Embedding theorems for Lipschitz and Lorentz spaces on lower Ahlfors regular sets

by

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Abstract. We prove norm inequalities between Lorentz and Besov–Lipschitz spaces of fractional smoothness.

1. Main results. In what follows we let \((F, \rho)\) be a metric space with a positive \(\sigma\)-finite Borel measure \(\mu\). By \(B(x,r)\) we denote the open ball centred at \(x\) with radius \(r\). We always assume that there exist \(d > 0\) and \(C_1 > 0\) such that

\[\mu(B(x,r)) \geq C_1 r^d \quad \text{for all } 0 < r \leq 1 \text{ and } x \in F,\]

i.e., the lower Ahlfors \(d\)-regularity of \(F\). In particular, \(F\) may be a \(d\)-set in \(\mathbb{R}^n\) and \(\mu\) the \(d\)-dimensional Hausdorff measure, or \(F\) may be an \(h\)-set with \(h(r) \geq r^d\) for \(0 < r \leq 1\) and \(\mu\) an \(h\)-measure [10, 11, 5, 6, 18].

We denote \(L^p = L^p(F, \mu)\). We obtain the following inequality of Sobolev type.

**Theorem 1.** If \(0 < p < \infty\) and \(0 < \alpha < d/p\), then there exists a constant \(c = c(d, C_1, p, \alpha)\) such that

\[\|u\|_{L^p(d/(d-\alpha p))} \leq c \left( \|u\|_{L^p} + \left( \int_{\rho(x,y) < 1} \frac{|u(x) - u(y)|^p}{\rho(x,y)^{d+\alpha p}} \mu(dy) \mu(dx) \right)^{1/p} \right)\]

for all \(u \in L^p\).

Under certain additional assumptions we can get rid of the \(L^p\) norm on the right hand side.

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Corollary 2. Let $F \subset \mathbb{R}^n$ and $F = aF := \{ax : x \in F\}$ for some $a > 1$. Let $\mu$ be the $d$-dimensional Hausdorff measure and assume that it is $\sigma$-finite on $F$. Let $0 < p < \infty$ and $0 < \alpha < d/p$. There exists a constant $c = c(d,C_1,p,\alpha)$ such that

\[
\|u\|_{L^{pd/(d-\alpha p)}} \leq c \left( \int_J \int_{F \cap F} \frac{|u(x) - u(y)|^p}{|x - y|^{d + \alpha p}} \mu(dy) \mu(dx) \right)^{1/p}
\]

for all $u \in L^p$.

The result applies e.g. if $F$ is a half-space in $\mathbb{R}^n$ (or more generally, an open cone) and $d = n$.

Inequality (2) for $p = 2$, $\alpha < 1$, and a $d$-set $F \subset \mathbb{R}^n$ was stated in [7, (2.3)] and applied in [7] to estimate the heat kernel of jump type processes (see also [4]). Such applications are our primary motivation to study such inequalities. They are also of interest in the study of function spaces on $d$-sets [17]. Furthermore, inequalities of this type have a close connection to Nash inequalities and heat kernel estimates (see [15, 8, 23, 1]).

Note that our proofs are different and more elementary than those in [7, 17]. Interestingly, in our inequalities we allow for all $p > 0$, rather than $p \geq 1$, $\alpha \in (0,d/p)$ may be larger than 1, and we only assume lower Ahlfors $d$-regularity. Moreover, our methods yield an extension to Besov–Lipschitz spaces, given below.

We recall the definition of Lorentz spaces $L_{p,q}$ [17, 2]. We define the decreasing rearrangement $u^*$ of $u$ in the usual way,

\[
u^*(t) = \inf \{s : \mu(\{x : |u(x)| > s\}) \leq t\}.
\]

For $0 < p, q < \infty$ we define

\[
\|u; L_{p,q}\| = \left( \int_0^\infty (t^{1/p}u^*(t))^q \frac{dt}{t} \right)^{1/q}, \quad \|u; L_{p,\infty}\| = \sup_{t > 0} (t^{1/p}u^*(t)).
\]

We say that $u \in L_{p,q}$ if $\|u; L_{p,q}\| < \infty$.

For $0 < p < \infty$, $0 < q \leq \infty$ and $\alpha > 0$ we define the Besov–Lipschitz type space $\text{Lip}_0(\alpha,p,q,F) = \{u \in L^p : \|u; \text{Lip}_0(\alpha,p,q,F)\| < \infty\}$, where

\[
\|u; \text{Lip}_0(\alpha,p,q,F)\| = \|u\|_{L^p} + \|(b_{\nu})_{\nu=0}^\infty\|_{\ell_1},
\]

and the sequence $\{b_{\nu}\}_{\nu=0}^\infty$ is defined by

\[
b_{\nu} = 2^{\nu \alpha} \left( \int_\mathbb{R} 2^{\nu d} \int_{\rho(x,y)<2^{-\nu}} |u(x) - u(y)|^p \mu(dy) \mu(dx) \right)^{1/p}.
\]

If $p, q \geq 1$, then (4) is a genuine norm.

The main result of this note is the following embedding theorem, which extends Proposition 6 in [17, p. 216].
Theorem 3. Let $F$, $\mu$, $\rho$ and $d$ be as in Theorem [1]. Let $0 < p < \infty$, $p \leq q \leq \infty$ and $0 < \alpha < d/p$. Then there exists a constant $c = c(d, C_1, p, q, \alpha)$ such that for all $u \in \text{Lip}_0(\alpha, p, q, F)$,
\begin{equation}
\|u; L_{p^\ast, q}\| \leq c\|u; \text{Lip}_0(\alpha, p, q, F)\|,
\end{equation}
where $p^\ast = pd/(d - \alpha p)$.

We may regard Theorem 3 as a subcritical case of a limiting embedding (see [22], Remark 11.5 for definitions and a further discussion).

We mention that the Hardy inequality of [12, 9, 3] is similar to (3), except that it estimates the weighted $L^p$ norm (and not $L^{p^\ast}$) by $E$.

We note that the definition of $\text{Lip}_0(\alpha, p, q, F)$ is very similar to the definition of the space $A_{p,q}^{d,\alpha}$ of Grigor’yan [13]. By the definition
\begin{equation}
\|u; \text{Lip}_0(\alpha, p, q, F)\| \leq c\|u; A_{p,q}^{d,\alpha}\|,
\end{equation}
and these two norms are equivalent for bounded $d$-sets $F$. Correspondingly, (6) holds with the norm $\text{Lip}_0(\alpha, p, q, F)$ replaced by the norm of $A_{p,q}^{d,\alpha}$ in Theorem 3. See [13, 14] for a further discussion.

We now recall the definition of $\text{Lip}(\alpha, p, q, F)$ of Jonsson and Wallin [17]. Assume that $F \subset \mathbb{R}^n$ and $\rho$ is the Euclidean distance. Let $\alpha > 0$ and $k \in \mathbb{Z}$ satisfy $k < \alpha \leq k + 1$. Let $\{f^{(j)}\}_{|j| \leq k}$ be a family of functions defined $\mu$-a.e. on $F$, where $j = (j_1, \ldots, j_n)$ is a multiindex and $|j| = j_1 + \cdots + j_n$. We define $P_j$ and $R_j$ by requiring that
\begin{equation}
P_j(x, y) = \sum_{|j+l| \leq k} \frac{f^{(j+l)}(y)}{l!} (x - y)^l, \quad x, y \in F,
\end{equation}
and that $f^{(j)}(x) = P_j(x, y) + R_j(x, y)$. The collection $\{f^{(j)}\}_{|j| \leq k}$ belongs to the Lipschitz space $\text{Lip}(\alpha, p, q, F)$ if and only if $f^{(j)} \in L^p$ for $|j| \leq k$, and for $\nu = 0, 1, 2, \ldots$ and $|j| \leq k$,
\begin{equation}
\left(2^{\nu d} \int_{|x-y|<2^{-\nu}} |R_j(x, y)|^p \mu(dx) \mu(dy)\right)^{1/p} \leq 2^{-\nu(\alpha - |j|)} a_\nu
\end{equation}
for some sequence $(a_\nu) \in \ell^q$. The norm of $\{f^{(j)}\}_{|j| \leq k}$ in $\text{Lip}(\alpha, p, q, F)$ is
\begin{equation}
\sum_{|j| \leq k} \|f^{(j)}\|_{L^p} + \inf \|(a_\nu)\|_{\ell^q},
\end{equation}
where the infimum is taken over all possible sequences $(a_\nu)$. We see that the definition of $\text{Lip}(\alpha, p, q, F)$ uses (a substitute of) Taylor expansion of $k$th order, while $\text{Lip}_0(\alpha, p, q, F)$ uses only increments of the function (0-order Taylor expansion). This motivates the notation $\text{Lip}_0$.

For a function $f$ we put $\tilde{f}^{(0)} = f$ and $\tilde{f}^{(j)} = 0$ if $|j| > 0$. Clearly,
\begin{equation}
\|f^{(0)}; \text{Lip}_0(\alpha, p, q, F)\| = \|\{\tilde{f}^{(j)}\}; \text{Lip}(\alpha, p, q, F)\|.
\end{equation}
In particular, we have $\text{Lip}(\alpha, p, q, F) = \text{Lip}_0(\alpha, p, q, F)$ for $\alpha \leq 1$. 

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It seems that Lip$_0(\alpha, p, q, F)$ is more appropriate to study jump processes on metric spaces (see [16, 20, 21]). For a $d$-set $F$ the space Lip$_0(\alpha d_w/4, 2, 2, F)$ is the domain of the Dirichlet form of a symmetric $\alpha$-stable process on $F$ [21], where $\alpha \in (0, 2)$ and $d_w$ is the so-called walk dimension of $F$ [20]. Also, Lip$_0(d_w/2, 2, \infty, F)$ is the domain of the Dirichlet form of the Brownian motion e.g. on the Sierpiński gasket $F \subset \mathbb{R}^n$, (see [16]). Our results shed light on domains of non-local Dirichlet forms defined on more general sets.

Notation $c = c(a, b, \ldots, z)$ means that the constant $0 < c < \infty$ depends only on $a, b, \ldots, z$. All functions are assumed to be Borel measurable and complex-valued. In fact our results remain valid for Banach-space-valued functions $u$ (see (13), (14)).

2. Proof of Theorem 3. In the following lemma we adopt the convention that $\frac{0}{0} = 0$.

**Lemma 4.** For every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n \leq a_0 + 3 \cdot 4^\varepsilon \sum_{n=1}^{\infty} \frac{a_n^{1+\varepsilon}}{(a_{n-1} + a_n + a_{n+1})^\varepsilon}$$

if $a_n \geq 0$, $n = 0, 1, \ldots$ and $a_n = 0$ for large $n$.

**Proof.** Let

$$A = \left\{ n \in \{1, 2, \ldots\} : a_n \geq \frac{1}{3} (a_{n-1} + a_{n+1}) \right\}, \quad B = \{1, 2, \ldots\} \setminus A,$$

and let $N$ be such that $B \subset \{1, \ldots, N\}$. For $n \in A$ we have $a_{n-1} + a_n + a_{n+1} \leq 4a_n$, hence

$$\sum_{n \in A} a_n \leq 4^\varepsilon \sum_{n \in A} \frac{a_n^{1+\varepsilon}}{(a_{n-1} + a_n + a_{n+1})^\varepsilon}.$$

On the other hand, we have

$$\sum_{n \in B} a_n \leq \frac{1}{3} \sum_{n \in B} (a_{n-1} + a_{n+1}) \leq \frac{1}{3} a_0 + \frac{2}{3} \sum_{n=1}^{N} a_n + \frac{1}{3} a_{N+1},$$

thus

$$\frac{1}{3} \sum_{n \in B} a_n < \frac{1}{3} a_0 + \frac{2}{3} \sum_{n \in A, n \leq N} a_n + \frac{1}{3} a_{N+1}.$$

Since $N + 1 \in A$, we obtain from (12),

$$\sum_{n \in B} a_n < a_0 + 2 \sum_{n \in A, n \leq N+1} a_n,$$

and this together with (11) completes the proof. \[\blacksquare\]
Remark 1. We note that (10) does not hold for all sequences \( a_n \geq 0 \). Indeed, for \( a_n = \exp(b^n) \), the right hand side of (10) is finite if \( b \) is large enough, while the left hand side is infinite. One can prove that (10) holds, with some constant \( c = c(\varepsilon) \) instead of \( 3 \cdot 4^\varepsilon \) in (10), for all sequences \( a_n = o(q^n) \), where \( q > 0 \); however, the proof is more complicated and will be omitted.

Proof of Theorem 3. Let \( u \in \text{Lip}_0(\alpha, p, q, F) \). Our goal is to prove (6) with \( c \) independent of \( u \). Note that

\[
\| |u|; L_{p^*, q} \| = \| u; L_{p^*, q} \|
\]

and

\[
\| |u|; \text{Lip}_0(\alpha, p, q, F) \| \leq \| u; \text{Lip}_0(\alpha, p, q, F) \|
\]

hence it suffices to prove (6) for \( u \geq 0 \).

Furthermore, since for any \( t > 0 \) we have

\[
\| u \wedge t; \text{Lip}_0(\alpha, p, q, F) \| \leq \| u; \text{Lip}_0(\alpha, p, q, F) \|
\]

by the bounded convergence theorem we may also assume that \( u \) is bounded. Finally, we may and will assume that \( \| u \|_{L^p} = 1 \).

Let

\[
E_n = \{ x \in F : u(x) \in [2^n, 2^{n+1}) \}, \\
\mu_n = \mu(E_n), \quad n \in \mathbb{Z}.
\]

The idea of the proof is to estimate the norms in (6) by means of \( \mu_n \) only, and then use special inequalities for sequences, including (10) and the Hardy inequality. While estimates for the \( L^p \) and \( L_{p^*, q} \) norms of \( u \) by means of \( \mu_n \) are straightforward, this is not the case for the \( \ell^q \) norm of \( (b_\nu) \). This is the place where the somewhat unusual terms \( \mu_n/(\mu_{n-1} + \mu_n + \mu_{n+1}) \) arise, which result from considering \( x \) and \( y \) not in neighbouring sets \( E_n \) (see (5) and (16)). We estimate the terms by using Lemma 4. The assumption \( \| u \|_{L^p} = 1 \) implies that \( \mu_{n-1} + \mu_n + \mu_{n+1} \leq 2^{-(n-1)p} \), thus \( \mu_{n-1} + \mu_n + \mu_{n+1} \leq C_1/2 \) for \( n \geq n_0 = n_0(C_1, p) \).

We claim that for any \( n \geq n_0 \) there exists \( \nu \in \{0, 1, 2, \ldots\} \) (depending on \( n, u, \ldots \)) such that

\[
2^n \mu_n^{1/p}(\mu_{n-1} + \mu_n + \mu_{n+1})^{-\alpha/d}
\]

\[
\leq c2^{\nu p} \left( 2^{\nu d} \int_{E_n B(x, 2^{-\nu})} \int |u(x) - u(y)|^p \mu(dy) \mu(dx) \right)^{1/p}
\]

with constant \( c = c(d, C_1, p, \alpha) \) independent of \( n \). Here we adopt the convention that \( 0^a = 0 \) for \( a < 0 \), hence the claim is obvious if \( \mu_{n-1} + \mu_n + \mu_{n+1} = 0 \).
We now prove the claim in the case when \( \mu_{n-1} + \mu_n + \mu_{n+1} > 0 \). We have

\[
\begin{align*}
  b_{n,\nu} &:= \int_{E_n \setminus (E_{n-1} \cup E_n \cup E_{n+1})} |u(x) - u(y)|^p \mu(dy) \mu(dx) \\
  &\geq \int_{E_n \setminus (E_{n-1} \cup E_n \cup E_{n+1})} |u(x) - u(y)|^p \mu(dy) \mu(dx) \\
  &\geq 2^{(n-1)p} \mu_n \cdot \mu(B(x, 2^{-\nu}) \setminus (E_{n-1} \cup E_n \cup E_{n+1})) \\
  &\geq 2^{(n-1)p} \mu_n (C_1 2^{-\nu d} - (\mu_{n-1} + \mu_n + \mu_{n+1})).
\end{align*}
\]

We take \( \nu \in \{0, 1, 2, \ldots\} \) such that

\[
2(\mu_{n-1} + \mu_n + \mu_{n+1}) \leq C_1 2^{-\nu d} < 2^{d+1}(\mu_{n-1} + \mu_n + \mu_{n+1}).
\]

Then

\[
b_{n,\nu} \geq \frac{C_1}{2} 2^{(n-1)p} \mu_n 2^{-\nu d},
\]

hence

\[
2^{\nu \alpha}(2^{\nu d} b_{n,\nu})^{1/p} \geq c(d, C_1, p, \alpha)(\mu_{n-1} + \mu_n + \mu_{n+1})^{-\alpha/d} 2^n \mu_n^{1/p},
\]

and the claim is proven.

We will first prove (6) in the case when \( q < \infty \). Observe that \( 2^n \leq u^*(t) < 2^{n+1} \) if \( \sum_{k>n} \mu_k < t < \sum_{k>n} \mu_k \). Hence

\[
\|u; L_{p^*,q}\|^q = \int_0^\infty (t^{1/p^*} u^*(t))^{q} \frac{dt}{t} \leq 2^q \sum_{n \in \mathbb{Z}} \frac{\sum_{k \geq n} \mu_k}{\sum_{k \geq n} \mu_k} \int \frac{t^{q/p^*}}{2^{nq}} dt \leq \frac{2^q}{q} \sum_{n \in \mathbb{Z}} \left((\sum_{k \geq n} \mu_k)^{q/p^*} - (\sum_{k>n} \mu_k)^{q/p^*}\right) 2^{nq} \leq \frac{2^q}{q} \sum_{n \in \mathbb{Z}} (\sum_{k \geq n} \mu_k)^{q/p^*} 2^{nq}.
\]

We use the following variant of the Hardy inequality (\cite{17}, Lemma 3, p. 121, \cite{19}), valid for \( s, q > 0 \):

\[
\sum_{n=n_0}^{\infty} \left(\sum_{k \geq n} \mu_k\right)^s 2^{nq} \leq c(n_0, s, q) \sum_{n=n_0}^{\infty} \mu_n^s 2^{nq},
\]

and the estimate \( \sum_{k \geq n} \mu_k \leq 2^{-np} \), which follows from \( \|u\|_{L^p} = 1 \). We deduce
from (17) that

\[
\| u; L^p, q \|^q \leq c \sum_{n<n_0} 2^{-npq/p^*} 2^{nq} + c \sum_{n=n_0}^\infty \mu_n^{q/p^*} 2^{nq}
\]

\[
\leq c\| u \|_{L_p}^q + c \sum_{n=n_0}^\infty \mu_n^{q/p^*} 2^{nq},
\]

where \( c = c(d, C_1, p, q, \alpha) \).

Since \( u \) is bounded, \( \mu_n = 0 \) for all large \( n \). We are going to apply Lemma 4 to \( a_n = \mu_n^\gamma 2^{nq} \), where \( \gamma = q/p^* \) and \( \varepsilon = \alpha q / (\gamma d) > 0 \). Observe that \( \gamma (1 + \varepsilon) = q/p \). Note that

\[
a_{n+1} \leq c \left( \frac{\mu_n}{\mu_{n-1} + \mu_n + \mu_{n+1}} \right)^{\gamma (1+\varepsilon)} 2^{nq}
\]

with \( c = c(d, p, q, \alpha) \). Thus by Lemma 4 and the inequality (15) raised to the \( q \)th power we obtain

\[
\sum_{n=n_0}^\infty 2^{nq} \mu_n^\gamma \leq c \sum_{n=n_0}^\infty (2^{nq}) \mu_n^{q/p} \left( \frac{\mu_n}{\mu_{n-1} + \mu_n + \mu_{n+1}} \right)^{\alpha q / d} 2^{nq} + 2^{(n_0-1)q} \mu_{n_0-1}^\gamma
\]

\[
\leq c \sum_{n=n_0}^\infty 2^{(n_0-1)q} \mu_{n_0-1}^\gamma + c \| u \|_{L_p}^q
\]

(19)

(20)

with \( c = c(d, C_1, p, q, \alpha) \). We note that in (19) above \( \nu(n) \) depends also on \( n \) and \( u \), but the dependence vanishes in (20). The first term in (20) is now estimated as follows:

\[
\sum_{n=n_0}^\infty \sum_{\nu=0}^\infty 2^{\nu q} (2^{\nu d} b_{n, \nu})^{q/p}
\]

\[
= \sum_{\nu=0}^\infty 2^{\nu q} \sum_{n=n_0}^\infty (2^{\nu d} b_{n, \nu})^{q/p} \leq \sum_{\nu=0}^\infty 2^{\nu q} (2^{\nu d} \sum_{n=n_0}^\infty b_{n, \nu})^{q/p}
\]

\[
\leq \sum_{\nu=0}^\infty 2^{\nu q} \left( \int_{\rho(x, y) < 2^{-\nu}} |u(x) - u(y)|^p \mu(dy) \mu(dx) \right)^{q/p}
\]

\[
\leq \| u; \text{Lip}_0(\alpha, p, q, F) \|^q.
\]

Putting (18), (20) and (21) together we obtain (6).
It remains to show (6) in the case when \( q = \infty \). We have
\[
\| u; L_{p^*, \infty} \| \leq 2 \sup_n \left( \sum_{k \geq n} \mu_k \right)^{1/p^*} 2^n.
\]
Observe that for \( n \leq n_0 \),
\[
\left( \sum_{k \geq n} \mu_k \right)^{1/p^*} 2^n \leq (2^{-np})^{1/p^*} 2^n \leq 2^{n_0 (1-p/p^*)},
\]
hence
\[
\sup_{n \leq n_0} \left( \sum_{k \geq n} \mu_k \right)^{1/p^*} 2^n \leq c(d, C_1, p, \alpha) \| u \|_{L^p}.
\]
Now let
\[
S = \sup_{n \geq n_0} \left( \sum_{k \geq n} \mu_k \right)^{1/p^*} 2^n.
\]
We have \( S < \infty \), because \( u \) is bounded. Let \( N \geq n_0 \) be such that
\[
\left( \sum_{k \geq N} \mu_k \right)^{1/p^*} 2^N \geq \frac{3}{4} S.
\]
If \( N = n_0 \), then \( S \leq c(d, C_1, p, \alpha) \| u \|_{L^p} \) by (22). Henceforth we assume that \( N > n_0 \). By (15) we get
\[
\sup_{n \geq n_0} 2^n \mu_n^{1/p} (\mu_{n-1} + \mu_n + \mu_{n+1})^{-\alpha/d} \leq c \| u; \text{Lip}_0(\alpha, p, \infty, F) \| (\text{see (5)}).
\]
From (23) and the inequalities \( \left( \sum_{k \geq n} \mu_k \right)^{1/p^*} 2^n \leq S \) for \( n = N - 1 \) and \( n = N + 1 \), we obtain, respectively,
\[
\mu_{N-1} + \mu_N + \mu_{N+1} \leq \sum_{k \geq N-1} \mu_k \leq \left( \frac{8}{3} \right)^{p^*} \sum_{k \geq N} \mu_k,
\]
\[
\mu_N \geq \left( \left( \frac{3}{2} \right)^{p^*} - 1 \right) \sum_{k \geq N+1} \mu_k.
\]
Thus \( \sum_{k \geq N} \mu_k \leq c(p^*) \mu_N \), hence by \( 1/p^* = -\alpha/d + 1/p \) and (24),
\[
\frac{3}{4} S \leq \left( \sum_{k \geq N} \mu_k \right)^{1/p^*} 2^N \leq c(\mu_{N-1} + \mu_N + \mu_{N+1})^{-\alpha/d} 2^N \mu_N^{1/p}
\]
\[
\leq c \| u; \text{Lip}_0(\alpha, p, \infty, F) \|.
\]

Proof of Theorem 7. By Theorem 3 applied to \( p = q < \infty \) we have
\[
\| u \|_{L_{p^*, p}} \leq c \| u; \text{Lip}_0(\alpha, p, F) \|
\]
\[
\leq c \left( \| u \|_{L^p} + \left( \int_{\rho(x,y) < 1} \frac{|u(x) - u(y)|^p}{\rho(x,y)^{d+\alpha p}} \mu(dy, \mu(dx)) \right)^{1/p} \right),
\]
and the theorem follows from the embedding $L_{p^*,p} \subset L_p^*$ for $p < p^*$ [2].

**Proof of Corollary 2.** Denote $u^{(a)}(x) = u(ax)$ and

$$
\mathcal{E}(u) = \int_{\mathbb{F}\mathbb{F}} \frac{|u(x) - u(y)|^p}{|x - y|^{d + \alpha}} \mu(dy) \mu(dx).
$$

It is easy to check that $\|u^{(a)}\|_{L^s} = a^{-d/s} \|u\|_{L^s}$ and $\mathcal{E}(u^{(a)}) = a^{-d + \alpha p} \mathcal{E}(u)$. Hence by (2) applied to $u^{(a^n)}$ we obtain

$$
\|u\|_{L^{pd/(d - \alpha p)}} \leq c \left( r_0^{-\alpha} \|u\|_{L^p} + \mathcal{E}(u)^{1/p} \right)
$$

and the corollary follows by letting $n \to \infty$. ■

**Note.** One can simplify the proof of Corollary 2 to get a stronger result. Namely, assume instead of (1) that for some $C_1, d, r_0 > 0$,

$$
\mu(B(x, r)) \geq C_1 r^d \quad \text{for all } 0 < r \leq r_0 \text{ and } x \in \mathbb{F}.
$$

Then the new measure $\tilde{\mu}(A) := \mu(A) r_0^{-d}$ and the new metric $\tilde{\rho}(x, y) := \rho(x, y)/r_0$ satisfy [1], hence [2] holds. Coming back to $\mu$ and $\rho$ we get the following corollary.

**Corollary 5.** Assume that [25] holds. If $0 < p < \infty$ and $0 < \alpha < d/p$, then there exists a constant $c = c(d, C_1, p, \alpha)$ such that

$$
\|u\|_{L^{pd/(d - \alpha p)}} \leq c \left( r_0^{-\alpha} \|u\|_{L^p} + \left( \int_{\rho(x,y) < r_0} \frac{|u(x) - u(y)|^p}{\rho(x,y)^{d + \alpha}} \mu(dy) \mu(dx) \right)^{1/p} \right)
$$

for all $u \in L^p$. In particular, if [25] holds for all $r_0 > 0$, then

$$
\|u\|_{L^{pd/(d - \alpha p)}} \leq c \left( \int_{\mathbb{F}\mathbb{F}} \frac{|u(x) - u(y)|^p}{\rho(x,y)^{d + \alpha}} \mu(dy) \mu(dx) \right)^{1/p}
$$

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