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The joint essential numerical range, compact perturbations, and the Olsen problem

by

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Dedicated to Professor Muneo Chō on his 60th birthday

Abstract. Let $T_1, \dots, T_n$ be bounded linear operators on a complex Hilbert space H. Then there are compact operators $K_1, \dots, K_n \in B(H)$ such that the closure of the joint numerical range of the n-tuple $(T_1 - K_1, \dots, T_n - K_n)$ equals the joint essential numerical range of $(T_1, \dots, T_n)$. This generalizes the corresponding result for $n = 1$.

We also show that if $S \in B(H)$ and $n \in \mathbb{N}$ then there exists a compact operator $K \in B(H)$ such that $\|(S - K)^n\| = \|S^n\|_e$. This generalizes results of C. L. Olsen.

1. Introduction. Let T be a bounded linear operator acting on a complex Hilbert space H. The properties of T can be frequently improved by adding to it a compact perturbation. For example:

- [S] there exists a compact operator $K_1 \in B(H)$ such that
$$\sigma(T + K_1) = \sigma_W(T),$$
where $\sigma_W(T) = \bigcap\{\sigma(T + L) : L \in B(H) \text{ compact}\}$ denotes the Weyl spectrum of T;
- [CSSW] there exists a compact operator K_2 such that $\overline{W(T + K_2)} = W_e(T)$, where W denotes the numerical range and W_e the essential numerical range;
- [OP] if p is a polynomial and $p(T)$ is compact, then there exists a compact operator K_3 such that $p(T + K_3) = 0$.

The present paper studies variations of the second and third result mentioned above.

In the next section we generalize the second result to n-tuples of operators.

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In Section 3 we study the following problem of C. L. Olsen which is still open: if $S \in B(H)$ and p a polynomial, does there exist a compact operator L such that $\|p(S+L)\| = \|p(S)\|_e$?

A positive answer is known only for some special polynomials. In [O] it was proved for $p(z) = z, z^2, z^3$. In [CLSW] a positive answer was given for all linear polynomials.

We improve the result of [O] and give a positive answer for all powers $p(z) = z^n$.

2. Joint numerical range and compact perturbations. Let $T = (T_1, \dots, T_n)$ be an n-tuple of operators on a complex Hilbert space H. The *joint numerical range* $W(T)$ is defined by

$$W(T) = \{(\langle T_1 x, x\rangle, \dots, \langle T_n x, x\rangle) : x \in H, \|x\| = 1\}.$$

It is well known that the numerical range $W(T)$ is convex for $n = 1$ but not convex in general for $n \geq 2$.

Apart from the (spatial) numerical range $W(T)$ it is also possible to define an algebraic numerical range. Let $\mathcal{A}$ be a Banach algebra with unit e, let $a = (a_1, \dots, a_n) \in \mathcal{A}^n$. The *joint algebraic numerical range* $V(a, \mathcal{A})$ is defined by

$$V(a, \mathcal{A}) = \{(f(a_1), \dots, f(a_n)) : f \in \mathcal{A}^*,\ \|f\| = 1 = f(e)\}.$$

It is well known that $V(a, \mathcal{A})$ is always a compact convex subset of $\mathbb{C}^n$; see [BD1, p. 24].

For n-tuples of Hilbert space operators the spatial and algebraic numerical ranges are closely connected. Denote by $B(H)$ the algebra of all bounded linear operators on a Hilbert space H. By $\overline{\operatorname{conv}}\, M$ we denote the closed convex hull of a set M.

THEOREM 1. *Let $T = (T_1, \dots, T_n) \in B(H)^n$. Then*

$$V(T, B(H)) = \overline{\operatorname{conv}}\, W(T).$$

Proof. The statement is well known for $n = 1$ (see [BD1, p. 83]). For $n \geq 2$ it is more or less folklore. For the sake of convenience we briefly indicate the proof.

Clearly, $W(T) \subset V(T, B(H))$. Since $V(T, B(H))$ is closed and convex, we have $\overline{\operatorname{conv}}\, W(T) \subset V(T, B(H))$.

Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n \setminus \overline{\operatorname{conv}}\, W(T)$. Then there are $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that

$$\begin{aligned}
\operatorname{Re}\sum_{j=1}^{n}\alpha_j z_j > {} & \sup\Big\{\operatorname{Re}\sum_{j=1}^{n}\alpha_j\lambda_j : (\lambda_1,\dots,\lambda_n)\in W(T)\Big\}\\
= {} & \sup\Big\{\operatorname{Re}\Big\langle\sum_{j=1}^{n}\alpha_j T_j x, x\Big\rangle : x\in H,\ \|x\|=1\Big\}\\
= {} & \sup\Big\{\operatorname{Re}\mu : \mu\in W\Big(\sum_{j=1}^{m}\alpha_j T_j\Big)\Big\}\\
= {} & \sup\Big\{\operatorname{Re}\mu : \mu\in V\Big(\sum_{j=1}^{n}\alpha_j T_j, B(H)\Big)\Big\}\\
= {} & \sup\Big\{\operatorname{Re}\sum_{j=1}^{n}\alpha_j f(T_j) : f\in B(H)^*,\ \|f\|=1=f(I)\Big\}\\
= {} & \sup\Big\{\operatorname{Re}\sum_{j=1}^{n}\alpha_j\lambda_j : (\lambda_1,\dots,\lambda_n)\in V(T,B(H))\Big\}.
\end{aligned}$$

Hence $(z_1,\dots,z_n)\notin V(T,B(H))$ and $V(T,B(H))=\overline{\operatorname{conv}}\,W(T)$. ∎

Let H be an infinite-dimensional Hilbert space. The *joint essential numerical range* $W_e(T)$ of an n-tuple $T=(T_1,\dots,T_n)\in B(H)^n$ is the set of all $(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ such that there exists an orthonormal sequence $(x_k)\subset H$ with

$$\lambda_j=\lim_{k\to\infty}\langle T_j x_k, x_k\rangle \qquad (j=1,\dots,n).$$

The essential numerical range for $n=1$ was introduced and studied in [FSW]. The joint essential numerical range was studied e.g. in [TC], [B], [LP].

Although $W(T)$ is not convex in general for $n\geq 2$, it was shown in [LP] that $W_e(T)$ is always convex. In fact, $W_e(T)$ is equal to the algebraic numerical range of the corresponding classes in the Calkin algebra $B(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the ideal of all compact operators on H. Denote by $\pi: B(H)\to B(H)/\mathcal{K}(H)$ the canonical projection. As usual, we write $\pi(T)=(\pi(T_1),\dots,\pi(T_n))$.

We summarize the basic properties of the joint essential numerical range $W_e(T)$ in the following theorem:

THEOREM 2. *Let H be an infinite-dimensional Hilbert space and let $T=(T_1,\dots,T_n)\in B(H)^n$. Then $W_e(T)$ is a compact convex subset of $\mathbb{C}^n$ and*

$$\begin{aligned}
W_e(T) = {} & \bigcap_{K\in\mathcal{K}(H)^n}\overline{W(T+K)} = \bigcap_{K\in\mathcal{K}(H)^n}\overline{\operatorname{conv}}\,W(T+K)\\
= {} & V(\pi(T),B(H)/\mathcal{K}(H)).
\end{aligned}$$

Proof. The first two equalities were proved in [LP]. They also imply the compactness and convexity of $W_e(T)$.

If $f \in (B(H)/\mathcal{K}(H))^*$ and $\|f\| = 1 = f(I + \mathcal{K}(H))$ then $f \circ \pi \in B(H)^*$ and $\|f \circ \pi\| = 1 = (f \circ \pi)(I)$. Moreover, $f \circ \pi$ annihilates compact operators. So

$$V(\pi(T), B(H)/\mathcal{K}(H)) \subset V(T + K, B(H))$$

for all $K \in \mathcal{K}(H)^n$. Hence

$$\begin{aligned} V(\pi(T), B(H)/\mathcal{K}(H)) &\subset \bigcap_{K \in \mathcal{K}(H)^n} V(T + K, B(H)) \\ &= \bigcap_{K \in \mathcal{K}(H)^n} \overline{\operatorname{conv}}\, W(T + K) = W_e(T). \end{aligned}$$

Conversely, let $\lambda = (\lambda_1, \dots, \lambda_n) \in W_e(T)$, i.e., there exists an orthonormal sequence $(x_k) \subset H$ such that $\lambda_j = \lim_{k\to\infty} \langle T_j x_k, x_k \rangle$ for all $j = 1, \dots, n$. Let LIM be any Banach limit. Define $f \in B(H)^*$ by $f(S) = \mathrm{LIM}_{k\to\infty} \langle S x_k, x_k \rangle$ $(S \in B(H))$. Then f annihilates all compact operators and so it induces a functional $\tilde{f} \in (B(H)/\mathcal{K}(H))^*$. Clearly, $\|\tilde{f}\| = 1 = \tilde{f}(I + \mathcal{K}(H))$ and $\tilde{f}(\pi(T)) = \lambda$. Hence $\lambda \in V(\pi(T), B(H)/\mathcal{K}(H))$. ∎

The main result of this section is that for each n-tuple $(T_1, \dots, T_n)$ of Hilbert space operators there exists an n-tuple of compact operators $(K_1, \dots, K_n)$ such that $W_e(T_1, \dots, T_n) = W(T_1 - K_1, \dots, T_n - K_n)^-$. This improves the results of [LP] mentioned in Theorem 2.

For $n = 1$ the existence of the optimal compact perturbation was proved in [CSSW]. We use the operator-theoretical method of [CLSW] (however, this method is only sketched there in a not very clear way, the paper [CLSW] uses mainly another method based on the notion of M-ideals).

Remark 3. In fact, the joint essential numerical range $W_e(T)$ was studied in [LP] only for n-tuples of selfadjoint operators. However, every operator $S \in B(H)$ can be written as $S = \operatorname{Re} S + i \operatorname{Im} S$, where $\operatorname{Re} S = \frac{1}{2}(S + S^*)$ and $\operatorname{Im} S = \frac{1}{2i}(S - S^*)$ are selfadjoint operators.

Any n-tuple $T = (T_1, \dots, T_n) \in B(H)^n$ can be identified with the $(2n)$-tuple

$$(\operatorname{Re} T_1, \operatorname{Im} T_1, \dots, \operatorname{Re} T_n, \operatorname{Im} T_n)$$

of selfadjoint operators and the joint essential numerical range $W_e(T) \subset \mathbb{C}^n$ can be identified with $W_e(\operatorname{Re} T_1, \operatorname{Im} T_1, \dots, \operatorname{Re} T_n, \operatorname{Im} T_n) \subset \mathbb{R}^{2n}$. So all the statements concerning $W_e(T)$ can be reduced to the corresponding statements for tuples of selfadjoint operators.

An important property of selfadjoint operators is that their numerical range is real. The next proposition characterizes the numerical range of tuples of selfadjoint operators.

Proposition 4. *Let $S = (S_1, \dots, S_n)$ be an n-tuple of selfadjoint operators on a Hilbert space H. Let $z = (z_1, \dots, z_1) \in \mathbb{R}^n$. Then:*

(i) $z \in V(S, B(H))$ *if and only if*

$$\Big|\sum_{j=1}^{n} \alpha_j z_j + \lambda\Big| \le \Big\|\sum_{j=1}^{n} \alpha_j S_j + \lambda\Big\|$$

for all $\alpha_1, \dots, \alpha_n, \lambda \in \mathbb{R}$.

(ii) $z \in W_e(S)$ *if and only if*

$$\Big|\sum_{j=1}^{n} \alpha_j z_j + \lambda\Big| \le \Big\|\sum_{j=1}^{n} \alpha_j S_j + \lambda\Big\|_e$$

for all $\alpha_1, \dots, \alpha_n, \lambda \in \mathbb{R}$.

Proof. Let $z \in V(S, B(H))$. Then there exists $f \in B(H)^*$ such that $\|f\| = 1 = f(I)$ and $f(S_j) = z_j$ $(j = 1, \dots, n)$. Let $\alpha_1, \dots, \alpha_n, \lambda \in \mathbb{R}$. Then

$$\Big|\sum_{j=1}^{n} \alpha_j z_j + \lambda\Big| = \Big|f\Big(\sum_{j=1}^{n} \alpha_j S_j + \lambda I\Big)\Big| \le \Big\|\sum_{j=1}^{n} \alpha_j S_j + \lambda\Big\|.$$

Conversely, let $|\sum_{j=1}^{n} \alpha_j z_j + \lambda| \le \|\sum_{j=1}^{n} \alpha_j S_j + \lambda\|$ for all $\alpha_1, \dots, \alpha_n, \lambda \in \mathbb{R}$. Let $\mathcal{A}$ be the real subspace of $B(H)$ generated by $S_1, \dots, S_n, I$. Let $f : \mathcal{A} \to \mathbb{R}$ be the real functional defined by $f(\sum_{j=1}^{n} \alpha_j S_j + \lambda I) = \sum_{j=1}^{n} \alpha_j z_j + \lambda$. Then $\|f\| \le 1$ and $f(I) = 1$. So $\|f\| = 1$ and $f(S_j) = z_j$ $(j = 1, \dots, n)$.

By the Hahn–Banach theorem there exists a functional $\tilde{f} : B(H) \to \mathbb{R}$ extending f such that $\|\tilde{f}\| = \|f\| = 1$. Let $g \in B(H)^*$ be the complex functional defined by $g(V) = \tilde{f}(V) - i\tilde{f}(iV)$ $(V \in B(H))$. Then $\|g\| = \|\tilde{f}\| = 1$ and $\tilde{f} = \operatorname{Re} g$ (see [BD2, p. 3]). We have $|g(I)| \le 1$ and $\operatorname{Re} g(I) = \tilde{f}(I) = 1$, so $g(I) = 1$. Thus $g(S_1, \dots, S_n) \in V(S, B(H))$.

Moreover, $V(S, B(H)) \subset \prod_{j=1}^{n} V(S_j, B(H)) \subset \mathbb{R}^n$. So

$$\begin{aligned}(z_1, \dots, z_n) &= (\tilde{f}(S_1), \dots, \tilde{f}(S_n)) = (\operatorname{Re} g(S_1), \dots, \operatorname{Re} g(S_n)) \\ &= (g(S_1), \dots, g(S_n)) \in V(S, B(H)).\end{aligned}$$

Part (ii) can be proved similarly using $W_e(S) = V(\pi(S), B(H)/\mathcal{K}(H))$. ∎

Lemma 5. *Let H be a Hilbert space and let $S \in B(H)$, $x \in H$, $\|x\| = 1$, $t \ge 5\|S\|$. Then*

$$\big|\|(S + t)x\| - t - \operatorname{Re}\langle Sx, x\rangle\big| \le \frac{2\|S\|^2}{t}.$$

Proof. We have

$$\begin{aligned}\|(S + t)x\| &= (\|S + t)x\|^2)^{1/2} = (t^2 + 2t \operatorname{Re}\langle Sx, x\rangle + \|Sx\|^2)^{1/2} \\ &= t(1 + s)^{1/2},\end{aligned}$$

where $s = 2\operatorname{Re}\langle Sx,x\rangle/t + \|Sx\|^2/t^2$. Then

$$s \le \frac{2\|S\|}{t} + \frac{\|S\|^2}{t^2} \le \frac{11\|S\|}{5t} \le \frac{1}{2}.$$

It is easy to verify that for $s \le 1/2$ we have

$$1 + \frac{s}{2} - \frac{s^2}{4} \le (1+s)^{1/2} \le 1 + \frac{s}{2}.$$

So

$$\begin{aligned}\big|\|(S+t)x\| - t - \operatorname{Re}\langle Sx,x\rangle\big| &= |t(1+s)^{1/2} - t - \operatorname{Re}\langle Sx,x\rangle| \\ &\le \left|t(1+s)^{1/2} - t - \frac{ts}{2}\right| + \left|\frac{ts}{2} - \operatorname{Re}\langle Sx,x\rangle\right| \\ &\le t\left|(1+s)^{1/2} - 1 - \frac{s}{2}\right| + \frac{\|Sx\|^2}{2t} \\ &\le \frac{ts^2}{4} + \frac{\|S\|^2}{2t} \le \frac{\|S\|^2}{t}\left(\frac{11^2}{4\cdot 5^2} + \frac{1}{2}\right) \le \frac{2\|S\|^2}{t}. \ \blacksquare\end{aligned}$$

For $S \in B(H)$ denote by $\|S\|_e$ the essential norm of S, $\|S\|_e = \inf\{\|S+L\| : L \in \mathcal{K}(H)\}$. For a subspace $M \subset H$ denote by P_M the orthogonal projection onto M.

Proposition 6. *Let H be a separable infinite-dimensional Hilbert space, and let $(e_1, e_2, \dots)$ be an orthonormal basis in H. Let $S \in B(H)$. Then*

$$\|S\|_e = \lim_{k\to\infty} \|P_{H_k^\perp} S P_{H_k^\perp}\|,$$

where $H_k = \bigvee_{j=1}^k e_j$ $(k \in \mathbb{N})$.

Proof. For each $k \in \mathbb{N}$, the operator $S - P_{H_k^\perp} S P_{H_k^\perp}$ is of finite rank, so $\|S\|_e \le \|P_{H_k^\perp} S P_{H_k^\perp}\|$.

Clearly,

$$\|P_{H_1^\perp} P_{H_1^\perp}\| \ge \|P_{H_2^\perp} S P_{H_2^\perp}\| \ge \cdots,$$

so the limit $\lim_{k\to\infty} \|P_{H_k^\perp} S P_{H_k^\perp}\|$ exists and is greater than or equal to $\|S\|_e$.

Suppose on the contrary that $\lim_{k\to\infty} \|P_{H_k^\perp} S P_{H_k^\perp}\| > \|S\|_e$. Then there exists $\varepsilon > 0$ such that $\|P_{H_k^\perp} S P_{H_k^\perp}\| > \|S\|_e + \varepsilon$ for all $k \in \mathbb{N}$. Find $x_k \in H_k^\perp$, $\|x_k\| = 1$ such that $\|Sx_k\| \ge \|P_{H_k^\perp} S x_k\| > \|S\|_e + \varepsilon$. Clearly, $x_k \to 0$ weakly. Let $L \in B(H)$ be a compact operator. Then $\|Lx_k\| \to 0$ and $\|S+L\| \ge \sup_{k\in\mathbb{N}} \|(S+L)x_k\| \ge \|S\|_e + \varepsilon$. So $\|S\|_e = \inf_{L\in\mathcal{K}(H)} \|S+L\| \ge \|S\|_e + \varepsilon$, a contradiction. ∎

Corollary 7. *Let H be a separable infinite-dimensional Hilbert space, $S \in B(H)$, $t, t' \ge 5\|S\|^2$. Then:*

(i) $\big|\|S+t\| - \|S+t'\| - t + t'\big| \le 4\|S\|^2/\min\{t,t'\}$;

(ii) $\big|\|S+t\|_e - \|S+t'\|_e - t + t'\big| \le 4\|S\|^2/\min\{t,t'\}$.

Proof. Let $x \in H$, $\|x\| = 1$. By Lemma 5,

$$\big|\|(S+t)x\| - t - \operatorname{Re}\langle Sx, x\rangle\big| \le \frac{2\|S\|^2}{t},$$
$$\big|\|(S+t')x\| - t' - \operatorname{Re}\langle Sx, x\rangle\big| \le \frac{2\|S\|^2}{t'}.$$

So

$$\big|\|(S+t)x\| - \|(S+t')x\| - t + t'\big| \le \frac{4\|S\|^2}{\min\{t, t'\}}.$$

Let $(x_k) \subset H$ be a sequence of unit vectors such that $\|(S+t)x_k\| \to \|S+t\|$. Without loss of generality we may assume that $\lim_{k\to\infty} \|(S+t')x_k\|$ exists. Then

$$\begin{aligned}\|S+t\| - \|S+t'\| - t + t' &\le \lim_{k\to\infty} (\|(S+t)x_k\| - \|(S+t')x_k\| - t + t') \\ &\le \frac{4\|S\|^2}{\min\{t, t'\}}.\end{aligned}$$

By symmetry, we get (i).

By Proposition 6, we have $\|S+t\|_e = \lim_{k\to\infty} \|P_{H_k^\perp}(S+t)P_{H_k^\perp}\|$, where $H_k = \bigvee_{j=1}^k e_j$ and $(e_1, e_2, \dots)$ is an orthonormal basis in H. This together with (i) gives

$$\begin{aligned}&\big|\|S+t\|_e - \|S+t'\|_e - t + t'\big| \\ &\qquad = \lim_{k\to\infty} \big|\|P_{H_k^\perp}(S+t)P_{H_k^\perp}\| - \|P_{H_k^\perp}(S+t')P_{H_k^\perp}\| - t + t'\big| \\ &\qquad \le \lim_{k\to\infty} \frac{4\|P_{H_k^\perp} S P_{H_k^\perp}\|^2}{\min\{t, t'\}} \le \frac{4\|S\|^2}{\min\{t, t'\}}. \ \blacksquare\end{aligned}$$

LEMMA 8. *Let $T = (T_1, \dots, T_n)$ be an n-tuple of operators on a separable infinite-dimensional Hilbert space H. Then there exist mutually orthogonal finite-dimensional subspaces $F_k \subset H$ such that $H = \bigoplus_{k=1}^\infty F_k$ and $P_r T_j P_s = 0$ for all $r, s \in \mathbb{N}$, $|r-s| \ge 2$ and $j = 1, \dots, n$ (i.e., the operators $T_1, \dots, T_n$ are simultaneously block 3-diagonal).*

Proof. Let $(e_1, e_2, \dots)$ be an orthonormal basis in H. Let $F_1 = \bigvee\{e_1\}$. Let $G_2 = \bigvee\{F_1, T_j F_1, T_j^* F_1 \ (1 \le j \le n), e_2\}$ and $F_2 = G_2 \ominus F_1$. Then $\dim F_2 < \infty$, $T_j F_1 \subset F_1 \oplus F_2$ and $T_j^* F_1 \subset F_1 \oplus F_2$ for all $j = 1, \dots, n$.

We continue this construction for $k \in \mathbb{N}$ inductively. If $k \ge 3$ and the subspaces $F_1, \dots, F_{k-1}$ have already been constructed, then set

$$G_k = \bigvee\{F_1, \dots, F_{k-1}, T_j F_{k-1}, T_j^* F_{k-1} \ (j = 1, \dots, n), e_k\}$$

and $F_k = G_k \ominus (F_1 \oplus \dots \oplus F_{k-1})$. Then $\dim F_k < \infty$, $T_j F_{k-1} \subset (F_1 \oplus \dots \oplus F_k)$ and $T_j^* F_{k-1} \subset (F_1 \oplus \dots \oplus F_k)$ for all $j = 1, \dots, n$.

If we continue this construction for all $k \in \mathbb{N}$ we get the required decomposition (note that $\bigoplus_{k=1}^{\infty} F_k = H$ since $e_k \in F_1 \oplus \cdots \oplus F_k$ for each k). ∎

The following result was proved in [CLSW, Lemma 6]. We formulate it in a more explicit form.

LEMMA 9. *Let $S \in B(H)$ be block 3-diagonal, i.e., there are finite-dimensional subspaces F_j such that $H = \bigoplus_{j=1}^{\infty} F_j$ and $P_{F_r} S P_{F_s} = 0$ whenever $|r-s| \geq 2$. Denote by Q_k the orthogonal projection onto $\bigoplus_{j=k+1}^{\infty} F_j$. Let $l, d \in \mathbb{N}$, $k = l + 2d$ and let $V \in B(H)$ satisfy $V = Q_k V Q_k$. Then $\|S+V\| \leq \max\{\|S\|, \|Q_l(S+V)Q_l\|\} + \|S\|/\sqrt{d}$.*

Proof. Let $x \in H$, $\|x\| = 1$. Then

$$\|P_{F_{l+1}}x + P_{F_{l+2}}x\|^2 + \|P_{F_{l+3}}x + P_{F_{l+4}}x\|^2 + \cdots + \|P_{F_{l+2d-1}}x + P_{F_{l+2d}}x\|^2 \leq \|x\|^2 = 1.$$

So there exists j_0, $l+1 \leq j_0 \leq l+2d-1 \leq k-1$, such that $\|P_{F_{j_0}}x + P_{F_{j_0+1}}x\|^2 \leq d^{-1}$.

Write $x = u + v + w$, where $u \in \bigoplus_{j=1}^{j_0-1} F_j$, $v = P_{F_{j_0}}x + P_{F_{j_0+1}}x$ and $w \in \bigoplus_{j=j_0+2}^{\infty} F_j$. We have $(S+V)x = Su + Sv + (S+V)w$, where $\|Sv\| \leq \|S\| \cdot \|v\| \leq \|S\|/\sqrt{d}$, $Su \in \bigoplus_{j=1}^{j_0} F_j$ and $(S+V)w = Q_l(S+V)Q_l w \in \bigoplus_{j_0+1}^{\infty} F_j$. Thus $Su \perp (S+V)w$ and

$$\|Su + (S+V)w\| \leq \max\{\|S\|, \|Q_l(S+V)Q_l\|\}.$$

Hence $\|(S+V)x\| \leq \max\{\|S\|, \|Q_l(S+V)Q_l\|\} + \|S\|/\sqrt{d}$. ∎

Let $T_1, \ldots, T_n \in B(H)$ be selfadjoint operators. We show that there exist compact selfadjoint operators $K_1, \ldots, K_n \in B(H)$ such that

$$\Big\|\sum_{j=1}^{n} \alpha_j T_j + \lambda\Big\|_e = \Big\|\sum_{j=1}^{n} \alpha_j (T_j - K_j) + \lambda\Big\| \tag{1}$$

for all $\alpha_1, \ldots, \alpha_n, \lambda \in \mathbb{R}$. Consequently, $W_e(T_1, \ldots, T_n) = W(T_1 - K_1, \ldots, T_n - K_n)^-$.

We show (1) first under the technical assumptions that the space H is separable and $\operatorname{Int}_{\mathbb{R}} W_e(T_1, \ldots, T_n) \neq \emptyset$, where $\operatorname{Int}_{\mathbb{R}}$ denotes the interior in the sense of $\mathbb{R}^n$. However, these assumptions are not necessary.

Let $T = (T_1, \ldots, T_n) \in B(H)^n$. For $\alpha = (\alpha_1, \ldots, \alpha_n)$ write $\|\alpha\|_1 = \sum_{j=1}^{n} |\alpha_j|$ and $\alpha T = \sum_{j=1}^{n} \alpha_j T_j$.

THEOREM 10. *Let H be a separable infinite-dimensional Hilbert space, let $T = (T_1, \ldots, T_n) \in B(H)^n$ be an n-tuple of selfadjoint operators such that $\operatorname{Int}_{\mathbb{R}} W_e(T) \neq \emptyset$. Then there exist compact selfadjoint operators $K_1, \ldots, K_n$*

in $B(H)$ such that

$$\Big\|\sum_{j=1}^{n}\alpha_j T_j+\lambda\Big\|_e=\Big\|\sum_{j=1}^{n}\alpha_j(T_j-K_j)+\lambda\Big\|$$

for all $\alpha_1,\dots,\alpha_n,\lambda\in\mathbb{R}$.

Proof. Without loss of generality we may assume that $\|T_j\|=1$ $(j=1,\dots,n)$ and $(0,\dots,0)\in\operatorname{Int}_{\mathbb{R}} W_e(T)$. Let $\varepsilon>0$ be such that $(\mu_1,\dots,\mu_n)\in W_e(T_1,\dots,T_n)$ for all $\mu_1,\dots,\mu_n\in\mathbb{R}$, $|\mu_j|\le\varepsilon$ $(j=1,\dots,n)$.

By Lemma 8, there exist finite-dimensional subspaces $F_1,F_2,\dots$ such that $H=\bigoplus_{j=1}^{\infty}F_j$ and the operators $T_1,\dots,T_n$ are simultaneously block 3-diagonal with respect to this decomposition. Denote by Q_k the orthogonal projection onto the space $\bigoplus_{j=k+1}^{\infty}F_j$.

For $m=0,1,\dots$ we construct inductively n-tuples $S^{(m)}=(S_1^{(m)},\dots,S_n^{(m)})\in B(H)^n$ of selfadjoint operators and an increasing sequence (k_m) of non-negative integers such that

$$Q_{k_m}S_j^{(m)}Q_{k_m}=2^{-m}Q_{k_m}T_jQ_{k_m}\qquad(j=1,\dots,n),\tag{2}$$

$$\Big\|\sum_{j=1}^{n}\alpha_j(T_j-S_j^{(m)})+\lambda\Big\|\le\Big\|\sum_{j=1}^{n}\alpha_jT_j+\lambda\Big\|_e-\frac{\varepsilon}{2^{m+1}}\tag{3}$$

for all $\alpha_1,\dots,\alpha_n,\lambda\in\mathbb{R}$, $\sum_{j=1}^{n}|\alpha_j|=1$, and

$$\|S_j^{(m+1)}-S_j^{(m)}\|\le\frac{\|T_j\|}{2^{m+1}}\qquad(j=1,\dots,n).\tag{4}$$

For $m=0$ set formally $k_0=0$ and $S_j^{(0)}=T_j$ $(j=1,\dots,n)$. Clearly, (2) is satisfied.

Let $\alpha_1,\dots,\alpha_n,\lambda\in\mathbb{R}$, $\sum_{j=1}^{n}|\alpha_j|=1$. For $\mu\in\mathbb{R}$ write $\operatorname{sign}\mu=1$ if $\mu\ge0$ and $\operatorname{sign}\mu=-1$ if $\mu<0$. We have

$$\Big\|\sum_{j=1}^{n}\alpha_j(T_j-S_j^{(0)})+\lambda\Big\|=|\lambda|$$

and

$$\begin{aligned}\Big\|\sum_{j=1}^{n}\alpha_jT_j+\lambda\Big\|_e&\ge\max\Big\{|\mu|:\mu\in W_e\Big(\sum_{j=1}^{n}\alpha_jT_j+\lambda\Big)\Big\}\\&\ge\max\Big\{\Big|\sum_{j=1}^{n}\alpha_j\nu_j+\lambda\Big|:(\nu_1,\dots,\nu_n)\in W_e(T_1,\dots,T_n)\Big\}\\&\ge\Big|\sum_{j=1}^{n}\frac{\varepsilon\alpha_j\operatorname{sign}\lambda}{\operatorname{sign}\alpha_j}+\lambda\Big|=\varepsilon+|\lambda|.\end{aligned}$$

Thus

$$|\lambda| \le \Big\|\sum_{j=1}^{n} \alpha_j T_j + \lambda\Big\|_e - \varepsilon \tag{5}$$

for all $\alpha_1, \dots, \alpha_n, \lambda \in \mathbb{R}$, $\|\alpha\|_1 = 1$. In particular, (3) is satisfied for $m = 0$.

Suppose that we have already constructed the n-tuples $S^{(0)}, \dots, S^{(m)} \in B(H)^n$ and numbers $k_0, \dots, k_m \in \mathbb{N}$ satisfying (2)–(4). Set $r = 2^{m+6}/\varepsilon$ and $\delta = \varepsilon/2^{m+4}$. By Proposition 6, for all $\alpha \in \mathbb{R}^n$, $\|\alpha\|_1 = 1$ and $\lambda \in \mathbb{R}$, $|\lambda| \le r$, we have

$$\|Q_k(\alpha T + \lambda)Q_k\| \searrow \|\alpha T + \lambda\|_e$$

and the functions $(\alpha, \lambda) \mapsto \|Q_k(\alpha T + \lambda)Q_k\|$, $(\alpha, \lambda) \mapsto \|\alpha T + \lambda\|_e$, are continuous. By the Dini theorem the convergence is uniform on the set $A := \{(\alpha, \lambda) \in \mathbb{R}^n \times \mathbb{R} : \|\alpha\|_1 = 1,\ |\lambda| \le r\}$.

Let $l > k_m$ satisfy

$$\|Q_l(\alpha T + \lambda)Q_l\| - \|\alpha T + \lambda\|_e \le \delta$$

for all $(\alpha, \lambda) \in A$. Find $d \in \mathbb{N}$ such that $(r+1)/\sqrt{d} \le \delta$ and set $k_{m+1} = l + 2d$. For $j = 1, \dots, n$ set

$$S_j^{(m+1)} = S_j^{(m)} - \frac{1}{2^{m+1}}\, Q_{k_{m+1}} T_j Q_{k_{m+1}}.$$

Clearly, $S_j^{(m+1)}$ is selfadjoint, $\|S_j^{(m+1)} - S_j^{(m)}\| \le \|T_j\|/2^{m+1}$ and

$$\begin{aligned} Q_{k_{m+1}} S_j^{(m+1)} Q_{k_{m+1}} &= 2^{-m} Q_{k_{m+1}} T_j Q_{k_{m+1}} - 2^{-(m+1)} Q_{k_{m+1}} T_j Q_{k_{m+1}} \\ &= \frac{1}{2^{m+1}}\, Q_{k_{m+1}} T_j Q_{k_{m+1}}. \end{aligned}$$

To show (3), let first $(\alpha, \lambda) \in A$. We have

$$\begin{aligned} &\|Q_l(\alpha(T - S^{(m+1)}) + \lambda)Q_l\| \\ &\quad = \Big\|Q_l((1 - 2^{-m})\alpha T + \lambda)Q_l + \frac{1}{2^{m+1}}\, Q_{k_{m+1}} \alpha T Q_{k_{m+1}}\Big\| \\ &\quad \le (1 - 2^{-m})\|Q_l(\alpha T + \lambda)Q_l\| + \frac{1}{2^{m+1}}\, \|Q_l \lambda Q_l\| \\ &\qquad + \Big\|\frac{1}{2^{m+1}}\, (Q_l - Q_{k_{m+1}})\lambda(Q_l - Q_{k_{m+1}}) + \frac{1}{2^{m+1}}\, Q_{k_{m+1}}(\alpha T + \lambda)Q_{k_{m+1}}\Big\| \\ &\quad \le (1 - 2^{-m})(\|\alpha T + \lambda\|_e + \delta) + \frac{|\lambda|}{2^{m+1}} + \frac{1}{2^{m+1}} \max\{|\lambda|, \|\alpha T + \lambda\|_e + \delta\} \\ &\quad \le \Big(1 - \frac{1}{2^{m+1}}\Big)(\|\alpha T + \lambda\|_e + \delta) + \frac{1}{2^{m+1}}\, (\|\alpha T + \lambda\|_e - \varepsilon) \\ &\quad \le \|\alpha T + \lambda\|_e + \delta - \frac{\varepsilon}{2^{m+1}} = \|\alpha T + \lambda\|_e - \frac{\varepsilon}{2^{m+1}} + \frac{\varepsilon}{2^{m+4}} \end{aligned}$$

(where we used (5)).

To estimate $\|\alpha(T - S^{(m+1)}) + \lambda\|$, we apply Lemma 9. Define $S = \alpha(T - S^{(m)}) + \lambda$ and $V = 2^{-(m+1)} Q_{k_{m+1}} \alpha T Q_{k_{m+1}}$. By (3), we have $\|S\| \leq \|\alpha T + \lambda\| \leq r + 1$. Then

$$\begin{aligned}\|\alpha(T - S^{(m+1)}) + \lambda\| = \|S + V\| &\leq \max\{\|S\|, \|Q_l(S+V)Q_l\|\} + \frac{r+1}{\sqrt{d}} \\ &\leq \max\left\{\|\alpha T + \lambda\|_e - \frac{\varepsilon}{2^{m+1}}, \|\alpha T + \lambda\|_e - \frac{\varepsilon}{2^{m+1}} + \frac{\varepsilon}{2^{m+4}}\right\} + \delta \\ &= \|\alpha T + \lambda\|_e - \frac{\varepsilon}{2^{m+1}} + \frac{\varepsilon}{2^{m+3}}.\end{aligned}$$

For $|\lambda| > r$, by Corollary 7, we have

$$\begin{aligned}&\|\alpha T + \lambda\|_e - \|\alpha(T - S^{(m+1)}) + \lambda\| \\ &\qquad = \left\|\frac{\alpha}{\operatorname{sign}\lambda} T + \frac{\lambda}{\operatorname{sign}\lambda}\right\|_e - \left\|\frac{\alpha}{\operatorname{sign}\lambda}(T - S^{(m+1)}) + \frac{\lambda}{\operatorname{sign}\lambda}\right\| \\ &\qquad \geq \left\|\frac{\alpha}{\operatorname{sign}\lambda} T + r\right\|_e - \left\|\frac{\alpha}{\operatorname{sign}\lambda}(T - S^{(m+1)}) + r\right\| + \frac{8}{r} \\ &\qquad \leq \frac{\varepsilon}{2^{m+1}} - \frac{\varepsilon}{2^{m+3}} + \frac{8}{r} = \frac{\varepsilon}{2^{m+2}}.\end{aligned}$$

So $S^{(m+1)}$ satisfies (2)–(4).

Suppose that the n-tuples $S^{(m)} \in B(H)^n$ have been constructed for all $m \in \mathbb{N}$. For $j = 1, \ldots, n$, the sequence $(S_j^{(m)})_m$ is norm convergent; denote by K_j its limit. We have

$$\|K_j\|_e = \lim_{m\to\infty} \|S_j^{(m)}\|_e \leq \lim_{m\to\infty} \|Q_{k_m} S_j^{(m)} Q_{k_m}\| \leq \lim_{m\to\infty} 2^{-m}\|T_j\| = 0.$$

Hence the operator K_j is compact for all $j = 1, \ldots, n$. Obviously, K_j is selfadjoint.

For $\alpha \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\|\alpha\|_1 = 1$ we have

$$\|\alpha(T - K) + \lambda\| = \lim_{m\to\infty} \|\alpha(T - S^{(m)}) + \lambda\| \leq \|\alpha T + \lambda\|_e.$$

The opposite inequality is clear, so $\|\alpha(T - K) + \lambda\| = \|\alpha T + \lambda\|_e$ for all $\alpha \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\|\alpha\|_1 = 1$. Consequently, $\|\alpha(T - K) + \lambda\| = \|\alpha T + \lambda\|_e$ for all $\alpha \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. ∎

COROLLARY 11. *Let H be a separable Hilbert space, let $T = (T_1, \ldots, T_n) \in B(H)^n$ be an n-tuple of selfadjoint operators. Then there exists an n-tuple $K = (K_1, \ldots, K_n) \in B(H)^n$ of compact selfadjoint operators such that*

$$\|\alpha T + \lambda\|_e = \|\alpha(T - K) + \lambda\|$$

for all $\alpha \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

Proof. We prove the statement by induction on n. Let $T = (T_1, \ldots, T_n) \in B(H)^n$ be an n-tuple of selfadjoint operators.

If $\operatorname{Int}_{\mathbb{R}} W_e(T) \neq \emptyset$, then the statement follows from Theorem 10. Suppose on the contrary that $\operatorname{Int}_{\mathbb{R}} W_e(T) = \emptyset$.

Since $W_e(T)$ is a convex set, there are numbers $\beta_1, \dots, \beta_n \in \mathbb{R}$, $(\beta_1, \dots, \beta_n) \neq (0, \dots, 0)$, such that $\sum_{j=1}^n \beta_j \nu_j = 0$ for all $(\nu_1, \dots, \nu_n) \in W_e(T)$. So $W_e(\sum_{j=1}^n \beta_j T_j) = \{0\}$. Thus $\|\sum_{j=1}^n \beta_j T_j\|_e = 0$ and $L := \sum_{j=1}^n \beta_j T_j$ is a compact operator. Without loss of generality we may assume that $\beta_n \neq 0$. Thus $T_n = \beta_n^{-1}(L - \sum_{j=1}^{n-1} \beta_j T_j)$.

From the induction assumption, there exist selfadjoint operators $K_1, \dots, K_{n-1} \in \mathcal{K}(H)$ such that

$$\Big\|\sum_{j=1}^{n-1} \alpha_j (T_j - K_j) + \lambda\Big\| = \Big\|\sum_{j=1}^{n-1} \alpha_j T_j + \lambda\Big\|_e$$

for all $\alpha_1, \dots, \alpha_{n-1}, \lambda \in \mathbb{R}$. Set $K_n = \beta_n^{-1}(L - \sum_{j=1}^{n-1} \beta_j K_j)$. Then K_n is a compact selfadjoint operator.

Let $\alpha_1, \dots, \alpha_n, \lambda \in \mathbb{R}$. Then

$$\sum_{j=1}^{n} \alpha_j (T_j - K_j) = \sum_{j=1}^{n-1} \alpha_j (T_j - K_j) - \alpha_n \beta_n^{-1} \sum_{j=1}^{n-1} \beta_j (T_j - K_j).$$

By the induction assumption,

$$\begin{aligned}\Big\|\sum_{j=1}^{n} \alpha_j (T_j - K_j) + \lambda\Big\| &= \Big\|\sum_{j=1}^{n-1} \alpha_j T_j - \alpha_n \beta_n^{-1} \sum_{j=1}^{n-1} \beta_j T_j + \lambda\Big\|_e \\ &= \Big\|\sum_{j=1}^{n} \alpha_j T_j + \lambda\Big\|_e. \ \blacksquare\end{aligned}$$

The assumption of separability is not essential and can be omitted.

THEOREM 12. *Let H be an infinite-dimensional Hilbert space, and let $T = (T_1, \dots, T_n) \in B(H)^n$ be an n-tuple of selfadjoint operators. Then there exists an n-tuple $K = (K_1, \dots, K_n) \in B(H)^n$ of compact selfadjoint operators such that*

$$\|\alpha T + \lambda\|_e = \|\alpha(T - K) + \lambda\|$$

for all $\alpha \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

Proof. There exists a decomposition $H = \bigoplus_{\nu \in J} H^{(\nu)}$ such that all the subspaces $H^{(\nu)}$ are separable and reducing for the operators $T_1, \dots, T_n$. Write $T_j^{(\nu)} = P_{H^{(\nu)}} T_j P_{H^{(\nu)}}$.

For all $\alpha \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ there are only finitely many $\nu \in J$ such that $\|\alpha T^{(\nu)} + \lambda\| > \|\alpha T + \lambda\|_e + \varepsilon$. So there are only countably many $\nu \in J$ such that $\|\alpha T^{(\nu)} + \lambda\| > \|\alpha T + \lambda\|_e$. Hence there exists a countable subset

$J_0 \subset J$ such that $\|\alpha T^{(\nu)} + \lambda\| \le \|\alpha T + \lambda\|_e$ for all $\nu \notin J_0$ and all rational $\alpha_1, \dots, \alpha_n, \lambda$. Let $H_0 = \bigoplus_{\nu \in J_0} H^{(\nu)}$. Then H_0 is a separable subspace reducing for $T_1, \dots, T_n$ and $\|P_{H \ominus H_0}(\alpha T + \lambda) P_{H \ominus H_0}\| \le \|\alpha T + \lambda\|_e$ for all $\alpha \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. So we can use Corollary 11 for the operators $P_{H_0} T_j P_{H_0}$. ∎

COROLLARY 13. *Let $T = (T_1, \dots, T_n) \in B(H)$ be an n-tuple of selfadjoint operators. Then there exists an n-tuple $K = (K_1, \dots, K_n) \in B(H)$ of compact selfadjoint operators such that $W_e(T) = \overline{W(T-K)}$. In particular, $\overline{W(T-K)}$ is convex.*

Proof. Let $K = (K_1, \dots, K_n) \in B(H)^n$ be an n-tuple of compact selfadjoint operators such that $\|\alpha(T-K)+\lambda\| = \|\alpha T+\lambda\|_e$ for all $\alpha \in \mathbb{R}^n, \lambda \in \mathbb{R}$. By Proposition 4,

$$W_e(T) = V(T-K, B(H)) = \overline{\operatorname{conv}}\, W(T-K) \supset \overline{W(T-K)} \supset W_e(T).$$

So $W_e(T) = \overline{W(T-K)} = \overline{\operatorname{conv}}\, W(T-K)$. ∎

By Remark 3, the same statement is true for an arbitrary n-tuple of operators.

COROLLARY 14. *Let $T = (T_1, \dots, T_n) \in B(H)^n$. Then there exists an n-tuple $K = (K_1, \dots, K_n) \in B(H)$ of compact operators such that $W_e(T) = \overline{W(T-K)}$.*

We do not know whether Theorem 12 remains true for nonselfadjoint operators.

PROBLEM 15. Let $T \in B(H)^n$. Does there exist $K \in \mathcal{K}(H)^n$ such that

$$\|\alpha T + \lambda\|_e = \|\alpha(T-K) + \lambda\|$$

for all $\alpha \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$?

Using the method of Theorem 10, it is possible to obtain a positive answer if $\operatorname{Int} W_e(T) \neq \emptyset$. However, there are technical problems if $W_e(T)$ is a set of a lower dimension.

3. Olsen's problem. The method from the previous section can also be used to improve the results of C. L. Olsen [O].

THEOREM 16. *Let $S \in B(H)$, $n \in \mathbb{N}$ and $\|S^n\|_e \neq 0$. Then there exists a compact operator $K \in B(H)$ such that $\|(S-K)^j\| = \|S^j\|_e$ for $j = 1, \dots, n$.*

Proof. Without loss of generality we may assume that $\|S\| = 1$. Furthermore, we may assume that the space H is separable (cf. proof of Theorem 12).

By Lemma 8, there exist finite-dimensional subspaces $F_1, F_2, \dots$ such that $H = \bigoplus_{j=1}^{\infty} F_j$ and the operators $S, S^2, \dots, S^n$ are block 3-diagonal with respect to this decomposition. Denote by Q_k the orthogonal projection onto the space $\bigoplus_{j=k+1}^{\infty} F_j$.

Let $0 < \varepsilon < \|S^n\|_e/2$. For $1 \le j \le n$ we have $2\varepsilon \le \|S^n\|_e \le \|S^j\|_e \cdot \|S^{n-j}\|_e \le \|S^j\|_e$. Let $c = 2^n\varepsilon^{-1}$. For $m = 0, 1, \dots$ set $s_m = c/(c+m)$. So $s_0 = 1$, $s_m \searrow 0$ and

$$s_m - s_{m+1} = \frac{c}{(c+m)(c+m+1)} \le \frac{s_m}{c} = \frac{s_m\varepsilon}{2^n}.$$

We construct inductively operators $S_m \in B(H)$ and an increasing sequence (k_m) of nonnegative integers such that:

$$Q_{k_m} S_m Q_{k_m} = (1 - s_m) Q_{k_m} S Q_{k_m}, \tag{6}$$

$$\|S_{m+1} - S_m\| \le s_m - s_{m+1}, \tag{7}$$

$$\|S_m^j\| \le \|S^j\|_e - \varepsilon s_m \quad (j = 1, \dots, n). \tag{8}$$

For $m = 0$ set $S_0 = 0$ and $k_0 = 0$. Then clearly (6) and (8) are satisfied.

Let $m \ge 0$ and suppose that S_m and k_m have already been constructed. Choose a positive number $\delta < \min\{(s_m - s_{m+1})\varepsilon, s_m\varepsilon 2^{-n}\}$.

By Proposition 6, there exists $l > k_m$ such that

$$\|Q_l S^j Q_l\| \le \|S^j\|_e + \delta \quad (j = 1, \dots, n).$$

Let $k_{m+1} > l + 2/\delta^2 + 2$. Set

$$S_{m+1} = S_m + (s_m - s_{m+1}) Q_{k_{m+1}} S Q_{k_{m+1}}.$$

Clearly (6) and (7) are satisfied.

To show (8), fix $j \in \{1, \dots, n\}$. Write $W = (s_m - s_{m+1}) Q_{k_{m+1}} S Q_{k_{m+1}}$ and $V = S_{m+1}^j - S_m^j = (S_m + W)^j - S_m^j$. So V can be expressed as the sum of $2^j - 1$ operators, each being a product of j elements of the pair $\{S_m, W\}$, where W appears at least once. Since $\|W\| \le s_m - s_{m+1}$ and $\|S_m\| \le 1$ by (8), we have $\|V\| \le (2^j - 1)(s_m - s_{m+1})$.

By Lemma 9, $\|S_{m+1}^j\| \le \max\{\|S_m^j\|, \|Q_l S_{m+1}^j Q_l\|\} + \delta$, where $\|S_m^j\| \le \|S^j\|_e - \varepsilon s_m$ by the induction assumption. Further,

$$\begin{aligned}
\|Q_l S_{m+1}^j Q_l\| &\le \|Q_l S_m^j Q_l\| + \|V\| = (1 - s_m)^j \|Q_l S^j Q_l\| + \|V\| \\
&\le (1 - s_m)(\|S^j\|_e + \delta) + (2^j - 1)(s_m - s_{m+1}) \\
&\le (1 - s_m)\|S^j\|_e + \delta + \frac{(2^n - 1)s_m\varepsilon}{2^n} \\
&\le \|S^j\|_e + s_m(-\|S^j\|_e + \varepsilon) \le \|S^j\|_e - \varepsilon s_m.
\end{aligned}$$

Hence $\|S_{m+1}^j\| \le \|S^j\|_e - \varepsilon s_m + \delta \le \|S^j\|_e - \varepsilon s_{m+1}$.

Suppose that we have constructed operators S_m and integers k_m satisfying (6)–(8). Then the sequence (S_m) is norm-convergent. Denote its limit

by U. For $j = 1, \dots, n$ we have $\|U^j\| = \lim_{m\to\infty} \|S_m^j\| \le \|S^j\|_e$. Further,

$$\begin{aligned}\|S - U\|_e &= \lim_{m\to\infty} \|Q_{k_m}(S-U)Q_{k_m}\| \\ &\le \lim_{m\to\infty} (\|Q_{k_m}(S-S_m)Q_{k_m}\| + \|Q_{k_m}(S_m-U)Q_{k_m}\|) \\ &\le \lim_{m\to\infty} (s_m + \|S_m - U\|) = 0.\end{aligned}$$

So the operator $K := S - U$ is compact and we have $\|(S-K)^j\| = \|U^j\| = \|S^j\|_e$ for all $j = 1, \dots, n$. ∎

COROLLARY 17. *Let $S \in B(H)$, $n \in \mathbb{N}$. Then there exists a compact operator $K \in B(H)$ such that $\|(S-K)^n\| = \|S^n\|_e$.*

Proof. If $\|S^n\|_e \neq 0$ then the statement was proved in the previous theorem. If $\|S^n\|_e = 0$ then the statement was proved in [OP]. ∎

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