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Lipschitz equivalence of graph-directed fractals

by

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Abstract. This paper studies the geometric structure of graph-directed sets from the point of view of Lipschitz equivalence. It is proved that if $\{E_i\}_i$ and $\{F_j\}_j$ are dust-like graph-directed sets satisfying the transitivity condition, then E_{i_1} and E_{i_2} are Lipschitz equivalent, and E_i and F_j are quasi-Lipschitz equivalent when they have the same Hausdorff dimension.

1. Introduction. Two metric spaces (A, d_A) and (B, d_B) are called *Lipschitz equivalent*, denoted by $A \simeq B$, if there exists a bijection $f\colon A \to B$ satisfying

$$c^{-1}\mathrm{d}_A(x,y) \le \mathrm{d}_B(f(x), f(y)) \le c\mathrm{d}_A(x,y) \quad \text{for all } x, y \in A,$$

where $c \ge 1$ is a constant.

One of interesting topics in fractal geometry is to classify fractals under Lipschitz equivalence since bi-Lipschitz mappings preserve many "fractal properties" of sets. Many works have been devoted to the related topics. Cooper and Pignataro [1], Falconer and Marsh [4, 5], David and Semmes [2], Xi [10, 11] studied the shape of Cantor sets, nearly Lipschitz equivalence, BPI equivalence and quasi-Lipschitz equivalence. Recently, Xi et al. [8, 13, 14] studied Lipschitz equivalence of self-similar sets.

It is well-known that $E \simeq F$ implies $\dim_{\mathrm{H}} E = \dim_{\mathrm{H}} F$, where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. For quasi-self-similar circles, Falconer and Marsh [4] pointed out that two quasi-self-similar circles have the same Hausdorff dimension if and only if they are Lipschitz equivalent.

Then a natural question is to characterize the Lipschitz equivalence for self-similar sets with the same Hausdorff dimension. For a family of similitudes $\{S_i\colon \mathbb{R}^m \to \mathbb{R}^m\}_{i=1}^n$, suppose $E = \bigcup_i S_i(E)$ is a self-similar set [6]. We say E is *dust-like* [5] if $\bigcup_i S_i(E)$ is a disjoint union. A number r is the *ratio*

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of similitude of S, if $|S(x)-S(y)| = r|x-y|$ for all $x, y \in \mathbb{R}^m$. Fix a ratio set $\mathcal{R} = \{r_i\}_{i=1}^n$, and let $\mathcal{M}_{\mathcal{R}}$ be the collection of dust-like self-similar sets defined by $\mathcal{M}_{\mathcal{R}} = \{E = \bigcup_{i=1}^n S_i(E) : E$ is dust-like and S_i has ratio r_i for all $i\}$. Suppose $\mathcal{R} = \{r_i\}_{i=1}^n$ and $\mathcal{T} = \{t_j\}_{j=1}^m$ are ratio sets with $\sum_i r_i^s = \sum_j t_j^s = 1$. Given $\mathcal{T} = \{t_j\}_{j=1}^m$, an algorithm is constructed in [12] to calculate every ratio set $\mathcal{R}$ satisfying $\mathcal{M}_{\mathcal{R}} \simeq \mathcal{M}_{\mathcal{T}}$. It is proved in [5] that if $\mathcal{M}_{\mathcal{R}} \simeq \mathcal{M}_{\mathcal{T}}$, then $\mathbb{Q}(r_1^s, \ldots, r_n^s) = \mathbb{Q}(t_1^s, \ldots, t_m^s)$ and there are positive integers p, q such that $\mathrm{sgp}(r_1^p, \ldots, r_n^p) \subset \mathrm{sgp}(t_1, \ldots, t_m)$, $\mathrm{sgp}(t_1^q, \ldots, t_m^q) \subset \mathrm{sgp}(r_1, \ldots, r_m)$, where $\mathrm{sgp}(a_1, \ldots, a_k)$ is the multiplicative semigroup generated by $\{a_1, \ldots, a_k\}$. The following example follows from this necessary condition (see also [3, Proposition 8.9]): Let C be the middle-third Cantor set, and $F = \beta F \cup [\beta F + (1-\beta)/2] \cup [\beta F + (1-\beta)]$ the self-similar set with $\beta = 3^{-\log 3/\log 2}$. Then C and F have the same dimension $\log 2/\log 3$, but are not Lipschitz equivalent.

If self-similar sets are *not dust-like*, for example self-similar arcs, then the issue of their Lipschitz equivalence is complicated. It is proved in [9] that if two self-similar arcs are quasi-arcs with the same Hausdorff dimension, then they are Lipschitz equivalent. [9] also constructs two self-similar arcs γ_1 and γ_2 such that $\dim_{\mathrm{H}} \gamma_1 = \dim_{\mathrm{H}} \gamma_2$ and $\gamma_1 \not\simeq \gamma_2$. Other *overlapping* cases, for example the $\{1,3,5\}$-$\{1,4,5\}$ problem and its generalizations, are studied in [8, 13, 14].

In this paper, we study the geometric structure of *graph-directed sets*, which generalizes the notion of self-similar sets, from the point of view of Lipschitz equivalence. For convenience, we recall the definition of graph-directed sets (see [7]).

Definition 1. Let $G = (\mathcal{V}, \mathcal{E})$ be a directed graph with vertex set $\mathcal{V}$ and directed-edge set $\mathcal{E}$. Suppose that for each edge $e \in \mathcal{E}$, there is a corresponding similitude $T_e \colon \mathbb{R}^n \to \mathbb{R}^n$ of ratio $t_e \in (0,1)$. We also assume the *transitivity condition*: for any vertex pair $(i,j) \in \mathcal{V} \times \mathcal{V}$, there is a sequence of $k(i,j)$ edges $(e_1, \ldots, e_{k(i,j)})$ which form a directed path from vertex i to vertex j. The *graph-directed sets* on G with contracting similitudes $\{T_e\}_{e \in \mathcal{E}}$ are non-empty compact subsets $\{E_i\}_{i \in \mathcal{V}}$ of $\mathbb{R}^n$ satisfying

$$E_i = \bigcup_{j \in \mathcal{V}} \bigcup_{e \in \mathcal{E}_{i,j}} T_e(E_j) \quad \text{for } i \in \mathcal{V}, \tag{1.1}$$

where $\mathcal{E}_{i,j}$ is the set of edges starting at i and ending at j. In particular, if (1.1) is a disjoint union for each $i \in \mathcal{V}$, we say that $\{E_i\}_{i \in \mathcal{V}}$ are *dust-like* graph-directed sets on $(\mathcal{V}, \mathcal{E})$.

Remark 1. The graph with respect to a self-similar set only contains one vertex.

Now we state our first result about the Lipschitz equivalence between *dust-like* graph-directed sets.

THEOREM 1. *Let $\{E_i\}_{i\in\mathcal{V}}$ be dust-like graph-directed sets on $G=(\mathcal{V},\mathcal{E})$ satisfying the transitivity condition (see Definition 1). Then for all $i,j\in\mathcal{V}$,*

$$E_i \simeq E_j.$$

The following classical result in [1] can also be considered as a corollary of the above theorem.

COROLLARY 1. *Suppose $E\subset\mathbb{R}^m$ is a dust-like self-similar set. Let $F=\bigcup_{i=1}^k g_i(E)$ be a disjoint union with a family of similitudes $\{g_i\colon\mathbb{R}^m\to\mathbb{R}^m\}_{i=1}^k$. Then E and F are Lipschitz equivalent.*

By the example mentioned above, it is difficult to find a bi-Lipschitz bijection between self-similar sets. However, we can construct some bijection which satisfies the "quasi-Lipschitz" condition (see Definition 2) between two dust-like graph-directed sets of equal dimension. The definition below was introduced by Xi [11].

DEFINITION 2. Two compact sets E and F of Euclidean spaces are said to be *quasi-Lipschitz equivalent* if there is a bijection $f\colon E\to F$ such that for every $\varepsilon>0$, there exists $\delta>0$ satisfying

$$\left|\frac{\log|f(x)-f(y)|}{\log|x-y|}-1\right|<\varepsilon \tag{1.2}$$

whenever $x,y\in E$ with $0<|x-y|<\delta$.

The quasi-Lipschitz equivalence is stronger than "nearly Lipschitz equivalence" ([5]) and weaker than "Lipschitz equivalence". There are some related results: Suppose E,F are dust-like C^1 self-conformal sets in Euclidean spaces. Then $\dim_{\mathrm{H}}E=\dim_{\mathrm{H}}F$ if and only if E and F are *nearly* Lipschitz equivalent ([5, 10]). In fact, $\dim_{\mathrm{H}}E=\dim_{\mathrm{H}}F$ if and only if E and F are *quasi*-Lipschitz equivalent ([11]).

Now suppose two graph-directed sets have the same Hausdorff dimension; a question is to characterize the quasi-Lipschitz equivalence between them, although they may not be Lipschitz equivalent. We can state our second result.

THEOREM 2. *Let $\{E_i\}_{i=1}^m$ and $\{F_j\}_{j=1}^n$ be dust-like graph-directed sets satisfying the transitivity condition. If $\dim_{\mathrm{H}}E_i=\dim_{\mathrm{H}}F_j$, then E_i and F_j are quasi-Lipschitz equivalent.*

This paper is organized as follows. Section 2 brings the proofs of Theorem 1 and Corollary 1. In Section 3, the proof of Theorem 2 is provided.

2. The proof of Theorem 1. In this section, we *always assume* that the sets $\{E_i\}_{i\in\mathcal{V}}$ are dust-like graph-directed sets on $G=(\mathcal{V},\mathcal{E})$ satisfying the *transitivity condition* (see Definition 1). We begin with two lemmas which follow immediately from the definitions of dust-like graph-directed sets and the transitivity condition.

LEMMA 1. *Suppose that $E \in \{E_i\}_{i\in\mathcal{V}}$. Then there are contracting similitudes S_0, S_1 and a compact set F such that*

$$E = S_0(E) \cup S_1(E) \cup F,$$

where the union is disjoint.

LEMMA 2. *Suppose that $E \in \{E_i\}_{i\in\mathcal{V}}$. Then there are* non-empty *families $\{\Gamma_j\}_{j\in\mathcal{V}}$ consisting of contracting similitudes such that*

$$E = \bigcup_{j\in\mathcal{V}} \bigcup_{S\in\Gamma_j} S(E_j),$$

where the union is disjoint.

We skip the straightforward proofs of the above two lemmas. The lemma below is the key point in the proof of Theorem 1 and may be of interest in itself.

LEMMA 3. *Suppose that $E \in \{E_i\}_{i\in\mathcal{V}}$. Then for any similitudes $\{T_i\}_{i=0}^k$ such that $\{T_i(E)\}_{i=0}^k$ are pairwise disjoint*,

$$E \simeq T_0(E) \cup T_1(E) \cup \cdots \cup T_k(E).$$

Proof. By induction, it suffices to verify the conclusion for $k = 1$, i.e.,

$$E \simeq T_0(E) \cup T_1(E).$$

By Lemma 1, we have

$$E = S_0(E) \cup S_1(E) \cup F,$$

where the union is disjoint. For a finite word $i_1 \ldots i_k \in \{0,1\}^k$, put $S_{i_1\ldots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}$, where S_w equals the identity mapping if w is the empty word. We also use 1^k as an abbreviation of $1\ldots 1$ (k ones). With this notation,

$$\begin{aligned} E &= (S_0E \cup F) \cup (S_{10}E \cup S_1F \cup S_{11}E) \\ &= (S_0E \cup F) \cup (S_{10}E \cup S_1F) \cup (S_{110}E \cup S_{11}F \cup S_{111}E) \\ &= \bigcup_{k=0}^{\infty} (S_{1^k0}E \cup S_{1^k}F) \cup \{\omega\}, \end{aligned}$$

where ω is the fixed point of the similitude S_1. Consequently, we can write

$$E = S_0E \cup \bigcup_{k=0}^{\infty} S_{1^{k+1}0}E \cup \Big(\bigcup_{k=0}^{\infty} S_{1^k}F \cup \{\omega\}\Big) =: S_0E \cup E' \cup F',$$

where $E' \cup F' = S_1E \cup F$, and

$$T_0E \cup T_1E = T_0E \cup \bigcup_{k=0}^{\infty} T_1S_{1^k0}E \cup \Big(\bigcup_{k=0}^{\infty} T_1S_{1^k}F \cup \{T_1\omega\}\Big) =: T_0E \cup E'' \cup F'',$$

where $E'' \cup F'' = T_1E$. We define a bijection $f\colon E \to T_0E \cup T_1E$ by

$$f(x) = \begin{cases} T_0S_0^{-1}(x) & \text{if } x \in S_0E, \\ T_1S_1^{-1}(x) & \text{if } x \in E' = \bigcup_{k=0}^{\infty} S_{1^{k+1}0}E, \\ T_1(x) & \text{if } x \in F' = \bigcup_{k=0}^{\infty} S_{1^k}F \cup \{\omega\}. \end{cases}$$

It remains to show that f is bi-Lipschitz.

Since

$$d(S_0E, E' \cup F') > 0 \quad \text{and} \quad d(T_0E, E'' \cup F'') > 0,$$

where d is the Hausdorff distance, we only need to consider the restriction of f to $E' \cup F'$ (the corresponding image is $E'' \cup F''$). Suppose s_0, s_1, t_1 are the ratios of S_0, S_1, T_1, respectively. Put

$$\Delta := \min\{d(S_0E, S_1E), d(S_0E, F), d(S_1E, F)\} > 0.$$

For $x \in E'$ and $y \in F'$, suppose that $x \in S_{1^{m+1}0}E = S_{1^{m+1}}(S_0E)$ with $m \geq 0$ and $y \in S_{1^k}F$ with $k \geq 0$ or $k = \infty$. Here $S_{1^\infty}F = \{\omega\}$. Then $f(x) \in T_1S_{1^m0}E$ and $f(y) \in T_1S_{1^k}F$. Let $|E|$ be the diameter of E. Then

$$s_1^{\min(m+1,k)}\Delta \leq |x-y| \leq s_1^{\min(m+1,k)}|E|,$$
$$t_1s_1^{\min(m,k)}\Delta \leq |f(x)-f(y)| \leq t_1s_1^{\min(m,k)}|E|.$$

Therefore, for any $x \in E'$ and $y \in F'$,

$$\frac{t_1\Delta}{|E|} \leq \frac{s_1^{\min(m,k)}}{s_1^{\min(m+1,k)}}\frac{t_1\Delta}{|E|} \leq \frac{|f(x)-f(y)|}{|x-y|} \leq \frac{s_1^{\min(m,k)}}{s_1^{\min(m+1,k)}}\frac{t_1|E|}{\Delta} \leq \frac{t_1|E|}{s_1\Delta}. \ \blacksquare$$

Proof of Theorem 1. Let $\{\Psi_j\}_{j\in\mathcal{V}}$ be a family of similitudes such that the sets $\{\Psi_j(E_j)\}_{j\in\mathcal{V}}$ are pairwise disjoint. Let $E \in \{E_i\}_{i\in\mathcal{V}}$ and $\{\Gamma_j\}_{j\in\mathcal{V}}$ be as in Lemma 2. Then

$$E = \bigcup_{j\in\mathcal{V}} \bigcup_{S\in\Gamma_j} S(E_j),$$

where the union is disjoint. According to Lemma 3, for any $j \in \mathcal{V}$,

$$\bigcup_{S\in\Gamma_j} S(E_j) \simeq \Psi_j(E_j).$$

Then for any $E \in \{E_i\}_{i\in\mathcal{V}}$,

$$E = \bigcup_{j\in\mathcal{V}} \bigcup_{S\in\Gamma_j} S(E_j) \simeq \bigcup_{j\in\mathcal{V}} \Psi_j(E_j),$$

which implies $E_i \simeq E_j$ for all $i, j \in \mathcal{V}$. $\blacksquare$

Corollary 1 follows from Lemma 3, since any self-similar set has a special graph-directed construction with its graph containing only one point. We also give another proof by using Theorem 1 as follows.

Proof of Corollary 1. By induction, it suffices to show $E \simeq g_1(E) \cup g_2(E)$ for the dust-like self-similar set $E = \bigcup_{i=1}^m S_i(E)$, which is a disjoint union. Since $g_1(E) \simeq g_2(E) \simeq S_1(E) \simeq S_2(E) \simeq E$, we only need to prove $E \simeq S_1(E) \cup S_2(E)$. If $m = 2$, then $E = S_1(E) \cup S_2(E)$. Without loss of generality, we assume that $m \geq 3$.

Let ϱ_i be the ratio of S_i for any i. Take $\{r_i\}_{i=2}^{m-1}$ such that

$$\max\{\varrho_1, \ldots, \varrho_m\} < r_2 < \cdots < r_{m-1} < 1.$$

Let $E_1 = E$ and $E_k = r_k^{-1}[S_1(E) \cup \cdots \cup S_k(E)]$ for $1 < k < m$. Then we get a dust-like graph-directed construction satisfying the transitivity condition:

$$\begin{aligned} &E_2 = r_2^{-1}[S_1(E_1) \cup S_2(E_1)], \\ &E_k = r_k^{-1}[(r_{k-1}E_{k-1}) \cup (S_k E_1)] \quad \text{for } k \in \mathbb{N} \cap (2, m-1], \\ &E_1 = r_{m-1}E_{m-1} \cup S_m(E_1). \end{aligned}$$

Therefore, it follows from Theorem 1 that $E_2 \simeq E_1$. Here $E_1 = E$ and $E_2 \simeq S_1(E) \cup S_2(E)$, which implies $E \simeq S_1(E) \cup S_2(E)$. ∎

3. The proof of Theorem 2. Let $\{E_i\}_{i\in\mathcal{V}}$ be dust-like graph-directed sets satisfying the transitivity condition. Denote by $\{t_e\}_{e\in\mathcal{V}}$ the ratio set of $\{E_i\}_{i\in\mathcal{V}}$. Write $t_* = \min\{t_e : e \in \mathcal{V}\} > 0$. By iterating (1.1) we can obtain the following lemma.

Lemma 4. *Suppose that $0 < r < t_*$. Then there are families $\{\Gamma_{i,j}^r\}_{i,j\in\mathcal{V}}$ of similitudes such that*

$$E_i = \bigcup_{j\in\mathcal{V}} \bigcup_{S\in\Gamma_{i,j}^r} S(E_j) \quad \textit{for all } i, j \in \mathcal{V},$$

where the union is disjoint, and $r(S) \in (t_* r, r]$ for any $S \in \bigcup_{i,j\in\mathcal{V}} \Gamma_{i,j}^r$.

Instead of proving Theorem 2 directly, we will prove the following proposition.

Proposition 1. *Let $\{E_i\}_{i\in\mathcal{V}}$ be dust-like graph-directed sets satisfying the transitivity condition. Suppose $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ is the symbolic space of two letters equipped with the usual distance*

$$d(\sigma, \sigma') = 2^{-\min\{k : \sigma_k \neq \sigma'_k\}} \quad \textit{for } \sigma \neq \sigma'. \tag{3.1}$$

If $E \in \{E_i\}_{i\in\mathcal{V}}$, then there is a bijection $f : E \to \Sigma_2$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\left|\frac{\log d(f(x),f(y))}{\dim_{\mathrm H} E\cdot\log|x-y|}-1\right|<\varepsilon$$

whenever $x,y\in E$ *with* $0<|x-y|<\delta$.

As in [11], we can see that Theorem 2 is a corollary of Proposition 1. In fact, for dust-like graph-directed sets $\{E_i\}_i$ and $\{F_j\}_j$ satisfying the transitivity condition, if $E\in\{E_i\}_i$ and $F\in\{F_j\}_j$ have the same Hausdorff dimension s, then it follows from Proposition 1 that there are bijections $f\colon E\to\Sigma_2$ and $g\colon F\to\Sigma_2$ such that

$$\frac{\log d(f(x_1),f(x_2))}{s\log|x_1-x_2|}\to 1,\qquad \frac{\log d(g(y_1),g(y_2))}{s\log|y_1-y_2|}\to 1$$

uniformly whenever $|x_1-x_2|,|y_1-y_2|\to 0$. Then $g^{-1}\circ f:E\to F$ is a bijection satisfying (1.2).

Proof of Proposition 1. We begin with some notation for symbolic systems. Given $w=w_1\dots w_k\in\{0,1\}^k$ and $w'=w'_1\dots w'_{k'}\in\{0,1\}^{k'}$, write $w*w':=w_1\dots w_kw'_1\dots w'_{k'}\in\{0,1\}^{k+k'}$, and let $[w]$ denote the cylinder with respect to w, i.e.,

$$[w]:=\{\sigma\in\Sigma_2\colon \sigma_1\dots\sigma_k=w\}.$$

Given w, we can split $[w]$ into two cylinders $[w*0]$ and $[w*1]$.

Let $r_k=t_*\cdot 2^{-k}$ $(<t_*)$ for $k\ge 1$. Then it follows from Lemma 4 that there are corresponding families $\{\Gamma_{i,j}^{r_k}\}_{i,j\in\mathcal V}$ for all $k\ge 1$ such that $E_i=\bigcup_{j\in\mathcal V}\bigcup_{S\in\Gamma_{i,j}^{r_k}}S(E_j)$ with $r(S)\in(t_*r_k,r_k]$ for any $S\in\Gamma_{i,j}^{r_k}$. Write

$$\Xi_i^k=\bigcup_{j\in\mathcal V}\Gamma_{i,j}^{r_k}.$$

We will estimate $\#\Xi_i^k$, the cardinality of Ξ_i^k. Let $\overline{M}=\max\{\mathcal H^s(E_i): i\in\mathcal V\}$ and $\underline{M}=\min\{\mathcal H^s(E_i):i\in\mathcal V\}$, where $s=\dim_{\mathrm H}E_i$ for any $i\in\mathcal V$. Notice that $0<\underline{M}\le\overline{M}<\infty$. It follows from Lemma 4 that

$$\mathcal H^s(E_i)=\sum_{j\in\mathcal V,\,S\in\Gamma_{i,j}^{r_k}}\mathcal H^s(S(E_j)),$$

where $r(S)\in(t_*r_k,r_k]$, which implies

$$(t_*r_k)^s\underline{M}\le(t_*r_k)^s\mathcal H^s(E_j)\le\mathcal H^s(S(E_j))\le r_k^s\mathcal H^s(E_j)\le r_k^s\overline{M}.$$

Therefore,

$$\frac{\underline{M}}{r_k^s\overline{M}}\le\#\Xi_i^k\le\frac{\overline{M}}{t_*^sr_k^s\underline{M}}.$$

For each Ξ_i^k, there is an integer $n(i,k)$ such that $2^{n(i,k)} \le \#\Xi_i^k < 2^{n(i,k)+1}$, which implies

$$\frac{t_*^s \underline{M}}{2\overline{M}}\, r_k^s \le 2^{-[n(i,k)+1]} \le 2^{-n(i,k)} \le \frac{2\overline{M}}{\underline{M}}\, r_k^s. \tag{3.2}$$

By splitting $\#\Xi_i^k - 2^{n(i,k)}$ cylinders with respect to the word of length $n(i,k)$, we can find a family Σ_i^k consisting of finite words such that

(i) any word in Σ_i^k is of length $n(i,k)$ or $n(i,k)+1$;
(ii) $\Sigma_2 = \bigcup_{w\in\Sigma_i^k}[w]$ is a disjoint union;
(iii) $\#\Sigma_i^k = \#\Xi_i^k$.

Thus, we can find a one-to-one mapping $\pi_i^k\colon \Xi_i^k \to \Sigma_i^k$ for all $k \ge 1$ and $i \in \mathcal{V}$.

Now, for each $E \in \{E_i\}_{i\in\mathcal{V}}$, we can construct a bijection $f\colon E \to \Sigma_2$. Let $x \in E$; according to the construction of graph-directed sets, there are corresponding $i_k \in \mathcal{V}$ and $S_k \in \Gamma_{i_k,i_{k+1}}^{r_k}$ $(\subset \Xi_{i_k}^k)$ for all $k \ge 1$ such that

$$E = E_{i_1} \quad \text{and} \quad x \in S_1 \circ \cdots \circ S_k(E_{i_{k+1}}) \quad \text{for } k \ge 1.$$

The bijection $f\colon E \to \Sigma_2$ is defined by $f(x) = \sigma$ where for all $k \ge 1$,

$$\sigma \in [\pi_{i_1}^1(S_1) * \cdots * \pi_{i_k}^k(S_k)].$$

To show that f is as desired, we need some more notation. Put $\overline{D} = \max_{i\in\mathcal{V}}\{|E_i|\}$; let $\underline{D}_i = \min\{d(T_e(E_j), T_{e'}(E_{j'})) : e \ne e'$ with $e \in \mathcal{E}_{i,j}$, $e' \in \mathcal{E}_{i,j'}\}$ and $\underline{D} = \min_{i\in\mathcal{V}}\{\underline{D}_i\} > 0$.

Without loss of generality, suppose $x, y \in E$ are distinct points such that

$$x \in S_1 \circ \cdots \circ S_{N-1} \circ S_N(E_{i_{N+1}}), \qquad y \in S_1 \circ \cdots \circ S_{N-1} \circ S'_N(E_{i'_{N+1}}),$$

where $S_N, S'_N \in \Xi_{i_N}^N$ with $S_N \ne S'_N$. Then

$$d(S_N(E_{i_{N+1}}), S'_N(E_{i'_{N+1}})) \ge t_* r_N \underline{D}. \tag{3.3}$$

It follows from Lemma 4 and (3.3) that

$$t_*^N r_1 \cdots r_N \cdot \underline{D} \le |x-y| \le r_1 \cdots r_{N-1} \cdot \overline{D}. \tag{3.4}$$

On the other hand,

$$f(x) \in [\pi_{i_1}^1(S_1) * \cdots * \pi_{i_{N-1}}^{N-1}(S_{N-1}) * \pi_{i_N}^N(S_N)],$$
$$f(y) \in [\pi_{i_1}^1(S_1) * \cdots * \pi_{i_{N-1}}^{N-1}(S_{N-1}) * \pi_{i_N}^N(S'_N)],$$

where $\pi_{i_N}^N(S_N) \ne \pi_{i_N}^N(S'_N)$. Together with (3.1), (3.2) and condition (i) about Σ_i^k, we get

$$\left(\frac{t_*^s \underline{M}}{2\overline{M}}\right)^N (r_1 \cdots r_N)^s \le d(f(x), f(y)) \le \left(\frac{2\overline{M}}{\underline{M}}\right)^{N-1} (r_1 \cdots r_{N-1})^s, \tag{3.5}$$

where $s = \dim_{\mathrm{H}} E$. In view of (3.4), (3.5) and the fact that $r_k = t_* \cdot 2^{-k}$, we have $N \to \infty$ uniformly as $|x - y| \to 0$, and thus

$$\frac{\log d(f(x), f(y))}{\log |x - y|} \to s = \dim_{\mathrm{H}} E \quad \text{uniformly as } |x - y| \to 0,$$

where

$$\frac{\log(r_1 \cdots r_N)}{-(N^2/2)\log 2} \to 1. \ \blacksquare$$

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