The structure of Lindenstrauss–Pełczyński spaces

by

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Abstract. Lindenstrauss–Pełczyński (for short LP) spaces were introduced by these authors [Studia Math. 174 (2006)] as those Banach spaces $X$ such that every operator from a subspace of $c_0$ into $X$ can be extended to the whole $c_0$. Here we obtain the following structure theorem: a separable Banach space $X$ is an LP-space if and only if every subspace of $c_0$ is placed in $X$ in a unique position, up to automorphisms of $X$. This, in combination with a result of Kalton [New York J. Math. 13 (2007)], provides a negative answer to a problem posed by Lindenstrauss and Pełczyński [J. Funct. Anal. 8 (1971)]. We show that the class of LP-spaces does not have the 3-space property, which corrects a theorem in an earlier paper of the authors [Studia Math. 174 (2006)]. We then solve a problem in that paper showing that $L_\infty$ spaces not containing $l_1$ are not necessarily LP-spaces.

1. LP-spaces have all subspaces of $c_0$ in a unique position. In [6] we introduced the class of Lindenstrauss–Pełczyński spaces (for short LP) as those Banach spaces $E$ such that all operators from subspaces of $c_0$ into $E$ can be extended to $c_0$. The spaces are so named because Lindenstrauss and Pełczyński first proved in [9] that $C(K)$-spaces have this property. In [6] it was shown that every LP-space is an $L_\infty$-space, that not all $L_\infty$-spaces are LP-spaces, and that complemented subspaces of Lindenstrauss spaces (see also [9, 7]), separably injective spaces and $L_\infty$-spaces not containing $c_0$ are LP-spaces.

We now prove a fundamental structure theorem for this class; namely, separable LP-spaces are characterized as those $L_\infty$ Banach spaces having all subspaces of $c_0$ placed in a unique position. Precisely, let $Y, X$ be Banach spaces. Following [5] we say that $X$ is $Y$-automorphic if any isomorphism between two subspaces of $X$ isomorphic to $Y$ can be extended to an automorphism of $X$. We agree that if $X$ contains no copies of $Y$ then it is $Y$-automorphic. Lindenstrauss and Pełczyński prove in [9] that $C[0,1]$ is $H$-automorphic for all subspaces $H$ of $c_0$ and pose the question of whether

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this property characterizes the subspaces of $c_0$. Kalton shows in [8] that the answer is no since $C[0,1]$ is also $l_1$-automorphic. In the opposite direction, there is the question of whether the property of being $H$-automorphic for all subspaces of $c_0$ characterizes separable $C(K)$-spaces. The answer is no. In fact, amongst separable Banach spaces which contain an isomorphic copy of $c_0$, it characterizes being $L\mathcal{P}$.

**Theorem 1.**

(i) A Banach space that contains $c_0$ and is $H$-automorphic for all subspaces $H$ of $c_0$ is an $L\mathcal{P}$-space.

(ii) Every separable $L\mathcal{P}$-space is $H$-automorphic for all subspaces $H$ of $c_0$.

Before entering into the proof, recall (see [4, 6]) the identification of exact sequences $0 \to Y \to X \to Z \to 0$ of Banach spaces with $z$-linear maps $F : Z \to Y$, i.e. homogeneous maps such that for some constant $K > 0$ and every finite set $x_1, \ldots, x_n$ one has $\|F(\sum x_j) - \sum Fx_j\| \leq K \sum \|x_j\|$. The identification will be written as $0 \to Y \to X \to Z \to 0 \equiv F$. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y' \to X' \to Z \to 0$ of Banach spaces are said to be equivalent if there is a continuous linear operator $T : X \to X'$ providing a commutative diagram

$$
\begin{array}{c}
0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0 \\
\| \downarrow T \| \\
0 \longrightarrow Y' \longrightarrow X' \longrightarrow Z \longrightarrow 0
\end{array}
$$

Two $z$-linear maps $F, G$ are said to be equivalent, and written $F \equiv G$, when the associated exact sequences are equivalent. Under these identifications, given an exact sequence $0 \to Y \to X \to Z \to 0 \equiv F$ and an operator $k : Z' \to Z$, the upper sequence in the associated pull-back diagram

$$
\begin{array}{c}
0 \longrightarrow Y \longrightarrow PB \longrightarrow Z' \longrightarrow 0 \equiv Fk \\
\| \downarrow k \\
0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0 \equiv F
\end{array}
$$

corresponds to the $z$-linear map $Fk$ (standard composition of maps). We will need the following lemma of independent interest.

**Lemma 1.** Let $F : Z \to Y$ be a $z$-linear map and let $k : Z \to Z$ be a compact operator. Then $Fk \equiv F$ implies $F \equiv 0$.

**Proof.** If $Fk \equiv F$ then $F - Fk \equiv F(1 - k) \equiv 0$. If $1$ is not an eigenvalue of $k$ then $1 - k$ is an automorphism of $Z$, so $F(1 - k) \equiv 0$ implies $F \equiv 0$. If $1$ is an eigenvalue of $k$ then let $z_1^1, \ldots, z_{n_1}^1$ be a basis for the associated space of eigenvectors. Let $Z_1 = [z_1^1, \ldots, z_{n_1}^1]$ and consider the exact sequence
0 \to Z_1 \to Z \xrightarrow{q_1} Z/Z_1 \to 0.

Let \( s_1 : Z/Z_1 \to Z \) be a continuous linear section for \( q_1 \). The operator \( q_1 k s_1 : Z/Z_1 \to Z/Z_1 \) is compact. If \( 1 \) is not an eigenvalue of \( q_1 k s_1 \) then \( 1 \) is not an automorphism of \( Z/Z_1 \). Since \( Z_1 \) is finite-dimensional, \( F_1 = 0 \) and there exists a \( z \)-linear map \( F_1 : Z/Z_1 \to Y \) such that \( F_1 q \equiv F \). Now, \( F(1 - k) \equiv 0 \) implies \( F_1 q_1 (1 - k) s_1 = 0 \), and therefore \( F_1 = 0 \). Hence \( F \equiv 0 \). It remains to treat the case where \( 1 \) is an eigenvalue of \( q_1 k s_1 \). Take then a basis \( q_1(z_1^2), \ldots, q_1(z_n^2) \) for the associated space of eigenvectors and form the closed linear span

\[
Z_2 = [z_1^1, \ldots, z_n^1, s_1 q_1 z_1^2, \ldots, s_1 q_1 z_n^2].
\]

The exact sequence

\[
0 \to Z_2 \to Z \xrightarrow{q_2} Z/Z_2 \to 0
\]

admits a continuous linear section \( s_2 \). If \( 1 \) is not an eigenvalue of the operator \( q_2 k s_2 \) the argument as before yields \( F \equiv 0 \). It remains to treat the case where \( 1 \) is an eigenvalue of \( q_2 k s_2 \). We then proceed as follows. Assume that after \( n \) steps, \( 1 \) is an eigenvalue of \( q_n k s_n \). Take a basis \( q_n(z_1^{n+1}), \ldots, q_n(z_n^{n+1}) \) for the associated space of eigenvectors and form the closed linear span

\[
Z_{n+1} = [z_1^1, \ldots, z_n^1, s_1 q_1 z_1^2, \ldots, s_1 q_1 z_n^2, \ldots, s_n q_1 z_1^{n+1}, \ldots, s_n q_1 z_n^{n+1}].
\]

The exact sequence

\[
0 \to Z_{n+1} \to Z \xrightarrow{q_{n+1}} Z/Z_{n+1} \to 0
\]

admits a continuous linear section \( s_{n+1} \). If \( 1 \) is not an eigenvalue of \( q_{n+1} k s_{n+1} \) the same argument as before yields \( F \equiv 0 \). It remains to treat the case where \( 1 \) is an eigenvalue of \( q_{n+1} k s_{n+1} \).

The process must stop because \( Z_n \subset \ker (1 - k)^n \) and \( k \) has finite ascent, i.e. there is a natural \( N \) such that \( \ker (1 - k)^N = \ker (1 - k)^{N+1} \). □

**Proof of Theorem 1.** To prove (i) we adapt the arguments of [10, Thm. 3.2]. Let \( X \) be \( H \)-automorphic for all subspaces \( H \) of \( c_0 \), and assume that there is an embedding \( j : c_0 \to X \). Assume there is a subspace \( i : H \subset c_0 \) and a norm one operator \( T : H \to X \) that cannot be extended to \( c_0 \) through \( i \). Then for small \( \varepsilon > 0 \) the operator \( j i + \varepsilon T : H \to X \) is an into isomorphism that cannot be extended to an operator \( R : X \to X \) through \( j i \), as otherwise \( R j i = j i + \varepsilon T \) and \( \varepsilon^{-1} (R j - j) \) would be an extension of \( T \) through \( i \).

We show (ii). Let \( X \) be a separable \( L \)-space. If \( X \) does not contain \( c_0 \), then the result is (vacuously) true. So let \( i : H \to X \) be an embedding where \( j : H \to c_0 \) is a subspace of \( c_0 \). The extension \( J : c_0 \to X \), which exists
because \( X \) is \( \mathcal{L}P \), yields the commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \\
\| & & \| & & \downarrow J & & \downarrow J' & & \\
0 & \longrightarrow & H & \xrightarrow{i} & X & \xrightarrow{q} & X/H & \longrightarrow & 0
\end{array}
\]

We now show that the operator \( qJ \) is not weakly compact. Otherwise it would be compact, hence \( J'p = qJ \) would be compact and thus \( J' \) would also be compact. Since \( X \) is separable, the embedding \( i \) can be extended to \( c_0 \), which yields a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H & \xrightarrow{i} & X & \xrightarrow{q} & X/H & \longrightarrow & 0 \\
\| & & \| & & \downarrow I & & \downarrow I' & & \\
0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0
\end{array}
\]

Putting the two diagrams together one gets a commutative pull-back diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \\
\| & & \| & & \downarrow IJ & & \downarrow I'J'=k & & \\
0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0
\end{array}
\]

in which \( k = I'J' \) is compact. Lemma 1 shows that is impossible.

A Banach space \( C \) is said to have Pełczyński’s property \((V)\) if every operator on \( C \) is either weakly compact or an isomorphism on a copy of \( c_0 \). Since \( C(K) \)-spaces have property \((V)\) \cite{11} and we have shown that the operator \( qJ \) is not weakly compact, it must be an isomorphism on a subspace isomorphic to \( c_0 \). Therefore \( q \) is also an isomorphism on a subspace isomorphic to \( c_0 \), which will necessarily be complemented in both \( X/H \) and \( X \). This means the existence of a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H & \xrightarrow{i} & X & \xrightarrow{q} & X/H & \longrightarrow & 0 \\
\| & & \| & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & H & \xrightarrow{(i,0)} & X \oplus c_0 & \longrightarrow & X/H \oplus c_0 & \longrightarrow & 0
\end{array}
\]

in which both \( \beta \) and \( \gamma \) are isomorphisms. An application of the diagonal principles developed in \cite{5} to the diagrams (1) and (2) yields a commutative diagram
in which both $\sigma, \mu$ are isomorphisms. Now, starting with a different embedding $i' : H \to X$ would lead to a similar diagram with $j$ replaced by some embedding $j' : H \to c_0$. But since $c_0$ is $H$-automorphic by the classical Lindenstrauss–Pełczyński theorem, we are done.

The conclusion of Theorem 1(ii) clearly fails for nonseparable spaces since $c_0 \oplus l_\infty$ contains a complemented and an uncomplemented copy of $c_0$.

TheLP-space $X$ was defined in [10] to be extensible if every operator $Y \to X$ from a subspace $Y$ of $X$ can be extended to $X$. It is clear that an extensible space that contains $c_0$ must be an LP-space; hence

**Corollary 1.** Every $L_\infty$-space which is $H$-automorphic for every subspace $H$ of $c_0$ is an $\mathcal{LP}$-space.

A Banach space $X$ was defined in [10] to be extensible if every operator $Y \to X$ from a subspace $Y$ of $X$ can be extended to $X$. It is clear that an extensible space that contains $c_0$ must be an LP-space; hence

**Corollary 2.** An extensible $L_\infty$-space is an $\mathcal{LP}$-space.

Thus, even the product of separable automorphic spaces such as $l_2 \oplus c_0$ may fail to be extensible.

**2. Counterexamples.** Our first counterexample is to show that, unlike $C[0, 1]$, separable LP-spaces may fail to be $l_1$-automorphic:

A separable $\mathcal{LP}$-space that is not $l_1$-automorphic. Consider an embedding $i : l_1 \to C[0, 1]$ and another embedding $j : l_1 \to X$ of $l_1$ into its corresponding Bourgain–Pisier space $L_\infty$-space $X$ (see [2]). It was proved in [6] that $X$ is a Schur $\mathcal{LP}$-space. This means that $j$ cannot be extended through $i$ to the whole $C[0, 1]$ since Pełczyński’s property (V) of $C[0, 1]$ would make such an extension a weakly compact operator. The $\mathcal{LP}$-space $X \oplus C[0, 1]$ is not $l_1$-automorphic because the embeddings $(0, i) : l_1 \to X \oplus C[0, 1]$ and $(j, 0) : l_1 \to X \oplus C[0, 1]$ cannot be transformed to each other by an automorphism $\sigma : X \oplus C[0, 1] \to X \oplus C[0, 1]$. Otherwise, if $\sigma(0, i) = (j, 0)$, and $\pi : X \oplus C[0, 1] \to X$ is the projection, the operator $\pi \sigma|_{C[0, 1]} : C[0, 1] \to X$ would be an extension of $j$ through $i$.

Our second counterexample shows that the statement of [6, Thm. 2] that “the class of $\mathcal{LP}$-spaces has the 3-space property” is wrong.
Proposition 2.1. For every subspace $H$ of $c_0$ different from $c_0$, there is a twisted sum

$$0 \to C(\omega^\omega) \to \Omega_H \to c_0 \to 0,$$

and an operator $H \to \Omega_H$ that cannot be extended to $c_0$. Hence the space $\Omega_H$ is not an $\mathcal{LP}$-space.

Proof. Consider the exact sequence $0 \to C(\omega^\omega) \to \Omega \to c_0 \to 0 \equiv M$ constructed in [3] which has the additional property of having the quotient map $q : \Omega \to c_0$ strictly singular. Since every quotient of $c_0$ is isomorphic to a subspace of $c_0$, we can assume that there is an embedding $u_H : c_0/H \to c_0$. The pull-back sequence $0 \to C(\omega^\omega) \to P_H \xrightarrow{p} c_0/H \to 0 \equiv Gu_H$ also has strictly singular quotient map. We form the commutative diagram

$$
\begin{array}{cccccccc}
0 & = & 0 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & P_H & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & \Omega_H & \xrightarrow{Q} & c_0 & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & = & 0 \\
\end{array}
$$

To show that $\Omega_H$ is not an $\mathcal{LP}$-space we show that $j$ cannot be extended to $c_0$ through $i$. Indeed, suppose $J$ is such an extension, and denote by $\nu$ the induced operator between the quotient spaces. There is a commutative diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & H & \longrightarrow & c_0 & \longrightarrow & c_0/H & \longrightarrow & 0 \equiv F \\
\| & & \| & & \| & & \| & & \\
0 & \longrightarrow & H & \xrightarrow{j} & \Omega_H & \longrightarrow & P_H & \longrightarrow & 0 \\
\| & & \| & & J & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & c_0 & \longrightarrow & c_0/H & \longrightarrow & 0 \equiv F \\
\end{array}
$$

The diagram means that $Fp\nu \equiv F$. But since $p$ is strictly singular, $p\nu$ is also strictly singular, hence compact. Lemma 1 can be used to conclude the argument.
The previous example also provides a negative answer to a question posed in [6, p. 227]: Is every $L_\infty$-space not containing $l_1$ an $L_P$-space? The space $\Omega_H$ does not contain $l_1$ since “not containing $l_1$” is a 3-space property (see [4, Thm. 3.2.d]).

Our last example provides a partial answer to the question of whether the original space $\Omega$ constructed in [3] and which is the starting point in the proof of Proposition 2.1 is an $L_P$-space.

**Proposition 2.2.** There exists an $L_P$-space $X$ admitting two nontrivial exact sequences

$$0 \to X \to A_i \xrightarrow{q_i} c_0 \to 0$$

such that

1. $A_1$ is an $L_P$-space and $q_1$ is strictly singular.
2. $A_2$ is not an $L_P$-space.

**Proof.** Consider the projective presentation of $c_0$,

$$0 \to K \to \ell_1 \to c_0 \to 0,$$

and embed $K$ into its corresponding Bourgain–Pisier space $L_\infty$-space $X$ (see [2]). It was proved in [6] that $X$ is an $L_P$-space. To construct $A_1$ we consider the push-out diagram

$$
\begin{array}{ccc}
0 & \to & X \\
\downarrow & & \downarrow \\
0 & \to & A_1 \\
\downarrow & & \downarrow \\
S & = & S \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
$$

Since the Schur property is a 3-space property and $L_\infty$-spaces with the Schur property are $L_P$-spaces, $A_1$ is an $L_P$-space, and the quotient $A_1 \to c_0$ must be strictly singular.

To obtain $A_2$, let $0 \to X \to A_1 \to c_0 \to 0$ be the previously constructed sequence having strictly singular quotient, and let $0 \to H \to c_0 \to c_0 \to 0$ be the nontrivial sequence constructed by Bourgain in [1]. We form the pull-back diagram

$$
\begin{array}{ccc}
0 & \to & X \\
\downarrow & & \downarrow \\
0 & \to & A_1 \\
\downarrow & & \downarrow \\
S & = & S \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
$$
and then follow the argument in the proof of Proposition 2.1 to show that $A_2$ is not an $\mathcal{L}P$-space. ■

3. Positive results. The first positive result exhibits two situations in which a twisted sum of two $\mathcal{L}P$-spaces is an $\mathcal{L}P$-space. The counterexamples in Section 2 show that these results are optimal.

**Proposition 3.1.** Let $0 \rightarrow Y \xrightarrow{i} X \xrightarrow{q} Z \rightarrow 0$ be an exact sequence in which both $Y, Z$ are $\mathcal{L}P$-spaces. Then $X$ is an $\mathcal{L}P$-space in the following cases:

1. $Z$ does not contain $c_0$.
2. $Y$ is separably injective.

**Proof.** Let $j : H \rightarrow c_0$ be a subspace of $c_0$ and let $t : H \rightarrow X$ be an operator, and consider an extension $(qt)^e$ of $qt$ to $c_0$. To prove (1) observe that $(qt)^e$ is weakly compact, hence compact. Since $Y$ is an $\mathcal{L}_\infty$-space, $(qt)^e$ can be lifted to an operator $E : c_0 \rightarrow X$ through $q$, so $qE = (qt)^e$. The operator $Ej - t$ thus takes values in $Y$, and can therefore be extended to an operator $(Ej - t)^e : c_0 \rightarrow Y$. The operator $E - i(Ej - t)^e : c_0 \rightarrow X$ is the desired extension of $t$: $(E - i(Ej - t)^e)j = Ej - i(Ej - t) = t$.

The proof for (2) is analogous: in this case $(qt)^e$ can be lifted to an operator $E : c_0 \rightarrow X$ through $q$ since $Y$ is separably injective. ■

The second positive result is a correct statement and proof of the 3-space result presented in [6]. The argument there touches the poorly developed topic of relative homology with respect to an operator ideal. Precisely, classical Banach space homology works with the ideal $\mathfrak{I}$ of continuous linear operators in the sense that, given an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$ and a Banach space $E$, it produces the homology sequence
formed by the derived functors of $\mathcal{L}$. One could expect that a surjective and injective operator ideal $A$ would also produce a relative homology sequence

$$0 \to A(Z, E) \to A(X, E) \to A(Y, E) \to \cdots$$

formed by derived functors of $A$. The problem, however, is that derivation is a process that can be done via injective or projective presentations, and the results of the two processes might not coincide. In the classical setting, the injective and projective derivations of $\mathcal{L}$ are equivalent; in the relative setting the equivalence depends on the following extra property of the ideal $A$.

**Definition.** An injective and surjective operator ideal $A$ will be called balanced if any commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & l_1(\Gamma) & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow{\alpha} & & \downarrow & & \downarrow{\gamma} & & \\
0 & \longrightarrow & Y & \longrightarrow & l_\infty(\mathcal{A}) & \longrightarrow & l_\infty(\mathcal{A})/Y & \longrightarrow & 0
\end{array}
$$

has the property that there is an operator $\alpha' \in A$ such that $\alpha - \alpha'$ can be extended to $l_1(\Gamma)$ if and only if there exists $\gamma' \in A$ such that $\gamma - \gamma'$ can be lifted to $l_\infty(\mathcal{A})$.

The condition is clearly equivalent to the fact that projective and injective derivations coincide. Let us now define a Banach space $X$ to be $A$-injective (resp. separably $A$-injective) if $A'(\cdot, X) = 0$ (resp. $A'(S, X) = 0$ for every separable space $S$). The following proposition contains the right statement of Theorem 2 in [6].

**Proposition 3.2.** For any surjective and injective balanced operator ideal $A$, being $A$-injective (resp. separably $A$-injective) is a 3-space property.

**Proof.** We include the proof (an abstract version of results in [6]) for the sake of completeness. Let

$$0 \to A \overset{i}{\longrightarrow} B \overset{p}{\longrightarrow} C \to 0$$

be an exact sequence. By assumption, both $A(\cdot, A)$ and $A(\cdot, C)$ are exact functors and we need to prove that also $A(\cdot, B)$ is exact. We construct the commutative diagram
The rows are exact by the surjectivity of $A$, while the first three columns are also exact by injectivity of $A$. The fourth column is exact because $A$ is balanced. By hypothesis,

$$A'(Z, A) = A'(Z, C) = 0,$$

and the exactness of the fourth column implies that

$$A'(Z, B) = 0,$$

hence $A(\cdot, B)$ is exact. ■

In [6] it is established that $\mathcal{L}P$-spaces are precisely the relatively separably injective objects associated with the ideal $\Gamma_0$ of operators that factorize through a subspace of $c_0$. The mistake in the proof there is that the ideal $\Gamma_0$ is not balanced.

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