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Orbits in symmetric spaces, II

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Abstract. Suppose E is fully symmetric Banach function space on $(0,1)$ or $(0,\infty)$ or a fully symmetric Banach sequence space. We give necessary and sufficient conditions on $f \in E$ so that its orbit $\Omega(f)$ is the closed convex hull of its extreme points. We also give an application to symmetrically normed ideals of compact operators on a Hilbert space.

1. Introduction. Let I be either the interval $(0,1)$ or the semi-axis $(0,\infty)$ and suppose $f \in L_1(I) + L_\infty(I)$. We define the *orbit* $\Omega(f)$ of f to be the set of Tf where $T : L_1 + L_\infty \to L_1 + L_\infty$ is an operator with $\|T\|_{L_1\to L_1}, \|T\|_{L_\infty\to L_\infty} \le 1$ (see [9, 12]). Then it follows from the Calderón–Mityagin theorem [1, 3, 9, 11] that $\Omega(f)$ can be characterized as the set of $g \in L_1 + L_\infty$ such that

$$(1)\qquad \int_0^t g^*(s)\,ds \le \int_0^t f^*(s)\,ds, \qquad 0 < t < \infty,$$

where as usual f^* is the decreasing rearrangement of $|f|$ (see §2 for definitions). This may be written $g \preceq f$ where $\preceq$ is the Hardy–Littlewood–Pólya ordering. Thus E is an exact interpolation space if and only if it is fully symmetric (see §2).

The extreme points of $\Omega(f)$, which we denote $\partial_e\Omega(f)$, were obtained in [13] (for the case of spaces on $(0,1)$) and [4] (for the general case). Except in the special case when $I = (0,\infty)$ and $E \supset L_\infty$ these are given by $\partial_e\Omega(f) = \{g : g^* = f^*\}$ (see §2 for full details; in the exceptional cases the extreme points form a subset of this set). Let $\mathcal{Q}(f)$ be the convex hull of the set $\{g : g^* \le f^*\}$. Then it is clear that if E is fully symmetric and $f \in E$, the closure $\mathcal{Q}_E(f)$ of $\mathcal{Q}(f)$ in E coincides with the closed convex hull of $\partial_e\Omega(f)$.

In [12] it was shown for the case of $I = (0,1)$ that the orbit $\Omega(f)$ is always weakly compact in $L_1(0,1)$. It follows from results in [6] that if E is an

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order-continuous (equivalently, separable) symmetric function space (which is necessarily fully symmetric) and $f \in E$ then $\Omega(f)$ is weakly compact in E. Thus it is an immediate consequence of the Kreĭn–Milman theorem that $\mathcal{Q}_E(f)$ coincides with $\Omega(f)$.

For the case of nonseparable fully symmetric spaces the situation is less clear. The example $E = L_\infty$ and $f = 1$ shows that $\mathcal{Q}_E(f)$ may still coincide with $\Omega(f)$. This problem was first investigated by Braverman and Mekler [2] for the unit interval i.e. $I = (0,1)$. They gave a sufficient condition for $\Omega(f) = \mathcal{Q}_E(f)$ in terms of the behavior of the dilation operators σ_τ (see §2 for the appropriate definitions). Precisely, they showed that if E is a fully symmetric Banach function space on $(0,1)$ such that

$$\lim_{\tau\to\infty} \frac{\|\sigma_\tau\|_{E\to E}}{\tau} = 0$$

then $\Omega(f) = \mathcal{Q}_E(f)$ for every $f \in E$. This condition is, however, not necessary since it may fail in separable symmetric spaces (e.g. $E = L_1$).

Recently two of the current authors [14] found a necessary and sufficient condition for the similar problem concerning the positive part of the orbit. If $f \geq 0$ we denote by $\Omega_+(f)$ the set $\{g : g \in \Omega(f),\ g \geq 0\}$. In [14] it was shown, for a fully symmetric Banach function space E with a Fatou norm (sometimes called a weak Fatou property), that if $f \in E_+$ then $\Omega_+(f)$ coincides with the closed convex hull of its extreme points if and only if a local Braverman–Mekler type condition holds. If $I = (0,1)$, or if $I = (0,\infty)$ and E is not contained in $L_1(0,\infty)$, this condition takes the form

$$\lim_{\tau\to\infty} \frac{\|\sigma_\tau(f^*)\|_E}{\tau} = 0. \tag{2}$$

If $I = (0,\infty)$ and $E \subset L_1$, we must replace (2) by

$$\lim_{\tau\to\infty} \frac{\|\chi_{(0,1)}\sigma_\tau(f^*)\|_E}{\tau} = 0. \tag{3}$$

The results of [14] imply, under the same hypotheses on E (full symmetry and a Fatou norm), that (2) and (3) are sufficient for $\mathcal{Q}_E(f) = \Omega(f)$.

Our main result in this paper is to show that, indeed, if E is a fully symmetric Banach space with a Fatou norm on $(0,1)$ or $(0,\infty)$, then (2) and (3) are necessary and sufficient for $\Omega(f) = \mathcal{Q}_E(f)$. These results are Theorems 4.1, 4.2 and 4.3 below. We also establish the corresponding result for sequence spaces in Theorem 4.5; sequence spaces were not covered in [14] so we are also able to complete the picture for the positive part of the orbit.

We conclude the paper with an application to orbits in symmetrically normed ideals of compact operators on a Hilbert space.

2. Preliminaries. In this section we present some definitions from the theory of symmetric spaces. For more details on the latter theory we refer to [1, 9, 10].

Let I denote either $(0,1)$ or on $(0,\infty)$ with Lebesgue measure μ. If $f \in L_1(I) + L_\infty(I)$ we denote by f^* the decreasing rearrangement of f, i.e.

$$f^*(t) = \inf_{\mu A = t} \sup_{s \in I \setminus A} |f(s)|.$$

If f, g are functions in $L_1 + L_\infty$ we write $g \preceq f$ if

$$\int_0^t g^*(s)\, ds \le \int_0^t f^*(s)\, ds, \qquad t \in I.$$

This defines the *Hardy–Littlewood–Pólya ordering.*

A *symmetric Banach function space* E on I is a linear space with $L_1 \cap L_\infty \subset E \subset L_1 + L_\infty$, with an associated norm $\|\cdot\|_E$ such that $(E, \|\cdot\|_E)$ is complete and if $f \in E$, $g \in L_1 + L_\infty$ with $g^* \le f^*$ then $g \in E$ and $\|g\|_E \le \|f\|_E$. We will use E_+ to denote the positive cone of E, i.e. $\{f : f \in E, f \ge 0 \text{ a.e.}\}$. We will also assume the normalization $\|\chi_{(0,1)}\| = 1$. Let $\varphi_E(t) = \|\chi_{(0,t)}\|_E$ be the *fundamental function* of E.

E is said to have a *Fatou norm* if for every sequence $(f_n)_{n=1}^\infty$ of nonnegative functions such that $f_n \uparrow f$ a.e. with $f \in E$ we have $\lim_{n\to\infty} \|f_n\|_E = \|f\|_E$.

A symmetric Banach function space E is said to be *fully symmetric* if and only if, whenever $f \in E$, $g \in L_1 + L_\infty$ with $g \preceq f$, then $g \in E$ and $\|f\|_E \le \|g\|_E$. The space E is fully symmetric precisely when E is an exact interpolation space for the couple $(L_\infty(I), L_1(I))$ by the Calderón–Mityagin theorem [3, 11]. In this paper we will only consider fully symmetric Banach function spaces.

We will need the following inequality which can be found in [9, Theorem II.3.1]. If $f, g \in L_1 + L_\infty$, then

$$(f^* - g^*) \preceq (f - g)^*. \tag{4}$$

As a consequence, if E is fully symmetric and $f, g \in E$ we have

$$\|f^* - g^*\|_E \le \|f - g\|_E. \tag{5}$$

If E is a fully symmetric Banach function space and $f \in E$, we define the *orbit* of f by $\Omega(f) = \{g : g^* \preceq f^*\} \subset E$. The set of the extreme points of $\Omega(f)$ is well-known (see [4, 13]), and if $I = (0,1)$ or $I = (0,\infty)$, and E does not contain L_∞, it is given by

$$\partial_e(\Omega(f)) = \{g \in L_1 + L_\infty : f^* = g^*\}.$$

If $I = (0,\infty)$ and E contains L_∞ we must make a small correction:

$$\partial_e(\Omega(f)) = \{g \in L_1 + L_\infty : f^* = g^*,\ |g(t)| \ge \lim_{s\to\infty} f^*(s) \text{ a.e.}\}.$$

We define $\mathcal{Q}(f)$ to be the convex hull of the set $\{g \in L_1 + L_\infty : g^* \le f^*\}$. We will denote by $\mathcal{Q}_E(f)$ the closure in E of $\mathcal{Q}(f)$. This is easily seen to coincide with the closed convex hull of $\partial_e \Omega(f)$. Thus $\mathcal{Q}_E(f) \subset \Omega(f)$.

We next define the dilation operators on E. If $\tau > 0$ and $I = (0,\infty)$ the dilation operator σ_τ is defined by setting

$$(\sigma_\tau(f))(s) = f(s/\tau), \qquad s > 0.$$

In the case of the interval $(0,1)$ the operator σ_τ is defined by

$$(\sigma_\tau f)(s) = \begin{cases} f(s/\tau), & s \le \min\{1,\tau\}, \\ 0, & \tau < s \le 1. \end{cases}$$

The operators σ_τ $(\tau \ge 1)$ have the semigroup property $\sigma_{\tau_1}\sigma_{\tau_2} = \sigma_{\tau_1\tau_2}$. If E is a symmetric space and if $\tau > 0$, then the dilation operator σ_τ is a bounded operator on E and

$$\|\sigma_\tau\|_{E\to E} \le \max\{1,\tau\}.$$

If E is a fully symmetric function space on $(0,\infty)$ then $E + L_\infty$ is also a fully symmetric function space under the norm

$$\|f\|_{E+L_\infty} = \|f^*\chi_{(0,1)}\|_E.$$

The next lemma will be used later.

Lemma 2.1. *Let E be a symmetric function space on $(0,\infty)$ such that $E \setminus L_1 \ne \emptyset$, and suppose $f \in L_1 \cap E$. Then*

$$\lim_{\tau\to\infty} \tau^{-1}\|\sigma_\tau(f)\|_E = \lim_{\tau\to\infty} \tau^{-1}\|\sigma_\tau(f)\|_{E+L_\infty}.$$

Proof. We may suppose f is nonnegative and decreasing. Let $\varphi = \varphi_E$ be the fundamental function of E and let ψ be the least concave majorant of φ. Since $E \setminus L_1 \ne \emptyset$ we have $\lim_{t\to\infty} \psi'(t) = 0$. For any $\tau > 1$ we have, using Theorem II.5.5 of [9],

$$\begin{aligned}\|(\sigma_\tau f)\chi_{(1,\infty)}\|_E &\le \|f(\tau^{-1})\chi_{(0,1)} + (\sigma_\tau f)\chi_{(1,\infty)}\|_E \\ &\le f(\tau^{-1})\int_0^1 \psi'(s)\,ds + \int_1^\infty \psi'(s) f(\tau^{-1}s)\,ds \\ &\le \psi(1)f(\tau^{-1}) + \tau\int_{\tau^{-1}}^\infty \psi'(\tau s) f(s)\,ds.\end{aligned}$$

Now, since $f \in L_1$, we have

$$\lim_{\tau\to\infty} \tau^{-1} f(\tau^{-1}) = 0,$$

and by the Dominated Convergence Theorem,

$$\lim_{\tau\to\infty} \int_{\tau^{-1}}^\infty \psi'(\tau s) f(s)\,ds = \lim_{\tau\to\infty} \int_0^\infty \chi_{(\tau^{-1},\infty)}(s)\psi'(\tau s) f(s)\,ds = 0.$$

Hence

$$\lim_{\tau\to\infty} \tau^{-1}\|(\sigma_\tau f)\chi_{(1,\infty)}\|_E = 0,$$

and the lemma follows. ∎

We next discuss the corresponding notions for sequence spaces. If $\xi = (\xi_n)_{n=1}^\infty$ is a sequence then ξ^* denotes its decreasing rearrangement:

$$\xi_n^* = \inf_{|\mathbb{A}|=n-1} \sup_{k\in\mathbb{N}\setminus\mathbb{A}} |\xi_k|.$$

A Banach sequence space E is called *symmetric* if $\xi \in E$ and $\eta^* \le \xi^*$ implies that $\eta \in E$ and $\|\eta\|_E \le \|\xi\|_E$. We write $\eta \preceq \xi$ if

$$\sum_{k=1}^n \eta_k^* \le \sum_{k=1}^n \xi_k^*, \qquad n \in \mathbb{N}.$$

E is called *fully symmetric* if $\xi \in E$ and $\eta \preceq \xi$ implies that $\eta \in E$ and $\|\eta\|_E \le \|\xi\|_E$. If ξ is any bounded sequence we define its *orbit* $\Omega(\xi) = \{\eta : \eta \preceq \xi\}$.

In this context, we define the dilation operators σ_m only for $m \in \mathbb{N}$. Then

$$\sigma_m(\xi) = (\xi_1, \ldots, \xi_1, \xi_2, \ldots, \xi_2, \ldots)$$

where each ξ_j is repeated m times.

3. Approximation of the orbit. Our first proposition gives a simple criterion which will enable us to check when $\mathcal{Q}_E(f) = \Omega(f)$.

PROPOSITION 3.1. *Let E be a fully symmetric Banach space on $(0,\infty)$. Suppose f, g are nonnegative decreasing functions in E. Then $g \in \mathcal{Q}_E(f)$ if and only if, given $\epsilon > 0$, there exists a nonnegative decreasing function $h \in E$ and an integer p such that $0 \le h \le g$ and*

$$\|g - h\|_E < \epsilon \tag{6}$$

and

$$\int_{pa}^{b} h(t)\,dt \le \int_a^b f(t)\,dt, \qquad 0 < pa < b < \infty. \tag{7}$$

Proof. Suppose first $g \in \mathcal{Q}_E(f)$. Then given $\epsilon > 0$ there exist $f_1, \ldots, f_p$ in E such that $f_j^* \le f$ for $1 \le j \le p$ and

$$\left\| g - \frac{1}{p}(f_1 + \cdots + f_p) \right\|_E < \epsilon.$$

Let

$$u = \frac{1}{p}(f_1 + \cdots + f_p), \qquad v = \frac{1}{p}(|f_1| + \cdots + |f_p|).$$

Then if $h = g \wedge v^*$, using (5) we get

$$\|g - h\|_E \le \|g - g \wedge u^*\|_E \le \|g - u^*\|_E \le \|g - u\|_E < \epsilon.$$

It remains to observe that (7) holds by Lemma 4.1 of [8].

The converse is easy. If h satisfies (6) and (7) then $h \in \alpha\mathcal{Q}(f)$ for every $\alpha > 1$ by Theorem 6.3 of [8]. Hence $h \in \mathcal{Q}_E(f)$ and so $d(g, \mathcal{Q}_E(f)) < \epsilon$. ∎

The next lemma is surely well-known but we use it in the main result and include a proof for completeness.

LEMMA 3.2. *Let F be a continuous nonnegative increasing concave function on $[0,\infty)$ with $F(0) = 0$. Suppose that $(\alpha_n)_{n\in\mathbb{Z}}$ is an increasing doubly infinite sequence of distinct positive reals with*

$$\lim_{n\to-\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} \alpha_n = \infty.$$

Suppose that $(\beta_n)_{n\in\mathbb{Z}}$ is any sequence with

$$0 \le \beta_n \le F(\alpha_n), \qquad n \in \mathbb{Z}.$$

(i) *There is a least concave function G on $[0,\infty)$ such that $G(0) \ge 0$, and $G(\alpha_n) \ge \beta_n$ for $n \in \mathbb{Z}$. The function G is continuous nonnegative and increasing and $G(0) = 0$.*

(ii) *Furthermore, if $n \in \mathbb{Z}$ then either*

$$G(t) = G(\alpha_n)t/\alpha_n, \qquad 0 \le t \le \alpha_n,$$

or there exists $m < n$ so that

$$G(t) = \beta_m + \frac{G(\alpha_n) - \beta_m}{\alpha_n - \alpha_m}(t - \alpha_m), \qquad \alpha_m \le t \le \alpha_n.$$

Proof. (i) is almost immediate. G is defined as the infimum of the collection $\mathcal{C}$ of all increasing concave functions H on $[0,\infty)$ such that $H(\alpha_n) \ge \beta_n$ for all $n \in \mathbb{Z}$ and $H(0) \ge 0$. This collection is non-empty since $F \in \mathcal{C}$. Next, G is affine on each interval $[\alpha_n, \beta_{n+1}]$ and since $G \le F$, G is continuous at 0.

For (ii), assume G is not affine on $[0, \alpha_n]$. Then there exists a least $p < n$ so that g is affine on $[\alpha_p, \alpha_n]$. Let G_0 be the function equal to G on $[0, \alpha_{p-1}]$ and $[\alpha_n, \infty)$, and affine on $[\alpha_{p-1}, \alpha_n]$. Then for any $0 < \lambda < 1$ we have $(1-\lambda)G + \lambda G_0 \notin \mathcal{C}$. Hence there exists $k(\lambda) \in \{p, p+1, \ldots, n-1\}$ so that

$$(1-\lambda)G(\alpha_{k(\lambda)}) + \lambda G_0(\alpha_{k(\lambda)}) < \beta_{k(\lambda)}.$$

Letting $\lambda \to 0$ through a suitable sequence where $k(\lambda) = m < n$ is constant we obtain $G(\alpha_m) = \beta_m$ and the second alternative holds. ∎

We now prove one of our main results.

THEOREM 3.3. *Let E be a fully symmetric Banach function space on $(0,\infty)$ with a Fatou norm. Suppose $f \in E_+ \setminus L_1$ is such that $\Omega(f) = \mathcal{Q}_E(f)$. Then*

$$\lim_{\tau\to\infty} \tau^{-1}\|\sigma_\tau(f^*)\|_E = 0.$$

Proof. We may suppose that f is decreasing. We let

$$F(t) = \int_0^t f(s)\,ds.$$

Let us define a doubly infinite sequence $(a_n)_{n\in\mathbb{Z}}$ by $F(a_n) = (5/4)^n$.

We next introduce the family $\mathcal{K}$ of doubly infinite sequences $\kappa = (\kappa_n)_{n\in\mathbb{Z}}$ such that either $\kappa_n \in \mathbb{N}$ with $1 \le \kappa_n < a_{n+1}/a_n$ or $\kappa_n = \infty$. Then $\mathcal{K}$ is a complete lattice under the order $\kappa \le \kappa'$ if $\kappa_n \le \kappa'_n$ for all n. We may define the lattice operations $(\kappa \vee \kappa')_n = \max(\kappa_n, \kappa'_n)$ and $(\kappa \wedge \kappa')_n = \min(\kappa_n, \kappa'_n)$.

For each $\kappa \in \mathcal{K}$ we define $\psi_\kappa \in E$ as follows. Let $\Psi(t) = \Psi_\kappa(t)$ be the least increasing concave function such that $\Psi(0) \ge 0$,

$$\begin{aligned} \Psi(\kappa_n a_n) &\ge F(a_n) && \text{if } \kappa_n < \infty, \\ \Psi(a_n) &\ge 0 && \text{if } \kappa_n = \infty. \end{aligned}$$

The existence and properties of Ψ are guaranteed by applying Lemma 3.2 when $\alpha_n = \kappa_n a_n$ if $\kappa_n < \infty$ and $\alpha_n = a_n$ if $\kappa_n = \infty$ and $\beta_n = F(a_n)$ if $\kappa_n < \infty$ and $\beta_n = 0$ if $\kappa_n = \infty$. Since $F(a_n) \le F(\kappa_n a_n)$ it is clear from Lemma 3.2 that Ψ exists and $\Psi \le F$. Furthermore, Ψ is piecewise affine on $(0,\infty)$ and we may define $\psi_\kappa = \Psi'$, which is a nonnegative piecewise constant decreasing function on $(0,\infty)$. Clearly, $\psi_\kappa \in \Omega(f) \subset E$.

We note some elementary properties of the map $\kappa \mapsto \psi_\kappa$.

Lemma 3.4.

$$\psi_\kappa \preceq \psi_{\kappa'} \qquad \textit{if } \kappa' \le \kappa, \tag{8}$$

$$\psi_{\kappa\wedge\kappa'} \preceq \psi_\kappa \vee \psi_{\kappa'} \qquad \textit{for } \kappa, \kappa' \in \mathcal{K}. \tag{9}$$

Proof. (8) is quite trivial.

To see (9) note that

$$\int_0^t \max(\psi_\kappa(s), \psi_{\kappa'}(s))\,ds \ge \max(\Psi_\kappa(t), \Psi_{\kappa'}(t)).$$

Now if $\kappa_n \wedge \kappa'_n < \infty$ and $\kappa_n \le \kappa'_n$ we have

$$\int_0^{\kappa_n a_n} \max(\psi_\kappa(s), \psi_{\kappa'}(s))\,ds \ge \Psi_\kappa(\kappa_n a_n) \ge F(a_n),$$

and with a similar inequality when $\kappa'_n < \kappa_n$ we obtain, from the definition of $\Psi_{\kappa\wedge\kappa'}$,

$$\int_0^t \max(\psi_\kappa(s), \psi_{\kappa'}(s))\,ds \ge \Psi_{\kappa\wedge\kappa'}(t), \qquad 0 \le t < \infty.$$

This proves (9). ∎

LEMMA 3.5. *Suppose $\kappa \in \mathcal{K}$ satisfies*

$$\max(\kappa_n, \kappa_{n+1}) = \infty, \qquad n \in \mathbb{Z}. \tag{10}$$

Then for any $n \in \mathbb{Z}$ such that $\kappa_n < \infty$ we have

$$\psi_\kappa(t) \geq \frac{9F(a_n)}{25\kappa_n a_n}, \qquad a_n \leq t \leq \kappa_n a_n. \tag{11}$$

Proof. If f is not identically zero then ψ_κ is only identically zero when κ is identically ∞; we exclude this case, so that $\Psi_\kappa(t) > 0$ for $t > 0$. Observe first that ψ_κ is constant on $(a_n, \kappa_n a_n)$. If for every $m < n$ such that $\kappa_m < \infty$ we have $\Psi_\kappa(\kappa_m a_m) > F(a_m)$ then

$$\psi_\kappa(t) \geq \frac{F(a_n)}{\kappa_n a_n}, \qquad 0 < t \leq \kappa_n a_n.$$

Otherwise, since $\Psi_\kappa(t) > 0$ for all $t > 0$, we see that, by Lemma 3.2, there exists $m < n$ so that $\kappa_m < \infty$ and

$$\psi_\kappa(t) = \frac{\Psi_\kappa(\kappa_n a_n) - F(a_m)}{\kappa_n a_n - \kappa_m a_m}, \qquad \kappa_m a_m < t < \kappa_n a_n.$$

Then

$$\psi_\kappa(t) \geq \frac{F(a_n) - F(a_m)}{\kappa_n a_n - \kappa_m a_m}, \qquad a_n \leq t \leq \kappa_n a_n.$$

Noting that $m \leq n-2$ by (10), so that $F(a_m) \leq (4/5)^2 F(a_n)$, this implies that (11) holds for either alternative. ∎

For $\kappa \in \mathcal{K}$ and $r \in \mathbb{N}$ we will define a $\kappa^{[r]} \geq \kappa$ by suppressing the values of κ which are less than r. Precisely,

$$\kappa_n^{[r]} = \begin{cases} \kappa_n & \text{if } \kappa_n \geq r, \\ \infty & \text{if } \kappa_n < r. \end{cases}$$

We next prove the following lemma, which is the heart of the argument for Theorem 3.3:

LEMMA 3.6. *Under the hypotheses of the theorem, for any $\kappa \in \mathcal{K}$ we have*

$$\lim_{r\to\infty} \|\psi_{\kappa^{[r]}}\|_E = 0.$$

Proof. We will first prove the lemma under the additional assumption that (10) holds. Since $\|\psi_{\kappa^{[r]}}\|_E$ is decreasing in r (by (8)) it suffices to show that for given $\epsilon > 0$ we can find r so that $\|\psi_{\kappa^{[r]}}\|_E < \epsilon$. By Proposition 3.1 for any $\epsilon > 0$ we can find a nonnegative decreasing function $h \leq \psi_\kappa$ and an integer p so that

$$\int_{pa}^{b} h(t)\,dt \leq \int_{a}^{b} f(t)\,dt, \qquad 0 < pa < b < \infty, \tag{12}$$

and

$$\|\psi_\kappa - h\|_E < \epsilon/10. \tag{13}$$

We shall take $r = 36p$. Let $v = 10(\psi_\kappa - h)$. We will show that $\psi_{\kappa^{[r]}} \preceq v$. In order to do this we must show that if $\kappa_n^{[r]} < \infty$ we have

$$F(a_n) \le \int_0^{\kappa_n a_n} v^*(t)\,dt. \tag{14}$$

If $\kappa_n^{[r]} < \infty$ then

$$\begin{aligned}\int_0^{\kappa_n a_n} v^*(t)\,dt \ge \int_{pa_n}^{\kappa_n a_n} v(t)\,dt &= 10\Big(\int_{pa_n}^{\kappa_n a_n} \psi_\kappa(t)\,dt - \int_{pa_n}^{\kappa_n a_n} h(t)\,dt\Big)\\ &\ge 10\Big(\int_{pa_n}^{\kappa_n a_n} \psi_\kappa(t)\,dt - \int_{a_n}^{\kappa_n a_n} f(t)\,dt\Big),\end{aligned}$$

by (13). Hence by (11) of Lemma 3.5,

$$\begin{aligned}\int_0^{\kappa_n a_n} v^*(t)\,dt &\ge 10\bigg(\frac{9(\kappa_n a_n - pa_n)F(a_n)}{25\kappa_n a_n} - \int_{a_n}^{a_{n+1}} f(t)\,dt\bigg)\\ &\ge 10\bigg(\frac{35}{36}\,\frac{9}{25}\,F(a_n) - \frac{1}{4}\,F(a_n)\bigg) = F(a_n).\end{aligned}$$

This shows that (14) holds and so $\psi_{\kappa^{[r]}} \preceq v$ and $\|\psi_{\kappa^{[r]}}\|_E < \epsilon$. This completes the proof when (10) holds.

For the general case let us define $\kappa(0)_n = \kappa_n$ if n is even and $\kappa(0)_n = \infty$ if n is odd. Similarly, $\kappa(1)_n = \kappa_n$ if n is odd and $\kappa(1)_n = \infty$ if n is even. Both $\kappa(0)$ and $\kappa(1)$ satisfy (10). Then for an arbitrary κ we have $\kappa^{[r]} = \kappa(0)^{[r]} \wedge \kappa(1)^{[r]}$ and so by (9),

$$\limsup_{r\to\infty} \|\psi_{\kappa^{[r]}}\|_E \le \limsup_{r\to\infty} \|\psi_{\kappa(0)^{[r]}}\|_E + \limsup_{r\to\infty} \|\psi_{\kappa(1)^{[r]}}\|_E = 0. \quad \blacksquare$$

Next, for any integer p we define $\gamma_n^p = p$ if $pa_n < a_{n+1}$, and $\gamma_n^p = \infty$ otherwise. For each $q > p$ we define $\gamma_n^{p,q} = p$ if $pa_n < a_{n+1}$ and $|n| \le q$, and $\gamma_n^{p,q} = \infty$ otherwise. Let $\psi_p = \psi_{\gamma^p}$ and $\psi_{p,q} = \psi_{\gamma^{p,q}}$.

LEMMA 3.7. *Under the hypotheses of the theorem,*

$$\lim_{p\to\infty} \|\psi_p\|_E = 0.$$

Proof. Clearly, $\|\psi_p\|_E$ is decreasing in p. Assume $\|\psi_p\|_E > \epsilon > 0$ for all $p \in \mathbb{N}$. Since E has a Fatou norm, for each p there exists $q(p) > p$ so that $\|\psi_{p,q(p)}\|_E > \epsilon$. Let

$$\kappa = \textstyle\bigwedge_p \gamma^{p,q(p)}.$$

Thus κ is given by the formula

$$\kappa_n = \inf\{p : \ p < a_{n+1}/a_n, \ |n| \le q(p)\}$$

and κ has the properties that $\kappa \le \gamma_{p,q(p)}$ for all p and $\lim_{|n|\to\infty} \kappa_n = \infty$.

By Lemma 3.6 there exists $r \in \mathbb{N}$ so that $\|\psi_{\kappa^{[r]}}\|_E < \epsilon$. But then the set $\{n : \kappa_n < r\}$ is finite and so there is a choice of p such that $p > a_{n+1}/a_n$ whenever $\kappa_n < r$. Thus $\gamma_n^p = \infty$ if $\kappa_n < r$. Therefore

$$\kappa^{[r]} \le \gamma^{p,q(p)}$$

and so by (8),

$$\|\psi_{p,q(p)}\|_E < \epsilon,$$

which gives a contradiction. ∎

We can now complete the proof of Theorem 3.3. We will show that if $p \in \mathbb{N}$, then

$$F(t) \le \frac{4}{5} F(p^2 t) + \frac{5}{4} \int_0^{p^2 t} \psi_p(s)\, ds, \qquad 0 < t < \infty. \tag{15}$$

Indeed, if (15) fails for some t, we can assume $a_n \le t < a_{n+1}$ for some $n \in \mathbb{Z}$. We first argue that $a_{n+1} \le p a_n$. Suppose, on the contrary, that $a_{n+1} > p a_n$. Then we have

$$\frac{5}{4} \int_0^{p^2 t} \psi_p(s)\, ds \ge \frac{5}{4} \int_0^{p a_n} \psi_p(s)\, ds \ge \frac{5}{4} F(a_n) = F(a_{n+1}) \ge F(t),$$

which contradicts our hypothesis. Next we show that $a_{n+2} \le p a_{n+1}$. Indeed, if $a_{n+2} > p a_{n+1}$, then $p^2 t \ge p a_{n+1}$ and

$$\frac{5}{4} \int_0^{p^2 t} \psi_p(s)\, ds \ge \frac{5}{4} \int_0^{p a_{n+1}} \psi_p(s)\, ds \ge \frac{5}{4} F(a_{n+1}) > F(t).$$

But then $a_{n+2} \le p^2 a_n$ and so

$$\frac{4}{5} F(p^2 t) \ge \frac{4}{5} F(a_{n+2}) = F(a_{n+1}) > F(t)$$

and we have a contradiction. This establishes (15).

Now if $\tau \ge 1$ we replace t in (15) by t/τ and interpret the inequality in the form

$$\frac{1}{\tau} \sigma_\tau f \preceq \frac{1}{p^{-2}\tau} \sigma_{p^{-2}\tau} \left(\frac{4}{5} f + \frac{5}{4} \psi_p \right).$$

Hence

$$\lim_{\tau\to\infty} \tau^{-1} \|\sigma_\tau f\|_E \le \frac{4}{5} \lim_{\tau\to\infty} \tau^{-1} \|\sigma_\tau f\|_E + \frac{5}{4} \lim_{\tau\to\infty} \tau^{-1} \|\sigma_\tau \psi_p\|_E$$

so that

$$\lim_{\tau\to\infty} \tau^{-1} \|\sigma_\tau f\|_E \le \frac{5^2}{4} \|\psi_p\|_E.$$

Combining with Lemma 3.7 we obtain the theorem. ∎

The case when $f \in L_1$ is handled by reduction to the previous case:

THEOREM 3.8. *Let E be a fully symmetric Banach function space on $(0,\infty)$ with a Fatou norm. Suppose f is a decreasing nonnegative function such that $f \in E_+ \cap L_1$ and $\Omega(f) = \mathcal{Q}_E(f)$. Then*

$$\lim_{\tau\to\infty} \tau^{-1}\|\sigma_\tau(f^*)\|_{E+L_\infty} = 0.$$

Proof. An easy computation shows that $\mathcal{Q}(f+1) = \mathcal{Q}(f)+\mathcal{Q}(1)$. Hence $\mathcal{Q}_{E+L_\infty}(f+1) \supset \mathcal{Q}_E(f)+\mathcal{Q}_{L_\infty}(1) = \Omega(f)+\Omega(1)$. If $0 \le g \in \Omega(f+1)$ then $g - g\wedge 1 \in \Omega(f)$ and $g\wedge 1 \in \Omega(1)$ so that $\Omega(f)+\Omega(1) = \Omega(f+1)$. Hence $\mathcal{Q}_{E+L_\infty}(f+1) = \Omega(f+1)$ and we can apply Theorem 3.3. ∎

4. The main results. We can next state our main results:

THEOREM 4.1. *Let E be a fully symmetric Banach function space on $(0,\infty)$ with a Fatou norm, and such that $E \setminus L_1 \neq \emptyset$. Suppose $f \in E$. Then $\Omega(f) = \mathcal{Q}_E(f)$ if and only if* $\lim_{\tau\to\infty}\tau^{-1}\|\sigma_\tau(f^*)\|_E = 0$.

Proof. If $\lim_{\tau\to\infty}\tau^{-1}\|\sigma_\tau(f^*)\|_E = 0$ then $\Omega_+(f) \subset \mathcal{Q}_E(f)$ by Theorem 25 of [14]; thus $\Omega(f) = \mathcal{Q}_E(f)$. Conversely, if $\Omega(f) = \mathcal{Q}_E(f)$ we have either

$$\lim_{\tau\to\infty} \tau^{-1}\|\sigma_\tau(f)\|_E = 0$$

(when $f \notin L_1$ by Theorem 3.3), or

$$\lim_{\tau\to\infty} \tau^{-1}\|\sigma_\tau(f)\|_{E+L_\infty} = 0$$

(when $f \in L_1$ by Theorem 3.8). Then Lemma 2.1 shows that in both cases we have $\lim_{\tau\to\infty}\tau^{-1}\|\sigma_\tau(f)\|_E = 0$. ∎

THEOREM 4.2. *Let E be a fully symmetric Banach function space on $(0,\infty)$ with a Fatou norm, and such that $E \subset L_1$. If $f \in E$ then $\Omega(f) = \mathcal{Q}_E(f)$ if and only if* $\lim_{\tau\to\infty}\tau^{-1}\|\sigma_\tau(f^*)\|_{E+L_\infty} = \lim_{\tau\to\infty}\tau^{-1}\|\sigma_\tau(f^*)\chi_{(0,1)}\|_E = 0$.

Proof. The proof is very similar to that of Theorem 4.1 using instead Theorem 24 of [14]. ∎

We first give the extension to function spaces on $(0,1)$.

THEOREM 4.3. *Let E be a fully symmetric Banach function space on $(0,1)$ with a Fatou norm. Suppose $f \in E$. Then $\Omega(f) = \mathcal{Q}_E(f)$ if and only if* $\lim_{\tau\to\infty}\tau^{-1}\|\sigma_\tau(f^*)\|_E = 0$.

Proof. We define a function space F on $(0,\infty)$ by $f \in F$ if and only if $f^*\chi_{(0,1)} \in E$ and $f \in L_1$, with the norm

$$\|f\|_F = \max(\|f^*\chi_{(0,1)}\|_E, \|f^*\|_{L_1}).$$

Suppose $f \in E$ is nonnegative and decreasing. We will show that, regarding f as a member of F, we have $\Omega(f) = \mathcal{Q}_F(f)$. Note that the hypothesis $\Omega(f) = \mathcal{Q}_E(f)$ on $(0,1)$ implies only that if $g \in F$ and $g \in \Omega(f)$ then $g \in$

$\mathcal{Q}_F(f)$ provided $\mu(\operatorname{supp} g) \le 1$. We will show, however, that $\Omega(f) = \mathcal{Q}_F(f)$, and then the theorem follows.

We will need the following lemma:

LEMMA 4.4. *Let $h \in F$ be nonnegative and decreasing. Suppose $g \in F$ is nonnegative and decreasing, and $g \preceq h$ and $g(x) = 0$ for some $0 < x < \infty$. If there exists $c > 0$ such that $g(t) \le h(t)$ for $0 < t \le c$ then $g \in \mathcal{Q}_F(h)$.*

Proof. For any $\theta > 1$ we may pick $p > x/c$ so that

$$\int_0^c h(s)\,ds \le \theta \int_{x/p}^c h(s)\,ds.$$

Then if $0 < pa < b < \infty$ with $c \le b \le x$ we have

$$\int_{pa}^b g(s)\,ds \le \int_0^b h(s)\,ds \le \theta \int_{x/p}^b h(s)\,ds \le \theta \int_{pa}^b h(s)\,ds.$$

The same inequality holds trivially if $b > x$ or $b < c$. Thus by Theorem 6.3 of [8] we have $g \in \lambda Q(h)$ for any $\lambda > 1$, and the lemma follows. ∎

We continue the proof of the theorem. We will assume without loss of generality that $\int_0^1 f(t)\,dt = 1$. First suppose $g \in \Omega(f)$ is nonnegative, not identically zero, and decreasing and satisfies $g(x) = 0$ for some $0 < x < \infty$. Given $\epsilon > 0$ we may find $c_0 > 0$ so that

$$\int_0^{c_0} f(s)\,ds < \frac{\epsilon}{2}.$$

Let

$$\alpha = \sup_{0<t\le c_0} \frac{\int_0^t g(s)\,ds}{\int_0^t f(s)\,ds}.$$

We have $\alpha > 0$ and we may pick $0 < \beta < \alpha$ with $\alpha - \beta < \epsilon/2$ and then $0 < c < c_0$ with

$$g(c) > \beta f(c).$$

Let $c' \ge c$ be the least solution of

$$\alpha \int_0^{c'} f(s)\,ds = \int_0^c g(s)\,ds + (c' - c)g(c).$$

We now define

$$h(t) = \begin{cases} g(t) + (1-\alpha)f(t), & 0 < t \le c, \\ g(c) + (1-\alpha)f(t), & c < t \le \min(c', 1), \\ f(t), & \min(c', 1) < t \le 1, \\ 0, & t \ge 1. \end{cases}$$

From the construction we have $h \in \Omega(f)$. Thus $h \in \mathcal{Q}_F(f)$. For any $t \le \min(c', 1)$ we have

$$\int_0^t g(s)\,ds \le \int_0^t h(s)\,ds.$$

If $c' < 1$ then

$$\int_0^t h(s)\,ds = \int_0^t f(s)\,ds \ge \int_0^t g(s)\,ds, \qquad t > c'.$$

If $c' \ge 1$ then

$$\begin{aligned}\int_0^1 h(s)\,ds &\ge (1-\alpha)\int_0^1 f(s)\,ds + \int_0^c g(s)\,ds + g(c)(1-c)\\ &\ge (1-\alpha)\int_0^1 f(s)\,ds + \beta\int_0^c f(s)\,ds - \frac{\epsilon}{2} + \beta f(c)(1-c)\\ &\ge (1-\alpha)\int_0^1 f(s)\,ds + \beta\int_0^1 f(s)\,ds - \frac{\epsilon}{2} \ge 1-\epsilon.\end{aligned}$$

Hence $(1-\epsilon)g \preceq h$ and by Lemma 4.4 we have $(1-\epsilon)g \in \mathcal{Q}_F(h)$. Since $\epsilon > 0$ is arbitrary we have $g \in \mathcal{Q}_F(h) \subset \mathcal{Q}_F(f)$.

Finally, let us note that for general nonnegative decreasing $g \in F$ we have $\lim_{m\to\infty} \|g - g\chi_{(0,m)}\|_F = 0$ so that $\Omega_F(f) = \mathcal{Q}_F(f)$.

Now the result reduces to Theorems 4.1 and 4.2. ∎

The extension to sequence spaces requires a similar type of argument:

THEOREM 4.5. *Let E be a fully symmetric Banach sequence space with a Fatou norm and such that $E \setminus \ell_1 \ne \emptyset$. Suppose $\xi \in E$. Then $\Omega(\xi) = \mathcal{Q}_E(\xi)$ if and only if $\lim_{m\to\infty} m^{-1}\|\sigma_m(\xi^*)\|_E = 0$.*

Proof. We consider the Banach function space F of all bounded functions such that $(f^*(0), f^*(1), \ldots) \in E$ with the norm

$$\|f\|_F = f^*(0) + \|(a_n)_{n=1}^\infty\|_E,$$

where $f^*(0) = \|f\|_{L_\infty}$ and $a_n := \int_{n-1}^n f^*(s)\,ds$, $n \ge 1$. Then let $F(\mathbb{N})$ be the subspace of F of all functions f which are constant on each interval $(n-1, n]$. Clearly, the Banach spaces $(F(\mathbb{N}), \|\cdot\|_F)$ and $(E, \|\cdot\|_E)$ are linearly isomorphic, in particular

$$\|\xi\|_E \le \|\xi\|_F \le 2\|\xi\|_E \quad \forall \xi \in E = F(\mathbb{N}).$$

Let $\mathbb{E}$ denote the conditional expectation operator

$$\mathbb{E}f = \sum_{n\in\mathbb{N}} \chi_{(n-1,n]} \int_{n-1}^n f(t)\,dt.$$

Suppose ξ is a nonnegative decreasing sequence and let

$$f = \sum_{j=1}^{\infty} \xi_j \chi_{(j-1,j]} \in F.$$

The result will follow from:

THEOREM 4.6. *$\Omega(\xi) = \mathcal{Q}_E(\xi)$ if and only if $\Omega(f) = \mathcal{Q}_F(f)$.*

Proof. Suppose that $\Omega(\xi) = \mathcal{Q}_E(\xi)$. We may assume ξ has infinite support. Suppose $g \in \Omega(f)$ is nonnegative and decreasing. We will show that $g \in \mathcal{Q}_F(f)$ and then it follows that $\mathcal{Q}_F(f) = \Omega(f)$.

Suppose $\epsilon > 0$. Then we may pick an integer $m \in \mathbb{N}$ so that

$$g^*(m) - \lim_{n\to\infty} g^*(n) < \epsilon/4.$$

Now $\mathbb{E}g \in \mathcal{Q}_F(f)$ since $\Omega(\xi) = \mathcal{Q}_E(\xi)$. Hence, by Proposition 3.1 there is a nonnegative decreasing function h with $0 \le h \le \mathbb{E}g$ such that $\|\mathbb{E}g - h\|_E < \epsilon/4$ and, for some $p \in \mathbb{N}$,

$$\int_{pa}^{b} h(s)\,ds \le \int_a^b f(s)\,ds, \qquad 0 < pa < b < \infty.$$

Next we define

$$\varphi(s) = \begin{cases} g(s), & 0 < s \le m, \\ h(s), & m < s < \infty. \end{cases}$$

Note that $0 \le \varphi \preceq g \preceq f$. We show that $\varphi \in \mathcal{Q}_F(f)$. Suppose $r > p$ and $0 < ra < b$. Then if $m \le ra$ we clearly have

$$\int_{ra}^{b} \varphi(s)\,ds \le \int_a^b f(s)\,ds.$$

On the other hand, if $0 < ra < m$, let $c = \min(b, m)$. Then

$$\begin{aligned} \int_{ra}^{b} \varphi(s)\,ds &\le \int_0^b f(s)\,ds \le \int_a^b f(s)\,ds + c\xi_1/r \\ &\le \int_a^b f(s)\,ds + \frac{\xi_1}{(r-1)\xi_m} \int_{c/r}^{c} f(s)\,ds \\ &\le \left(1 + \frac{\xi_1}{(r-1)\xi_m}\right) \int_a^b f(s)\,ds. \end{aligned}$$

Since r is arbitrary these estimates show that $\varphi \in \lambda\mathcal{Q}(f)$ for every $\lambda > 1$ (Theorem 6.4 of [8]) and hence $\varphi \in \mathcal{Q}_F(f)$.

Now

$$\|g - \varphi\|_F = \|(g - \varphi)\chi_{(m,\infty)}\|_F \le \|(g - \mathbb{E}g)\chi_{(m,\infty)}\|_F + \|\mathbb{E}g - h\|_F.$$

However,

$$\|(g-\mathbb{E}g)\chi_{(m,\infty)}\|_F \le \sum_{j=m}^{\infty} 2(g(j)-g(j+1)) < \epsilon/2.$$

Hence

$$d(g,\mathcal{Q}_E(f)) \le \|g-\varphi\|_F < \epsilon.$$

Since $\epsilon>0$ is arbitrary we have $g\in\mathcal{Q}_F(f)$. This shows that $\Omega(f)=\mathcal{Q}_F(f)$.

We next turn to the converse. Assume $\mathcal{Q}_E(f)=\Omega(f)$ and that $\eta\in\Omega(\xi)$ is a decreasing sequence. Let $g=\sum_{n\in\mathbb{N}}\eta_n\chi_{(n-1,n]}$. Then $g\in\Omega(f)$ and so, by Proposition 3.1, given $\epsilon>0$, there exists a decreasing $0\le h\le g$ with $\|g-h\|_F<\epsilon$ and such that for some $p\in\mathbb{N}$ we have

$$\int_{pa}^{b} h(s)\,ds \le \int_{a}^{b} f(s)\,ds, \qquad 0<pa<b<\infty.$$

Let $\zeta\in E$ be defined by $\zeta_n=\int_{n-1}^{n} h(s)\,ds$. Then for $0\le pm\le n$ we have

$$\sum_{k=pm+1}^{n}\zeta_k=\int_{pm}^{n} h(s)\,ds \le \int_{m}^{n} f(s)\,ds=\sum_{k=m+1}^{n}\xi_k.$$

Hence $\zeta\in\lambda\mathcal{Q}(\xi)$ for every $\lambda>1$, by Theorem 5.5 of [8], so that $\zeta\in\mathcal{Q}_E(\xi)$. Furthermore,

$$\|\eta-\zeta\|_E \le \|g-\mathbb{E}h\|_F \le \|g-h\|_F < \epsilon.$$

It now follows that $\eta\in\mathcal{Q}_E(\xi)$ and the proof of the lemma is complete. ■

Theorem 4.5 now follows directly from Theorem 4.6. ■

Let us observe that the argument of Theorem 4.6 allows us to complete the picture for positive orbits in [14]:

THEOREM 4.7. *Let E be a fully symmetric sequence space with a Fatou norm. Then for any $\xi\in E_+$ the set $\Omega_+(\xi)=\Omega(\xi)\cap E_+$ coincides with the closed convex hull of its extreme points if and only if*

$$\lim_{m\to\infty} m^{-1}\|\sigma_m(\xi)\|_E=0.$$

In fact, we can prove by the same argument as in Theorem 4.6 that $\Omega_+(\xi)=\mathcal{Q}_E(\xi)\cap E_+$ if and only if $\Omega_+(f)=\mathcal{Q}_F(f)\cap F_+$.

We remark that in [14] some examples of Marcinkiewicz spaces and Orlicz spaces are discussed in the context of Theorems 4.1, 4.2, 4.3 and 4.5. We refer the reader to that paper for details. We take the opportunity to improve Proposition 33 of [14]:

PROPOSITION 4.8. *Let M be an Orlicz function. Then for any $f\in L_M(0,\infty)$ we have $\Omega(f)=\mathcal{Q}_{L_M}(f)$. Similarly, for any $\xi\in\ell_M$ we have $\Omega(\xi)=\mathcal{Q}_{\ell_M}(\xi)$.*

Proof. We give the proof only for $L_M(0,\infty)$. Suppose first that $M(t) = o(t)$ when $t \to 0$. We show that $\|\lim_{\tau\to\infty} \tau^{-1}\sigma_\tau f\|_{L_M} = 0$ whenever $f \in (L_M)_+$. Suppose $\alpha > 0$. Then

$$\int_0^\infty M\left(\frac{\alpha f(s/\tau)}{\tau}\right) ds = \int_0^\infty \tau M\left(\frac{\alpha f(s)}{\tau}\right) ds$$

for any $\tau > 1$. Since $f \in L_M$ there exists τ_0 such that

$$\int_0^\infty \tau_0 M\left(\frac{\alpha f(s)}{\tau_0}\right) ds < \infty.$$

Now letting $\tau \to \infty$ we deduce from the Dominated Convergence Theorem that

$$\lim_{\tau\to\infty} \int_0^\infty \tau M\left(\frac{\alpha f(s)}{\tau}\right) ds = 0,$$

so that $\lim_{\tau\to\infty} \tau^{-1}\|\sigma_\tau f\|_{L_M} = 0$ and we can apply Theorem 4.1.

Now if $M(t) \geq ct$ for all $t > 0$ where $c > 0$, then $L_M \subset L_1$. For any $\alpha > 0$ and $\tau > 1$,

$$\int_0^1 M\left(\frac{\alpha f^*(s/\tau)}{\tau}\right) ds = \int_0^1 \tau\chi_{(0,\tau^{-1})}(s) M\left(\frac{\alpha f^*(s)}{\tau}\right) ds.$$

As before, the right-hand side is integrable for some $\tau = \tau_0$ and we can apply the Dominated Convergence Theorem to deduce that $\tau^{-1}\|(\sigma_\tau f^*)\chi_{(0,1)}\|_{L_M}$ tends to 0 as τ approaches infinity. Now one can apply Theorem 4.2. ∎

5. A noncommutative analog. Let $\mathcal{H}$ be a separable complex Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$ and by $\mathcal{K}(\mathcal{H})$ the ideal of compact operators on $\mathcal{H}$. For any $T \in \mathcal{B}(\mathcal{H})$ we define the singular values

$$s_n(T) = \inf \|T(I-P)\|,$$

where the infimum is taken over all orthogonal projections P such that $\operatorname{rank}(P) < n$.

If E is a symmetric sequence space then we can define a Banach ideal of compact operators on $\mathcal{H}$ by $T \in \mathcal{S}_E$ if and only if $(s_k(T))_{k=1}^\infty \in E$, and then the norm is given by $\|T\|_E = \|(s_k(T))_{k=1}^\infty\|_E$. For fully symmetric spaces this is well-known (e.g. see [7]) but for symmetric spaces it follows from [8].

Let $\mathcal{H}$ be a separable Hilbert space and suppose $T \in \mathcal{K}(\mathcal{H})$. Let $\mathcal{Q}(T)$ be the convex hull of the set $\{ATB : \|A\|, \|B\| \leq 1\}$. We define its orbit $\Omega(T)$ to be the closure of $\mathcal{Q}(T)$ in $\mathcal{K}(\mathcal{H})$. It is easy to check from the definition that $R \in \Omega(T)$ if and only if

$$\sum_{k=1}^{n} s_k(R) \leq \sum_{k=1}^{n} s_k(T), \qquad n = 1, 2, \ldots.$$

For any symmetric Banach sequence space E we may define $\mathcal{Q}_E(T)$ to be the closure of $\mathcal{Q}(T)$ in $\mathcal{S}_E$.

THEOREM 5.1. *Let $\mathcal{E}$ be a fully symmetric sequence space with a Fatou norm. Suppose $\mathcal{S}_E$ is the corresponding ideal of compact operators. Then for $T \in \mathcal{S}_E$ we have $\Omega(T) = \mathcal{Q}_E(T)$ if and only if*

$$\lim_{m\to\infty} m^{-1} \|\sigma_m(s_k(T))_{k=1}^{\infty}\|_E = 0.$$

Proof. Let $\xi = (s_k(T))_{k=1}^{\infty}$. Let $R \in \mathcal{K}(\mathcal{H})$ and let $\eta = (s_k(R))_{k=1}^{\infty}$. If $R \in \mathcal{Q}(T)$ then it follows from Proposition 8.6 and Theorem 5.5 of [8] that $\eta \in \lambda\mathcal{Q}(\xi)$ for every $\lambda > 1$.

First suppose that $\Omega(T) = \mathcal{Q}_E(T)$. If $S \in \Omega(T)$ then given $\epsilon > 0$ there exists $R \in \mathcal{Q}(T)$ with $\|R - S\|_E < \epsilon$. Let $\zeta = (s_k(S))_{k=1}^{\infty}$. Then by the submajorization inequality of [5],

$$\eta - \zeta \preceq (s_k(R - S))_{k=1}^{\infty},$$

so that $\|\eta - \zeta\|_E < \epsilon$. Since $\eta \in \mathcal{Q}_E(\xi)$ and $\epsilon > 0$ is arbitrary, this implies that $\zeta \in \mathcal{Q}_E(\xi)$ and so $\mathcal{Q}_E(\xi) = \Omega(\xi)$. Theorem 4.5 can then be applied.

The converse direction is immediate. ∎

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