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# Rademacher functions in BMO

by

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**Abstract.** The Rademacher sums are investigated in the $BMO$ space on $[0,1]$. They span an uncomplemented subspace, in contrast to the dyadic $BMO_d$ space on $[0,1]$, where they span a complemented subspace isomorphic to $l_2$. Moreover, structural properties of infinite-dimensional closed subspaces of the span of the Rademacher functions in $BMO$ are studied and an analog of the Kadec–Pełczyński type alternative with $l_2$ and $c_0$ spaces is proved.

**1. Introduction.** In 1961, when studying some problems concerning partial differential equations, F. John and L. Nirenberg introduced the space $BMO$ of functions of bounded mean oscillation. In 1971 Fefferman [6] announced that the dual to the real Hardy space $H^1$ on $\mathbb{R}^n$ is $BMO$. Next year, the proof was published by Fefferman and Stein in their paper [7, Theorem 2] (see also Garnett [8, Theorem 4.4], Grafakos [12, Theorem 7.2.2], Kashin and Saakyan [15, Theorem 5.5], and Stein [30, pp. 142–144]). This duality result of Fefferman called considerable attention to the $BMO$ space and after 1971 many results were proved about this space (see e.g. Garnett [8, Chapter VI], Grafakos [12, Chapter 7] and Stein [30, Chapter IV]).

There is also a larger dyadic counterpart $BMO_d$ of the space of functions of bounded mean oscillation, $BMO_d \supsetneq BMO$. This dyadic space related to $BMO$ was studied already by Garnett and Jones [9]. Since $BMO$ is translation invariant and $BMO_d$ is not, $BMO$ is more important in analysis. On the other hand, it is much easier to work with $BMO_d$ because of the fact that dyadic cubes are nested (if two open dyadic cubes intersect then one of them is contained in the other).

Consider the Rademacher functions on $[0,1]$ defined by

$$r_k(t) = \operatorname{sign}[\sin(2^k \pi t)], \qquad k \in \mathbb{N},\ t \in [0,1],$$

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and the set of Rademacher sums

$$R_n(t) = \sum_{k=1}^{n} a_k r_k(t), \qquad a_k \in \mathbb{R},\ \text{for } k = 1, \dots, n \text{ and } n \in \mathbb{N}.$$

The behaviour of Rademacher sums in the spaces $L_p = L_p[0,1]$ is well known and it is described by the classical *Khintchine inequalities*: there exist constants $A_p, B_p > 0$ such that for every sequence $\{a_k\}_{k=1}^n$ of real numbers and any $n \in \mathbb{N}$ we have

$$(1) \qquad A_p\Big(\sum_{k=1}^{n} |a_k|^2\Big)^{1/2} \le \|R_n\|_{L_p[0,1]} \le B_p\Big(\sum_{k=1}^{n} |a_k|^2\Big)^{1/2}, \qquad 0 < p < \infty.$$

Hence, the Rademacher functions $\{r_n\}$ span an isomorphic copy of $l_2$ in $L_p$ for every $0 < p < \infty$. Moreover, the subspace $[r_n]_{n=1}^{\infty}$ is complemented in $L_p$ for $1 < p < \infty$ and it is not complemented in $L_1$ since no complemented infinite-dimensional subspace of $L_1$ can be reflexive. In $L_\infty$ we have $\|R_n\|_{L_\infty[0,1]} = \sum_{k=1}^n |a_k|$ and so the Rademacher functions span an isometric copy of $l_1$, which is known to be uncomplemented in $L_\infty$. Investigations of Rademacher sums in general symmetric (rearrangement invariant) spaces are well presented in the books by Lindenstrauss–Tzafriri [21], Kreĭn–Petunin–Semenov [17] and Astashkin [2], where also the definition and several properties of symmetric spaces can be found.

The purpose of this paper is to investigate sums of Rademacher functions in the $BMO$ space on $[0,1]$. Some results are contained in Leibov's dissertation [18] (Proposition 2 with some estimates of type (7), which we correct in this paper, and partly Theorem 5) but, in fact, they are not known to a wide audience since they were not published in any journal and the dissertation is not available.

The paper is organized as follows: after the introduction in Section 1, we collect some necessary definitions, notation, and auxiliary results in Section 2. The main result in Section 3 is Theorem 2 describing the behaviour of Rademacher sums in $BMO[0,1]$. In Section 4 we discuss the complementability of Rademacher subspaces $\mathcal{R}_d$ and $\mathcal{R}$ in $BMO_d$ and in $BMO$, respectively. Namely, it is well-known that $\mathcal{R}_d$ is complemented in $BMO_d$ (Theorem 3). At the same time we prove that $\mathcal{R}$ is not complemented in $BMO$ (Theorem 4). Finally, in Section 5, we investigate the structure of infinite-dimensional subspaces of $\mathcal{R}$. In particular, in Theorem 5 we state the following analogue of the Kadec–Pełczyński type alternative for $\mathcal{R}$: every infinite-dimensional closed subspace $X \subset \mathcal{R}$ is either isomorphic to $l_2$ and complemented in $BMO$, or contains a subspace $Y$ isomorphic to $c_0$ and complemented in $\mathcal{R}$. Then, in Examples 1 and 2, we construct block bases of the Rademacher system whose span is $l_2$ and $c_0$, respectively.

**2. Preliminaries and auxiliary results.** For any function $f \in L_1[0,1]$ and arbitrary interval $I \subset [0,1]$ we denote

$$f_I = \frac{1}{|I|}\int_I f(s)\,ds,$$

where $|I|$ is the Lebesgue measure of $I$. Then, as usual, the space $BMO = BMO[0,1]$ consists of all $f \in L_1[0,1]$ such that

$$\|f\|_{BMO} := \sup_I \frac{1}{|I|}\int_I |f(s) - f_I|\,ds < \infty. \tag{2}$$

The quantity $\|f\|_{BMO}$ is only a seminorm, since $\|f\|_{BMO} = 0$ if $f$ equals a constant a.e. To turn $BMO$ into a Banach space we can either restrict (2) to the linear space

$$\Big\{f \in L_1[0,1] : \int_0^1 f(x)\,dx = 0 \text{ with } \|f\|_{BMO} < \infty\Big\}$$

(with identification of functions equal a.e.) or consider in $BMO$ one of the norms $\|f\|'_{BMO} = \|f\|_{BMO} + |\int_0^1 f(x)\,dx|$ or $\|f\|''_{BMO} = \|f\|_{BMO} + \|f\|_{L_1[0,1]}$.

We also introduce a dyadic version of $BMO$. If $I_n^k = (k/2^n, (k+1)/2^n]$, $k = 0, 1, \dots, 2^n - 1$, $n = 0, 1, 2, \dots$, are dyadic intervals in $[0,1]$, then the space $BMO_d = BMO_d[0,1]$ consists of all $f \in L_1[0,1]$ such that

$$\|f\|_d = \|f\|_{BMO_d} := \sup_{k,n} \frac{1}{|I_n^k|}\int_{I_n^k} |f(s) - f_{I_n^k}|\,ds < \infty.$$

It is clear that $BMO \subset BMO_d$ and $\|f\|_d \le \|f\|_{BMO}$ for all $f \in BMO$. Moreover, $L_\infty \subset BMO$ and, for $f \in L_\infty[0,1]$, we have

$$\|f\|_{BMO} \le \sup_I \bigg(\frac{1}{|I|}\int_I (f(s) - f_I)^2\,ds\bigg)^{1/2} \le \sup_I \bigg(\frac{1}{|I|}\int_I |f(s)|^2\,ds\bigg)^{1/2} = \|f\|_{L_\infty}.$$

At the same time, $BMO \ne L_\infty$ and $BMO_d \ne BMO$. For example, we have $\ln|s - 1/2|\chi_{[0,1]}(s) \in BMO \setminus L_\infty$ and $\ln|s - 1/2|\chi_{[1/2,1]}(s) \in BMO_d \setminus BMO$.

To find a connection between the $BMO$- and $BMO_d$-norms, we introduce the functional

$$A(f) := \sup_{I_1, I_2} |f_{I_1} - f_{I_2}|, \qquad f \in L_1[0,1],$$

where the supremum is taken over all adjacent dyadic intervals $I_1, I_2$ of the same length.

The following assertion is an exercise from Garnett's book (cf. [8, Problem 12(b), p. 266]). We present the proof with concrete constants for the sake of completeness.

Proposition 1. *For any $f \in L_1[0,1]$ we have*

$$\frac{1}{3}[\|f\|_d + A(f)] \le \|f\|_{BMO} \le 32[\|f\|_d + A(f)]. \tag{3}$$

*Proof.* It is clear that the left-hand inequality of (3) is an immediate consequence of the estimate

$$A(f) \le 2\|f\|_{BMO}. \tag{4}$$

To prove (4), take two adjacent dyadic intervals $I_1$ and $I_2$ of the same length. Then

$$f_{I_1 \cup I_2} = \frac{1}{2}\bigg(\frac{1}{|I_1|}\int_{I_1} f\,ds + \frac{1}{|I_2|}\int_{I_2} f\,ds\bigg) = \frac{1}{2}(f_{I_1} + f_{I_2}). \tag{5}$$

Therefore, for $I := I_1 \cup I_2$ we have

$$\begin{aligned}\frac{1}{|I_1|}\int_{I_1} |f - f_I|\,ds &= \frac{1}{2|I_1|}\int_{I_1} |(f - f_{I_1}) + (f - f_{I_2})|\,ds \\ &\ge \frac{1}{2}\bigg|\frac{1}{|I_1|}\int_{I_1}(f - f_{I_1})\,ds + \frac{1}{|I_1|}\int_{I_1}(f - f_{I_2})\,ds\bigg| \\ &= \frac{1}{2}|f_{I_1} - f_{I_2}|,\end{aligned}$$

and similarly

$$\frac{1}{|I_2|}\int_{I_2} |f - f_I|\,ds \ge \frac{1}{2}|f_{I_1} - f_{I_2}|.$$

Thus,

$$\begin{aligned}\frac{1}{|I|}\int_I |f - f_I|\,ds &= \frac{1}{2}\bigg(\frac{1}{|I_1|}\int_{I_1} |f - f_I|\,ds + \frac{1}{|I_2|}\int_{I_2} |f - f_I)|\,ds\bigg) \\ &\ge \frac{1}{2}|f_{I_1} - f_{I_2}|,\end{aligned}$$

which implies (4).

Let us prove the right-hand inequality of (3). For any $I \subset [0,1]$ we can find adjacent dyadic intervals $I_1$ and $I_2$ of the same length such that $I \subset I_1 \cup I_2$ and

$$\frac{1}{2}|I_1| \le |I| \le 2|I_1|.$$

Then

$$\begin{aligned}\frac{1}{|I|}\int_I |f(s)-f_I|\,ds &= \frac{1}{|I|}\int_I \Big|f(s)-\frac{1}{|I|}\int_I f(t)\,dt\Big|\,ds\\
&\le \frac{1}{|I|^2}\int_I\int_I |f(s)-f(t)|\,dt\,ds\\
&\le \frac{16}{|I_1\cup I_2|^2}\int_{I_1\cup I_2}\int_{I_1\cup I_2} |f(s)-f(t)|\,dt\,ds\\
&\le \frac{16}{|I_1\cup I_2|^2}\int_{I_1\cup I_2}\int_{I_1\cup I_2} |f(s)-f_{I_1\cup I_2}|\,ds\,dt\\
&\quad+\frac{16}{|I_1\cup I_2|^2}\int_{I_1\cup I_2}\int_{I_1\cup I_2} |f(t)-f_{I_1\cup I_2}|\,ds\,dt\\
&= \frac{32}{|I_1\cup I_2|}\int_{I_1\cup I_2} |f(s)-f_{I_1\cup I_2}|\,ds.\end{aligned}$$

The above estimate and equality (5) imply that

$$\begin{aligned}\frac{1}{|I|}\int_I |f(s)-f_I|\,ds &\le \frac{16}{|I_1|}\int_{I_1} |f(s)-\tfrac{1}{2}(f_{I_1}+f_{I_2})|\,ds\\
&\quad+\frac{16}{|I_2|}\int_{I_2} |f(s)-\tfrac{1}{2}(f_{I_1}+f_{I_2})|\,ds\\
&\le \frac{16}{|I_1|}\int_{I_1} |f(s)-f_{I_1}|\,ds+\frac{16}{|I_2|}\int_{I_2} |f(s)-f_{I_2}|\,ds\\
&\quad+16|f_{I_1}-f_{I_2}| \le 32\|f\|_d+16A(f).\end{aligned}$$

Hence,

$$\|f\|_{BMO}\le 32\|f\|_d+16A(f)\le 32[\|f\|_d+A(f)]. \ \blacksquare$$

**3. Rademacher sums in** ***BMO*** **spaces.** The main purpose of this paper is to investigate the behaviour of Rademacher sums in the $BMO$ and $BMO_d$ spaces.

PROPOSITION 2. *For any $a_k\in\mathbb{R}$, $k=1,\dots,n$, we have*

$$\frac{1}{\sqrt{2}}\,\|(a_k)_{k=1}^n\|_{l_2}\le \Big\|\sum_{k=1}^n a_k r_k\Big\|_d \le \|(a_k)_{k=1}^n\|_{l_2} \tag{6}$$

*and*

$$\frac{2}{3}\max_{0\le j<m\le n}\Big|\sum_{k=j+1}^m a_k\Big|\le A\Big(\sum_{k=1}^n a_k r_k\Big)\le 4\max_{0\le j<m\le n}\Big|\sum_{k=j+1}^m a_k\Big|. \tag{7}$$

*Proof.* Set $f = \sum_{k=1}^n a_k r_k$ and let $I$ be a dyadic interval of length $2^{-m}$, that is, $I = I_m^i = (i/2^m, (i+1)/2^m]$. Then

$$(r_k)_I = \begin{cases} \operatorname{sgn}(r_k|_I) & \text{if } k \le m, \\ 0 & \text{if } k > m, \end{cases}$$

and so

$$f_I = \sum_{k=1}^n a_k (r_k)_I = \sum_{k=1}^{\min(m,n)} a_k \operatorname{sgn}(r_k|_I).$$

Thus, if $m \ge n$, then $f|_I$ is a constant and therefore the oscillation of $f$ on $I$ vanishes, i.e., $O_I(f) := |I|^{-1} \int_I |f(x) - f_I|\,dx = 0$. Otherwise, if $m < n$, we have

$$\begin{aligned} O_I(f) &= \frac{1}{|I|} \int_I |f(x) - f_I|\,dx = \frac{1}{|I|} \int_I \Big| \sum_{k=1}^n a_k r_k(x) - \sum_{k=1}^m a_k \operatorname{sgn}(r_k|_I) \Big|\,dx \\ &= \frac{1}{|I|} \int_I \Big| \sum_{k=m+1}^n a_k r_k(x) \Big|\,dx = \int_0^1 \Big| \sum_{k=m+1}^n a_k r_{k-m}(x) \Big|\,dx \\ &= \Big\| \sum_{k=m+1}^n a_k r_{k-m} \Big\|_{L_1}. \end{aligned}$$

Using Khintchine's inequality (1) for the space $L_1[0,1]$ with the sharp constant $A_1 = 1/\sqrt{2}$ (cf. [31]), we obtain

$$\frac{1}{\sqrt{2}} \Big( \sum_{k=m+1}^n a_k^2 \Big)^{1/2} \le \Big\| \sum_{k=m+1}^n a_k r_{k-m} \Big\|_{L_1} \le \Big( \sum_{k=m+1}^n a_k^2 \Big)^{1/2}.$$

Thus,

$$\frac{1}{\sqrt{2}} \Big( \sum_{k=1}^n a_k^2 \Big)^{1/2} \le \Big\| \sum_{k=1}^n a_k r_k \Big\|_d \le \Big( \sum_{k=1}^n a_k^2 \Big)^{1/2}.$$

Let now $I_1$ and $I_2$ be adjacent dyadic intervals of length $2^{-m}$ each. Then by the above observation,

$$f_{I_1} - f_{I_2} = \sum_{k=1}^{\min(m,n)} a_k \operatorname{sgn}(r_k|_{I_1}) - \sum_{k=1}^{\min(m,n)} a_k \operatorname{sgn}(r_k|_{I_2}).$$

Let $I$ be the smallest dyadic interval containing $I_1 \cup I_2$; let $I$ have length $2^{-j}$. Of course, $j < m$ and $r_k|_{I_1} = r_k|_{I_2}$ if $k \le j$. Then for $j > n$ we have $f_{I_1} = f_{I_2}$, and for $j \le n$,

$$f_{I_1} - f_{I_2} = \sum_{k=j+1}^{\min(m,n)} a_k [\operatorname{sgn}(r_k|_{I_1}) - \operatorname{sgn}(r_k|_{I_2})].$$

From the definition of $I$ it follows that $I_1 \cup I_2$ is in the middle of $I$. Suppose that $I_1$ lies to the left of $I_2$. Then it is easy to see that $r_{j+1}|_{I_1} = 1$, $r_{j+1}|_{I_2} = -1$ and $r_k|_{I_1} = -1$, $r_k|_{I_2} = 1$ if $j+2 \le k \le m$. Thus,

$$|f_{I_1} - f_{I_2}| = 2\Big|\sum_{k=j+2}^{\min(m,n)} a_k - a_{j+1}\Big|$$

and

$$A(f) = 2 \max_{0\le j<m\le n}\Big|\sum_{k=j+2}^{m} a_k - a_{j+1}\Big|.$$

It is not hard to check that

$$\frac{1}{3}\max_{0\le j<m\le n}\Big|\sum_{k=j+1}^{m} a_k\Big| \le \max_{0\le j<m\le n}\Big|\sum_{k=j+2}^{m} a_k - a_{j+1}\Big| \le 2\max_{0\le j<m\le n}\Big|\sum_{k=j+1}^{m} a_k\Big|.$$

Combining this inequality with the previous equality, we obtain (7). ∎

The following well-known assertion is an immediate consequence of inequalities (6) from Proposition 2. It was already proved by Garsia [10], [11] and even for martingale *BMO* spaces. It was also obtained by Müller and Schechtman [25, Theorem 1] and Kochneff, Sagher and Zhou [16, Theorem 1], who also gave an example showing that the similar result for *BMO* is not true.

THEOREM 1. *The sequence* $\{r_k\}_{k=1}^{\infty}$ *of Rademacher functions is equivalent in* $BMO_d$ *to the standard unit basis in* $l_2$.

From Propositions 1 and 2 and the elementary observation that

$$\max_{1\le m\le n}\Big|\sum_{k=1}^{m} a_k\Big| \le \max_{0\le j<m\le n}\Big|\sum_{k=j+1}^{m} a_k\Big| \le 2\max_{1\le m\le n}\Big|\sum_{k=1}^{m} a_k\Big|,$$

we obtain the following assertion:

THEOREM 2. *For any* $a_k \in \mathbb{R}$, $k = 1,\dots,n$, *and* $n \in \mathbb{N}$ *we have*

$$\frac{2}{9}\Big[\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2} + \max_{1\le m\le n}\Big|\sum_{k=1}^{m} a_k\Big|\Big] \le \Big\|\sum_{k=1}^{n} a_k r_k\Big\|_{BMO} \le 256\Big[\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2} + \max_{1\le m\le n}\Big|\sum_{k=1}^{m} a_k\Big|\Big].$$

*In particular, the following equivalence holds:*

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{BMO} \asymp \|(a_k)_{k=1}^{\infty}\|_{l_2} + \sup_{n\in\mathbb{N}}\Big|\sum_{k=1}^{n} a_k\Big|. \tag{8}$$

Corollary 1. *The sequence $\{r_k\}_{k=1}^{\infty}$ of Rademacher functions is equivalent in BMO to each of its subsequences.*

Corollary 2. *The system $\{r_k\}_{k=1}^{\infty}$ does not contain an unconditional basic subsequence in BMO.*

For example, $\sum_{k=1}^{\infty}(-1)^k r_k/k \in \mathcal{R}$ and $\sum_{k=1}^{\infty} r_k/k \notin \mathcal{R}$.

Corollary 3. *$L_\infty[0,1]$ is a unique (up to equivalence of norms) symmetric space on $[0,1]$ which is embedded into $BMO_d$.*

*Proof.* For arbitrary $\delta \in (0,1)$ let us introduce the "$\delta$-translation" of the dyadic $BMO$, that is, the space $BMO_d(\delta)$ with the norm $\|f\|_{d,\delta} := \|f_\delta\|_d$, where

$$f_\delta(s) := f(s-\delta)\chi_{[\delta,1]}(s) + f(s-\delta+1)\chi_{[0,\delta]}(s), \qquad s \in [0,1]. \tag{9}$$

Mei proved in [23] that there exists a $\delta_0 \in (0,1)$ such that

$$BMO = BMO_d \cap BMO_d(\delta_0).$$

If $X$ is a symmetric space on $[0,1]$ such that $X \subset BMO_d$, then $X \subset BMO_d(\delta_0)$ as well. Thus, $X \subset BMO$. Next, since $\{r_k\}_{k=1}^{\infty}$ is an unconditional basic sequence with constant 1 in an arbitrary symmetric space (see, for example, [2, Corollary 1.7]), by Theorem 2 we obtain

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_X = \Big\|\sum_{k=1}^{\infty} |a_k| r_k\Big\|_X \ge c_1 \Big\|\sum_{k=1}^{\infty} |a_k| r_k\Big\|_{BMO} \ge c_2 \|(a_k)\|_{l_1}.$$

On the other hand, we have

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_X \le C\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{L_\infty} = C\|(a_k)\|_{l_1}.$$

Therefore, $\{r_k\}_{k=1}^{\infty}$ is equivalent in $X$ to the standard unit basis in $l_1$ and, hence, by the Rodin–Semenov theorem [27, Theorem 7], we conclude that $X = L_\infty$ with equivalent norms. ∎

Corollary 4. *The sequence $\{r_k\}_{k=1}^{\infty}$ of Rademacher functions is not weakly convergent to zero in BMO.*

*Proof.* Define a linear functional on $\mathcal{R}$ by

$$\varphi_0\Big(\sum_{k=1}^{n} a_k r_k\Big) = \sum_{k=1}^{n} a_k, \qquad a_k \in \mathbb{R},\ k = 1,\dots,n,\ n \in \mathbb{N}.$$

Then, by the Hahn–Banach theorem, it can be extended to a functional $\varphi_0 \in \mathcal{R}^*$, because in view of Theorem 2 we have

$$\Big|\varphi_0\Big(\sum_{k=1}^{\infty} a_k r_k\Big)\Big| = \Big|\sum_{k=1}^{\infty} a_k\Big| \le C\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{BMO}.$$

Since $\varphi_0(r_n) = 1 \not\to 0$, we see that $r_n \not\to 0$ weakly in $BMO$. ∎

REMARK 1. Astashkin and Maligranda [3] recently proved an equivalence completely similar to (8) for Cesàro function spaces $K_p$ on $[0,1]$ defined by the norms $\|f\|_{K_p} = \sup_{0<x\le 1}(x^{-1}\int_0^x |f(t)|^p\,dt)^{1/p}$ $(1 \le p < \infty)$:

$$\Big\|\sum_{k=1}^{n} a_k r_k\Big\|_{K_p} \asymp \|\{a_k\}_{k=1}^{n}\|_{l_2} + \max_{1\le m\le n}\Big|\sum_{k=1}^{m} a_k\Big|.$$

It is worth noting that the spaces $K_p$ and $BMO$ are not comparable, that is, neither is embedded in the other one.

**4. On complementability of Rademacher subspaces in** $BMO$ **and** $BMO_d$**.** We investigate the geometrical properties of the subspaces $\mathcal{R}_d := [r_k]_{BMO_d}$ and $\mathcal{R} := [r_k]_{BMO}$ generated by the Rademacher system in $BMO_d$ and $BMO$, respectively. In fact, complementability of $\mathcal{R}_d$ in $BMO_d$ is well-known (see, for example, Garsia [10], [11], Müller and Schechtman [25]). However, we present a simple proof.

THEOREM 3. *The subspace $\mathcal{R}_d$ is complemented in $BMO_d$.*

*Proof.* In view of a dyadic version of the John–Nirenberg theorem, which can be proved in the same way as the classical John–Nirenberg theorem (see, for example, [12, pp. 124–127]), for an arbitrary $f \in BMO_d$, any dyadic interval $I_n^k$ and $\tau > 0$ we have

$$m\{x \in I_n^k : |f(x) - f_{I_n^k}| > \tau\} \le e|I_n^k| \exp\Big(-\frac{\tau}{2e\|f\|_d}\Big).$$

This inequality implies that, for any $1 \le p < \infty$,

$$\|f\|_d \asymp \|f\|_{d,p} := \sup_{k,n}\bigg(\frac{1}{|I_n^k|}\int\limits_{I_n^k} |f(x) - f_{I_n^k}|^p\,dx\bigg)^{1/p},$$

with a constant depending on $p$ (see [12, p. 128] and [26, p. 525]). More precisely,

$$\|f\|_d \le \|f\|_{d,p} \le 2e[p\Gamma(p)e]^{1/p}\|f\|_d.$$

Therefore, for every $1 \le p < \infty$, we have $BMO_d \subset L_p[0,1]$. It is well-known that the orthogonal projector $P$ generated by the Rademacher system is bounded from $L_p[0,1]$ onto $[r_k]_{k=1}^{\infty}$ if $1 < p < \infty$. Then, by (6) and the Khintchine inequality (1), for all $p \in (1,\infty)$ and $f \in BMO_d$,

$$\|Pf\|_d \asymp \|Pf\|_{L_p} \le \|P\|\,\|f\|_{L_p} \le C_p\|P\|\,\|f\|_d.$$

Thus, $P$ is bounded in $BMO_d$ and the proof is complete. ∎

In contrast with Theorem 3 and with the remarkable theorem of Maurey [22] (cf. also [24, pp. 229–242]) which states that $BMO[0,1]$ is isomorphic to dyadic $BMO_d[0,1]$, the subspace $\mathcal{R}$ is an uncomplemented subspace of $BMO$. To prove this, we will need an auxiliary assertion.

Denote by $\mathcal{U} = \{u_n\}_{n=1}^{\infty}$ a block basis of the Rademacher system, that is,

$$u_n = \sum_{k=m_n+1}^{m_{n+1}} a_k r_k \quad (n = 1, 2, \ldots),$$

where $1 \le m_1 < m_2 < \cdots$ and $a_k \in \mathbb{R}$. Moreover, let

$$\gamma_n(\mathcal{U}) = \sum_{k=m_n+1}^{m_{n+1}} a_k, \quad n = 1, 2, \ldots.$$

Proposition 3. *The subspace $\mathcal{R}$ contains a complemented subspace $E$ isomorphic to $c_0$.*

*Proof.* Take a block basis $\mathcal{U} = \{u_n\}_{n=1}^{\infty}$ satisfying the following assumptions:

(a) $\|u_n\|_{BMO} = 1,\ n = 1, 2, \ldots.$
(b) $\|u_n\|_d \asymp (\sum_{k=m_n+1}^{m_{n+1}} a_k^2)^{1/2} \le 2^{-n},\ n = 1, 2, \ldots.$
(c) $|\gamma_n(\mathcal{U})| \le 2^{-n},\ n = 1, 2, \ldots.$

It is clear that such a block basis exists. Let us prove that $[u_n]_{BMO}$ is isomorphic to $c_0$.

If $f = \sum_{n=1}^{\infty} \beta_n u_n \in \mathcal{R}$ with $\beta_n \in \mathbb{R}$, then

$$f = \sum_{n=1}^{\infty} \Big( \sum_{k=m_n+1}^{m_{n+1}} \beta_n a_k r_k \Big) = \sum_{k=1}^{\infty} \gamma_k r_k,$$

where $\gamma_k = \beta_n a_k$ if $k = m_n + 1, \ldots, m_{n+1}$. Assuming that $p, q \in \mathbb{N}$ satisfy

$$m_{n-1} \le p < m_n < m_{n+l} < q \le m_{n+l+1}$$

with some positive integers $n$ and $l$, we will estimate the sum $\sum_{k=p}^{q} \gamma_k$. Using (c), (a), Proposition 2 and inequality (4), we have

$$\begin{aligned}
\Big|\sum_{k=p}^{q} \gamma_k\Big| &= \Big|\sum_{k=p}^{m_n} \gamma_k + \sum_{k=m_n+1}^{m_{n+l}} \gamma_k + \sum_{k=m_{n+l}+1}^{q} \gamma_k\Big| \\
&= \Big|\sum_{k=p}^{m_n} \beta_{n-1} a_k + \sum_{i=n}^{n+l-1} \sum_{k=m_i+1}^{m_{i+1}} \beta_i a_k + \sum_{k=m_{n+l}+1}^{q} \beta_{n+l} a_k\Big| \\
&\le |\beta_{n-1}| \Big|\sum_{k=p}^{m_n} a_k\Big| + \sum_{i=n}^{n+l-1} |\beta_i| \Big|\sum_{k=m_i+1}^{m_{i+1}} a_k\Big| + |\beta_{n+l}| \Big|\sum_{k=m_{n+l}+1}^{q} a_k\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_n |\beta_n| \Big( \Big| \sum_{k=p}^{m_n} a_k \Big| + \sum_{i=n}^{n+l-1} \Big| \sum_{k=m_i+1}^{m_{i+1}} a_k \Big| + \Big| \sum_{k=m_{n+l}+1}^{q} a_k \Big| \Big) \\
&\le \sup_n |\beta_n| \Big( \frac{3}{2} A(u_{n-1}) + \frac{3}{2} A(u_{n+l}) + \sum_{i=n}^{n+l-1} 2^{-i} \Big) \\
&\le (3\|u_{n-1}\|_{BMO} + 3\|u_{n+l}\|_{BMO} + 1) \|\{\beta_n\}\|_{c_0} = 7\|\{\beta_n\}\|_{c_0}.
\end{aligned}
$$

Then from Theorem 2 and (b) it follows that $\|f\|_{BMO} \le C\|\{\beta_n\}\|_{c_0}$.

On the other hand, by (a), (b) and Theorem 2, there is a $\delta > 0$ such that $A(u_n) \ge \delta$ for all $n \in \mathbb{N}$. Therefore, by (4) and Proposition 2, we obtain

$$
\begin{aligned}
\|f\|_{BMO} \ge \frac{1}{2} A(f) &\ge \frac{1}{3} \sup_{n\in\mathbb{N}} \sup_{m_n+1 \le p < q \le m_{n+1}} \Big| \sum_{k=p}^{q} \beta_n a_k \Big| \\
&\ge \frac{1}{12} \sup_{n\in\mathbb{N}} |\beta_n| A(u_n) \ge \frac{\delta}{12} \|\{\beta_n\}\|_{c_0}.
\end{aligned}
$$

Thus, we have proved that $E := [u_n]_{BMO} \approx c_0$. Since $\mathcal{R}$ is separable, the Sobczyk theorem (see, for example, [1, Corollary 2.5.9]) implies that $E$ is a complemented subspace in $\mathcal{R}$. ∎

THEOREM 4. *The subspace $\mathcal{R}$ is not complemented in BMO.*

*Proof.* Assume on the contrary that $\mathcal{R}$ is complemented in $BMO$ and let $P_1 : BMO \to \mathcal{R}$ be a bounded linear projection whose range is $\mathcal{R}$. By Proposition 3, there is a subspace $E$ complemented in $\mathcal{R}$ and such that $E \approx c_0$. Let $P_2 : \mathcal{R} \to E$ be a bounded linear projection. Then $P := P_2 \circ P_1$ is a linear projection bounded in $BMO$ with image $E$. Thus, $BMO$ contains a complemented subspace $E \approx c_0$. Since $BMO$ is a conjugate space (more precisely, $BMO = (\operatorname{Re} H_1)^*$—see, for example, [15, p. 195]), this contradicts the well known result due to Bessaga–Pełczyński that a conjugate space cannot contain a complemented subspace isomorphic to $c_0$ (see [5, Corollary 4], which follows from Theorem 4 and its proof in [4]). This contradiction proves the theorem. ∎

**5. Structure of subspaces of $\mathcal{R}$.** Sarason [29] introduced the $VMO = VMO[0,1]$ space (space of vanishing mean oscillation in $[0,1]$) consisting of all $f \in BMO[0,1]$ for which $\lim_{|I|\to 0} |I|^{-1} \int_I |f(x) - f_I|\,dx = 0$. This is a closed subspace of $BMO$ containing the space $C[0,1]$ of continuous functions and is equal to the $BMO$-closure of $C[0,1]$. The space $VMO$ was investigated by several authors. For example, it is known that $VMO$ is not complemented in $BMO$ (see, e.g., [13]). Structural properties of closed subspaces of $VMO$ were considered by Leibov [18], [19] who proved an analog of the Kadec–Pełczyński theorem for $VMO$ (Kadec–Pełczyński type alternative [14]): an infinite-dimensional closed subspace of $VMO$ is either complemented in $BMO$ and isomorphic to $l_2$, or, for every $\varepsilon > 0$, it contains a subspace which is complemented in $VMO$ and $(1+\varepsilon)$-isometric to $c_0$ (cf. [18, Theorem 3.4] and [19]). A similar dichotomy for the dyadic $VMO$ space was proved by Müller and Schechtman [25].

Our purpose here is to prove the Kadec–Pełczyński type alternative for the Rademacher subspace $\mathcal{R}$ of $BMO$.

THEOREM 5. *Every infinite-dimensional closed subspace $X \subset \mathcal{R}$ is either isomorphic to $l_2$ and complemented in BMO, or contains a subspace $Y$ isomorphic to $c_0$ and complemented in $\mathcal{R}$.*

In the proof of this theorem we will need some auxiliary result. Since $r_n \nrightarrow 0$ weakly in $BMO$ (cf. Corollary 4), it follows that the corresponding system of functionals biorthogonal to $\{r_k\}_{k=1}^{\infty}$ is not a basis of the space $\mathcal{R}^*$. Nevertheless, the following assertion holds.

PROPOSITION 4. *The space $\mathcal{R}^*$ has a basis.*

*Proof.* Consider the sequence

$$s_n = r_n - r_{n-1}, \quad n = 1, 2, \ldots, \quad \text{with } r_0 = 0.$$

If $f = \sum_{n=1}^{\infty} \beta_n s_n$, then

$$f = \sum_{n=1}^{\infty} \beta_n (r_n - r_{n-1}) = \sum_{n=0}^{\infty} (\beta_n - \beta_{n+1}) r_n, \quad \text{where } \beta_0 = 0.$$

Therefore, by Theorem 2,

$$\|f\|_{BMO} \asymp \Big(\sum_{n=0}^{\infty} (\beta_n - \beta_{n+1})^2\Big)^{1/2} + \sup_{0 \le m < n < \infty} |\beta_m - \beta_n| \tag{10}$$

with a constant $c > 0$. This implies, in particular, that

$$|\beta_n| \le |\beta_n - \beta_1| + |\beta_1| \le 2c\|f\|_{BMO}, \quad n = 1, 2, \ldots. \tag{11}$$

Let us prove that $\{s_n\}_{n=1}^{\infty}$ is a shrinking basis in $\mathcal{R}$, that is, for any $\varphi \in \mathcal{R}^*$,

$$\|\varphi|_{[s_n]_{n=m}^{\infty}}\| \to 0 \quad \text{as } m \to \infty. \tag{12}$$

Assume that (12) does not hold. Then there exist $\varepsilon \in (0,1)$, a functional $\varphi \in \mathcal{R}^*$ with $\|\varphi\| = 1$, and a sequence of functions $f_n = \sum_{k=m_n}^{\infty} a_k^{m_n} s_k$, where $m_1 < m_2 < \cdots$, such that $\|f_n\|_{BMO} = 1$ and $\varphi(f_n) \ge \varepsilon > 0$ $(n = 1, 2, \ldots)$.

We construct two sequences of positive integers, $\{q_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$, $1 < q_1 < p_1 < q_2 < p_2 < \cdots$, in the following way: $q_1 = m_1$ and $p_1$ is chosen so that $\|\sum_{k=p_1+1}^{\infty} a_k^{q_1} s_k\|_{BMO} < \varepsilon/2$; $q_2$ is the least $m_n > p_1$ and $p_2$ is such that $\|\sum_{k=p_2+1}^{\infty} a_k^{q_2} s_k\|_{BMO} < \varepsilon/2$; $q_3$ is the least $m_n > p_2$ and $p_3$ is such that $\|\sum_{k=p_3+1}^{\infty} a_k^{q_3} s_k\|_{BMO} < \varepsilon/2$; and so on.

Set $a_k^{q_i} = 0$ if $p_i < k < q_{i+1}$, $i = 1, 2, \ldots$. Then $\mathcal{U} = \{u_i\}$, where $u_i = \sum_{k=q_i}^{q_{i+1}-1} a_k^{q_i} s_k$, is a block basis of $\{s_n\}_{n=1}^\infty$. Moreover, by the definition of $u_i$,

$$\sup_{i\in\mathbb{N}} \|u_i\|_{BMO} \le 2 \tag{13}$$

and

$$\varphi(u_i) = \varphi(f_i) - \varphi\Big(\sum_{k=p_i+1}^\infty a_k^{q_i} s_k\Big) \ge \varphi(f_i) - \Big\|\sum_{k=p_i+1}^\infty a_k^{q_i} s_k\Big\|_{BMO} \ge \frac{\varepsilon}{2}. \tag{14}$$

Let us show that for every nonnegative sequence $\{\gamma_n\}_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty \gamma_n^2 < \infty \quad \text{and} \quad \sum_{n=1}^\infty \gamma_n = \infty$$

the series $\sum_{n=1}^\infty \gamma_n u_n$ converges in $BMO$.

Let $b_k = a_k^{q_i}\gamma_i$ for $q_i \le k < q_{i+1}$, $i = 1, 2, \ldots$. If $q_i \le k < q_{i+1}$, then, in view of (11) and (13), we obtain $|b_k| \le 4c\gamma_i$, and hence

$$\lim_{k\to\infty} b_k = 0. \tag{15}$$

Moreover,

$$\sum_{k=1}^\infty (b_k - b_{k+1})^2 = \sum_{i=1}^\infty \sum_{k=q_i}^{q_{i+1}-2} (a_k^{q_i}\gamma_i - a_{k+1}^{q_i}\gamma_i)^2 + \sum_{i=1}^\infty (a_{q_{i+1}-1}^{q_i}\gamma_i - a_{q_{i+1}}^{q_{i+1}}\gamma_{i+1})^2 = A_1 + A_2.$$

We will estimate $A_1$ and $A_2$ separately. By (10),

$$A_1 = \sum_{i=1}^\infty \gamma_i^2 \sum_{k=q_i}^{q_{i+1}-2} (a_k^{q_i} - a_{k+1}^{q_i})^2 \le c^2 \sum_{i=1}^\infty \gamma_i^2,$$

and, by (11),

$$\begin{aligned} A_2 &\le 2\sum_{i=1}^\infty \big((a_{q_{i+1}-1}^{q_i})^2\gamma_i^2 + (a_{q_{i+1}}^{q_{i+1}})^2\gamma_{i+1}^2\big) \\ &\le 4\sup_{i\in\mathbb{N}} \max_{q_i\le k<q_{i+1}} |a_k^{q_i}|^2 \sum_{i=1}^\infty \gamma_i^2 \le 16c^2 \sum_{i=1}^\infty \gamma_i^2. \end{aligned}$$

Therefore, according to (10) and (15), the series

$$\sum_{n=1}^\infty \gamma_n u_n = \sum_{k=1}^\infty b_k s_k$$

converges in $BMO$. On the other hand, taking into account (14), we have

$$\varphi\Big(\sum_{n=1}^{\infty}\gamma_n u_n\Big)=\sum_{n=1}^{\infty}\gamma_n\,\varphi(u_n)\ge\frac{\varepsilon}{2}\sum_{n=1}^{\infty}\gamma_n=\infty,$$

which contradicts $\varphi\in\mathcal{R}^*$. Thus, (12) is proved.

Finally, by Proposition 1.b.1 in [20], the biorthogonal system $\{s_n^*\}_{n=1}^{\infty}$ of functionals is a basis in the space $\mathcal{R}^*$ and the proof is complete. ∎

*Proof of Theorem 5.* Assume that there is an $\varepsilon>0$ such that $\|f\|_d\ge\varepsilon\|f\|_{BMO}$ for every $f\in X$. Then, according to (6) and (1), for every $1\le p<\infty$ and all $f=\sum_{k=1}^{\infty}a_k r_k\in X$,

$$\|f\|_{BMO}\le\varepsilon^{-1}\|f\|_d\le\varepsilon^{-1}\Big(\sum_{k=1}^{\infty}a_k^2\Big)^{1/2}\le\varepsilon^{-1}A_p^{-1}\|f\|_{L_p[0,1]}. \tag{16}$$

On the other hand, $BMO\subset L_p$ for every $1\le p<\infty$ (see the proof of Theorem 3). Therefore, the $BMO$-norm on $X$ is equivalent to the $L_p$-norm for every $1\le p<\infty$. In particular, this implies that $X$ is isomorphic to $l_2$. Since the subspace $\mathcal{R}_p$ generated by the Rademacher system is complemented in $L_p$ if $1<p<\infty$, and $\mathcal{R}_p$ is isomorphic to $l_2$, it follows that $X$ is complemented in $L_p$ as well. Denote by $P$ a linear projection bounded from $L_p$ onto $X$. Then, by (16),

$$\begin{aligned}\|Pf\|_{BMO}&\le\varepsilon^{-1}A_p^{-1}\|Pf\|_{L_p}\le A_p^{-1}\varepsilon^{-1}\|P\|_{L_p\to L_p}\|f\|_{L_p}\\&\le C_p(\varepsilon)\|P\|_{L_p\to L_p}\|f\|_{BMO}.\end{aligned}$$

Therefore, $X$ is complemented in $BMO$.

Now, assume that there is a sequence $\{f_n\}_{n=1}^{\infty}\subset X$ with $\|f_n\|_{BMO}=1$ $(n=1,2,\dots)$ such that $\|f_n\|_d\to0$ as $n\to\infty$. Then $\{f_n\}_{n=1}^{\infty}$ does not contain any subsequence converging in $BMO$. In fact, if $\lim_{k\to\infty}f_{n_k}=f\in BMO$ for some $\{f_{n_k}\}\subset\{f_n\}$, then we have both $\|f\|_{BMO}=1$ and $\|f\|_d=0$ (and therefore $f=0$), which is impossible.

Hence, passing to a subsequence if necessary, we may assume that for some $\delta>0$,

$$\|f_m-f_n\|_{BMO}\ge\delta>0\quad\text{for all } m,n=1,2,\dots,\ m\ne n. \tag{17}$$

Let $\{s_k^*\}_{k=1}^{\infty}$ be the basis of $\mathcal{R}^*$ constructed in the proof of Proposition 4. Using the diagonal process, it is not hard to choose a subsequence $\{f_{n_i}\}_{i=1}^{\infty}\subset\{f_n\}$ such that for any $k\in\mathbb{N}$ we have

$$s_k^*(f_{n_{i+1}}-f_{n_i})\to0\quad\text{as } i\to\infty.$$

Since $\{s_k^*\}_{k=1}^{\infty}$ is a basis of $\mathcal{R}^*$ and $\|f_n\|_{BMO}=1$ $(n=1,2,\dots)$, it follows that

$$f_{n_{i+1}}-f_{n_i}\to0\quad\text{weakly in } BMO.$$

Then, by (17) and by Proposition 1.a.12 in [20], there is a block basis $\mathcal{U} = \{u_k\}_{k=1}^{\infty}$ of the Rademacher system, which is equivalent to some subsequence of $\{f_{n_{i+1}} - f_{n_i}\}_{i=1}^{\infty}$ (denoted $\{f_{n_{i+1}} - f_{n_i}\}$ as well). In particular,

$$\|u_k - (f_{n_{k+1}} - f_{n_k})\|_{BMO} \le 2^{-k}, \quad k = 1, 2, \ldots.$$

This implies that both $u_k \to 0$ weakly in $BMO$ and $\|u_k\|_d \to 0$ as $k \to \infty$.

Let $\varphi_0 \in \mathcal{R}^*$ be as in the proof of Corollary 4. If $u_k = \sum_{i=m_k+1}^{m_{k+1}} a_i r_i$, $1 \le m_1 < m_2 < \cdots$, then

$$\varphi_0(u_k) = \sum_{i=m_k+1}^{m_{k+1}} a_i \to 0 \quad \text{as } k \to \infty.$$

Therefore, using the same arguments as in the proof of Proposition 3, we may choose a subsequence $\{u_{k_i}\}_{i=1}^{\infty}$ such that $[u_{k_i}]_{i=1}^{\infty} \approx c_0$ and $[u_{k_i}]_{i=1}^{\infty}$ is complemented in $\mathcal{R}$. ∎

Using Theorem 5, we are able to describe the structure of subspaces of $\mathcal{R}$ in the following way.

Corollary 5. *Let $X \subset \mathcal{R}$ be an infinite-dimensional closed subspace of $BMO$. The following conditions are equivalent:*

(1) *$X$ does not contain a subspace isomorphic to $c_0$.*
(2) *$X$ is isomorphic to a dual space.*
(3) *$X$ is reflexive.*
(4) *$X$ is isomorphic to $l_2$.*
(5) *$X$ is complemented in $BMO$.*
(6) *The $BMO$-norm on $X$ is equivalent to the $L_1$-norm.*

*Proof.* By Theorem 5, condition (1) implies either of conditions (2)–(6). Conversely, it is obvious that each of the conditions (3), (4) and (6) implies (1). The implications (2)⇒(1) and (5)⇒(1) are consequences of the above mentioned results of Bessaga–Pełczyński. ∎

Recall that the function $f_\delta$ was defined in (9).

Corollary 6. *There is a $\delta \in (0,1)$ such that no bounded linear projection $P$ in $BMO_d$ with range $\mathcal{R}_d$ commutes with the $\delta$-shift on $K_\delta := BMO_d \cap BMO_d(\delta)$. This means that for every such projection there is a function $f \in K_\delta$ such that $P(f_\delta) \neq (Pf)_\delta$.*

*Proof.* Suppose, on the contrary, that for any $\delta \in (0,1)$ there exists a bounded linear projection $P : BMO_d \to \mathcal{R}_d$ such that $P(f_\delta) = (Pf)_\delta$ for every $f \in K_\delta$. By Mei's theorem (cf. [23]) there is $\delta_0 \in (0,1)$ such that $K_{\delta_0} = BMO$ and

$$\|f\|_{BMO} \asymp \max(\|f\|_d, \|f\|_{d,\delta_0}).$$

Then for any $f \in BMO$, by assumption,

$$\begin{aligned}\|Pf\|_{BMO} &\asymp \max(\|Pf\|_d, \|Pf\|_{d,\delta_0}) \le \max(C\|f\|_d, \|(Pf)_{\delta_0}\|_d)\\ &= \max(C\|f\|_d, \|P(f_{\delta_0})\|_d) \le C\max(\|f\|_d, \|f_{\delta_0}\|_d)\\ &= C\max(\|f\|_d, \|f\|_{d,\delta_0}) \asymp \|f\|_{BMO},\end{aligned}$$

which implies that $P$ is bounded in $BMO$, contrary to Theorem 4. ■

To end the paper, we consider some examples of block bases of the Rademacher system whose span is $l_2$ and $c_0$, respectively.

EXAMPLE 1. *A block basis of the Rademacher system which spans $l_2$ in $BMO$.* Let $u_k := r_{2k+1} - r_{2k}$ and $f = \sum_{k=1}^n a_k u_k$, $n = 1, 2, \ldots$. Then

$$f = \sum_{k=1}^n a_k(r_{2k+1} - r_{2k}) = \sum_{k=1}^n a_k r_{2k+1} - \sum_{k=1}^n a_k r_{2k}$$

and, by (7),

$$A(f) \asymp \max_{1\le k\le n} |a_k| = \|\{a_k\}_{k=1}^n\|_{c_0}.$$

On the other hand, according to (6),

$$\frac{1}{\sqrt{2}}\|\{a_k\}_{k=1}^n\|_{l_2} \le \|f\|_d \le 2\|\{a_k\}_{k=1}^n\|_{l_2}.$$

Combining these relations with Theorem 2, we obtain

$$\Big\|\sum_{k=1}^n a_k u_k\Big\|_{BMO} \asymp \|\{a_k\}_{k=1}^n\|_{l_2}, \qquad n = 1, 2, \ldots.$$

EXAMPLE 2. *A block basis of the Rademacher system which spans $c_0$ in $BMO$.* Take a block basis $\mathcal{U} = \{u_n\}_{n=1}^\infty$, where

$$u_n = \sum_{k=m_n+1}^{m_{n+1}} a_k r_k \quad \text{with} \quad m_{n+1} - m_n = 2^{2n}, \qquad n = 1, 2, \ldots,$$

and

$$a_k = \begin{cases} 2^{-2n} & \text{if } m_n + 1 \le k \le (m_n + m_{n+1})/2,\\ -2^{-2n} & \text{if } (m_n + m_{n+1})/2 + 1 \le k \le m_{n+1}.\end{cases}$$

Then, by (6) and (8),

$$\|u_n\|_d \asymp \Big(\sum_{k=m_n+1}^{m_{n+1}} 2^{-4n}\Big)^{1/2} = 2^{-n}, \qquad n = 1, 2, \ldots,$$

and

$$\|u_n\|_{BMO} \asymp 2^{-2n}\frac{m_{n+1} - m_n}{2} + 2^{-n} \asymp \frac{1}{2}, \qquad n = 1, 2, \ldots.$$

Moreover,

$$\gamma_n(\mathcal{U}) = \sum_{k=m_n+1}^{(m_n+m_{n+1})/2} 2^{-2n} - \sum_{k=(m_n+m_{n+1})/2+1}^{m_{n+1}} 2^{-2n} = 0.$$

Then (see the proof of Proposition 3) $[u_n]_{BMO} \approx c_0$.

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### References

[1] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Grad. Texts in Math. 233, Springer, New York, 2006.

[2] S. V. Astashkin, *Rademacher functions in symmetric spaces*, Sovrem. Mat. Fundam. Napravl. 32 (2009), 3–161 (in Russian); English transl.: J. Math. Sci. (New York) 169 (2010), 725–886.

[3] S. V. Astashkin and L. Maligranda, *Rademacher functions in Cesàro type spaces*, Studia Math. 198 (2010), 235–247.

[4] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, ibid. 17 (1958), 151–164.

[5] —, —, *Some remarks on conjugate spaces containing subspaces isomorphic to the space* $c_0$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 6 (1958), 249–250.

[6] C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. 77 (1971), 587–588.

[7] C. Fefferman and E. M. Stein, *$H^p$ spaces of several variables*, Acta Math. 129 (1972), 137–193.

[8] J. B. Garnett, *Bounded Analytic Functions*, Springer, New York, 2007.

[9] J. B. Garnett and P. W. Jones, *BMO from dyadic BMO*, Pacific J. Math. 99 (1982), 351–371.

[10] A. M. Garsia, *The Burgess Davis inequalities via Fefferman's inequality*, Ark. Mat. 11 (1973), 229–237.

[11] —, *Martingale Inequalities: Seminar Notes on Recent Progress*, Benjamin, Reading, MA, 1973.

[12] L. Grafakos, *Modern Fourier Analysis*, 2nd ed., Springer, New York, 2009.

[13] D. Hadwin and H. Yousefi, *A general view of BMO and VMO*, in: Contemp. Math. 454, Amer. Math. Soc., Providence, 2008, 75–91.

[14] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces $L_p$*, Studia Math. 21 (1962), 161–176.

[15] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow, 1984 (in Russian); English transl.: Amer. Math. Soc., Providence, 1989.

[16] E. Kochneff, Y. Sagher and K. C. Zhou, *BMO estimates for lacunary series*, Ark. Mat. 28 (1990), 301–310.

[17] S. G. Kreĭn, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (in Russian); English transl.: Amer. Math. Soc., Providence, 1982.
[18] M. V. Leibov, *The Geometry of the Function Space BMO*, Candidate's Dissertation, Rostov-na-Donu, 1985, 133 pp. (in Russian).
[19] —, *Subspaces of the space VMO*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 46 (1986), 51–54 (in Russian); English transl.: J. Soviet Math. 48 (1990), 536–538.
[20] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, I. Sequence Spaces*, Springer, Berlin, 1977.
[21] —, —, *Classical Banach Spaces, II. Function Spaces*, Springer, Berlin, 1979.
[22] B. Maurey, *Isomorphismes entre espaces $H_1$*, Acta Math. 145 (1980), 79–120.
[23] T. Mei, *BMO is the intersection of two translates of dyadic BMO*, C. R. Math. Acad. Sci. Paris 336 (2003), 1003–1006.
[24] P. F. Müller, *Isomorphisms between $H^1$ Spaces*, IMPAN Monogr. Mat. 66, Birkhäuser, Basel, 2005.
[25] P. F. Müller and G. Schechtman, *On complemented subspaces of $H^1$ and VMO*, in: Lecture Notes in Math. 1376, Springer, Berlin, 1989, 113–125.
[26] J. Pipher and L. A. Ward, *BMO from dyadic BMO on the bidisc*, J. London Math. Soc. (2) 77 (2008), 524–544.
[27] V. A. Rodin and E. M. Semyonov, *Rademacher series in symmetric spaces*, Anal. Math. 1 (1975), 207–222.
[28] —, —, *The complementability of a subspace that is generated by the Rademacher system in a symmetric space*, Funktsional. Anal. i Prilozhen. 13 (1979), no. 2, 91–92 (in Russian); English transl.: Funct. Anal. Appl. 13 (1979), no. 2, 150–151.
[29] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. 207 (1975), 391–405.
[30] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, 1993.
[31] S. J. Szarek, *On the best constants in the Khinchin inequality*, Studia Math. 58 (1976), 197–208.

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