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# The structure of Lindenstrauss–Pełczyński spaces

by

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**Abstract.** Lindenstrauss–Pełczyński (for short $\mathcal{LP}$) spaces were introduced by these authors [Studia Math. 174 (2006)] as those Banach spaces $X$ such that every operator from a subspace of $c_0$ into $X$ can be extended to the whole $c_0$. Here we obtain the following structure theorem: a separable Banach space $X$ is an $\mathcal{LP}$-space if and only if every subspace of $c_0$ is placed in $X$ in a unique position, up to automorphisms of $X$. This, in combination with a result of Kalton [New York J. Math. 13 (2007)], provides a negative answer to a problem posed by Lindenstrauss and Pełczyński [J. Funct. Anal. 8 (1971)]. We show that the class of $\mathcal{LP}$-spaces does not have the 3-space property, which corrects a theorem in an earlier paper of the authors [Studia Math. 174 (2006)]. We then solve a problem in that paper showing that $\mathcal{L}_\infty$ spaces not containing $l_1$ are not necessarily $\mathcal{LP}$-spaces.

**1. $\mathcal{LP}$-spaces have all subspaces of $c_0$ in a unique position.** In [6] we introduced the class of Lindenstrauss–Pełczyński spaces (for short $\mathcal{LP}$) as those Banach spaces $E$ such that all operators from subspaces of $c_0$ into $E$ can be extended to $c_0$. The spaces are so named because Lindenstrauss and Pełczyński first proved in [9] that $C(K)$-spaces have this property. In [6] it was shown that every $\mathcal{LP}$-space is an $\mathcal{L}_\infty$-space, that not all $\mathcal{L}_\infty$-spaces are $\mathcal{LP}$-spaces, and that complemented subspaces of Lindenstrauss spaces (see also [9, 7]), separably injective spaces and $\mathcal{L}_\infty$-spaces not containing $c_0$ are $\mathcal{LP}$-spaces.

We now prove a fundamental structure theorem for this class; namely, separable $\mathcal{LP}$-spaces are characterized as those $\mathcal{L}_\infty$ Banach spaces having all subspaces of $c_0$ placed in a unique position. Precisely, let $Y, X$ be Banach spaces. Following [5] we say that $X$ is $Y$-*automorphic* if any isomorphism between two subspaces of $X$ isomorphic to $Y$ can be extended to an automorphism of $X$. We agree that if $X$ contains no copies of $Y$ then it is $Y$-automorphic. Lindenstrauss and Pełczyński prove in [9] that $C[0,1]$ is $H$-automorphic for all subspaces $H$ of $c_0$ and pose the question of whether

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this property characterizes the subspaces of $c_0$. Kalton shows in [8] that the answer is no since $C[0,1]$ is also $l_1$-automorphic. In the opposite direction, there is the question of whether the property of being $H$-automorphic for all subspaces of $c_0$ characterizes separable $C(K)$-spaces. The answer is no. In fact, amongst separable Banach spaces which contain an isomorphic copy of $c_0$, it characterizes being $\mathcal{LP}$.

THEOREM 1.

(i) *A Banach space that contains $c_0$ and is $H$-automorphic for all subspaces $H$ of $c_0$ is an $\mathcal{LP}$-space.*

(ii) *Every separable $\mathcal{LP}$-space is $H$-automorphic for all subspaces $H$ of $c_0$.*

Before entering into the proof, recall (see [4, 6]) the identification of exact sequences $0 \to Y \to X \to Z \to 0$ of Banach spaces with *z-linear maps* $F : Z \to Y$, i.e. homogeneous maps such that for some constant $K > 0$ and every finite set $x_1, \ldots, x_n$ one has $\|F(\sum x_j) - \sum Fx_j\| \le K \sum \|x_j\|$. The identification will be written as $0 \to Y \to X \to Z \to 0 \equiv F$. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X' \to Z \to 0$ of Banach spaces are said to be *equivalent* if there is a continuous linear operator $T : X \to X'$ providing a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

Two $z$-linear maps $F, G$ are said to be equivalent, and written $F \equiv G$, when the associated exact sequences are equivalent. Under these identifications, given an exact sequence $0 \to Y \to X \to Z \to 0 \equiv F$ and an operator $k : Z' \to Z$, the upper sequence in the associated pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & PB & \longrightarrow & Z' & \longrightarrow & 0 \equiv Fk \\ & & \| & & \big\downarrow & & \big\downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F \end{array}$$

corresponds to the $z$-linear map $Fk$ (standard composition of maps). We will need the following lemma of independent interest.

LEMMA 1. *Let $F : Z \to Y$ be a $z$-linear map and let $k : Z \to Z$ be a compact operator. Then $Fk \equiv F$ implies $F \equiv 0$.*

*Proof.* If $Fk \equiv F$ then $F - Fk \equiv F(1-k) \equiv 0$. If 1 is not an eigenvalue of $k$ then $1-k$ is an automorphism of $Z$, so $F(1-k) \equiv 0$ implies $F \equiv 0$. If 1 is an eigenvalue of $k$ then let $z_1^1, \ldots, z_{n_1}^1$ be a basis for the associated space of eigenvectors. Let $Z_1 = [z_1^1, \ldots, z_{n_1}^1]$ and consider the exact sequence

$$0 \to Z_1 \to Z \xrightarrow{q_1} Z/Z_1 \to 0.$$

Let $s_1 : Z/Z_1 \to Z$ be a continuous linear section for $q_1$. The operator $q_1ks_1 : Z/Z_1 \to Z/Z_1$ is compact. If 1 is not an eigenvalue of $q_1ks_1$ then $1_{Z/Z_1} - q_1ks_1 = q_1(1_Z - k)s_1$ is an automorphism of $Z/Z_1$. Since $Z_1$ is finite-dimensional, $F_{|Z_1} \equiv 0$ and there exists a $z$-linear map $F_1 : Z/Z_1 \to Y$ such that $F_1q \equiv F$. Now, $F(1-k) \equiv 0$ implies $F_1q_1(1_Z - k)s_1 \equiv 0$, and therefore $F_1 \equiv 0$. Hence $F \equiv 0$. It remains to treat the case where 1 is an eigenvalue of $q_1ks_1$. Take then a basis $q_1(z_1^2), \ldots, q_1(z_{n_2}^2)$ for the associated space of eigenvectors and form the closed linear span

$$Z_2 = [z_1^1, \ldots, z_{n_1}^1, s_1q_1z_1^2, \ldots, s_1q_1z_{n_2}^2].$$

The exact sequence

$$0 \to Z_2 \to Z \xrightarrow{q_2} Z/Z_2 \to 0$$

admits a continuous linear section $s_2$. If 1 is not an eigenvalue of the operator $q_2ks_2$ the argument as before yields $F \equiv 0$. It remains to treat the case where 1 is an eigenvalue of $q_2ks_2$. We then proceed as follows. Assume that after $n$ steps, 1 is an eigenvalue of $q_nks_n$. Take a basis $q_n(z_1^{n+1}), \ldots, q_n(z_{n_{n+1}}^{n+1})$ for the associated space of eigenvectors and form the closed linear span

$$Z_{n+1} = [z_1^1, \ldots, z_{n_1}^1, s_1q_1z_1^2, \ldots, s_1q_1z_{n_2}^2, \ldots, s_nq_nz_1^{n+1}, \ldots, s_nq_nz_{n_{n+1}}^{n+1}].$$

The exact sequence

$$0 \to Z_{n+1} \to Z \xrightarrow{q_{n+1}} Z/Z_{n+1} \to 0$$

admits a continuous linear section $s_{n+1}$. If 1 is not an eigenvalue of $q_{n+1}ks_{n+1}$ the same argument as before yields $F \equiv 0$. It remains to treat the case where 1 is an eigenvalue of $q_{n+1}ks_{n+1}$.

The process must stop because $Z_n \subset \ker(1-k)^n$ and $k$ has finite ascent, i.e. there is a natural $N$ such that $\ker(1-k)^N = \ker(1-k)^{N+1}$. ∎

*Proof of Theorem 1.* To prove (i) we adapt the arguments of [10, Thm. 3.2]. Let $X$ be $H$-automorphic for all subspaces $H$ of $c_0$, and assume that there is an embedding $j : c_0 \to X$. Assume there is a subspace $i : H \subset c_0$ and a norm one operator $T : H \to X$ that cannot be extended to $c_0$ through $i$. Then for small $\varepsilon > 0$ the operator $ji + \varepsilon T : H \to X$ is an into isomorphism that cannot be extended to an operator $R : X \to X$ through $ji$, as otherwise $Rji = ji + \varepsilon T$ and $\varepsilon^{-1}(Rj - j)$ would be an extension of $T$ through $i$.

We show (ii). Let $X$ be a separable $\mathcal{LP}$-space. If $X$ does not contain $c_0$, then the result is (vacuously) true. So let $i : H \to X$ be an embedding where $j : H \to c_0$ is a subspace of $c_0$. The extension $J : c_0 \to X$, which exists because $X$ is $\mathcal{LP}$, yields the commutative diagram

$$(1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle J'} & & \\ 0 & \longrightarrow & H & \xrightarrow{i} & X & \xrightarrow{q} & X/H & \longrightarrow & 0 \end{array}$$

We now show that the operator $qJ$ is not weakly compact. Otherwise it would be compact, hence $J'p = qJ$ would be compact and thus $J'$ would also be compact. Since $X$ is separable, the embedding $i$ can be extended to $c_0$, which yields a commutative diagram

$$(2)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow[i]{} & X & \xrightarrow[q]{} & X/H & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I'} & & \\ 0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \end{array}$$

Putting the two diagrams together one gets a commutative pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle IJ} & & \downarrow{\scriptstyle I'J'=k} & & \\ 0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \end{array}$$

in which $k = I'J'$ is compact. Lemma 1 shows that is impossible.

A Banach space $C$ is said to have *Pełczyński's property* $(V)$ if every operator on $C$ is either weakly compact or an isomorphism on a copy of $c_0$. Since $C(K)$-spaces have property $(V)$ [11] and we have shown that the operator $qJ$ is not weakly compact, it must be an isomorphism on a subspace isomorphic to $c_0$. Therefore $q$ is also an isomorphism on a subspace isomorphic to $c_0$, which will necessarily be complemented in both $X/H$ and $X$. This means the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow[i]{} & X & \xrightarrow[q]{} & X/H & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & H & \xrightarrow{(i,0)} & X \oplus c_0 & \longrightarrow & X/H \oplus c_0 & \longrightarrow & 0 \end{array}$$

in which both $\beta$ and $\gamma$ are isomorphisms. An application of the diagonal principles developed in [5] to the diagrams (1) and (2) yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow[(i,0)]{} & X\oplus c_0 & \longrightarrow & X/H\oplus c_0 & \longrightarrow & 0\\ && \| && \big\downarrow{\scriptstyle\sigma} && \big\downarrow{\scriptstyle\mu} &&\\ 0 & \longrightarrow & H & \xrightarrow{(j,0)} & c_0\oplus X & \longrightarrow & c_0/H\oplus X & \longrightarrow & 0\end{array}$$

in which both $\sigma, \mu$ are isomorphisms. Now, starting with a different embedding $i' : H \to X$ would lead to a similar diagram with $j$ replaced by some embedding $j' : H \to c_0$. But since $c_0$ is $H$-automorphic by the classical Lindenstrauss–Pełczyński theorem, we are done. ∎

The conclusion of Theorem 1(ii) clearly fails for nonseparable spaces since $c_0 \oplus l_\infty$ contains a complemented and an uncomplemented copy of $c_0$. From the proof one is tempted to believe that $\mathcal{LP}$-spaces containing $c_0$ have Pełczyński's property $(V)$, which is not the case: let $X$ be a Schur $\mathcal{LP}$-space (see [6]) and select a quotient map $q : X \to c_0$ to construct the quotient $Q : X \oplus c_0 \to c_0$ given by $Q(x,y) = q(x)$. An immediate consequence of Theorem 1 and the fact that $\mathcal{L}_\infty$-spaces not containing $c_0$ are $\mathcal{LP}$-spaces is

COROLLARY 1. *Every $\mathcal{L}_\infty$-space which is $H$-automorphic for every subspace $H$ of $c_0$ is an $\mathcal{LP}$-space.*

A Banach space $X$ was defined in [10] to be *extensible* if every operator $Y \to X$ from a subspace $Y$ of $X$ can be extended to $X$. It is clear that an extensible space that contains $c_0$ must be an $\mathcal{LP}$-space; hence

COROLLARY 2. *An extensible $\mathcal{L}_\infty$-space is an $\mathcal{LP}$-space.*

Thus, even the product of separable automorphic spaces such as $l_2 \oplus c_0$ may fail to be extensible.

**2. Counterexamples.** Our first counterexample is to show that, unlike $C[0,1]$, separable $\mathcal{LP}$-spaces may fail to be $l_1$-automorphic:

*A separable $\mathcal{LP}$-space that is not $l_1$-automorphic.* Consider an embedding $i : l_1 \to C[0,1]$ and another embedding $j : l_1 \to X$ of $l_1$ into its corresponding Bourgain–Pisier space $\mathcal{L}_\infty$-space $X$ (see [2]). It was proved in [6] that $X$ is a Schur $\mathcal{LP}$-space. This means that $j$ cannot be extended through $i$ to the whole $C[0,1]$ since Pełczyński's property $(V)$ of $C[0,1]$ would make such an extension a weakly compact operator. The $\mathcal{LP}$-space $X \oplus C[0,1]$ is not $l_1$-automorphic because the embeddings $(0,i) : l_1 \to X \oplus C[0,1]$ and $(j,0) : l_1 \to X \oplus C[0,1]$ cannot be transformed to each other by an automorphism $\sigma : X \oplus C[0,1] \to X \oplus C[0,1]$. Otherwise, if $\sigma(0,i) = (j,0)$, and $\pi : X \oplus C[0,1] \to X$ is the projection, the operator $\pi\sigma_{|C[0,1]} : C[0,1] \to X$ would be an extension of $j$ through $i$.

Our second counterexample shows that the statement of [6, Thm. 2] that "the class of $\mathcal{LP}$-spaces has the 3-space property" is wrong.

Proposition 2.1. *For every subspace $H$ of $c_0$ different from $c_0$, there is a twisted sum*

$$0 \to C(\omega^\omega) \to \Omega_H \to c_0 \to 0,$$

*and an operator $H \to \Omega_H$ that cannot be extended to $c_0$. Hence the space $\Omega_H$ is not an $\mathcal{LP}$-space.*

*Proof.* Consider the exact sequence $0 \to C(\omega^\omega) \to \Omega \to c_0 \to 0 \equiv M$ constructed in [3] which has the additional property of having the quotient map $q : \Omega \to c_0$ strictly singular. Since every quotient of $c_0$ is isomorphic to a subspace of $c_0$, we can assume that there is an embedding $u_H : c_0/H \to c_0$. The pull-back sequence $0 \to C(\omega^\omega) \to P_H \xrightarrow{p} c_0/H \to 0 \equiv Gu_H$ also has strictly singular quotient map. We form the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & = & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & P_H & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle t} & & \\
0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & \Omega_H & \underset{Q}{\longrightarrow} & c_0 & \longrightarrow & 0 \\
 & & & & {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} & & \\
 & & & & H & = & H & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

To show that $\Omega_H$ is not an $\mathcal{LP}$-space we show that $j$ cannot be extended to $c_0$ through $i$. Indeed, suppose $J$ is such an extension, and denote by $\nu$ the induced operator between the quotient spaces. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & c_0 & \longrightarrow & c_0/H & \longrightarrow & 0 \equiv F \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle p} & & \\
0 & \longrightarrow & H & \xrightarrow{j} & \Omega_H & \longrightarrow & P_H & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & H & \longrightarrow & c_0 & \longrightarrow & c_0/H & \longrightarrow & 0 \equiv F
\end{array}$$

The diagram means that $Fp\nu \equiv F$. But since $p$ is strictly singular, $p\nu$ is also strictly singular, hence compact. Lemma 1 can be used to conclude the argument. ■

The previous example also provides a negative answer to a question posed in [6, p. 227]: Is every $\mathcal{L}_\infty$-space not containing $l_1$ an $\mathcal{L}P$-space? The space $\Omega_H$ does not contain $l_1$ since "not containing $l_1$" is a 3-space property (see [4, Thm. 3.2.d]).

Our last example provides a partial answer to the question of whether the original space $\Omega$ constructed in [3] and which is the starting point in the proof of Proposition 2.1 is an $\mathcal{LP}$-space.

PROPOSITION 2.2. *There exists an $\mathcal{LP}$-space $X$ admitting two nontrivial exact sequences*

$$0 \to X \to A_i \xrightarrow{q_i} c_0 \to 0$$

*such that*

(1) *$A_1$ is an $\mathcal{LP}$-space and $q_1$ is strictly singular.*
(2) *$A_2$ is not an $\mathcal{LP}$-space.*

*Proof.* Consider the projective presentation of $c_0$,

$$0 \to K \to \ell_1 \to c_0 \to 0,$$

and embed $K$ into its corresponding Bourgain–Pisier space $\mathcal{L}_\infty$-space $X$ (see [2]). It was proved in [6] that $X$ is an $\mathcal{LP}$-space. To construct $A_1$ we consider the push-out diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & l_1 & \longrightarrow & c_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & A_1 & \longrightarrow & c_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since the Schur property is a 3-space property and $\mathcal{L}_\infty$-spaces with the Schur property are $\mathcal{LP}$-spaces, $A_1$ is an $\mathcal{LP}$-space, and the quotient $A_1 \to c_0$ must be strictly singular.

To obtain $A_2$, let $0 \to X \to A_1 \to c_0 \to 0$ be the previously constructed sequence having strictly singular quotient, and let $0 \to H \to c_0 \to c_0 \to 0$ be the nontrivial sequence constructed by Bourgain in [1]. We form the pull-back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & = & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X & \longrightarrow & A_1 & \longrightarrow & c_0 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X & \longrightarrow & A_2 & \longrightarrow & c_0 & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & H & = & H & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and then follow the argument in the proof of Proposition 2.1 to show that $A_2$ is not an $\mathfrak{LP}$-space. ■

**3. Positive results.** The first positive result exhibits two situations in which a twisted sum of two $\mathfrak{LP}$-spaces is an $\mathfrak{LP}$-space. The counterexamples in Section 2 show that these results are optimal.

PROPOSITION 3.1. *Let* $0 \to Y \xrightarrow{i} X \xrightarrow{q} Z \to 0$ *be an exact sequence in which both* $Y, Z$ *are* $\mathfrak{LP}$*-spaces. Then* $X$ *is an* $\mathfrak{LP}$*-space in the following cases*:

(1) $Z$ *does not contain* $c_0$.
(2) $Y$ *is separably injective.*

*Proof.* Let $j : H \to c_0$ be a subspace of $c_0$ and let $t : H \to X$ be an operator, and consider an extension $(qt)^e$ of $qt$ to $c_0$. To prove (1) observe that $(qt)^e$ is weakly compact, hence compact. Since $Y$ is an $\mathfrak{L}_\infty$-space, $(qt)^e$ can be lifted to an operator $E : c_0 \to X$ through $q$, so $qE = (qt)^e$. The operator $Ej - t$ thus takes values in $Y$, and can therefore be extended to an operator $(Ej - t)^e : c_0 \to Y$. The operator $E - i(Ej - t)^e : c_0 \to X$ is the desired extension of $t$: $(E - i(Ej - t)^e)j = Ej - i(Ej - t) = t$.

The proof for (2) is analogous: in this case $(qt)^e$ can be lifted to an operator $E : c_0 \to X$ through $q$ since $Y$ is separably injective. ■

The second positive result is a correct statement and proof of the 3-space result presented in [6]. The argument there touches the poorly developed topic of relative homology with respect to an operator ideal. Precisely, classical Banach space homology works with the ideal $\mathfrak{L}$ of continuous linear operators in the sense that, given an exact sequence $0 \to Y \to X \to Z \to 0$ and a Banach space $E$, it produces the homology sequence

$$0 \to \mathfrak{L}(Z,E) \to \mathfrak{L}(X,E) \to \mathfrak{L}(Y,E) \\ \to \operatorname{Ext}(Z,E) \to \operatorname{Ext}(X,E) \to \operatorname{Ext}(Y,E) \to \cdots$$

formed by the derived functors of $\mathfrak{L}$. One could expect that a surjective and injective operator ideal $\mathcal{A}$ would also produce a relative homology sequence

$$0 \to \mathcal{A}(Z,E) \to \mathcal{A}(X,E) \to \mathcal{A}(Y,E) \\ \to \mathcal{A}'(Z,E) \to \mathcal{A}'(X,E) \to \mathcal{A}'(Y,E) \to \cdots$$

formed by derived functors of $\mathcal{A}$. The problem, however, is that derivation is a process that can be done via injective or projective presentations, and the results of the two processes might not coincide. In the classical setting, the injective and projective derivations of $\mathfrak{L}$ are equivalent; in the relative setting the equivalence depends on the following extra property of the ideal $\mathcal{A}$.

Definition. An injective and surjective operator ideal $\mathcal{A}$ will be called *balanced* if any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & l_1(\Gamma) & \longrightarrow & Z & \longrightarrow & 0 \\ & & \alpha\downarrow & & \downarrow & & \downarrow\gamma & & \\ 0 & \longrightarrow & Y & \longrightarrow & l_\infty(\Lambda) & \longrightarrow & l_\infty(\Lambda)/Y & \longrightarrow & 0 \end{array}$$

has the property that there is an operator $\alpha' \in \mathcal{A}$ such that $\alpha - \alpha'$ can be extended to $l_1(\Gamma)$ if and only if there exists $\gamma' \in \mathcal{A}$ such that $\gamma - \gamma'$ can be lifted to $l_\infty(\Lambda)$.

The condition is clearly equivalent to the fact that projective and injective derivations coincide. Let us now define a Banach space $X$ to be $\mathcal{A}$-*injective* (resp. *separably* $\mathcal{A}$-*injective*) if $\mathcal{A}'(\cdot, X) = 0$ (resp. $\mathcal{A}'(S, X) = 0$ for every separable space $S$). The following proposition contains the right statement of Theorem 2 in [6].

Proposition 3.2. *For any surjective and injective balanced operator ideal $\mathcal{A}$, being $\mathcal{A}$-injective* (*resp. separably $\mathcal{A}$-injective*) *is a* 3-*space property.*

*Proof.* We include the proof (an abstract version of results in [6]) for the sake of completeness. Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be an exact sequence. By assumption, both $\mathcal{A}(\cdot, A)$ and $\mathcal{A}(\cdot, C)$ are exact functors and we need to prove that also $\mathcal{A}(\cdot, B)$ is exact. We construct the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A}(Z,A) & \xrightarrow{q^*} & \mathcal{A}(X,A) & \xrightarrow{j^*} & \mathcal{A}(Y,A) & \longrightarrow & \mathcal{A}'(Z,A) \\
 & & {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{A}(Z,B) & \xrightarrow{q^*} & \mathcal{A}(X,B) & \xrightarrow{j^*} & \mathcal{A}(Y,B) & \longrightarrow & \mathcal{A}'(Z,B) \\
 & & {\scriptstyle p^*}\downarrow & & {\scriptstyle p^*}\downarrow & & {\scriptstyle p^*}\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{A}(Z,C) & \xrightarrow{q^*} & \mathcal{A}(X,C) & \xrightarrow{j^*} & \mathcal{A}(Y,C) & \longrightarrow & \mathcal{A}'(Z,C)
\end{array}$$

The rows are exact by the surjectivity of $\mathcal{A}$, while the first three columns are also exact by injectivity of $\mathcal{A}$. The fourth column is exact because $\mathcal{A}$ is balanced. By hypothesis,

$$\mathcal{A}'(Z,A) = \mathcal{A}'(Z,C) = 0,$$

and the exactness of the fourth column implies that

$$\mathcal{A}'(Z,B) = 0,$$

hence $\mathcal{A}(\cdot, B)$ is exact. ∎

In [6] it is established that $\mathcal{LP}$-spaces are precisely the relatively separably injective objects associated with the ideal $\Gamma_0$ of operators that factorize through a subspace of $c_0$. The mistake in the proof there is that the ideal $\Gamma_0$ is not balanced.

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