t,|\Omega|)}(s))}.$$

If $\Omega \supset E_k \downarrow \emptyset$ and $\|f\|_{M_{\phi_R}} \le 1$, then

$$\begin{aligned}\sigma(\chi_{E_k}f) &= \sup_{\overline{\sigma}'(g)\le 1} \int_0^{|\Omega|} (\chi_{E_k}f)^*(t)g^*(t)\,dt \le \sup_{\overline{\sigma}'(g)\le 1} \int_0^{|E_k|} f^*(t)g^*(t)\,dt \\ &\le \sup_{\overline{\sigma}'(g)\le 1} \int_0^{|E_k|} \phi_R(t)g^*(t)\,dt \qquad \text{(by the HLP principle)} \\ &\le \overline{\sigma}(\chi_{(0,|E_k|)}\phi_R) \downarrow 0 \qquad \text{(by (5.8))}.\end{aligned}$$

Therefore, the functions in the unit ball of $M_{\phi_R}(\Omega)$ have uniformly absolutely continuous norms in $L_\sigma(\Omega)$.

In view of Lemma 5.2,

$$\|\chi_{(0,t)}\|_{M_{\phi_R}} = \frac{t}{c(t)} \approx \frac{t}{\overline{\varrho}'(s^{m/n-1}\chi_{(t,|\Omega|)}(s))} \approx \overline{\sigma}_\varrho(\chi_{(0,t)}), \qquad 0 < t < |\Omega|/2,$$

so, by [2, Chapter 2, Theorem 5.13], $L_{\sigma_\varrho}(\Omega) \hookrightarrow M_{\phi_R}(\Omega)$. This means the functions in the unit ball of $L_{\sigma_\varrho}(\Omega)$ have uniformly absolutely continuous norms in $L_\sigma(\Omega)$. Thus, by Theorem 5.1, $W^{m,\varrho}(\Omega) \hookrightarrow\hookrightarrow L_\sigma(\Omega)$. ∎

Our next example involves a generalization of the Lorentz norms treated in Section 2, namely the Lorentz Gamma norms.

Definition 5.4. For a nonnegative, measurable (weight) function $\phi$ on $(0,|\Omega|)$ and an index $p$, $1 \le p < \infty$, the $\Gamma_p(\phi)$ *norm* is defined as

$$\varrho_{p,\phi}(f) := \Big(\int_0^{|\Omega|} f^{**}(t)^p \phi(t)\,dt\Big)^{1/p}, \qquad f \in \mathfrak{M}(\Omega).$$

We denote $L_{\varrho_{p,\phi}}(\Omega)$ by $\Gamma_p(\phi) = \Gamma_p(\phi,\Omega)$.

To ensure that $\varrho_{p,\phi}$ is neither trivial nor equivalent to $\varrho_1$, we require, respectively,

$$\int_0^{|\Omega|} \phi(t)\,dt < \infty \quad \text{and} \quad \int_0^{|\Omega|} t^{-p}\phi(t)\,dt = \infty. \tag{5.9}$$

The Köthe dual of $\varrho_{p,\phi}$ has

$$\varrho'_{p,\phi}(g) \approx \varrho_{p',\psi}(g) = \Big(\int_0^{|\Omega|} g^{**}(t)^{p'}\psi(t)\,dt\Big)^{1/p'}, \qquad g \in \mathfrak{M}(0,|\Omega|),$$

where, for $0 < t < |\Omega|$,

$$\psi(t) := \begin{cases} \dfrac{t^{p'-1}(\int_0^t \phi(s)\,ds)(t^p\int_t^{|\Omega|} s^{-p}\phi(s)\,ds)}{(\int_0^t \phi(s)\,ds + t^p\int_t^{|\Omega|} s^{-p}\phi(s)\,ds)^{p'+1}} & \text{when } 1 < p < \infty,\\[2ex] \dfrac{t}{\int_0^t \phi(s)\,ds + t\int_t^{|\Omega|} s^{-1}\phi(s)\,ds} & \text{when } p = 1. \end{cases}$$

(See [11, Theorem 6.2], [8, Theorem 2.7] or [10].)

Given two weight functions, $\phi_1$ and $\phi_2$, satisfying (5.9) for $1 < p < \infty$, it follows from [10] that

$$H_{n/m} : \Gamma_p(\phi_1) \to \Gamma_p(\phi_2) \quad (\Gamma_p(\phi_i) = \Gamma_p(\phi_i,(0,|\Omega|)),\ i = 1,2)$$

if and only if

$$(5.10)\qquad \int_0^t \phi_2(s)\,ds + t^p \int_t^{|\Omega|} s^{-p}\phi_2(s)\,ds$$
$$\lesssim \int_0^t \psi_1(s)\,ds + t^{(1-m/n)p} \int_t^{|\Omega|} s^{(m/n-1)p}\psi_1(s)\,ds, \qquad 0 < t < |\Omega|.$$

A straightforward calculation reveals that (5.10) is equivalent to

$$\sup_{0<a<|\Omega|} \varrho_{p,\phi_2}(\chi_{(0,a)})\varrho'_{p,\phi_1}(t^{m/n-1}\chi_{(a,|\Omega|)}(t)) < \infty.$$

Now, the methods of [7] show that this means

$$H_{n/m} : \Gamma_p(\phi_1) \to\to \Gamma_p(\phi_2)$$

if and only if

$$\lim_{a\to 0+} \varrho_{p,\phi_2}(\chi_{(0,a)})\varrho'_{p,\phi_1}(t^{m/n-1}\chi_{(a,|\Omega|)}(t)) = 0.$$

In view of Theorem 1.1, these considerations yield

THEOREM 5.5. *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ having $\partial\Omega \in \mathrm{Lip}_1$. Suppose $\phi_1$ and $\phi_2$ are weight functions on $(0,|\Omega|)$ satisfying*

$$\int_0^{|\Omega|} \phi_i(t)\,dt < \infty \quad \text{and} \quad \int_0^{|\Omega|} t^{-p}\phi_i(t)\,dt = \infty, \qquad i = 1,2,$$

*for fixed $p$, $1 < p < \infty$. Define*

$$\psi_1(t) := \frac{t^{p'-1}(\int_0^t \phi_1(s)\,ds)(t^p \int_t^{|\Omega|} s^{-p}\phi_1(s)\,ds)}{(\int_0^t \phi_1(s)\,ds + t^p \int_t^{|\Omega|} s^{-p}\phi_1(s)\,ds)^{p'+1}}.$$

*Then, in order to have*

$$W^{m,\varrho_{\Gamma_p(\phi_1)}}(\Omega) \hookrightarrow\hookrightarrow \Gamma_p(\phi_2)(\Omega),$$

*it is necessary and sufficient that*

$$\lim_{a\to 0+} \frac{\int_0^a \phi_2(t)\,dt + a^p \int_a^{|\Omega|} t^{-p}\phi_2(t)\,dt}{\int_0^a \psi_1(t)\,dt + a^{(1-m/n)p} \int_a^{|\Omega|} t^{(m/n-1)p}\psi_1(t)\,dt} = 0.$$

An earlier generalization of the Lebesgue spaces $L_p(\Omega)$, due to Orlicz, is defined in terms of an *N-function* $A(t) = \int_0^t a(s)\,ds$, in which $a(s)$ is increasing on $\mathbb{R}^+$, $a(0+) = 0$ and $\lim_{s\to\infty} a(s) = \infty$. Its so called gauge norm, $\varrho_A$, is defined as

$$\varrho_A(f) := \inf\Big\{\lambda > 0 : \int_\Omega A(|f(x)|/\lambda)\,dx \le 1\Big\}, \qquad f \in \mathfrak{M}(\Omega).$$

The Köthe dual of $\varrho_A$ is equivalent to the gauge norm, $\varrho_{\widetilde{A}}$, where

$$\widetilde{A}(t) := \int_0^t a^{-1}(s)\,ds, \qquad t > 0,$$

is the $N$-function *complementary* to $A$; in fact,

$$\varrho_{\widetilde{A}}(g) \le \varrho'_A(g) \le 2\varrho_{\widetilde{A}}(g), \qquad g \in \mathfrak{M}(\Omega).$$

The Orlicz spaces determined by $\varrho_A$ and $\varrho_{\widetilde{A}}$ are denoted by $L_A(\Omega)$ and $L_{\widetilde{A}}(\Omega)$, respectively.

Let $A$ and $\widetilde{A}$ be complementary $N$-functions. We will assume

$$\int_1^\infty \frac{\widetilde{A}(s)}{s^{1+n/(n-m)}}\,ds = \infty,$$

which is necessary and sufficient to guarantee

$$W^{m,\varrho_A}(\Omega) \not\hookrightarrow L_\infty(\Omega).$$

It essentially asserts that the function $t \mapsto t^{1-m/n}$ is not in $L_{\widetilde{A}}(0,|\Omega|)$. This was proved in [5] for $m = 1$; for general $m$ it follows from [13, Theorem A].

As shown in Lemma 5.2,

$$\overline{\sigma}_{\varrho_A}(\chi_{(0,t)}) \approx \frac{1}{\varrho_{\widetilde{A}}(s^{m/n-1}\chi_{(t,|\Omega|)}(s))}, \qquad 0 < t < \frac{|\Omega|}{2}.$$

The $N$-function $A_R$ associated to the same fundamental function satisfies

$$A_R^{-1}(t) \approx \varrho_{\widetilde{A}}(s^{m/n-1}\chi_{(1/t,|\Omega|)}(s)), \qquad t \ge |\Omega|^{-1};$$

see [2, Chapter 4, Lemma 8.17].

It is a simple exercise to get

$$\varrho_{\widetilde{A}}(s^{m/n-1}\chi_{(1/t,|\Omega|)}(s)) \approx \frac{t^{1-m/n}}{E^{-1}(t)},$$

where

$$E(t) := t^{n/(n-m)} \int_{|\Omega|}^t \frac{\widetilde{A}(s)}{s^{1+n/(n-m)}}\,ds, \qquad t \ge |\Omega|^{-1}. \tag{5.11}$$

As shown in [4],

$$W^{m,\varrho_A}(\Omega) \hookrightarrow L_{A_R}(\Omega). \tag{5.12}$$

Given (5.12), classical results [16, Theorems 3.17.8 and 3.17.10] ensure that the $N$-function $B$ satisfies

$$W^{m,\varrho_A}(\Omega) \hookrightarrow\hookrightarrow L_B(\Omega) \tag{5.13}$$

if $A_R$ grows essentially faster than $B$, that is,

$$\limsup_{t\to\infty} \frac{A_R(\lambda t)}{B(t)} = \infty \quad \text{for every } \lambda > 0. \tag{5.14}$$

We claim (5.14) is also necessary for (5.13). Indeed, by Theorem 3.8, (5.13) implies

$$0 = \lim_{a\to 0+} \varrho_B(\chi_{(0,a)})\varrho_{\widetilde{A}}(t^{m/n-1}\chi_{(a,|\Omega|)}(t)) = \lim_{a\to 0+} \frac{A_R^{-1}(1/a)}{B^{-1}(1/a)}.$$

Thus, for every $\lambda > 0$,

$$\lim_{t\to\infty} \frac{1}{\lambda}\,\frac{A_R^{-1}(t)}{B^{-1}(t)} = 0.$$

So, given $\varepsilon \in (0,1)$, there exists $t_0 > 1$ with

$$\frac{1}{\lambda}\,A_R^{-1}(t) < \varepsilon B^{-1}(t), \qquad t \geq t_0.$$

Dividing by $\varepsilon$ and applying the (nondecreasing) function $B$ to both sides of the inequality, we get

$$B\left(\frac{A_R^{-1}(t)}{\lambda\varepsilon}\right) < t, \qquad t \geq t_0.$$

Setting

$$y = \frac{A_R^{-1}(t)}{\lambda} \quad \text{and} \quad y_0 = \frac{A_R^{-1}(t_0)}{\lambda},$$

we have, by the convexity of $B$,

$$\frac{1}{\varepsilon}\,B(y) \leq B\left(\frac{y}{\varepsilon}\right) < A_R(\lambda y), \qquad y \geq y_0.$$

Therefore, $A_R(\lambda y)/B(y) > 1/\varepsilon$ for sufficiently large $y$, or

$$\lim_{y\to\infty} \frac{A_R(\lambda y)}{B(y)} = \infty.$$

To summarize, we have proved

THEOREM 5.6. *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ having $\partial\Omega \in \mathrm{Lip}_1$. Suppose $A$ and $\widetilde{A}$ are complementary $N$-functions and*

$$\int_1^\infty \frac{\widetilde{A}(s)}{s^{1+n/(n-m)}}\,ds = \infty.$$

*Define the $N$-function $A_R(t)$, for $t$ large, by*

$$A_R^{-1}(t) := \frac{t^{1-m/n}}{E^{-1}(t)},$$

*with $E$ as in* (5.11). *Then*

$$W^{m,\varrho_A}(\Omega) \hookrightarrow\hookrightarrow L_B(\Omega)$$

*for a given $N$-function $B$ if and only if*

$$\lim_{t\to\infty} \frac{A_R(\lambda t)}{B(t)} = \infty \quad \textit{for every } \lambda > 0. \tag{5.15}$$

Remark 5.7. In terms of the explicitly known functions $B$ and $E$, (5.15) can be expressed by

$$\lim_{t\to\infty} \frac{B\big(\frac{1}{\lambda t}E(t)^{1-m/n}\big)}{E(t)} = 0 \quad \text{for every } \lambda > 0. \tag{5.16}$$

We further recall that there is a direct and simple way to define an $N$-function equivalent to $\widetilde{A}$, namely,

$$\widetilde{\mathbf{A}}(t) := \sup_{s\in\mathbb{R}_+} [st - A(s)], \qquad t \in \mathbb{R}_+.$$

**6. Relations to other results.** In this section we compare, in some detail, our results to those in two papers we have recently learned about. The Sobolev spaces in both consist of the closure of the $C_0^\infty(\Omega)$ functions in the Sobolev norm, so we will denote them by $W_0^{m,\varrho}(\Omega)$.

Pustylnik, in [20], proves the equivalence of (1.5) and (1.6) in our principal Theorem 1.1, but under the restriction $1 < i_\varrho < I_\varrho < n/(m-1)$. In particular, this excludes Sobolev spaces $W_0^{m,\varrho}(\Omega)$ for which $L_\varrho(\Omega)$ is near $L_1(\Omega)$. A direct proof of our Theorem 5.1 is given for the case in which $L_\sigma(\Omega)$ is separable, though the condition (5.2) is not mentioned.

Curbera and Ricker [6] consider the special problem of when

$$W_0^{1,\varrho_\sigma}(\Omega) \hookrightarrow\hookrightarrow L_\sigma(\Omega). \tag{6.1}$$

Here, $\varrho_\sigma$ denotes the optimal r.i. norm $\varrho$ in

$$W_0^{1,\varrho}(\Omega) \hookrightarrow L_\sigma(\Omega),$$

and indeed in

$$W^{1,\varrho}(\Omega) \hookrightarrow L_\sigma(\Omega).$$

Their Theorem 3.1 is the inference (1.5)⇒(1.6) in Theorem 4.1 while their Theorem 3.7 is the inference (1.6)⇒(1.5) in the particular case $\varrho = \varrho_1$.

In both [6] and [20] the function $t \mapsto t^{m/n}\sigma(\chi_{(0,t)})$ (in [6] $m = 1$) plays an important role. We observe that from our expression (3.5) for $\varrho_\sigma$ one readily obtains

$$\varrho_\sigma(\chi_{(0,t)}) \approx t \sup_{t\le s<1} s^{m/n-1}\sigma(\chi_{(0,s)}), \tag{6.2}$$

with $t^{m/n}\sigma(\chi_{(0,t)})$ equal to its quasiconcave majorant in (6.2) when the function $t^{m/n-1}\sigma(\chi_{(0,t)})$ is nonincreasing. Given this, one can obtain Theorems 2.7 and 2.9 in [6] using our necessary condition (3.11) and our Theorem 5.1, respectively. Finally, the sufficiency for (6.1) of the condition

$$\frac{\int_0^1 t^{m/n}\sigma(\chi_{(0,t)})}{\varrho(\chi_{(0,t)})}\,\frac{dt}{t} < \infty \tag{6.3}$$

proved in Theorem 3 of [20] is a consequence of our Theorem 5.3, in that (6.3) implies our apparently weaker condition (5.8).

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