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Decomposing and twisting bisectorial operators

by

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Abstract. Bisectorial operators play an important role since exactly these operators lead to a well-posed equation $u'(t) = Au(t)$ on the entire line. The simplest example of a bisectorial operator A is obtained by taking the direct sum of an invertible generator of a bounded holomorphic semigroup and the negative of such an operator. Our main result shows that each bisectorial operator A is of this form, if we allow a more general notion of direct sum defined by an unbounded closed projection. As a consequence we can express the solution of the evolution equation on the line by an integral operator involving two semigroups associated with A.

1. Introduction. Let us first explain the motivation for investigating bisectorial operators. An invertible operator A on a Banach space X is called *bisectorial* if the imaginary line is in the resolvent set of A and $\lambda(\lambda I - A)^{-1}$ is bounded on that line. Such bisectorial operators were considered by McIntosh and Yagi [9] in the framework of spectral calculus. Mielke [10] showed in 1987 that, on Hilbert spaces, an operator A is bisectorial if and only if there exists $p \in (1, +\infty)$ such that for all $f \in L^p(\mathbb{R}; X)$ there is a unique solution $u \in W^{1,p}(\mathbb{R}; X)$ of

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}. \tag{1.1}$$

In that case, this property holds for all $p \in (1, +\infty)$. Thus, Mielke proved a result on maximal regularity for the evolution equation on the line for bisectorial operators on Hilbert space. He applied such results to non-linear equations and, in particular, to prove the existence of central manifolds. Mielke's result on maximal regularity was generalized to Banach spaces by Schweiker [11], and in [4] with the help of the operator-valued Fourier multiplier theorem due to Weis. Maximal regularity in Hölder spaces was considered in [1].

Most interesting is the spectral theory of bisectorial operators. An interesting problem is whether it is possible to decompose the Banach space X

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with the help of a spectral projection commuting with A so that the operator is the direct sum of an invertible generator of a bounded holomorphic semigroup and the negative of such an operator. There is always a natural spectral projection P_+ (see Section 3) defined by a contour integral, but this projection is unbounded in general as was shown by McIntosh and Yagi [9] (see also Dore and Venni [6]).

However, the spectral projection P_+ is always closed. This means that its kernel and its image are closed subspaces of X. Their sum is dense in X, if A is densely defined, but this sum is possibly different from the entire space. The part of A in these subspaces is the generator or the negative of the generator of a bounded holomorphic semigroup. In our main result we show that "twisting" A by its spectral projection, we obtain the generator of a bounded holomorphic semigroup on the entire space X. This is surprising since it shows that each bisectorial operator is, in fact, the "twisted version" (see Definition 2.9) of a sectorial operator.

The spectral projection was investigated before by Sybille Schweiker [11]. In particular, she associated with a bisectorial operator two semigroups which operate on the entire space X. These semigroups are holomorphic but singular as time goes to 0. However, the singularity can never be worse than logarithmic, as Schweiker showed. We now obtain these semigroups very simply from the semigroup generated by the twisted version of A. They allow one to give a representation formula for the solutions of (1.1) which are exploited further in [13].

Our main result (see Theorem 3.4) also holds for non-densely defined operators. For simplicity we do not consider more general operators which are merely bisectorial outside a compact set as in [3], where a spectral theory for these operators is developed.

2. Twisting bisectorial operators by unbounded projections. Let X be a Banach space. We start by defining unbounded projections.

DEFINITION 2.1. A *projection* P on X is a linear operator $P : D(P) \subset X \to X$ such that $P^2 = P$, i.e. $Px \in D(P)$ and $P^2x = Px$ for all $x \in D(P)$.

If P is a projection on X, then

$$\begin{aligned} \operatorname{im}(P) &:= \{Px : x \in D(P)\} = \{x \in D(P) : Px = x\}, \\ \ker(P) &:= \{x \in D(P) : Px = 0\} \end{aligned}$$

are subspaces of X such that $\operatorname{im}(P) \cap \ker(P) = \{0\}$. Moreover, it is easy to prove the following result.

LEMMA 2.2. *A projection P on X is closed if and only if* $\ker(P)$ *and* $\operatorname{im}(P)$ *are closed subspaces of X.*

Conversely, if X_1 and X_2 are subspaces of X such that $X_1 \cap X_2 = \{0\}$, then letting

$$D(P) := X_1 \oplus X_2, \quad P(x_1 + x_2) := x_1, \quad x_1 \in X_1,\, x_2 \in X_2,$$

defines a projection on X with $\mathrm{im}(P) = X_1$ and $\ker(P) = X_2$. This projection is closed if and only if X_1 and X_2 are closed.

REMARK 2.3 (Closability of projections).

(i) If P is closable, then $\overline{P}$ is a projection.
(ii) Let $X_1, X_2 \subset X$ be subspaces such that $X_1 \cap X_2 = \{0\}$. Then the projection onto X_1 defined above is closable if and only if $\overline{X_1} \cap \overline{X_2} = \{0\}$.
(iii) Let X_1 be a dense subspace of X which is different from X. Let X_2 be an algebraic complement. Then the projection onto X_1 with domain X is not closable.
(iv) Let A be a densely defined invertible operator which is not bounded. Let $x' \in X' \setminus D(A')$, and let $u \in D(A)$ be such that $\langle x', Au\rangle = 1$. Then $Px = \langle Ax, x'\rangle u$, with domain $D(P) = D(A)$, defines an unbounded non-closable projection.

Now, let A be an operator on X with non-empty resolvent set $\rho(A)$.

PROPOSITION 2.4. *Let P be a projection on X such that $D(A) \subset D(P)$. Then the following statements are equivalent.*

(i) *$PR(\mu, A)x = R(\mu, A)Px$ for all $x \in D(P)$ and some $\mu \in \rho(A)$.*
(ii) *If $y \in D(A)$ is such that $Ay \in D(P)$, then $Py \in D(A)$ and $PAy = APy$.*
(iii) *$PR(\mu, A)x = R(\mu, A)Px$, for all $x \in D(P)$ and $\mu \in \rho(A)$.*

Proof. (i)⇒(ii). Let $y \in D(A)$ be such that $Ay \in D(P)$. Then $x = \mu y - Ay \in D(P)$ and, by (i), $Py = PR(\mu, A)x = R(\mu, A)Px \in D(A)$. Moreover, $(\mu - A)Py = Px = \mu Py - PAy$. Thus $APy = PAy$.

(ii)⇒(iii). Let $\mu \in \rho(A)$, $x \in D(P)$ and $y = R(\mu, A)x$. Then $y \in D(A)$ and $\mu y - Ay = x$. Hence $Ay \in D(P)$. By assumption it follows that $Py \in D(A)$ and $(\mu - A)Py = P(\mu - A)y = Px$. Hence $R(\mu, A)Px = Py = PR(\mu, A)x$. ∎

DEFINITION 2.5. Let P be a projection on X and let $A : D(A) \subset X \to X$ be an operator with $\rho(A) \neq \emptyset$. We say that P *commutes with* A if $D(A) \subset D(P)$, and the equivalent conditions (i)–(iii) of Proposition 2.4 are satisfied.

Now let us give an example of a closed, unbounded and commuting projection.

Example 2.6. Let

$$X := \Big\{ x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{+\infty} \Big\{ \frac{1}{n^2} \left(|x_{2n}|^2 + |x_{2n-1}|^2\right) + |x_{2n} - x_{2n-1}|^2 \Big\} < +\infty \Big\}.$$

Then X is a Hilbert space with respect to the scalar product

$$(x \,|\, y) := \sum_{n=1}^{+\infty} \Big\{ \frac{1}{n^2} \left(x_{2n}\overline{y}_{2n} + x_{2n-1}\overline{y}_{2n-1}\right) + (x_{2n} - x_{2n-1})(\overline{y}_{2n} - \overline{y}_{2n-1}) \Big\}.$$

Define the operator A on X by

$$(Ax)_{2n} = -nx_{2n}, \qquad (Ax)_{2n-1} = -nx_{2n-1},$$

with maximal domain in X, i.e.,

$$D(A) = \Big\{ x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{+\infty} \{ |x_{2n}|^2 + |x_{2n-1}|^2 + n^2|x_{2n} - x_{2n-1}|^2 \} < +\infty \Big\}.$$

Then A is invertible. Let $X_+ := \{x \in X : x_{2n-1} = 0,\, n \in \mathbb{N}\}$ and $X_- := \{x \in X : x_{2n} = 0,\, n \in \mathbb{N}\}$. Then X_+ and X_- are closed subspaces of X (both isomorphic to ℓ_2) such that $X_+ \cap X_- = \{0\}$. The constant-1 sequence belongs to X but not to $X_+ \oplus X_-$. Let P be the projection given by $D(P) = X_+ \oplus X_-$,

$$(Px)_{2n} = x_{2n}, \qquad (Px)_{2n-1} = 0.$$

Then P is closed and it is immediate to check that it commutes with A.

Now we introduce the basic notion of this paper.

Definition 2.7. A closed, linear operator $A : D(A) \subset X \to X$ is called *bisectorial* if

(i) $i\mathbb{R} \setminus \{0\} \subset \rho(A)$,
(ii) $\sup_{s\in\mathbb{R}} \|sR(is, A)\|_{\mathcal{L}(X)} < +\infty$.

For $0 < \theta < \pi/2$ we consider the open horizontal sector $\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$, and the open vertical bisector $\Sigma'_\theta := \mathbb{C} \setminus \{\overline{\Sigma_\theta} \cup (-\overline{\Sigma_\theta})\}$. If A is a bisectorial operator on X then, by the usual geometric series expansion, one obtains $\omega \in (0, \pi/2)$ such that

$$\Sigma'_\omega \subset \rho(A), \tag{2.1}$$

and

$$\sup_{\lambda\in\Sigma'_\omega} \|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} < +\infty. \tag{2.2}$$

We say that an operator A *generates a bounded holomorphic semigroup* if $\lambda \in \rho(A)$ for $\mathrm{Re}(\lambda) > 0$ and

$$\sup_{\mathrm{Re}(\lambda)>0} \|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} < +\infty.$$

In fact, we can then construct a semigroup $(e^{tA})_{t>0} \subset \mathcal{L}(X)$, which has a bounded and holomorphic extension to a sector Σ_θ for some $0 < \theta < \pi/2$. This semigroup is a C_0-semigroup if and only if A is densely defined. Moreover, A is invertible if and only if the semigroup is *exponentially stable*, i.e. if

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq M e^{-\varepsilon t}, \qquad t > 0,$$

for some constants $\varepsilon, M > 0$. We refer to the monographs [2] and [8] for these properties. Thus, if A generates a bounded holomorphic semigroup, then A is, in particular, bisectorial.

Example 2.8. The operator A defined in Example 2.6 generates a bounded holomorphic C_0-semigroup on X. Indeed, for $\mathrm{Re}(\lambda) \geq 0$, the resolvent of A is given by $R(\lambda, A)y = \tilde{x}$, with

$$\tilde{x}_{2n} = \frac{1}{\lambda + n}\, y_{2n}, \qquad \tilde{x}_{2n-1} = \frac{1}{\lambda + n}\, y_{2n-1}.$$

Hence,

$$\begin{aligned}
&\|\lambda \tilde{x}\| \\
&\leq |\lambda| \Bigg(\sum_{n=1}^{+\infty} \bigg\{ \frac{1}{|\lambda + n|^2}\, \frac{1}{n^2}\, (|y_{2n}|^2 + |y_{2n-1}|^2) + \frac{1}{|\lambda + n|^2}\, |y_{2n} - y_{2n-1}|^2 \bigg\} \Bigg)^{1/2} \\
&\leq \sup_{\mathrm{Re}(\lambda) \geq 0,\, n \in \mathbb{N}} \frac{|\lambda|}{|\lambda + n|}\, \|y\|.
\end{aligned}$$

Moreover, since the operator A is invertible, it generates a semigroup which is also exponentially stable.

In the following, let A be an invertible (i.e. $0 \in \rho(A)$) bisectorial operator on X, and let P be a closed projection on X commuting with A. Then $X_+ := \mathrm{im}(P)$ and $X_- := \ker(P)$ are closed subspace on X. Consider the *parts* A_+ and A_- of A on X_+ and X_-, respectively, i.e.

$$D(A_\pm) = \{x \in D(A) \cap X_\pm : Ax \in X_\pm\}, \qquad A_\pm x = Ax, \qquad x \in D(A_\pm).$$

Then it follows from Proposition 2.4 that A_+ and A_- are both bisectorial on X_+ and X_-, respectively.

Now, let $Z := X_+ \oplus X_-$ with norm $\|x_1 + x_2\|_Z := \|x_1\|_X + \|x_2\|_X$, where $x_1 \in X_+$ and $x_2 \in X_-$. Then Z is a Banach space such that

$$D(A) \hookrightarrow Z \hookrightarrow X, \tag{2.3}$$

if $D(A)$ is considered as a Banach space with respect to the graph norm $\|x\|_{D(A)} := \|Ax\|_X$ (recall that $0 \in \rho(A)$). Moreover, the projections

$$P_+ = P_{|Z} \quad \text{and} \quad P_- = (I - P_+)_{|Z}$$

are bounded as operators on Z.

Now we can define the *twisted operator* $\tilde{A}$.

Definition 2.9. Let $A : D(A) \subset X \to X$ be an invertible bisectorial operator and let P be a projection on X commuting with A. Define the operator $\tilde{A}$ on X by

$$D(\tilde{A}) := \{x \in Z : -P_+x + P_-x \in D(A)\}, \qquad \tilde{A}x := A(-P_+x + P_-x).$$

We say that $\tilde{A}$ is *the operator A twisted by P* or that $\tilde{A}$ is *the P-twisted version of A.*

The part $\tilde{A}_{|Z}$ of $\tilde{A}$ in Z is just the direct sum of $-A_+$ and A_-. Thus $\tilde{A}_{|Z}$ is a bisectorial operator on Z. For $\tilde{A}$ itself we can show the following.

Proposition 2.10. *In the setting of Definition 2.9, let $\lambda \in \tilde{\rho} := \rho(A) \cap \rho(-A)$. Then $\lambda \in \rho(\tilde{A})$ and*

$$R(\lambda, \tilde{A}) = P_+R(\lambda, -A) + P_-R(\lambda, A).$$

In particular, $i\mathbb{R} \subset \rho(\tilde{A})$. Moreover,

$$\sup_{s\in\mathbb{R}} \|R(is, \tilde{A})\|_{\mathcal{L}(X)} < +\infty. \tag{2.4}$$

Finally, $\sigma(\tilde{A}) = -\sigma(A_+) \cup \sigma(A_-)$.

Proof. Let $\lambda \in \tilde{\rho}$. Define

$$\tilde{R}(\lambda) := P_+R(\lambda, -A) + P_-R(\lambda, A).$$

Then

$$\begin{aligned} -P_+\tilde{R}(\lambda) + P_-\tilde{R}(\lambda) &= -P_+R(\lambda, -A) + P_-R(\lambda, A) \\ &= -P_+R(\lambda, -A) - P_+R(\lambda, A) + R(\lambda, A) \\ &= P_+(R(-\lambda, A) - R(\lambda, A)) + R(\lambda, A) \\ &= 2\lambda P_+R(-\lambda, A)R(\lambda, A) + R(\lambda, A) \\ &= 2\lambda R(-\lambda, A)P_+R(\lambda, A) + R(\lambda, A), \end{aligned} \tag{2.5}$$

and the operator in (2.6) maps X into $D(A)$. Thus, $\tilde{R}(\lambda)$ maps X into $D(\tilde{A})$ and

$$\begin{aligned} (\lambda - \tilde{A})\tilde{R}(\lambda) &= \lambda\tilde{R}(\lambda) - A(-P_+\tilde{R}(\lambda) + P_-\tilde{R}(\lambda)) \\ &= \lambda\tilde{R}(\lambda) + AP_+(R(\lambda, -A) + R(\lambda, A)) - AR(\lambda, A) \\ &= \lambda P_+(R(\lambda, -A) - R(\lambda, A)) + \lambda R(\lambda, A) \\ &\quad + AP_+(R(\lambda, -A) + R(\lambda, A)) - AR(\lambda, A) \\ &= P_+\{\lambda R(\lambda, -A) - \lambda R(\lambda, A) + AR(\lambda, -A) + AR(\lambda, A)\} + I = I. \end{aligned} \tag{2.6}$$

Now, let $y \in D(\tilde{A})$, i.e. $y \in Z$ and $-P_+y + P_-y \in D(A)$. Then

$$\begin{aligned} \tilde{R}(\lambda)\tilde{A}y &= (P_+R(\lambda, -A) + P_-R(\lambda, A))A(-P_+y + P_-y) \\ &= AR(\lambda, -A)(-P_+y) + AR(\lambda, A)P_-y \\ &= A(-P_+R(\lambda, -A)y + P_-R(\lambda, A)y) = \tilde{A}\tilde{R}(\lambda)y. \end{aligned}$$

This shows, by (2.6), that

$$\tilde{R}(\lambda)(\lambda - \tilde{A})y = (\lambda - \tilde{A})\tilde{R}(\lambda)y = y, \qquad y \in D(\tilde{A}),$$

and the first claim is proved.

It follows from [2, Proposition 3.10.3] that $\sigma(\tilde{A}) = \sigma(\tilde{A}_{|Z})$. But $\tilde{A}_{|Z}$ is the direct sum of $-A_+$ and A_-. Hence $\sigma(\tilde{A}_{|Z}) = -\sigma(A_+) \cup \sigma(A_-)$.

Finally, in order to prove (2.4), observe first that

$$isR(is, \tilde{A}_{|Z}) = \tilde{A}_{|Z}R(is, \tilde{A}_{|Z}) + I_{|Z}.$$

Thus, $\sup_{s\in\mathbb{R}} \|\tilde{A}_{|Z}R(is, \tilde{A}_{|Z})\|_{\mathcal{L}(Z)} < +\infty$, since $\tilde{A}_{|Z}$ is bisectorial. Now consider $D(\tilde{A})$ with the graph norm $\|x\|_{D(\tilde{A})} := \|\tilde{A}x\|_X$. Then $D(\tilde{A})$ is a Banach space and the embeddings

$$D(\tilde{A}) \hookrightarrow Z \hookrightarrow X \tag{2.7}$$

are continuous. This follows immediately from the closed graph theorem. Thus, with appropriate constants $C_1, C_2, C_3 > 0$ we have, for any $x \in X$ and any $s \in \mathbb{R}$,

$$\begin{aligned}\|R(is, \tilde{A})x\|_X &\le C_1\|R(is, \tilde{A})x\|_Z = C_1\|\tilde{A}R(is, \tilde{A})\tilde{A}^{-1}x\|_Z \\ &\le C_1 \sup_{s\in\mathbb{R}} \|\tilde{A}R(is, \tilde{A})\|_{\mathcal{L}(Z)}\|\tilde{A}^{-1}x\|_Z \le C_2\|\tilde{A}^{-1}x\|_Z \\ &\le C_3\|\tilde{A}^{-1}x\|_{D(\tilde{A})} = C_3\|x\|_X. \ \blacksquare\end{aligned}$$

The results in Proposition 2.10 and, in particular, estimate (2.4), do not allow us to conclude that $\tilde{A}$ is bisectorial in general. In fact, this may not true, and the following example shows that estimate (2.4) cannot be essentially improved.

Example 2.11. Consider the operator A and the projection P defined in Example 2.6. We have shown that P commutes with A, so that we can define $\tilde{A}$, the P-twisted version of A. By the definition,

$$D(\tilde{A}) = \Big\{x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{+\infty}\{|x_{2n}|^2 + |x_{2n-1}|^2 + n^2|x_{2n} + x_{2n-1}|^2\} < +\infty\Big\},$$

and

$$(\tilde{A}x)_{2n} = nx_{2n}, \qquad (\tilde{A}x)_{2n-1} = -nx_{2n-1}.$$

It is not difficult to see that $\sigma(\tilde{A}) = \mathbb{N} \cup (-\mathbb{N})$, and that

$$(R(\lambda, \tilde{A})y)_{2n} = \frac{1}{\lambda - n}y_{2n}, \qquad (R(\lambda, \tilde{A})y)_{2n-1} = \frac{1}{\lambda + n}y_{2n-1},$$

for $\lambda \notin \mathbb{N} \cup (-\mathbb{N})$. Now, let $e_k = (0, \dots, 0, 1, 0, \dots)$ be the kth unit vector and $v_k = \frac{k}{\sqrt{2}}(e_{2k} + e_{2k-1})$. Then $\|v_k\| = 1$. Let

$$u_k := R(ik, \tilde{A})v_k = \frac{k}{\sqrt{2}}\left(\frac{1}{ik-k}\, e_{2k} + \frac{1}{ik+k}\, e_{2k-1}\right)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{i-1}\, e_{2k} + \frac{1}{i+1}\, e_{2k-1}\right).$$

Then

$$\|u_k\|^2 \geq \frac{1}{2}\left|\frac{1}{i-1} - \frac{1}{i+1}\right|^2 = \frac{1}{2}.$$

Hence $\|R(ik, \tilde{A})\|_{\mathcal{L}(X)} \geq 1/\sqrt{2}$ for all $k \in \mathbb{N}$, and

$$\sup_{s\in\mathbb{R}} \|sR(is, \tilde{A})\|_{\mathcal{L}(X)} = +\infty.$$

Thus $\tilde{A}$ is not bisectorial.

3. Twisting by the positive spectral projection. The simplest way to obtain a bisectorial operator is the following. Assume that $X = X_+ \oplus X_-$ is the direct sum of two closed subspaces. Let $-A_+$ and A_- be invertible generators of bounded and holomorphic semigroups on X_+ and X_-, respectively, and let $A := A_+ \oplus A_-$. Then A is bisectorial and, moreover, A_+ and A_- are the parts of A in X_+ and X_-, respectively. We want to give this simple situation a name.

Definition 3.1. A bisectorial operator A on X is called *decomposable* if X is the direct sum $X = X_+ \oplus X_-$ of closed subspaces such that $R(is, A)X_+ \subset X_+$ and $R(is, A)X_- \subset X_-$ for all $s \in \mathbb{R} \setminus \{0\}$ and

$$\sigma(A_+) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0\}, \qquad \sigma(A_-) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0\},$$

where A_+ is the part of A in X_+ and A_- is the part of A in X_-.

It is not difficult to see the following (see e.g. the appendix of [8]).

Proposition 3.2. *Let A be the generator of a holomorphic semigroup such that $i\mathbb{R} \subset \rho(A)$. Then A is bisectorial and decomposable.*

Even on Hilbert spaces there exist indecomposable invertible bisectorial operators. This was shown by McIntosh and Yagi (see [9, Theorem 3]).

Theorem 3.3 (McIntosh, Yagi). *Let X be a separable Hilbert space. Then there exists an invertible bisectorial operator A which is not decomposable.*

Our aim is to prove the following.

Theorem 3.4. *Let $A : D(A) \subset X \to X$ be an invertible bisectorial operator. Then there exists a (possibly unbounded) projection P_+, commuting* *with A, such that the operator $\tilde{A}$ obtained by twisting A by P_+ generates a bounded holomorphic semigroup.*

Theorem 3.4, whose proof will be given at the end of this section, says that if we allow a more general notion of direct sum, defined by a (possibly) unbounded projection, any invertible bisectorial operator can be obtained by an unbounded decomposition. In fact, we have $A = \tilde{\tilde{A}}$ and so A is the twisted version of $\tilde{A}$, which is the generator of a bounded holomorphic semigroup.

In order to prove Theorem 3.4, we start by defining the projection P_+ which will fulfill the requirement.

Let A be an invertible bisectorial operator and let $0 < \omega < \pi/2$ be such that $\Sigma'_\omega \subset \rho(A)$ and $\sup_{\lambda\in\Sigma'_\omega} \|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} < +\infty$ (see (2.1) and (2.2)). Let $\varepsilon > 0$ be such that $\{z \in \mathbb{C} : |z| \le \varepsilon\} \subset \rho(A)$. For $\omega < \theta < \pi/2$ we consider the contour $\Gamma^+_{\theta,\varepsilon}$ which consists of the line $\{re^{-i\theta} : r > \varepsilon\}$, the arc $\{\varepsilon e^{i\alpha} : -\theta \le \alpha \le \theta\}$, and the line $\{re^{i\theta} : r > \varepsilon\}$ oriented downwards. Let

$$Q_+ := \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda}\, d\lambda.$$

Then $Q_+ \in \mathcal{L}(X)$ and, by Cauchy's Theorem, it does not depend on the choice of θ and $\varepsilon > 0$ satisfying the requirement above.

Proposition 3.5. *Let $P_+ := AQ_+$ with domain $D(P_+) := \{x \in X : Q_+x \in D(A)\}$. Then P_+ is a closed projection commuting with A.*

Proof. Let $\omega < \theta' < \theta < \pi/2$, and $0 < \varepsilon < \varepsilon'$ be such that $\{z \in \mathbb{C} : |z| \le \varepsilon'\} \subset \rho(A)$. Then, using Cauchy's Theorem and the resolvent identity, we get

$$\begin{aligned}(Q_+)^2 &= \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda} \bigg(\frac{1}{2\pi i} \int_{\Gamma^+_{\theta',\varepsilon'}} \frac{1}{\lambda'(\lambda' - \lambda)}\, d\lambda'\bigg) d\lambda \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma^+_{\theta',\varepsilon'}} \frac{R(\lambda', A)}{\lambda'} \bigg(\frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{1}{\lambda(\lambda' - \lambda)}\, d\lambda\bigg) d\lambda' \\ &= \frac{1}{2\pi i} \int_{\Gamma^+_{\theta',\varepsilon'}} \frac{R(\lambda', A)}{(\lambda')^2}\, d\lambda' = \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda^2}\, d\lambda.\end{aligned}$$

Hence $Q_+ \in \mathcal{L}(X; D(A))$ and

$$A(Q_+)^2 = \frac{1}{2\pi i} \int_{\Gamma^+_{\varepsilon,\theta}} \frac{\lambda R(\lambda, A) - I}{\lambda^2}\, d\lambda = Q_+.$$

Now, let $x \in D(P_+)$, i.e. $Q_+x \in D(A)$. Then

$$Q_+P_+x = Q_+AQ_+x = A(Q_+)^2x = Q_+x.$$

Hence $P_+x \in D(P_+)$ and

$$(P_+)^2x = AQ_+P_+x = AQ_+x = P_+x.$$

Now, let $x \in D(A)$ be such that $Ax \in D(P_+)$, i.e. $Q_+Ax \in D(A)$. Then

$$Q_+Ax = \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda} Ax\, d\lambda = \frac{1}{2\pi i} A \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda} x\, d\lambda = AQ_+x.$$

Therefore $P_+x = AQ_+x = Q_+Ax \in D(A)$ and, by direct computation, it follows that $AP_+x = P_+Ax$ and so, by Proposition 2.4, P_+ commutes with A. ∎

Now, let $X_+ := \mathrm{im}(P)$. Then the following holds.

Proposition 3.6. *Let A_+ be the part of A in X_+. Then $\sigma(A_+) \subset \{\mu \in \mathbb{C} : \mathrm{Re}(\mu) > 0\}$ and*

$$R(\mu, A_+) = \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\mu - \lambda} d\lambda, \qquad \mathrm{Re}(\mu) < 0. \tag{3.1}$$

Moreover, there exists $M_+ > 0$ such that

$$\|R(\mu, A_+)\|_{\mathcal{L}(X_+)} \le M_+, \qquad \mathrm{Re}(\mu) < 0. \tag{3.2}$$

Proof. It follows from Proposition 2.4 that $\rho(A) \subset \rho(A_+)$ and $R(\mu, A)_{|X_+} = R(\mu, A_+)$ for all $\mu \in \rho(A)$. Now, let $\mathrm{Re}(\mu) < 0$ and, in order to show that $\mu \in \rho(A_+)$, define

$$R_+ := \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\mu - \lambda} d\lambda.$$

Then $R_+ \in \mathcal{L}(X)$ and, adapting the computations in the proof of Proposition 3.5, it follows that $R_+Q_+ \in \mathcal{L}(X; D(A))$ and that

$$(\mu - A)R_+Q_+ = Q_+.$$

Observe that, if $x \in D(A)$, then $R_+x \in D(A)$ and $AR_+x = R_+Ax$. It follows that $(\mu - A)R_+P_+x = P_+x$ for all $x \in D(P_+)$ and $R_+(\mu - A)x = x$ for all $x \in D(A) \cap X_+$. This shows that $\mu \in \rho(A_+)$ and $R(\mu, A_+) = R_+$.

Finally, formula (3.1) and the bisectoriality of A yield the existence of a constant $C > 0$ such that

$$\|R(\mu, A_+)\| \le \int_{\Gamma^+_{\theta,\varepsilon}} \frac{\|\lambda R(\lambda, A)\|}{|\lambda|\,|\mu - \lambda|} |d\lambda| \le C \int_{\Gamma^+_{\theta,\varepsilon}} \frac{|d\lambda|}{|\lambda|^2} =: M_+,$$

and conclude the proof. ∎

Since the spectrum of A_+ is included in the right half-plane, we call P_+ the *positive spectral projection* associated with A.

Similarly, we let $\Gamma^-_{\theta,\varepsilon} := -\Gamma^+_{\theta,\varepsilon}$ be oriented upwards,

$$Q_- := \frac{1}{2\pi i} \int_{\Gamma^-_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda}\, d\lambda,$$

and $P_- := AQ_-$. Observing that, for any integrable $f : \Gamma^+_{\theta,\varepsilon} \to X$, we have

$$-\int_{\Gamma^-_{\theta,\varepsilon}} f(-\lambda)\, d\lambda = \int_{\Gamma^+_{\theta,\varepsilon}} f(\lambda)\, d\lambda,$$

and taking into account that $R(\lambda, -A) = -R(-\lambda, A)$ for any $\lambda \in \mathbb{C}$, it follows immediately that P_- is the positive spectral projection associated with $-A$.

Then it follows from the residue theorem that $Q_+ + Q_- = A^{-1}$, and so $D(P_+) = D(P_-)$ and $P_+ = I - P_-$ in the domain of the projections.

Defining $X_- := \ker(P_+)$, and letting A_- be the part of A in X_-, we deduce from Proposition 3.6 that $\sigma(A_-)$ is in the left half-plane, and that there exists $M_- > 0$ such that

$$\|R(\mu, A_-)\|_{\mathcal{L}(X_-)} \le M_-, \qquad \mathrm{Re}(\mu) > 0. \tag{3.3}$$

Next we take advantage of the following theorem.

THEOREM 3.7 (Phragmén–Lindelöf, [5, Corollary 6.4.4]). *Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, and let $h : \overline{\mathbb{C}_+} \to X$ be continuous and holomorphic on $\mathbb{C}_+$. Assume that, for each $\delta > 0$, there exists $C > 0$ such that*

$$\|h(z)\| \le Ce^{\delta|z|}, \qquad z \in \mathbb{C}_+.$$

Moreover, assume that

$$\sup_{s\in\mathbb{R}} \|h(is)\| < +\infty.$$

Then there exists $M > 0$ such that $\|h(z)\| \le M$ for all $z \in \mathbb{C}_+$.

Define, for any $\mu \in \overline{\mathbb{C}_+}$,

$$h^-(\mu) := \mu R(\mu, A_-).$$

Then estimate (3.3), Theorem 3.7, and the bisectoriality of A_- (see Section 2) imply that there exists $\overline{M}_- > 0$ such that

$$\|\mu R(\mu, A_-)\|_{\mathcal{L}(X_-)} \le \overline{M}_-, \qquad \mathrm{Re}(\mu) > 0. \tag{3.4}$$

Moreover, Theorem 3.7 still holds, of course, if we replace everywhere $\mathbb{C}_+$ with $\mathbb{C}_- := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. Then, defining for any $\mu \in \overline{\mathbb{C}_-}$,

$$h^+(\mu) := \mu R(\mu, A_+),$$

and using also estimate (3.2) and the bisectoriality of A_+, we find that there exists $\overline{M}_+ > 0$ such that

$$\|\mu R(\mu, A_+)\|_{\mathcal{L}(X_+)} \le \overline{M}_+, \qquad \mathrm{Re}(\mu) < 0. \tag{3.5}$$

Now consider the operator $\tilde{A}$ obtained by twisting A by P_+. Then, by Proposition 2.10, one has

$$\sigma(\tilde{A}) = -\sigma(A_+) \cup \sigma(A_-) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}.$$

Furthermore, (3.4) and (3.5) imply that there exists $\overline{M} > 0$ such that

$$\|\mu R(\mu, \tilde{A}_{|Z})\|_{\mathcal{L}(Z)} \le \overline{M}$$

for any $\mu \in \mathbb{C}_+$. This implies that

$$\|R(\mu, \tilde{A})\|_{\mathcal{L}(X)} \le M_1, \qquad \mu \in \mathbb{C}_+,$$

for some $M_1 > 0$. Indeed, taking embeddings (2.7) into account, and recalling that $0 \in \rho(\tilde{A})$, we have with appropriate constants $C_1, C_2 > 0$, for any $x \in X$ and any $\mu \in \mathbb{C}^+$,

$$\begin{aligned}(3.6)\qquad \|R(\mu, \tilde{A})x\|_X &\le C_1 \|R(\mu, \tilde{A})x\|_Z = C_1 \|\tilde{A} R(\mu, \tilde{A}) \tilde{A}^{-1} x\|_Z \\ &= C_1 \|\mu R(\mu, \tilde{A}) \tilde{A}^{-1} x - \tilde{A}^{-1} x\|_Z \le C_1(\overline{M}+1)\|\tilde{A}^{-1}x\|_Z \\ &\le C_1 C_2 (\overline{M}+1) \|\tilde{A}^{-1} x\|_{D(\tilde{A})} = C_1 C_2 (\overline{M}+1)\|x\|_X.\end{aligned}$$

In particular, $\|R(is, \tilde{A})\|_{\mathcal{L}(X)} \le M_1$ for all $s \in \mathbb{R}$ (as we already proved in Proposition 2.10). In the context here, where the positive spectral projection is used, we can improve the estimate.

PROPOSITION 3.8. *Let $\tilde{A}$ be the P_+-twisted version of an invertible bisectorial operator A, where P_+ is the positive spectral projection defined in Proposition* 3.5. *Then*

$$(3.7)\qquad \sup_{s \in \mathbb{R}} \|s R(is, \tilde{A})\|_{\mathcal{L}(X)} < +\infty.$$

Proof. For $\mu \in \rho(A) \cap \rho(-A)$ we have

$$\begin{aligned}R(\mu, \tilde{A}) &= P_+ R(\mu, -A) + P_- R(\mu, A) = P_+(R(\mu, -A) - R(\mu, A)) + R(\mu, A) \\ &= S(\mu) + R(\mu, A),\end{aligned}$$

where $S(\mu) := P_+(R(\mu, -A) - R(\mu, A))$. Since A is bisectorial we only have to show the assertion for $S(\mu)$. For this purpose, let μ be to the left of $\Gamma^+_{\theta,\varepsilon}$. Then, by Cauchy's Theorem,

$$\begin{aligned}Q_+ R(\mu, A) &= \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A) R(\mu, A)}{\lambda}\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A) - R(\mu, A)}{\lambda(\mu - \lambda)}\, d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda(\mu - \lambda)}\, d\lambda.\end{aligned}$$

Since $P_+ R(\mu, A) = A Q_+ R(\mu, A)$ and $AR(\lambda, A) = \lambda R(\lambda, A) - I$ for any

$\lambda \in \rho(A)$, we have

$$\begin{aligned}P_+R(\mu, A) &= \frac{1}{2\pi i} \int\limits_{\Gamma^+_{\theta,\varepsilon}} \frac{AR(\lambda, A)}{\lambda(\mu - \lambda)}\, d\lambda \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\mu - \lambda}\, d\lambda - \frac{1}{2\pi i} \int\limits_{\Gamma^+_{\theta,\varepsilon}} \frac{d\lambda}{\lambda(\mu - \lambda)} \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\mu - \lambda}\, d\lambda.\end{aligned}$$

Now let μ be to the left of the curve Γ' consisting of the lines $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ oriented downwards. Then, by Cauchy's Theorem, we have

$$P_+R(\mu, A) = \frac{1}{2\pi i} \int\limits_{\Gamma'} \frac{R(\lambda, A)}{\mu - \lambda}\, d\lambda,$$

and, if $-\mu$ is to the left of Γ', then

$$\begin{aligned}P_+R(\mu, -A) = -P_+R(-\mu, A) &= -\frac{1}{2\pi i} \int\limits_{\Gamma'} \frac{R(\lambda, A)}{-\mu - \lambda}\, d\lambda \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma'} \frac{R(\lambda, A)}{\lambda + \mu}\, d\lambda.\end{aligned}$$

In particular, for $\mu = is$ $(s \neq 0)$, we have

$$\begin{aligned}S(is) &= P_+(R(is, -A) - R(is, A)) \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma'} \left(\frac{1}{is + \lambda} - \frac{1}{is - \lambda}\right) R(\lambda, A)\, d\lambda \\ &= -\frac{1}{i\pi} \int\limits_{\Gamma'} \frac{\lambda}{s^2 + \lambda^2} R(\lambda, A)\, d\lambda.\end{aligned}$$

Observe that

$$|a + be^{2i\theta}| \geq \sqrt{\frac{1 + \cos(2\theta)}{2}}\,(a + b), \qquad a, b \geq 0.$$

Therefore, taking (2.2) into account, there exist $M, C > 0$ such that

$$\begin{aligned}\|sS(is)\| &\leq \left\| \frac{1}{\pi i} \int\limits_{\Gamma'} \frac{\lambda s}{s^2 + \lambda^2} R(\lambda, A)\, d\lambda \right\| \leq \frac{2}{\pi} \int\limits_0^{+\infty} \frac{C|s|}{|s^2 + r^2 e^{2i\theta}|}\, dr \\ &\leq \frac{2C}{\pi} \sqrt{\frac{2}{1 + \cos(2\theta)}} \int\limits_0^{+\infty} \frac{|s|}{s^2 + r^2}\, dr = M \sqrt{\frac{2}{1 + \cos(2\theta)}}. \ \blacksquare\end{aligned}$$

Now, basing on the estimates (3.6) and (3.7) we can prove Theorem 3.4 with the help of the Phragmén–Lindelöf theorem.

Proof of Theorem 3.4. Let $h(\mu) := \mu R(\mu, \tilde{A})$. Then h is holomorphic and continuous on $\overline{\mathbb{C}_+}$. Moreover, h is bounded on $i\mathbb{R}$ by Proposition 3.8. The estimate (3.6) shows that h is of subexponential growth on $\mathbb{C}_+$. Thus Theorem 3.7 implies that h is bounded on $\mathbb{C}_+$. This implies that $\tilde{A}$ generates a bounded holomorphic C_0-semigroup on X. ∎

4. The semigroups associated with a bisectorial operator. Let A be an invertible, bisectorial operator on X. We consider the operators $Q_\pm$ defined in the previous section, and the spectral projections $P_\pm = AQ_\pm$. Let $\tilde{A}$ be the operator A twisted by P_+, and let $(\tilde{T}(t))$ be the holomorphic semigroup generated by $\tilde{A}$.

PROPOSITION 4.1. *Define, for any $t > 0$,*

$$T^+(t) := P_+\tilde{T}(t), \qquad T^-(t) := P_-\tilde{T}(t).$$

Then $T^\pm(t) \in \mathcal{L}(X)$ for all $t > 0$, and

$$T^\pm(t+s) = T^\pm(t)T^\pm(s), \qquad t, s > 0.$$

Moreover, $T^+(t)T^-(s) = T^-(t)T^+(s) = 0$ for all $t, s > 0$.

Proof. Since $\tilde{T}(t)X \subset D(\tilde{A}) \subset Z$, the operators $T^+(t)$ and $T^-(t)$ are bounded. Since Q_+ and Q_- commute with the resolvent of A, they also commute with $R(\mu, \tilde{A}) = P_+R(\mu, -A) + P_+R(\mu, A)$ (see Proposition 2.10). Consequently, Q_+ and Q_- also commute with $\tilde{T}(t)$. Hence also P_+ and P_- commute with $\tilde{T}$. This implies the semigroup property. Since

$$P_-x = x - P_+x, \qquad x \in D(P_+) = D(P_-),$$

we have $P_+P_-x = P_-P_+x = 0$. This implies $T^+(t)T^-(s) = T^-(s)T^+(t) = 0$. ∎

It follows from the definition that

$$\tilde{T}(t) = T^+(t) + T^-(t), \qquad t > 0.$$

Moreover, $T^\pm \in C^\infty((0, +\infty), X)$ and

$$\frac{d}{dt}T^\pm(t) = \mp AT^\pm(t), \qquad t > 0.$$

It follows that, for $x \in Z$,

$$\mp A\int_0^t T^\pm(s)x\,ds = T^\pm(t)x - x. \tag{4.1}$$

It is possible to express the semigroups $T^\pm$ directly by a contour integral, without using $\tilde{T}$. Let $\omega < \theta < \pi/2$ as in Section 3.

Proposition 4.2. *One has, for $t > 0$,*

$$T^+(t) = \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} e^{-\lambda t} R(\lambda, A)\, d\lambda, \tag{4.2}$$

$$T^-(t) = \frac{1}{2\pi i} \int_{\Gamma^-_{\theta,\varepsilon}} e^{\lambda t} R(\lambda, A)\, d\lambda. \tag{4.3}$$

Proof. For $t > 0$ let

$$S(t) := \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} e^{-\lambda t} R(\lambda, A)\, d\lambda \in \mathcal{L}(X).$$

If $x \in X_-$, then $R(\lambda, A)x$ has a holomorphic extension to $\mathbb{C}_+$, as a consequence of Proposition 3.6. Hence $S(t)x = 0$ by Cauchy's Theorem.

Now let $x \in X_+$. Then replacing λ by $-\lambda$ we have

$$\begin{aligned} S(t)x &= -\frac{1}{2\pi i} \int_{\Gamma^-_{\theta,\varepsilon}} e^{\lambda t} R(-\lambda, A)x\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma^-_{\theta,\varepsilon}} e^{\lambda t} R(\lambda, -A)x\, d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma^-_{\theta,\varepsilon}} e^{\lambda t} R(\lambda, \tilde{A})x\, d\lambda = \tilde{T}(t)x = T^+(t)x \end{aligned}$$

by the usual exponential formula for the holomorphic semigroup $\tilde{T}$. Hence $S(t)x = T^+(t)x$ for all $x \in Z$.

Now let $\lambda \in \rho(A)$. Since $R(\lambda, A)$ is injective from X into $D(A)$, and $R(\lambda, A)$ commutes with $T^+(t)$ and $S(t)$ for any $t > 0$, the statement follows by taking embeddings (2.3) into account. Indeed, for any $x \in X$ we have

$$S(t)x = (\lambda I - A)S(t)R(\lambda, A)x = (\lambda I - A)T^+(t)R(\lambda, A)x = T^+(t)x. \ \blacksquare$$

Even though $\tilde{T}(t) = T^+(t) + T^-(t)$ converges strongly to the identity as $t \to 0$ if A is densely defined, the norms of $T^+(t)$ and $T^-(t)$ blow up as t approaches to 0 whenever $P_\pm$ is unbounded. More precisely, the following holds.

Proposition 4.3. *Let $x \in X$ and let A be densely defined. Then the limit $\lim_{t\to 0} T^+(t)x$ exists if and only if $x \in D(P_+)$ and, in this case, $\lim_{t\to 0} T^+(t)x = P_+x$. If P_+ is unbounded, then $\lim_{t\to 0} \|T^+(t)\|_{\mathcal{L}(X)} = +\infty$.*

Proof. If $x \in D(P_+)$, then $\lim_{t\to 0} T^+(t)x = \lim_{t\to 0} \tilde{T}(t)P_+x = P_+x$. Conversely, assume that $\lim_{t\to 0} T^+(t)x = y$. Since $\lim_{t\to 0} \tilde{T}(t)x = x$ and P_+ is closed, it follows that $x \in D(P_+)$ and $P_+x = \lim_{t\to 0} P_+\tilde{T}(t)x = \lim_{t\to 0} T^+(t)x = y$.

Now, assume that there exists $t_n \to 0$ such that $\|T^+(t_n)\|_{\mathcal{L}(X)} \le C$. Then for $x \in D(P_+)$ one has $\|P_+x\| = \lim_{n\to +\infty} \|T^+(t_n)x\| \le C\|x\|$. Since $D(P_+)$ is dense, it follows that P_+ is bounded. $\blacksquare$

However, the following result (see [11, Lemma 1.2.3]) shows that the singularity of $T^{\pm}$ at 0 is mild.

PROPOSITION 4.4 (Schweiker). *There exists a constant $C > 0$ such that*

$$\|T^{\pm}(t)\|_{\mathcal{L}(X)} \le C|\log t|, \qquad 0 < t \le 1/2.$$

Proof. Since $0 \in \rho(A)$, there exists a constant $M_1 > 0$ such that

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{M_1}{1 + |\lambda|}$$

for all $\lambda = re^{\pm i\theta}$, $r \ge 0$. Let Γ' consist of the two rays $\{re^{\pm i\theta} : r \ge 0\}$, where θ is chosen as in Section 3, directed downwards. Then, by Cauchy's Theorem,

$$T^{+}(t) = \frac{1}{2\pi i} \int_{\Gamma'} e^{-\lambda t} R(\lambda, A)\, d\lambda.$$

Hence there exists a positive constant M_2 such that, for $0 < t \le 1/2$,

$$\begin{aligned}\|T^{+}(t)\|_{\mathcal{L}(X)} &\le \frac{1}{2\pi}\, 2M_1 \int_0^{+\infty} e^{-rt\cos(\theta)} \frac{1}{1+r}\, dr = \frac{M_1}{\pi} \int_1^{+\infty} e^{-(s-1)t\cos(\theta)} \frac{ds}{s} \\ &= \frac{M_1}{\pi}\, e^{t\cos(\theta)} \int_1^{\infty} e^{-st\cos(\theta)} \frac{ds}{s} \le \frac{M_1}{\pi}\, e^{\cos(\theta)/2} \int_t^{+\infty} e^{-r\cos(\theta)} \frac{dr}{r} \\ &\le \frac{M_1}{\pi}\, e^{\cos(\theta)/2} \Big(\int_1^{+\infty} e^{-r\cos(\theta)} \frac{dr}{r} + \int_t^1 \frac{dr}{r} \Big) \le M_1(M_2 - \log t). \ \blacksquare\end{aligned}$$

It is possible to define the semigroups T^{+} and T^{-} directly by the contour integrals (4.2) and (4.3), without using $\tilde{A}$. This is what Schweiker has done in [11], where also the semigroup properties of T^{+} and T^{-} are proved directly from these formulas and in particular the surprising estimate at the origin (cf. Proposition 4.4). Moreover, Schweiker proved in [11, Lemma 1.2.3] that, if $0 < \omega < \overline{\omega} := \inf\{|\mathrm{Re}(\lambda)| : \lambda \in \sigma(A)\}$, then there exist $M_0^{\pm} > 0$ such that

$$\|T^{\pm}(t)\|_{\mathcal{L}(X)} \le M_0^{\pm} e^{-\omega t}, \qquad t \ge 1. \tag{4.4}$$

5. Squares and roots. In this section we investigate the square of a bisectorial operator. We use the following notion (cf. [7, Section 2.1]). An operator B on X is called *sectorial* if $(-\infty, 0) \subset \rho(A)$ and

$$\|s(s + B)^{-1}\|_{\mathcal{L}(X)} \le M, \qquad s > 0,$$

for some $M > 0$. We denote by

$$\varphi_{\mathrm{sec}}(B) := \inf\{\theta \in (0, \pi] : \sigma(B) \subset \Sigma_\theta,\ \sup_{\lambda \notin \Sigma_\theta} \|\lambda R(\lambda, B)\|_{\mathcal{L}(X)} < +\infty\}$$

the *sectorial angle* of B. Then $0 \le \varphi_{\mathrm{sec}}(B) < \pi$ (by a geometric series argument, see [7, Section 2.1] or [2, Corollary 3.7.12]).

Thus, the operator $-B$ generates a bounded holomorphic semigroup if and only if B is sectorial and $\varphi_{\mathrm{sec}}(B) < \pi/2$. We also recall that for each sectorial operator B there exists a unique sectorial operator $B^{1/2}$ such that $(B^{1/2})^2 = B$. Moreover, $\varphi_{\mathrm{sec}}(B^{1/2}) = \varphi_{\mathrm{sec}}(B)^{1/2}$ (see [7, Proposition 3.1.2]).

PROPOSITION 5.1. *Let A be an operator.*

(i) *If A is bisectorial, then A^2 is sectorial.*
(ii) *If A^2 is sectorial, then $i\mathbb{R} \setminus \{0\} \subset \rho(A)$ and, if A is also invertible, then*

$$\|R(is, A)\|_{\mathcal{L}(X)} \leq M$$

for all $s \in \mathbb{R}$ and some $M > 0$.

Proof. For $s \in \mathbb{R} \setminus \{0\}$ we have

$$s^2 + A^2 = (A - is)(A + is). \tag{5.1}$$

(i) Assume that the operator A is bisectorial. Then it follows from (5.1) that $(s^2 + A^2)^{-1} = R(is, A)R(-is, A)$ and

$$\|s^2(s^2 + A^2)^{-1}\|_{\mathcal{L}(X)} \leq (\sup_{s\neq 0} \|sR(is, A)\|_{\mathcal{L}(X)})^2 < +\infty.$$

(ii) Assume that A^2 is sectorial and let $s \neq 0$. Then it follows from (5.1) that $(A - is)^{-1} = (A + is)(s^2 + A^2)^{-1}$. Assume, in addition, that A is invertible and let $M := \sup_{s\in\mathbb{R}} \|s^2(s^2 + A^2)^{-1}\|_{\mathcal{L}(X)}$. Then

$$\|A^2(s^2 + A^2)^{-1}\|_{\mathcal{L}(X)} = \|I - s^2(s^2 + A^2)^{-1}\|_{\mathcal{L}(X)} \leq M + 1.$$

Hence

$$\|A(s^2 + A^2)^{-1}\|_{\mathcal{L}(X)} = \|A^{-1}A^2(s^2 + A^2)^{-1}\|_{\mathcal{L}(X)} \leq \|A^{-1}\|_{\mathcal{L}(X)}(M + 1).$$

Thus $(A + is)^{-1} = A(s^2 + A^2)^{-1} + is(s^2 + A^2)^{-1}$ is bounded. ∎

However, if A^2 is sectorial, it does not follow that A is bisectorial. In fact, it may happen that $\|R(is, A)\|_{\mathcal{L}(X)} \geq C > 0$ for some constant C and all $s \in \mathbb{R}$.

EXAMPLE 5.2. Consider the operator $B = \tilde{A}$ from Example 2.11. Then $i\mathbb{R} \subset \rho(B)$, $\sup_{s\in\mathbb{R}} \|R(is, B)\|_{\mathcal{L}(X)} < +\infty$, but

$$\|R(ik, B)\|_{\mathcal{L}(X)} \geq 1/\sqrt{2}$$

for all $k \in \mathbb{N}$. Thus B is not bisectorial. An estimate similar to the one given in Example 2.8 shows that B^2 is sectorial.

In Proposition 5.1(i) we have shown that, if A is a bisectorial operator, then A^2 is sectorial so that is possible to define its square root. In the following proposition we show that if A is also invertible, the square root of A^2 is the negative of the operator A twisted by its positive spectral projection.

THEOREM 5.3. *For any invertible bisectorial operator A, the square root of A^2 is the negative of the P_+-twisted version of A:*

$$-\tilde{A} = (A^2)^{1/2}.$$

Proof. Let $Q_\pm$ and $P_\pm$ be defined as in Section 3. A change of variable and the resolvent identity show that

$$\begin{aligned}
Q_+ - Q_- &= \frac{1}{2\pi i}\Big\{ \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda}\, d\lambda - \int_{\Gamma^-_{\theta,\varepsilon}} \frac{R(\lambda, A)}{\lambda}\, d\lambda \Big\} \\
&= \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} \frac{R(\lambda, A) - R(-\lambda, A)}{\lambda}\, d\lambda \\
&= -2\Big(\frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} R(\lambda, A) R(-\lambda, A)\, d\lambda \Big) \\
&= 2\Big(\frac{1}{2\pi i} \int_{\Gamma^+_{\theta,\varepsilon}} R(\lambda^2, A^2)\, d\lambda \Big) = \frac{1}{2\pi i} \int_{(\Gamma^+_{\theta,\varepsilon})^2} R(w, A^2) w^{-1/2}\, dw \\
&= (A^2)^{-1/2},
\end{aligned}$$

where $(\Gamma^+_{\theta,\varepsilon})^2 = \{z^2 : z \in \Gamma^+_{\theta,\varepsilon}\}$, and the last identity is the well-known formula for the square root [2, (3.51), p. 166]. It follows from Proposition 2.10 that $\tilde{A}^{-1} = -P_+ A^{-1} + P_- A^{-1}$. Since P_+ and P_- commute with A, we have

$$\tilde{A}^{-1} = -P_+ A^{-1} + P_- A^{-1} = -(Q_+ - Q_-) = -(A^2)^{-1/2}.$$

Hence $\tilde{A} = -(A^2)^{1/2}$. ∎

6. Mild solutions. Let A be a closed, linear operator on X. Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$, in this section we study uniqueness of the solution for the problem

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}, \tag{6.1}$$

and we give a representation formula for this solution in terms of the semigroups $(T^\pm(t))$ associated with A.

We say that a continuous function $u : \mathbb{R} \to X$ is a *mild solution* of (6.1) if $\int_0^t u(s)\, ds \in D(A)$ and

$$u(t) = u(0) + A \int_0^t u(s)\, ds + \int_0^t f(s)\, ds, \qquad t \in \mathbb{R}.$$

In order to prove uniqueness of the solution of (6.1), we need a spectral condition on A and a growth condition on u.

DEFINITION 6.1. Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$. We say that g is *polynomially bounded* if

$$\|g(t)\| \le \alpha(1+|t|)^k, \qquad t \in \mathbb{R},$$

for some $k \in \mathbb{N}$ and some $\alpha > 0$. The function g is called *weakly polynomially bounded* if

$$\int_{\mathbb{R}} \|g(t)\|(1+|t|)^{-k}\, dt < +\infty$$

for some $k \in \mathbb{N}$.

The notion of weak polynomial boundedness is clearly weaker than that of polynomial boundedness. Note that g is weakly polynomially bounded whenever $g \in L^p(\mathbb{R}; X)$ for some $1 \le p \le \infty$. Now we can prove the following.

PROPOSITION 6.2. *Let A be a densely defined, closed and linear operator on X such that $i\mathbb{R} \subset \rho(A)$. Then there exists at most one weakly polynomially bounded mild solution u of* (6.1).

Proof. Let u be a weakly polynomially bounded mild solution of (6.1) for $f = 0$. Then the Carleman spectrum of u, as defined in [2, Section 4.6], is empty. This is proved as the last six lines of the proof of [4, Theorem 2.7]. It follows from [2, Theorem 4.8.2] that $u(t) = 0$ for all $t \in \mathbb{R}$. ∎

REMARK 6.3. Conversely, Schweiker [12, Theorem 1.1] showed the following. If, for each $f \in \mathrm{BUC}(\mathbb{R}; X)$, there is a unique mild solution $u \in \mathrm{BUC}(\mathbb{R}; X)$ of (6.1), then $i\mathbb{R} \subset \rho(A)$ and $\sup_{s\in\mathbb{R}} \|R(is, A)\| < +\infty$. She also showed that on Hilbert spaces this condition is sufficient for this type of well-posedness.

Now we assume that A is densely defined, bisectorial and invertible, and keep the notations of Sections 3 and 4. In particular, we consider the semigroups $(T^+(t))$ and $(T^-(t))$ associated with A. Recall (see Proposition 4.4 and estimate (4.4)) that there exist $\omega > 0$ and $C > 0$ such that

$$\|T^{\pm}(t)\| \le C(1 + |\log(t)|)e^{-\omega t}, \qquad t > 0.$$

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ be weakly polynomially bounded. Then the function $u : \mathbb{R} \to X$ given by

$$u(t) := \int_{-\infty}^{t} T^-(t-s)f(s)\, ds - \int_{t}^{+\infty} T^+(s-t)f(s)\, ds \tag{6.2}$$

is continuous and weakly polynomially bounded. Indeed, we have

$$\begin{aligned}\int_{\mathbb{R}} (1+|t|)^{-k-2}\Big\| \int_t^{+\infty} T^-(t-s)f(s)\,ds\Big\|\,dt \\ &\le \int_{\mathbb{R}} (1+|t|)^{-k-2} \int_{-\infty}^{t-1} \|T^-(t-s)\|\,\|f(s)\|\,ds\,dt \\ &\quad + \int_{\mathbb{R}} (1+|t|)^{-k-2} \int_{t-1}^{t} \|T^-(t-s)\|\,\|f(s)\|\,ds\,dt \\ &= I_1 + I_2.\end{aligned}$$

Now let

$$c(f) := \int_{-\infty}^{+\infty} \|f(t)\|(1+|t|)^{-k}\,dt < +\infty.$$

Then we have

$$\begin{aligned}I_1 \le C\int_{\mathbb{R}}(1+|t|)^{-k-2} \int_{-\infty}^{t-1} (1+|\log(t-s)|)e^{-\omega(t-s)} \\ \times(1+|s|)^k\|f(s)\|(1+|s|)^{-k}\,ds\,dt.\end{aligned}$$

Since

$$\begin{aligned}C_2 &:= \sup_{t\in\mathbb{R}} \sup_{s\le t-1} (1+|\log(t-s)|)e^{-\omega(t-s)}\,\frac{(1+|s|)^k}{(1+|t|)^k} \\ &\le \sup_{t\in\mathbb{R}} \sup_{r\ge 1} (1+\log(r))e^{-\omega r}\,\frac{(1+|t|+r)^k}{(1+|t|)^k} \\ &\le \sup_{r\ge 1} (1+\log(r))e^{-\omega r}(1+r)^k < +\infty,\end{aligned}$$

it follows that

$$I_1 \le Cc(f)C_2\int_{\mathbb{R}} (1+|t|)^{-2}\,dt = 2Cc(f)C_2.$$

On the other hand, we can estimate I_2 by changing the order of integration:

$$I_2 \le C\int_{\mathbb{R}} \|f(s)\| \int_s^{s+1} (1+|t|)^{-k-2}(1+|\log(t-s)|)e^{-\omega(t-s)}\,dt\,ds \le Cc(f)C_3,$$

where

$$\begin{aligned}C_3 &= \sup_{s\in\mathbb{R}} (1+|s|)^k \int_0^1 (1+|r+s|)^{-k-2}(1+|\log(r)|)\,dr \\ &\le \sup_{s\in\mathbb{R}} \sup_{0\le r\le 1} (1+|s|)^k(1+|r+s|)^{-k-2}\int_0^1 (1+|\log(r)|)\,dr < +\infty.\end{aligned}$$

Analogously, one can prove the same for the second term on the right-hand side of (6.2).

Finally, we can prove the following.

THEOREM 6.4. *Assume that* $f \in L^1_{\mathrm{loc}}(\mathbb{R};X)$ *is weakly polynomially bounded. Then the function* u *defined by* (6.2) *is the unique weakly polynomially bounded mild solution of problem* (6.1).

Proof. Let u be defined by (6.2). To show that u is a mild solution we consider the function v given by $v(t) := A^{-1}u(t)$. It suffices to show that

$$v(t) = v(0) + A\int_0^t v(s)\,ds + \int_0^t A^{-1}f(s)\,ds.$$

Note that, by definition,

$$v(s) = \int_{-\infty}^{s} T^-(s-r)A^{-1}f(r)\,dr - \int_{s}^{+\infty} T^+(r-s)A^{-1}f(r)\,dr.$$

Hence, by Fubini's Theorem,

$$\begin{aligned}\int_0^t v(s)\,ds = &\int_{-\infty}^{0}\int_0^t T^-(s-r)A^{-1}f(r)\,ds\,dr + \int_0^t\int_r^t T^-(s-r)A^{-1}f(r)\,ds\,dr\\ &-\int_0^t\int_0^r T^+(r-s)A^{-1}f(r)\,ds\,dr - \int_t^{+\infty}\int_0^t T^+(r-s)A^{-1}f(r)\,ds\,dr.\end{aligned}$$

Since A is closed we obtain, by (4.1), for $t>0$,

$$\begin{aligned}A\int_0^t v(s)\,ds = &\int_{-\infty}^{0} (T^-(t-r)A^{-1}f(r) - T^-(-r)A^{-1}f(r))\,dr\\ &+\int_0^t (T^-(t-r)A^{-1}f(r) - P_-A^{-1}f(r))\,dr\\ &-\int_0^t (P_+A^{-1}f(r) - T^+(r)A^{-1}f(r))\,dr\\ &-\int_t^{+\infty} (T^+(r-t)A^{-1}f(r) - T^+(r)A^{-1}f(r))\,dr\\ = v(t) - &\int_{-\infty}^{0} T^-(-r)A^{-1}f(r)\,dr - \int_0^t A^{-1}f(r)\,dr + \int_0^{+\infty} T^+(r)A^{-1}f(r)\,dr\\ = v(t) - &\int_0^t A^{-1}f(r)\,dr - v(0).\ \blacksquare\end{aligned}$$

Our point is the representation formula (6.2). In special cases it had been proved before. Lunardi [8, (4.4.26), p. 164] gave a proof when A generates a holomorphic semigroup, and Schweiker [11, Chapter 2] gave a different proof if $f \in \mathrm{BUC}(\mathbb{R}; X)$ and A is densely defined. Here we do not address the question of maximal regularity. This was done in previous work with the help of multiplier theorems. In fact, in [1] it is shown that for each $f \in C^{\alpha}(\mathbb{R}; X)$ there exists a unique classical solution $u \in C^{1+\alpha}(\mathbb{R}; X)$ of (6.1), where $\alpha \in (0,1)$. Since a classical solution is also a weak solution we now have a representation formula for this solution. On the other hand, with the help of the representation formula (6.2) one can prove that $u \in C^{1+\alpha}(\mathbb{R}; X)$ for $f \in C^{\alpha}(\mathbb{R}; X)$ more directly as in [8, Theorem 4.3.1] without making use of Fourier multiplier theorems. This is done in [13].

In the L^p-context the following is known. Let $p \in (1, +\infty)$. If X is a Hilbert space and $f \in L^p(\mathbb{R}; X)$, then there exists a unique strong solution $u \in W^{1,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$ of (6.2) (see [4] or [10]). Again we can deduce that u is given by 6.1. If X is a UMD-space this result remains true if A is R-bisectorial (instead of merely sectorial, see [4]).

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