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On partial isometries in C^*-algebras

by

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Abstract. We study similarity to partial isometries in C^*-algebras as well as their relationship with generalized inverses. Most of the results extend some recent results regarding partial isometries on Hilbert spaces. Moreover, we describe partial isometries by means of interpolation polynomials.

1. Introduction. This article is motivated by recent publications about partial isometries on Hilbert spaces and their relationship with generalized inverses (see [BM], [M], [MS1], [MS2] and references therein). Our goal, among others, is to investigate some of the results presented in these publications, but in the context of C^*-algebras.

One of the most illustrative results regarding the relationship between partial isometries and generalized inverses on Hilbert spaces is given by M. Mbekhta in [M]. In that article, the author establishes that a contraction on a Hilbert space is a partial isometry if and only if it has a contractive generalized inverse. In the present paper we extend this result to elements in a C^*-algebra. Furthermore, following this direction, we characterize normal partial isometries. We should emphasize that the proof presented here differs from that given by Mbekhta in [M], since that proof strongly used the Hilbert space structure. As an immediate consequence, we extend some results of Furuta et al. [FN] and Gupta [G] about partial isometries on Hilbert spaces. All these results can be found in Section 2.

Section 3 is devoted to describing partial isometries with the use of interpolation polynomials. More precisely, we provide diverse formulas for partial isometries coming from interpolation polynomials of the function $f(x) = x^{-1}$.

In Section 4 we obtain some results concerning similarity to partial isometries in C^*-algebras. This problem, but for operators on Hilbert spaces, has been considered in many articles (see [BM], [MS1], [MS2], inter alia). How-

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ever, an analysis of similarity to partial isometries and their relationship with generalized inverses in the context of C^*-algebras has not been developed so far. This analysis is our main goal in Section 4. One interesting result regarding similarity to partial isometries and generalized inverses on Hilbert spaces is given by C. Badea and M. Mbekhta in [BM]. Recall that a bounded linear operator T defined on a Hilbert space is a partial isometry if and only if its adjoint, T^*, is a generalized inverse of T (moreover, T^* coincides with the Moore–Penrose inverse of T). Hence, in [BM] the authors show that this relationship with generalized inverses still holds if similarity to partial isometries is considered. More precisely, they prove that T is similar to a partial isometry if and only if T^* is similar (by means of a positive operator) to a generalized inverse of T. In this paper we prove that this result still holds in C^*-algebras. Moreover, we note that the generalized inverse involved in the similarity condition is not, in general, the Moore–Penrose inverse of the element. Therefore, we finish Section 4 by studying under which conditions this is the case.

Finally, we introduce some concepts and notation. Along this article $\mathcal{A}$ denotes a C^*-algebra with identity 1 and invertible group $\mathcal{A}^{-1}$. Given $a \in \mathcal{A}$ we say that a is a *contraction* if $\|a\| \leq 1$, and a is *normal* if $aa^* = a^*a$. In addition, we say that $a = a^* \in \mathcal{A}$ is *positive* if $\sigma(a) \subseteq [0, \infty)$ where $\sigma(a)$ denotes the spectrum of a. Moreover, an element $a \in \mathcal{A}$ will be called *regular* if $a \in a\mathcal{A}a$. In that case, every element $b \in \mathcal{A}$ such that $a = aba$ will be called a *generalized inverse* of a. Given $a \in \mathcal{A}$ regular there always exists a unique generalized inverse b of a such that $a = aba$, $b = bab$, $(ba)^* = ba$ and $(ab)^* = ab$ (see Theorems 5 and 6 in [HM1]). It is called the *Moore–Penrose inverse* of a and will be denoted by $a^\dagger$. The reader is referred to [HM1] and [HM2] for several properties of generalized inverses in C^*-algebras. In particular, if $a \in \mathcal{A}$ satisfies $a = aa^*a$, i.e., $a^* = a^\dagger$, then we shall say that a is a *partial isometry.*

2. Partial isometries in C^*-algebras. We begin by extending Theorem 3.1 of [M] to the context of C^*-algebras.

THEOREM 2.1. *Let $a \in \mathcal{A}$ with $\|a\| \leq 1$. The following conditions are equivalent:*

(1) *a is a partial isometry;*
(2) *there exists $b \in \mathcal{A}$ with $\|b\| \leq 1$ such that $aba = a$;*
(3) *there exists $b \in \mathcal{A}$ with $\|b\| \leq 1$ such that $aba = a$ and $bab = b$.*

The next two results will be useful for the proof of this theorem. The reader is referred to [HM2] for their proofs.

THEOREM 2.2. *Let $a, b \in \mathcal{A}$ be such that $0 \neq a = aba$ and $bab = b$. Then*

$$1 \leq \|b\|\gamma(a) \leq \|ba\|\,\|ab\|, \tag{2.1}$$

where $\gamma(a) = \inf\{\|ax\| : \mathrm{dist}(x, a^{-1}(0)) \geq 1\}$.

COROLLARY 2.3. *Let $0 \neq a \in \mathcal{A}$. Then $\gamma(a) = \|a\|$ if and only if $a/\|a\|$ is a partial isometry.*

Now, we prove Theorem 2.1. It should be noted that this proof is notably different from that given by M. Mbekhta on Hilbert spaces.

Proof of Theorem 2.1. (1)⇒(2). Take $b = a^*$.

(2)⇒(3). Let $b' = bab \in \mathcal{A}$. Then $\|b'\| \leq 1$, $ab'a = ababa = a$ and $b'ab' = bababab = bab = b'$.

(3)⇒(1). Since $aba = a$ and a, b are contractions we have $\|a\| = \|aba\| \leq \|ba\| \leq \|a\|$, i.e., $\|a\| = \|ba\| \leq 1$. Similarly, from $b = bab$, we obtain $\|b\| = \|ab\| \leq 1$. On the other hand, as ab and ba are idempotents we have $1 \leq \|ab\|$ and $1 \leq \|ba\|$. Hence, $\|a\| = \|ba\| = 1$ and $\|b\| = \|ab\| = 1$ and, by Theorem 2.2, $\gamma(a) = 1$. Therefore, $\gamma(a) = 1 = \|a\|$ and so, by Corollary 2.3, a is a partial isometry. ∎

In the next proposition we shall relate normal partial isometries and generalized inverses.

PROPOSITION 2.4. *Let $a \in \mathcal{A}$ with $\|a\| \leq 1$. The following conditions are equivalent:*

(1) *a is a normal partial isometry;*
(2) *there exists $b \in \mathcal{A}$ with $\|b\| \leq 1$ such that $aba = a$ and $ba = ab$;*
(3) *there exists $b \in \mathcal{A}$ with $\|b\| \leq 1$ such that $aba = a$, $bab = b$ and $ba = ab$;*
(4) *$\|a^\dagger\| \leq 1$ and $aa^\dagger = a^\dagger a$.*

Proof. (1)⇒(2). Take $b = a^*$.

(2)⇒(3). Let $b' = bab$. By the proof of Theorem 2.1, it remains to show that $b'a = ab'$. Now, since $ba = ab$, we have $b'a = baba = abab = ab'$.

(3)⇒(4). Since $aba = a$, both ab and ba are idempotents. Moreover, as $\|ab\| \leq 1$ and $\|ba\| \leq 1$, both ba and ab are selfadjoint and so $b = a^\dagger$.

(4)⇒(1). If $\|a^\dagger\| \leq 1$ then, by Theorem 2.1, a is a partial isometry and so $a^* = a^\dagger$. Therefore, since $aa^\dagger = a^\dagger a$, we see that a is normal. ∎

The commutativity of a and $a^\dagger$, which appears in the above proposition, has been studied by Harte and Mbekhta in [HM2]. In particular, the reader is referred to Theorem 10 in [HM2] for conditions equivalent to $aa^\dagger = a^\dagger a$.

As a simple consequence of the previous proposition we obtain the next result which has been obtained by Furuta and Nakamoto, but in the context of operators on a Hilbert space (Theorem 4 of [FN]). The proof presented here differs from that given by Furuta and Nakamoto, which strongly used the Hilbert space structure.

Corollary 2.5. *If $a \in \mathcal{A}$ is a contraction such that $a^k = a$ for some $k \geq 2$ then a is a normal partial isometry.*

Proof. Let $b = a^{k-2}$. Since a is a contraction, so is b. Moreover, $aba = a^k = a$ and $ab = a^{k-1} = ba$. Therefore, the assertion follows by Proposition 2.4. ∎

Also as a consequence of Theorem 2.1 we obtain the next result which has been proved for operators by B. C. Gupta in [G]. For this, we define the *right annihilator* of a^k as $a^{-k}(0) := \{x \in \mathcal{A} : \ a^k x = 0\}$.

Corollary 2.6. *Let $a \in \mathcal{A}$ be a contraction such that $a^{-1}(0) = a^{-2}(0)$. If a^k is a partial isometry for some $k \geq 1$ then a is a partial isometry.*

Proof. First, note that if $a^{-1}(0) = a^{-2}(0)$ then $a^{-1}(0) = a^{-n}(0)$ for every $n \geq 1$. In fact, suppose that $a^{-1}(0) = a^{-2}(0) = a^{-(n-1)}(0)$. Thus, if $a^n x = 0$ then $ax \in a^{-(n-1)}(0) = a^{-1}(0)$. Hence, $a^2 x = a(ax) = 0$, i.e., $x \in a^{-2}(0) = a^{-1}(0)$ and so $a^{-1}(0) = a^{-n}(0)$.

Now, suppose that a^k is a partial isometry for some $k > 1$. Then $a(1 - (a^*)^k a^k) = 0$ and so $a = a(a^*)^k a^k = a(a^*)^k a^{k-1} a$. Defining $b := (a^*)^k a^{k-1}$ we find that b is a contraction such that $aba = a$ so, by Theorem 2.1, a is a partial isometry. ∎

Remark 2.7. (I) The following implications hold: $a \in \mathcal{A}$ is hyponormal (i.e., $aa^* \leq a^*a$) $\Rightarrow a^{-1}(0) \subseteq a^{*-1}(0) \Rightarrow a^{-1}(0) = a^{-2}(0)$.

(II) In Corollary 2.6, the condition on the annihilators of a and a^2 cannot be avoided. Indeed, consider $a = \left(\begin{smallmatrix} 0 & 1/2 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{M}_2(\mathbb{C})$. Then a is a contraction with $a^{-1}(0) \neq a^{-2}(0)$ and $a^2 = 0$ is a partial isometry, but a is not ($aa^*a \neq a$).

Remark 2.8. If $a \in \mathcal{A}$ is a contraction such that a^k is an isometry for some $k \geq 1$ (i.e., $(a^k)^* a^k = 1$) then a is an isometry. In fact, if a^k is an isometry then $a^{-k}(0) = \{0\}$ and so $a^{-1}(0) = a^{-k}(0) = \{0\}$. Thus, by Corollary 2.6, a is a partial isometry with $a^{-1}(0) = \{0\}$, i.e., an isometry.

3. Approximations of partial isometries. In this section we shall describe partial isometries by means of interpolation polynomials.

First we shall consider the divided differences interpolation polynomials of $f(x) = 1/x$, given by

$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k+1} \prod_{j=0}^{k-1} \left(1 - \frac{x}{j+1}\right)$$

where, by convention, we consider $\prod_{j=0}^{-1} \left(1 - \frac{x}{j+1}\right) = 1$.

Remark 3.1.

(1) It is easy to prove that

$$1 - xp_n(x) = \prod_{j=0}^{n} \left(1 - \frac{x}{j+1}\right). \tag{3.1}$$

(2) The following property is well-known:

$$\lim_{n\to\infty} p_n(x) = \frac{1}{x}, \tag{3.2}$$

where the convergence is uniform on every compact subset of $(0, \infty)$.

By means of these polynomials we obtain another characterization of partial isometries.

Theorem 3.2. *Let $a \in \mathcal{A}$ and $s_n(a) := p_n(aa^*)a$. The following conditions are equivalent:*

(1) *a is a partial isometry;*
(2) $\lim_{n\to\infty} s_n(a) = a$.

Before we prove Theorem 3.2, we present some technical results.

Lemma 3.3. *Let $a \in \mathcal{A}$, $a \geq 0$. Then*

$$\lim_{t\to\infty} e^{-ta}a = 0.$$

Proof. Since $e^{ta} = 1+ta+(ta)^2/2+\cdots \geq ta$, we have $e^{ta}-ta \geq 0$. On the other hand, $e^{-ta} = (e^{-\frac{1}{2}ta})(e^{-\frac{1}{2}ta})^* \geq 0$ and it commutes with $e^{ta} - ta \geq 0$. Thus, $1-tae^{-ta} = e^{-ta}(e^{ta}-ta) \geq 0$ and so $0 \leq tae^{-ta} \leq 1$ for $t > 0$. Hence, $\|e^{-ta}a\| \leq 1/t$ and $\lim_{t\to\infty} \|e^{-ta}a\| = 0$, i.e., $\lim_{t\to\infty} e^{-ta}a = 0$. ∎

Lemma 3.4. *Let $a \in \mathcal{A}$. Then*

$$\lim_{n\to\infty} \prod_{j=0}^{n} \left(1 - \frac{aa^*}{j+1}\right)^m a = 0$$

for every $m \in \mathbb{N}$.

Proof. For every $x \in \mathbb{R}^+$ we have $1 - x \leq e^{-x}$. Hence, $1 - \frac{x}{j+1} \leq e^{-x\frac{1}{j+1}}$ for every $x \in \mathbb{R}^+$ and $j \in \mathbb{N}$. So, there exists $J_0 \in \mathbb{N}$ such that for every $j \geq J_0$,

$$0 \leq 1 - \frac{x}{j+1} \leq e^{-x\frac{1}{j+1}}.$$

From this,

$$0 \leq \prod_{j=J_0}^{n} \left(1 - \frac{x}{j+1}\right) \leq e^{-x\sum_{j=J_0}^{n} \frac{1}{j+1}}.$$

Hence, setting $t_n := \sum_{j=J_0}^{n} \frac{1}{j+1}$ we obtain

$$0 \le \prod_{j=J_0}^{n} \Big(1 - \frac{x}{j+1}\Big)^{2m} x \le e^{-2mxt_n} x.$$

Now, since $aa^* \ge 0$, there exists $J_0 \in \mathbb{N}$ such that for every $j \ge J_0$,

$$0 \le \prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{2m} aa^* \le e^{-2maa^*t_n} aa^*. \tag{3.3}$$

On the other hand,

$$\begin{aligned}\Big\| \prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{m} a\Big\|^2 &= \Big\| \prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{m} a\Big(\prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{m} a)^*\Big\| \\ &= \Big\| \prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{m} aa^* \prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{m}\Big\| \\ &= \Big\| \prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{2m} aa^*\Big\|.\end{aligned}$$

Therefore, by (3.3) and Lemma 3.3, we obtain

$$\begin{aligned}\Big\| \prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{m} a\Big\|^2 &= \Big\| \prod_{j=J_0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big)^{2m} aa^*\Big\| \\ &\le \|e^{-2maa^*t_n} aa^*\| \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

Thus, $\lim_{n\to\infty} \prod_{j=J_0}^{n} \big(1 - \frac{aa^*}{j+1}\big)^{m} a = 0$ and so $\lim_{n\to\infty} \prod_{j=0}^{n} \big(1 - \frac{aa^*}{j+1}\big)^{m} a = 0$. ∎

Now, we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. (1)⇒(2). Suppose that a is a partial isometry, i.e., $aa^*a = a$. Hence, $\big(1 - \frac{aa^*}{j+1}\big)a = a\big(1 - \frac{1}{j+1}\big)$ and so

$$\begin{aligned}s_n(a) = p_n(aa^*)a &= \sum_{k=0}^{n} \frac{1}{k+1} \prod_{j=0}^{k-1} \Big(1 - \frac{aa^*}{j+1}\Big)a \\ &= a\sum_{k=0}^{n} \frac{1}{k+1} \prod_{j=0}^{k-1} \Big(1 - \frac{1}{j+1}\Big) = ap_n(1) = a.\end{aligned}$$

Then $s_n(a) = a$ for every $n \in \mathbb{N}$ and so $\lim_{n\to\infty} s_n(a) = a$.

(2)⇒(1). By (3.1), we have

$$1 - aa^*p_n(aa^*) = \prod_{j=0}^{n} \Big(1 - \frac{aa^*}{j+1}\Big).$$

Then

$$a - aa^* s_n(a) = \prod_{j=0}^{n} \left(1 - \frac{aa^*}{j+1}\right) a.$$

Now, by Lemma 3.4 and since $\lim_{n\to\infty} s_n(a) = a$, we obtain

$$a - aa^* a = 0,$$

i.e., a is a partial isometry. ■

Remark 3.5. The polynomials $p_n(x)$ can be rewritten in the following way:

$$p_0(x) = 1, \qquad p_{n+1}(x) = p_n(x) + \frac{1}{n+2}[1 - xp_n(x)], \qquad n \geq 1.$$

In fact,

$$p_{n+1}(x) = p_n(x) + \frac{1}{n+2} \prod_{j=0}^{n} \left(1 - \frac{x}{j+1}\right) = p_n(x) + \frac{1}{n+2}[1 - xp_n(x)].$$

Hence, given $a \in \mathcal{A}$ define

$$a_0 = a, \qquad a_{n+1} = a_n + \frac{1}{n+2}(a - aa^* a_n), \qquad n \geq 1.$$

Thus, as an immediate consequence of Theorem 3.2, we obtain:

Corollary 3.6. *Let $a \in \mathcal{A}$. The following statements are equivalent:*

(1) *a is a partial isometry;*
(2) $\lim_{n\to\infty} a_n = a$.

In a similar manner, partial isometries can also be described by means of the Hermite interpolation polynomials of $f(x) = 1/x$, given by

$$q_0(x) = 2 - x,$$

$$q_n(x) = \sum_{i=0}^{n} [2(1+i) - x] \frac{1}{(1+i)^2} \prod_{j=0}^{i-1} \left(1 - \frac{x}{1+j}\right)^2 \qquad \text{for } n \geq 1,$$

where $\prod_{j=0}^{-1} \left(1 - \frac{x}{1+j}\right)^2 = 1$. Now q_n is the unique polynomial of degree $2n+1$ such that $q_n(x_i) = f(x_i) = 1/x_i$ and $q_n'(x_i) = f'(x_i) = -1/x_i^2$ for $x_i = i + 1$ with $i = 0, 1, \ldots, n$ (see [QSS]). In the next remark we collect some properties of these polynomials. For the proof the reader is referred to [D].

Remark 3.7.

(1) It is straightforward that

$$1 - xq_n(x) = \prod_{j=0}^{n} \left(1 - \frac{x}{j+1}\right)^2. \tag{3.4}$$

(2) The following property is widely known:

$$\lim_{n\to\infty} q_n(x) = \frac{1}{x}, \tag{3.5}$$

where the convergence is uniform on every compact subset of $(0,\infty)$.

THEOREM 3.8. *Let $a \in \mathcal{A}$ and $w_n(a) := q_n(aa^*)a$. Then the following conditions are equivalent:*

(1) *a is a partial isometry;*
(2) $\lim_{n\to\infty} w_n(a) = a$.

Proof. (1)⇒(2). Suppose that a is a partial isometry, i.e., $aa^*a = a$. Then $\left(1 - \frac{aa^*}{1+j}\right)^2 a = a\left(1 - \frac{1}{1+j}\right)^2$ and so

$$\begin{aligned} w_n(a) = q_n(aa^*)a &= \sum_{i=0}^{n} [2(1+i) - aa^*] \frac{1}{(1+i)^2} \prod_{j=0}^{i-1} \left(1 - \frac{aa^*}{1+j}\right)^2 a \\ &= \sum_{i=0}^{n} [2(1+i) - aa^*] \frac{1}{(1+i)^2} a \prod_{j=0}^{i-1} \left(1 - \frac{1}{1+j}\right)^2 \\ &= \sum_{i=0}^{n} [2(1+i)a - aa^*a] \frac{1}{(1+i)^2} \prod_{j=0}^{i-1} \left(1 - \frac{1}{1+j}\right)^2 \\ &= a \sum_{i=0}^{n} [2(1+i) - 1] \frac{1}{(1+i)^2} \prod_{j=0}^{i-1} \left(1 - \frac{1}{1+j}\right)^2 \\ &= a q_n(1) = a. \end{aligned}$$

Thus, $w_n(a) = a$ for every $n \in \mathbb{N}$ and so $\lim_{n\to\infty} w_n(a) = a$.

(2)⇒(1). By Remark 3.7, we have

$$1 - aa^* q_n(aa^*) = \prod_{j=0}^{n} \left(1 - \frac{aa^*}{j+1}\right)^2.$$

Thus,

$$a - aa^* w_n(a) = \prod_{j=0}^{n} \left(1 - \frac{aa^*}{j+1}\right)^2 a.$$

Therefore, by Lemma 3.4 and since $\lim_{n\to\infty} w_n(a) = a$, we obtain

$$a - aa^*a = 0,$$

i.e., a is a partial isometry. ∎

REMARK 3.9. The Hermite polynomials can also be defined by

$$q_0(x) := 2 - x,$$
$$q_{n+1}(x) := q_n(x) + \frac{1}{n+2}\left(2 - \frac{x}{n+2}\right)[1 - xq_n(x)], \qquad n \geq 1.$$

In fact,

$$q_{n+1}(x) = q_n(x) + \frac{1}{(n+2)^2}[2(n+2) - x]\prod_{j=0}^{n}\left(1 - \frac{x}{1+j}\right)^2$$
$$= q_n(x) + \frac{1}{(n+2)}\left(2 - \frac{x}{n+2}\right)[1 - xq_n(x)].$$

Thus, given $a \in \mathcal{A}$ let

$$a_0 := (2 - aa^*)a, \qquad a_{n+1} := a_n + \frac{1}{n+2}\left(2 - \frac{aa^*}{n+2}\right)[a - aa^*a_n], \qquad n \geq 1.$$

With these definitions, Theorem 3.8 yields:

COROLLARY 3.10. *Let $a \in \mathcal{A}$. The following conditions are equivalent:*

(1) *a is a partial isometry;*
(2) $\lim_{n\to\infty} a_n = a$.

REMARK 3.11. The reader is referred to [CMQ] for other formulas for partial isometries of the type of those of Theorems 3.2 and 3.8.

With the above notation the following characterization of unitary elements of $\mathcal{A}$ can be obtained:

COROLLARY 3.12. *Let $a \in \mathcal{A}^{-1}$. The following statements are equivalent:*

(1) *a is unitary;*
(2) $\lim_{n\to\infty} p_n(aa^*) = 1$;
(3) $\lim_{n\to\infty} q_n(aa^*) = 1$.

Proof. (1)$\Leftrightarrow$(2). Clearly, if $a \in \mathcal{A}$ is unitary then $p_n(aa^*) = p_n(1) = 1$. On the other hand, suppose $\lim_{n\to\infty} p_n(aa^*) = 1$. Then $\lim_{n\to\infty} p_n(aa^*)a = a$, and so, by Theorem 3.2, a is a partial isometry. Thus, since a is invertible, we conclude that a is unitary.

(1)$\Leftrightarrow$(3). Analogously. ■

REMARK 3.13. Note that the previous corollary can also be proved by means of Remarks 3.1(2) and 3.7(2). Indeed, since $a \in \mathcal{A}^{-1}$ it follows that $\sigma(aa^*)$ is a compact set such that $\sigma(aa^*) \subseteq (0, \infty)$. Therefore, by functional calculus, $\lim_{n\to\infty} p_n(aa^*) = (aa^*)^{-1} = 1$ (resp. $\lim_{n\to\infty} q_n(aa^*) = (aa^*)^{-1} = 1$) and so a is unitary.

We emphasize that Theorems 3.2 and 3.8 cannot be proved by means of functional calculus if a is not invertible, since the condition $\sigma(aa^*) \subseteq (0,\infty)$ is not fulfilled.

4. Similarity to partial isometries in C^*-algebras. In this section we focus on describing similarity to partial isometries in C^*-algebras in terms of generalized inverses. Recall that given $a,b \in \mathcal{A}$ we say that a,b are *similar* if there exists $c \in \mathcal{A}^{-1}$ such that $a = cbc^{-1}$; in that case, we write $a \sim b$. If c is moreover positive then we write $a \sim_+ b$.

The next result, in the context of operators on Hilbert spaces, can be found in [BM].

THEOREM 4.1. *Let $a \in \mathcal{A}$. Then there exists $b \in \mathcal{A}$ such that $a = aba$ and $a^* \sim_+ b$ if and only if $a \sim v$ for some partial isometry $v \in \mathcal{A}$. Moreover, $ab = ba$ if and only if v is normal.*

Proof. Suppose $a^* = cbc^{-1}$ for some $c > 0$ and some generalized inverse b of a. Then

$$a = aba = ac^{-1}a^*ca.$$

Now, if $c^{1/2}$ denotes the positive square root of c then

$$c^{1/2}ac^{-1/2} = (c^{1/2}ac^{-1/2})(c^{-1/2}a^*c^{1/2})(c^{1/2}ac^{-1/2}).$$

So, $v = c^{1/2}ac^{-1/2}$ is a partial isometry and clearly $v \sim a$. If in addition $ab = ba$ then $ac^{-1}a^*c = c^{-1}a^*ca$ and so $vv^* = c^{1/2}ac^{-1}a^*c^{1/2} = c^{-1/2}a^*cac^{1/2} = v^*v$, i.e., v is normal.

Conversely, let $a = cvc^{-1}$ for some partial isometry v. Then

$$vv^*v = c^{-1}acc^*a^*(c^{-1})^*c^{-1}ac = v.$$

So,

$$acc^*a^*(c^{-1})^*c^{-1}a = cvc^{-1} = a.$$

Hence, $b = cc^*a^*(cc^*)^{-1}$ is a generalized inverse of a and $b \sim_+ a^*$. Moreover, if v is normal then $c^{-1}acc^*a^*(c^{-1})^* = c^*a^*(c^{-1})^*c^{-1}ac$, so $acc^*a^*(c^{-1})^*c^{-1} = cc^*a^*(c^{-1})^*c^{-1}a$, i.e., $ab = ba$. ■

REMARK 4.2. It should be stressed that the generalized inverse appearing in Theorem 4.1 is not, in general, the Moore–Penrose inverse of a. For example, consider $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$, V the partial isometry given by $V = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $L = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then $T = LVL^{-1} = \left(\begin{smallmatrix} 2 & -2 \\ 1 & -1 \end{smallmatrix}\right) \sim V$, but $T^* = \left(\begin{smallmatrix} 2 & 1 \\ -2 & -1 \end{smallmatrix}\right) \nsim T^\dagger = \frac{1}{5}\left(\begin{smallmatrix} 1 & 1/2 \\ -1 & -1/2 \end{smallmatrix}\right)$ since $\mathrm{Tr}(T^*) \neq \mathrm{Tr}(T^\dagger)$.

In the next proposition we describe the condition $a^* \sim_+ a^\dagger$ in terms of the invertible elements realizing the similarity $a \sim v$. For this, we denote by $[a,b]$ the commutator of $a,b \in \mathcal{A}$, i.e., $[a,b] = ab - ba$.

PROPOSITION 4.3. *Let $a \in \mathcal{A}$ be regular. Then $a^* \sim_+ a^\dagger$ if and only if $a = dvd^{-1}$ for some partial isometry $v \in \mathcal{A}$ such that $[vv^*, d^*d] = [v^*v, d^*d] = 0$.*

Proof. Suppose $a^* = ca^\dagger c^{-1}$ with $c > 0$. Then $a = aa^\dagger a = ac^{-1}a^*ca$, and so

$$c^{1/2}ac^{-1/2} = c^{1/2}ac^{-1/2}(c^{1/2}ac^{-1/2})^*c^{1/2}ac^{-1/2}.$$

Thus, $v = c^{1/2}ac^{-1/2}$ is a partial isometry. Furthermore, as $aa^\dagger$ is selfadjoint we have $ac^{-1}a^*c = cac^{-1}a^*$ or, on replacing a by $c^{-1/2}vc^{1/2}$, $c^{-1/2}vv^*c^{1/2} = c^{1/2}vv^*c^{-1/2}$. So, $c^{-1}vv^* = vv^*c^{-1}$. Similarly, as $a^\dagger a$ is selfadjoint we get $c^{-1}v^*v = v^*vc^{-1}$. Therefore, the result follows by taking $d = c^{-1/2}$.

Conversely, suppose that $a = dvd^{-1}$ for some partial isometry v such that $[vv^*, d^*d] = [v^*v, d^*d] = 0$. As $vv^*v = v$ we get $add^*a^*(dd^*)^{-1}a = a$, i.e., $b = dd^*a^*(dd^*)^{-1}$ is a generalized inverse of a and, clearly, $a^* \sim_+ b$. Let us prove that $b = a^\dagger$. For this, it is sufficient to show that ab and ba are selfadjoint. Now, since $d^*dvv^* = vv^*d^*d$, we have

$$\begin{aligned} ab &= add^*a^*(dd^*)^{-1} = dvd^{-1}dd^*(d^*)^{-1}v^*d^*(d^*)^{-1}d^{-1} \\ &= dvv^*d^{-1} = (d^*)^{-1}vv^*d^* = (d^*)^{-1}d^{-1}add^*a^*(d^*)^{-1}d^* \\ &= (dd^*)^{-1}add^*a^* = b^*a^* = (ab)^*. \end{aligned}$$

Similarly, from $d^*dv^*v = v^*vd^*d$, we deduce that ba is selfadjoint and the proof is complete. ∎

If a regular element $a \in \mathcal{A}$ is also normal then the condition $a^* \sim_+ a^\dagger$ turns out to be equivalent to a being a partial isometry:

PROPOSITION 4.4. *Let $a \in \mathcal{A}$ normal. Then the following conditions are equivalent:*

(1) $a^* \sim_+ a^\dagger$;
(2) *a is a partial isometry.*

Proof. If $a^* \sim_+ a^\dagger$ then, by Theorem 4.1, $a \sim v$ for some partial isometry and so $r(a) \le 1$, where $r(a)$ denotes the spectral radius of a. Moreover, as $a^* \sim v^*$ and v^* is also a partial isometry, we have $r(a^*) \le 1$. Thus, as $a^* \sim_+ a^\dagger$, it follows that $r(a^\dagger) \le 1$. Now, as a is normal, so is $a^\dagger$ (see Theorem 10 of [HM1]) and so $\|a\| = r(a) \le 1$ and $\|a^\dagger\| = r(a^\dagger) \le 1$. Therefore, by Theorem 2.1, a is a partial isometry. The converse implication is trivial. ∎

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