Operators whose adjoints are quasi $p$-nuclear

by

J. M. DELGADO, C. PIÑEIRO and E. SERRANO (Huelva)

Abstract. For $p \geq 1$, a set $K$ in a Banach space $X$ is said to be relatively $p$-compact if there exists a $p$-summable sequence $(x_n)$ in $X$ with $K \subseteq \{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_p^\prime}\}$. We prove that an operator $T : X \to Y$ is $p$-compact (i.e., $T$ maps bounded sets to relatively $p$-compact sets) iff $T^*$ is quasi $p$-nuclear. Further, we characterize $p$-summing operators as those operators whose adjoints map relatively compact sets to relatively $p$-compact sets.

1. Introduction. In [4], Grothendieck characterized the compact subsets of a Banach space as those sets lying in the closed convex hull of a null sequence. This result aroused interest in the study of sets sitting inside the convex hull of certain classes of null sequences.

In [13], Sinha and Karn introduced the notion of $p$-compact set ($p \geq 1$). A set $K$ of a Banach space $X$ is relatively $p$-compact if it is contained in the $p$-convex hull of a $p$-summable sequence $(x_n)$ in $X$, i.e. $K \subseteq \{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_p^\prime}\}$. This notion opens a new approach to the $p$-approximation property. The authors of [13] investigate when the identity map on $X$ can be approximated by finite rank operators on $p$-compact subsets of $X$ and they connect their results with the $p$-approximation properties defined by Saphar [12] and Reinov [10] (which were conceived via the tensor product route). To this end, there is a previous analysis of the ideal $\mathcal{K}_p$ of $p$-compact operators (the operators mapping bounded sets to relatively $p$-compact sets) and it is proved that the adjoint of a $p$-compact operator admits a factorization through a subspace of $\ell_p$. Using this factorization, a complete norm $\kappa_p$ is defined on the ideal $\mathcal{K}_p$. It is shown that $\mathcal{K}_p$ is contained in the ideal $\Pi^d_p$ of operators with $p$-summing adjoint [13, Proposition 5.3] and that $\mathcal{K}_p(X, Y)$ contains the space $N^d_p(X, Y)$ of operators with $p$-nuclear adjoint whenever $Y$ is reflexive (see the remark after [13, Proposition 5.3]).

2010 Mathematics Subject Classification: 47B07, 46B50, 47B10.
Key words and phrases: $p$-compact sets, $p$-compact operator, weakly $p$-compact operator, $p$-summing operator, quasi $p$-nuclear operator, $p$-nuclear operator.

DOI: 10.4064/sm197-3-6 [291] © Instytut Matematyczny PAN, 2010
The aim of this paper is to deepen the study of \( K_p \) and its possible applications. In Section 3, we show the close relationship between \( p \)-compact operators and quasi \( p \)-nuclear operators. Quasi \( p \)-nuclear operators, introduced by Persson and Pietsch in [6], are an important tool to obtain results and counterexamples related to the approximation property of order \( p \) (see [8]). We prove that an operator is quasi \( p \)-nuclear iff its adjoint is \( p \)-compact (Proposition 3.8); in fact, the dual result is also true, which improves Proposition 5.3 in [13]. Another important result of that section is the characterization of \( p \)-summing operators as those operators whose adjoints map relatively compact sets to relatively \( p \)-compact sets. In the last section, we deal with the Banach ideal \( V_p \) of \( p \)-completely continuous operators (operators mapping relatively weakly \( p \)-compact sets to relatively \( p \)-compact sets) and we show that, though \( \Pi_p \subset V_p \) [13, Proposition 5.4], the inclusion is strict in general for every \( p \geq 1 \).

2. Preliminaries and notations. Throughout this paper, \( X \) and \( Y \) will be Banach spaces. As usual, we denote the closed unit ball of \( X \) by \( B_X \), the dual of \( X \) by \( X^* \), and the space of all bounded (linear) operators from \( X \) into \( Y \) by \( \mathcal{L}(X,Y) \). The subspace of \( \mathcal{L}(X,Y) \) consisting of all compact (respectively, weakly compact) operators from \( X \) into \( Y \) is denoted by \( \mathcal{K}(X,Y) \) (respectively, \( \mathcal{W}(X,Y) \)).

Given a real number \( p \in [1, \infty) \) and an arbitrary set \( I, \ell_p(I) \) (respectively, \( \ell_\infty(I) \)) stands for the Banach space of all scalar functions \( \xi \) defined on \( I \) satisfying \( \sum_{i \in I} |\xi_i|^p < \infty \) (respectively, \( \sup_{i \in I} |\xi_i| < \infty \)) endowed with its natural norm. As usual, we write \( \ell_p \) instead of \( \ell_p(\mathbb{N}) \).

Let \( \ell_p^w(X) \) be the space of all weakly \( p \)-summable sequences \( (x_n) \) in \( X \). It is a Banach space with the norm
\[
\|(x_n)\|_p^w = \sup_{x^* \in B_{X^*}} \left( \sum_n \langle x_n, x^* \rangle^p \right)^{1/p} = \sup_{(\alpha_n) \in B_{\ell_p'}} \left\| \sum_n \alpha_n x_n \right\|.
\]
The subspace of \( \ell_p^w(X) \) consisting of the (strongly) \( p \)-summable sequences is denoted by \( \ell_p(X) \), which is also a Banach space endowed with the norm
\[
\|(x_n)\|_p = \left( \sum_n \|x_n\|^p \right)^{1/p}.
\]
We write \( \ell_\infty(X) \) for the Banach space of all bounded sequences \( (x_n) \) in \( X \) with the norm
\[
\|(x_n)\|_\infty = \sup_n \|x_n\|.
\]
We denote by \( c_0(X) \) the space of all norm null sequences in \( X \), which is a closed subspace of \( \ell_\infty(X) \) with the above norm.
In addition to the classical Banach ideals \([L, \|\cdot\|], [K, \|\cdot\|]\) and \([W, \|\cdot\|]\), we deal with the ideals \([\Pi_p, \pi_p]\) of all \(p\)-summing operators and \([N_p, \nu_p]\) of all \(p\)-nuclear operators. We also consider the injective hull of \([N_p, \nu_p]\), which has been treated in the literature under the name of the Banach ideal of \(p\)-nuclear operators \([\mathcal{Q}]\). We denote this Banach ideal by \(\Omega N_p\). So, an operator \(T: X \to Y\) is quasi \(p\)-nuclear iff \(j_Y \circ T \in N_p(X, \ell_\infty(B_{Y^*}))\), where \(j_Y\) is the natural isometric embedding from \(Y\) into \(\ell_\infty(B_{Y^*})\). It is well known that \(T \in \Omega N_p(X, Y)\) iff there exists a sequence \((x^*_n) \in \ell_p(X^*)\) such that
\[
\|Tx\| \leq \left( \sum_n |\langle x, x^*_n \rangle|^p \right)^{1/p}
\]
for all \(x \in X\). The quasi \(p\)-nuclear norm is
\[
\nu_p^Q(T) = \inf \{ \| (x^*_n) \|_p : \text{[1] holds for all } x \in X \}
\]
for all \(T \in \Omega N_p(X, Y)\). If \(A\) is a Banach ideal, then \(A^d\) denotes its dual ideal, that is, \(A^d(X, Y) = \{ T \in \mathcal{L}(X, Y) : T^* \in A(Y^*, X^*) \}\).

If \(p > 1\) and \(p' = p(p-1)^{-1}\), the map \(\Phi_p : (x_n) \in \ell_p^w(X) \mapsto \Phi_p(x_n) \in \mathcal{L}(\ell_{p'}, X)\), where \(\Phi_p(x_n)(\alpha_n) = \sum_n \alpha_n x_n\), is an isometric isomorphism which allows us to identify the spaces \(\ell_p^w(X)\) and \(\mathcal{L}(\ell_{p'}, X)\). For \(p = 1\), \(\ell_1^w(X)\) is isometrically isomorphic to \(\mathcal{L}(c_0, X)\) under the corresponding map \(\Phi_1\).

The following notions were introduced by Sinha and Karn in \([13]\) trying to extend the characterization of compact sets in Banach spaces as those sets lying inside of the closed convex hull of a norm null sequence \([4]\). If \(p \in [1, \infty)\), the \(p\)-convex hull of a sequence \((x_n) \in \ell_p^w(X)\) is
\[
p-co (x_n) = \Phi_p(x_n)(B_{\ell_{p'}}) = \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_{p'}} \right\}
\]
\((c_0\) instead of \(\ell_{p'}\) if \(p = 1\)). It is clear that the \(p\)-convex hull of a sequence is an absolutely convex set; if \(p > 1\), it is also weakly compact so, in particular, norm closed.

A set \(K \subset X\) is relatively \(p\)-compact if there exists a sequence \((x_n) \in \ell_p(X)\) such that \(K \subset p-co (x_n)\). Since \(p-co (x_n)\) is a relatively compact set when \((x_n) \in \ell_p(X)\), relatively \(p\)-compact sets in \(X\) are relatively compact. If compact sets are viewed as \(\infty\)-compact sets, then it is easy to show that \(p\)-compact sets are \(q\)-compact for \(1 \leq p < q \leq \infty\). Notice that the convex hull of a relatively \(p\)-compact set is relatively \(p\)-compact too.

A set \(K \subset X\) is relatively weakly \(p\)-compact if there exists a sequence \((x_n) \in \ell_p^w(X)\) such that \(K \subseteq p-co (x_n)\). If \(p > 1\), relatively weakly \(p\)-compact sets in \(X\) are relatively weakly compact. However, \(p = 1\) is a pathological case: \(B_{c_0}\) is weakly 1-compact since \(B_{c_0} = p-co (e_n)\), where \((e_n) \in \ell_1^w(c_0)\) is the unit vector basis in \(c_0\). Again, it is a standard argument to prove that weakly \(p\)-compact sets are weakly \(q\)-compact for \(1 < p < q < \infty\).
Finally, we recall that an operator \( T \in \mathcal{L}(X,Y) \) is said to be \( p \)-compact (respectively, weakly \( p \)-compact) if \( T(B_X) \) is relatively \( p \)-compact (respectively, weakly \( p \)-compact) in \( Y \). The set of \( p \)-compact (respectively, weakly \( p \)-compact) operators from \( X \) into \( Y \) is denoted by \( \mathcal{K}_p(X,Y) \) (respectively, \( \mathcal{W}_p(X,Y) \)).

### 3. Main results

The next propositions are the keys to connect \( p \)-compactness and quasi \( p \)-nuclearity.

**Proposition 3.1.** Let \( p \in [1, \infty) \), \( T \in \mathcal{L}(X,Y) \) and \( (y_n) \in \ell^w_p(Y) \). The following statements are equivalent:

(a) \( \|T^* y^*\| \leq \left( \sum_n |\langle y_n, y^* \rangle|^p \right)^{1/p} \) for all \( y^* \in Y^* \).

(b) \( T(B_X) \subseteq \overline{p\text{-co}} \left( y_n \right) \).

**Proof.** (a)\( \Rightarrow \) (b). By contradiction, assume that there exists \( x_0 \in B_X \) so that \( Tx_0 \not\in \overline{p\text{-co}} \left( y_n \right) \). As \( \overline{p\text{-co}} \left( y_n \right) \) is absolutely convex, we can separate \( Tx_0 \) and \( \overline{p\text{-co}} \left( y_n \right) \) strictly by a closed hyperplane; that is to say, there exist \( \alpha > 0 \) and \( y^* \in Y^* \) such that \( |\langle Tx_0, y^* \rangle| > \alpha \) and \( |\langle y, y^* \rangle| < \alpha \) for all \( y \in \overline{p\text{-co}} \left( y_n \right) \). Then

\[
\alpha < \langle Tx_0, y^* \rangle \leq \|T^* y^*\|
\]

\[
\leq \left( \sum_n |\langle y_n, y^* \rangle|^p \right)^{1/p} = \sup_{(\alpha_n) \in B_{\ell^p'}} \left| \left\langle \sum_n \alpha_n y_n, y^* \right\rangle \right| \leq \alpha,
\]

a contradiction.

(b)\( \Rightarrow \) (a). Given \( \varepsilon > 0 \) and \( y^* \in B_{Y^*} \), choose \( x \in B_X \) such that \( \|T^* y^*\| < |\langle x, T^* y^* \rangle| + \varepsilon/2 \). Now, take \( (\alpha_n) \in B_{\ell^p'} \), so that \( \|Tx - \sum_n \alpha_n y_n\| < \varepsilon/2 \). Then

\[
\|T^* y^*\| < |\langle x, T^* y^* \rangle| + \varepsilon/2
\]

\[
\leq \left| \left\langle Tx - \sum_n \alpha_n y_n, y^* \right\rangle \right| + \left| \left\langle \sum_n \alpha_n y_n, y^* \right\rangle \right| + \varepsilon/2
\]

\[
< \sum_n |\alpha_n| |\langle y_n, y^* \rangle| + \varepsilon \leq \| (\alpha_n) \|_{p'} \cdot \left( \sum_n |\langle y_n, y^* \rangle|^p \right)^{1/p} + \varepsilon
\]

\[
\leq \left( \sum_n |\langle y_n, y^* \rangle|^p \right)^{1/p} + \varepsilon
\]

and letting \( \varepsilon \to 0 \) we obtain the conclusion.

Arguing in a similar way, we obtain the dual version of the above result:

**Proposition 3.2.** Let \( p \in [1, \infty) \), \( T \in \mathcal{L}(X,Y) \) and \( (x_n^*) \in \ell^w_p(X^*) \). The following statements are equivalent:
(a) \[ \|Tx\| \leq (\sum_n |\langle x, x_n^* \rangle|^p)^{1/p} \text{ for all } x \in X. \]
(b) \[ T^*(B_{Y^*}) \subseteq \overline{p\text{-co}}(x_n^*). \]

**Remark 3.3.** In Proposition 3.1 we can use \( p\text{-co}(y_n) \) instead of \( \overline{p\text{-co}}(y_n) \) in case \( p > 1 \). On the other hand, if \( p = 1 \) and \( (y_n) \in \ell_1(Y) \), we have \( \{\sum_n \alpha_n y_n : (\alpha_n) \in B_{c_0}\} = \{\sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_\infty}\} \) and this set is 1-compact too. In fact, for \( \delta_n \rightarrow \infty \) such that \( \sum_n |\delta_n| \|y_n\| < \infty \), we have the obvious inclusion
\[ \left\{ \sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_\infty} \right\} \subseteq \left\{ \sum_n \alpha_n (\delta_n y_n) : (\alpha_n) \in B_{c_0} \right\}. \]

**Corollary 3.4.** Let \( T \in \mathcal{L}(X,Y) \). Then the following properties hold:

(I) If \( T \in \mathcal{K}_p(X,Y) \), then \( T^* \in \Omega \mathcal{N}_p(Y^*,X^*) \).

(II) \( T \in \Omega \mathcal{N}_p(X,Y) \) iff \( T^* \in \mathcal{K}_p(Y^*,X^*) \).

In other words, \( \mathcal{K}_p \subseteq \Omega \mathcal{N}^d_p \) and \( \Omega \mathcal{N}_p = \mathcal{K}^d_p \).

The converse of Corollary 3.4(I) cannot be deduced directly from Proposition 3.1. Indeed, if \( T^* \in \Omega \mathcal{N}_p(Y^*,X^*) \), then there exists a sequence \((y_n^*) \in \ell_p(Y^{**})\) such that \( \|T^*y^*\| \leq (\sum_n |\langle y_n^{**}, y^* \rangle|^p)^{1/p} \) for all \( y^* \in Y^* \), and consequently \( T(B_X) \subseteq p\text{-co}(y_n^*) \). In other words, \( T \in \mathcal{K}_p(X,Y^{**}) \) (although \( T(X) \subseteq Y \)). In addition, we will need to deal with the ideal of so-called \( N^p \)-operators. We recall that \( T \in N^p(X,Y) \) if there exist sequences \((x_n^*) \in \ell_{p'}(X^*)\) and \((y_n) \in \ell_p(Y)\) such that \( T \) admitts the representation \( T = \sum_n x_n^* \otimes y_n \) (note that \( N^p(X,Y) \subseteq \mathcal{K}_p(X,Y) \)). The norm in this ideal will be denoted by \( \nu^p \) and is defined by
\[ \nu^p(T) = \inf \|\langle y_n\rangle\|_p \cdot \|(x_n^*)\|_{p'}^{w} \]
where the infimum is taken over all representations of \( T \) as above (see [10]). We will make use of the following theorem:

**Theorem ([10, Theorem 1]).** Let \( p \in [1,\infty] \), \( T \in \mathcal{L}(X,Y) \) and suppose that either \( X^* \) or \( Y^{***} \) has the approximation property. If \( T \in N^p(X,Y^{**}) \), then \( T \in N^p(X,Y) \). In other words, under these conditions, the \( p \)-nuclearity of \( T^* \) implies that \( T \in N^p(X,Y) \).

Let \( K \) be a bounded subset of \( X \). We define the following bounded operators:
\[ u_K : \ell_1(K) \rightarrow X, \quad (\xi_x)_{x \in K} \mapsto \sum_{x \in K} \xi_x x, \]
\[ j_K : X^* \rightarrow \ell_\infty(K), \quad x^* \mapsto (\langle x, x^* \rangle)_{x \in K}. \]

Notice that \( u_K^* = j_K \). We write \( u_X \) and \( j_X \) instead of \( u_{B_X} \) and \( j_{B_X} \), respectively.

**Proposition 3.5.** Let \( K \) be a bounded subset of \( X \). The following statements are equivalent:
(a) $K$ is relatively $p$-compact.
(b) $u_K$ is $p$-compact.
(c) $j_K$ is $p$-nuclear.

Proof. (a)$\iff$(b). This follows from the inclusions $K \subseteq u_K(B_{\ell_1(K)}) \subseteq \overline{co}(K)$.
(b)$\iff$(c). Let $u_K$ be $p$-compact. By Corollary 3.4, $j_K$ is quasi $p$-nuclear, and since $\ell_\infty(K)$ is an injective space, $j_K$ is $p$-nuclear [6, Theorem 38]. For the converse, suppose $j_K$ is $p$-nuclear. According to [10, Theorem 1], the operator $u_K$ belongs to $N_p(\ell_1(K),X)$ and, a fortiori, it is $p$-compact.

Corollary 3.6. Let $K$ be a subset of $X$. If $K$ is relatively $p$-compact in $X^{**}$, then $K$ is $p$-compact in $X$. In particular, an operator $T \in \mathcal{L}(X,Y)$ is $p$-compact iff $T^{**}$ is $p$-compact.

Proof. By Proposition 3.5, $J_K : x^{**} \mapsto (\langle x,x^{**} \rangle)_{x \in K} \in \ell_\infty(K)$ is $p$-nuclear, hence so is $j_K = J_K|_{X^*} : x^* \in X^* \mapsto (\langle x,x^* \rangle)_{x \in K} \in \ell_\infty(K)$. Again a call to Proposition 3.5 tells us that $K$ is $p$-compact in $X$.

Remark 3.7. Let $A$ be a bounded subset of $X^*$. As in the proof of Proposition 3.5, $A$ is relatively $p$-compact iff the operator $j_A : x \in X \mapsto (\langle x,x^* \rangle)_{x^* \in A} \in \ell_\infty(A)$ is $p$-nuclear.

In Corollary 3.4, it is shown that $\mathcal{K}_p \subseteq \mathcal{QN}_p$. Now if $T \in \mathcal{L}(X,Y)$ is such that $T^* \in \mathcal{QN}_p(Y^*,X^*)$ then $T^{**} \in \mathcal{K}_p(X^{**},Y^{**})$ (Corollary 3.4). From the above result, it follows that $T \in \mathcal{K}_p(X,Y)$. This leads to the following proposition which improves Proposition 5.3 in [13].

Proposition 3.8. $\mathcal{K}_p = \mathcal{QN}_p^d$.

In a recent paper [14], Sinha and Karn have dealt with the Banach operator ideals $\mathcal{K}_p^d$ and $\mathcal{K}_p^{dd}$. The above results simplify the understanding of that paper, since $\mathcal{K}_p^d = \mathcal{QN}_p$ and $\mathcal{K}_p^{dd} = \mathcal{K}_p$.

Corollary 3.9. An operator $T \in \mathcal{L}(X,Y)$ is such that $T^* \in \mathcal{QN}_p(Y^*,X^*)$ if and only if there exists $(y_n) \in \ell_p(Y)$ such that $\|T^*y^*\| \leq (\sum_n |\langle y_n,y^* \rangle|^p)^{1/p}$ for all $y^* \in Y^*$.

As we have mentioned in the introduction, $p$-compact operators have been characterized as those operators whose adjoints factor through a subspace of $\ell_p$ [13, Theorem 3.1]. This factorization yields a complete norm defined on $\mathcal{K}_p(X,Y)$. Having in mind the preceding results, we have obtained the same factorization for the adjoints of $p$-compact operators in a much simpler way. In fact, Theorem 3.1 in [13] can be stated in the following manner:

Proposition 3.10. Let $X$ and $Y$ be Banach spaces and $p \in [1,\infty)$. The following statements are equivalent:
(a) $T \in \mathcal{K}_p(X,Y)$.
(b) There exists a closed subspace $H$ of $\ell_p$ and operators $R \in \mathcal{Q}N_p(Y^*,H)$ and $S \in \mathcal{L}(H,X^*)$ such that $T^* = S \circ R$.

Proof. (a) ⇒ (b). If $T \in \mathcal{K}_p(X,Y)$, there exists a sequence $(y_n) \in \ell_p(Y)$ such that $\|T^*y^*\| \leq \left(\sum_n |(y_n,y^*)|^p\right)^{1/p}$ for all $y^* \in Y^*$ (Proposition 3.1). Put
\[
H = \left\{ ((y_n,y^*)) : y^* \in Y^* \right\}
\]
and define the operators $R: y^* \in Y^* \mapsto ((y_n,y^*)) \in H$ and $S: ((y_n,y^*)) \in H \mapsto T^*y^* \in Y^*$. It is easy to check that $H$, $R$ and $S$ satisfy the required conditions. The converse is trivial via Proposition 3.8. \hfill \blacksquare

If $T \in \mathcal{K}_p(X,Y)$, we define
\[
k_p(T) = \inf \| (y_n) \|_p
\]
where the infimum is taken over all sequences $(y_n) \in \ell_p(Y)$ satisfying
\[
T(B_X) \subseteq \left\{ \sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_p'} \right\}.
\]

The inequality $k_p(T) \geq \nu^Q_p(T^*)$ (respectively, the equality $k_p(T^*) = \nu^Q_p(T)$) is a direct consequence of Proposition 3.1 (respectively, Proposition 3.2). Now, $[\mathcal{K}_p,k_p]$ becomes a Banach ideal and the proof is similar to that in [6, p. 31] showing that $[\mathcal{Q}N_p,\nu^Q_p]$ is a Banach ideal (both proofs can be connected via Proposition 3.1). According to [7, Theorem 6.1.8], the norm $k_p$ is equivalent to the norm $\kappa_p$ defined by Sinha and Karn in [13]. Moreover, at the end of this section we prove that these norms coincide (Proposition 3.15).

**Proposition 3.11.** $[\mathcal{K}_p,k_p]$ is the surjective hull of $[N^p,\nu^p]$ for all $p \in [1,\infty)$.

Proof. If $T \in \mathcal{L}(X,Y)$ and $T \circ u_X(B_X) \subseteq [\ell_1(B_X) \xrightarrow{u_X} X \xrightarrow{T} Y]$ is relatively $p$-compact, then so is $T(B_X)$. In other words, $\mathcal{K}_p$ is surjective, and since $N^p \subseteq \mathcal{K}_p$, we have $(N^p)^s \subseteq \mathcal{K}_p$.

On the other hand, if $T \in \mathcal{K}_p(X,Y)$, then $T^* \in \mathcal{Q}N_p(Y^*,X^*)$ (Corollary 3.4). Thus, $j_X \circ T^* \in \mathcal{Q}N_p(Y^*,\ell_\infty(B_X)) = N_p(Y^*,\ell_\infty(B_X))$, and since $j_X \circ T^* = (T \circ u_X)^*$ and $\ell_\infty(B_X)$ has the approximation property, it follows that $T \circ u_X \in N^p(\ell_1(B_X),Y)$ ([10, Theorem 1]). So, we have obtained the equality $(N^p)^s = \mathcal{K}_p$.

Now, a standard argument shows that
\[
(N^p(\ell_1(I),Y),\nu^p) = (\mathcal{K}_p(\ell_1(I),Y),k_p) \quad \text{(isometrically)}
\]
for all nonempty sets $I$. In particular, this proves that $k_p = (\nu^p)^s$. \hfill \blacksquare
Now, we can state our main result. We will need the following theorem:

**Theorem (III Proposition 6.14).** Let $1 \leq p < \infty$ and let $X$ and $Y$ be Banach spaces. An operator $T: X \to Y$ is $p$-summing if and only if there exists a positive constant $C$ such that for every finite-dimensional subspace $E$ of $X$ and every finite-codimensional subspace $F$ of $Y$, the finite-dimensional operator

$$q_F \circ T \circ i_E: E \to X \to Y \to Y/F$$

satisfies $\pi_p(q_F \circ T \circ i_E) \leq C$. Furthermore, we have $\pi_p(T) = \inf C$, where the infimum is taken over all such pairs $E, F$.

**Theorem 3.12.** Let $T \in \mathcal{L}(X,Y)$ and $p \in [1, \infty)$. The following statements are equivalent:

(a) $T$ is $p$-summing.
(b) $T^*$ maps relatively compact subsets of $Y^*$ to relatively $p$-compact subsets of $X^*$.

**Proof.** (a)$\Rightarrow$(b). Let $(y_n^*)$ be a null sequence in $Y^*$ and define $S: y \in Y \mapsto (\langle y, y_n^* \rangle) \in c_0$. Obviously, $S$ is $\infty$-nuclear; therefore, $S \circ T$ is $p$-nuclear and

$$\nu_p(S \circ T) \leq \nu_\infty(S) \pi_p(T) \leq \pi_p(T) \sup_n \|y_n^*\|$$

[16 Theorem 9.13]. Then $(S \circ T)^*: e_n \in \ell_1 \mapsto T^* y_n^* \in X^*$ belongs to $\mathcal{N}^p(\ell_1, X^*)$ and $\nu^p((S \circ T)^*) \leq \nu_p(S \circ T)$. As mentioned before, $\mathcal{K}_p(\ell_1, X^*)$ and $\mathcal{N}^p(\ell_1, X^*)$ are isometric, so

$$k_p((S \circ T)^*) \leq \nu_p(S \circ T) \leq \pi_p(T) \sup_n \|y_n^*\|.$$  

This proves that the linear map

$$U: c_0(Y^*) \to \mathcal{K}_p(\ell_1, X^*), \quad (y_n^*) \mapsto \sum_n e_n^* \otimes T^* y_n^*,$$

is well defined and $\|U\| \leq \pi_p(T)$ (this inequality will be used in the next proposition). Notice that, in particular, we have proved that the set $\{T^* y_n^* : n \in \mathbb{N}\}$ is relatively $p$-compact.

(b)$\Rightarrow$(a). To prove (a) we will use [III Proposition 6.14]. Let $E$ be a finite-dimensional subspace of $X$ and $F$ a subspace of $Y$ whose codimension is finite. Given the sequence

$$E \xrightarrow{i_E} X \xrightarrow{T} Y \xrightarrow{q_F} Y/F,$$

we obtain

$$F^\perp \xrightarrow{q_F^*} Y^* \xrightarrow{T^*} X^* \xrightarrow{i_E^*} X^*/E^\perp.$$
For simplicity, we identify the operator \( Q : e_n \in \ell_1 \mapsto y_n^* \in Y^* \) with the sequence \( (y_n^*) \). Now, consider the map

\[
\phi : \mathcal{K}(\ell_1, Y^*) \to \mathcal{K}_p(\ell_1, X^*), \quad (y_n^*) \mapsto (T^*y_n^*). 
\]

The map \( \phi \) is linear and has closed graph, so it is continuous. Thus, there exists a positive constant \( C \) such that \( k_p(T^*y_n^*) < C \) for every relatively compact sequence \( (y_n^*) \) in \( BY^* \).

Choose \( (y_n^*) \) dense in \( B_{F\perp} \). Since \( k_p(T^*y_n^*) < C \), there exists a sequence \( (x_n^*) \) in \( \ell_p(X^*) \) such that \( \|(x_n^*)\|_p < C \) and \( \{T^*y_n^*\} \subseteq p\text{-co}(x_n^*) \). By density, we also have \( T^*(B_{F\perp}) \subseteq p\text{-co}(x_n^*) \). This yields \( k_p(T^*q^*_f) \leq \|(x_n^*)\|_p \) and therefore \( k_p(i^*_E \circ T^* \circ q^*_f) < C \). Now, we can conclude that \( k_p(i^*_E \circ T^* \circ q^*_f) < C \) (see the comment after the definition of \( k_p \) on page 297). Finally, recall that \( \pi_p \leq \nu_p^Q \).

**Proposition 3.13.** Let \( X, Y \) and \( Z \) be Banach spaces and \( p \geq 1 \). If the operator \( T : X \to Y \) is \( p \)-summing and \( S : Z \to Y^* \) is compact, then \( T^* \circ S \) is \( p \)-compact and \( k_p(T^* \circ S) \leq \pi_p(T)\|S\| \).

**Proof.** Given \( S \in \mathcal{K}(Z, Y^*) \) and \( \varepsilon > 0 \) there exists a null sequence \( (y_n^*) \) such that \( S(B_Z) \subseteq \overline{co}(y_n^*) \) and

\[
\sup_n \|y_n^*\| < \sup_{\|z\| \leq 1} \|Sz\| + \varepsilon = \|S\| + \varepsilon.
\]

Now, we define the operator \( A : (\alpha_n) \in \ell_1 \mapsto \sum_n \alpha_n T^*y_n^* \in X^* \). In the above theorem we have proved that

\[
k_p(A) \leq \pi_p(T) \sup_n \|y_n^*\|.
\]

Thus, given \( \delta > 0 \), there exists \( (x_n^*) \) in \( \ell_p(X^*) \) such that \( \overline{co}(T^*y_n^*) \subseteq p\text{-co}(x_n^*) \) and \( \|(x_n^*)\|_p < \pi_p(T)\|y_n^*\|_{\infty} + \delta \). Consequently, \( T^*(S(B_Z)) \subseteq \overline{co}(T^*y_n^*) \subseteq \overline{co}(T^*x_n^*) \) and these inclusions yield

\[
k_p(T^* \circ S) \leq \|(x_n^*)\|_p < \pi_p(T)\|y_n^*\|_{\infty} + \delta.
\]

Letting \( \delta \to 0 \), we obtain \( k_p(T^* \circ S) \leq \pi_p(T)\|y_n^*\|_{\infty} \). Finally, since \( \|y_n^*\|_{\infty} \leq \|S\| + \varepsilon \) we deduce

\[
k_p(T^* \circ S) \leq \pi_p(T)(\|S\| + \varepsilon).
\]

The proof concludes by letting \( \varepsilon \to 0 \).

The dual version of the main theorem is also valid.

**Theorem 3.14.** Let \( T \in \mathcal{L}(X, Y) \) and \( p \in [1, \infty) \). The following statements are equivalent:

(a) \( T^* \) is \( p \)-summing.

(b) \( T \) maps relatively compact subsets of \( X \) to relatively \( p \)-compact subsets of \( Y \).
Proof. (a)⇒(b). This is an easy consequence of Theorem 3.12 and Corollary 3.6. (b)⇒(a). By Proposition 3.5, we can consider the linear map

$$V : c_0(X) \rightarrow \mathcal{N}_p(Y^*, \ell_\infty), \quad (x_n) \mapsto \sum_n T x_n \otimes e_n$$

$$(e_n)$$ is the canonical basis of $c_0). The operator $V$ is continuous because its graph is closed. Let $J$ be the restriction of $V^*$ to $\Pi_{p'}(\ell_\infty, Y^*)$. A straightforward argument shows that $J : \Pi_{p'}(\ell_\infty, Y^*) \rightarrow \ell_1(X^*)$ is the continuous linear map defined by $J(A) = (T^* \circ A(e_n))$. As $\pi_{p'}(A) \leq \nu_{p'}(A)$ for all $A \in \mathcal{N}_{p'}(\ell_\infty, Y^*)$, it follows that the map

$$J_0 : \mathcal{N}_{p'}(\ell_\infty, Y^*) \rightarrow \ell_1(X^*), \quad A \mapsto (T^* \circ A(e_n)),$$

is continuous. Now we consider $J_0^* : \ell_\infty(X^{**}) \rightarrow \Pi_p(Y^*, \ell_{\infty}^n)$ and $\phi = J_0^*|c_0(X^{**})$. If $(x_n^{**}) \in c_0(X^{**})$, $y^* \in Y^*$ and $\mu \in \ell_{\infty}^n$, then

$$\langle J_0^*(x_n^{**})(y^*), \mu \rangle = J_0^*(x_n^{**})(\mu \otimes y^*) = \langle (x_n^{**}), J_0(\mu \otimes y^*) \rangle$$

$$= \langle (x_n^{**}), (T^*[\mu \otimes y^*(e_n)]) \rangle = \sum_n \langle x_n^{**}, T^*([\mu, e_n]y^*) \rangle$$

$$= \sum_n \langle T^{**}x_n^{**}, y^* \rangle \langle \mu, e_n \rangle = \sum_n \langle T^{**}x_n^{**}, y^* \rangle e_n, \mu \rangle.$$

This proves that $\phi$ maps $c_0(X^{**})$ into $\Pi_p(Y^*, \ell_\infty)$ and $\phi(x_n^{**}) = \sum_n T^{**}x_n^{**} \otimes e_n$. Finally, we will show that $\phi(c_0(X^{**})) \subseteq \mathcal{N}_p(Y^*, \ell_\infty)$. First, for each $n \in \mathbb{N},$ we define

$$\frac{2}{2} \quad \phi_n : \ell_\infty^n(X^{**}) \rightarrow \Pi_p(Y^*, \ell_{\infty}^n), \quad (x_k^{**})_{k=1}^n \mapsto \sum_{k=1}^n T^{**}x_k^{**} \otimes e_k.$$

By the ideal properties, we have $\|\phi_n\| \leq \|\phi\|$ for all $n \in \mathbb{N}$. In view of Corollary 9.5, $\pi_p(u) = \nu_p(u)$ for all $u \in \mathcal{L}(Y^*, \ell_{\infty}^n)$. Thus, we can write (2) in the form

$$\frac{3}{3} \quad \phi_n : \ell_\infty^n(X^{**}) \rightarrow \mathcal{N}_p(Y^*, \ell_{\infty}^n), \quad (x_k^{**})_{k=1}^n \mapsto \sum_{k=1}^n T^{**}x_k^{**} \otimes e_k.$$

Let us prove that $(\phi(x_1^{**}, \ldots, x_n^{**}, 0, \ldots))_n$ is a Cauchy sequence in $\mathcal{N}_p(Y^*, \ell_\infty)$ for all $(x_k^{**}) \in c_0(X^{**})$. According to (3) and the ideal properties of $\mathcal{N}_p$ we have

$$\nu_p(\phi(x_1^{**}, \ldots, x_n^{**}, 0, \ldots) - \phi(x_1^{**}, \ldots, x_m^{**}, 0, \ldots))$$

$$= \nu_p(\phi(0, \ldots, 0, x_{m+1}^{**}, \ldots, x_n^{**}, 0, \ldots)) \leq \|\phi\| \cdot \sup_{m<k\leq n} \|x_k^{**}\|$$

for $n > m.$ Thus, $(\phi(x_1^{**}, \ldots, x_n^{**}, 0, \ldots))_n$ converges to an operator $S \in \mathcal{N}_p(Y^*, \ell_\infty)$ and this operator is necessarily equal to $\phi(x_n^{**}) = \sum_n T^{**}x_n^{**} \otimes e_n$. In particular, this implies that $T^{**}$ maps relatively compact sets in $X^{**}$ to
relatively $p$-compact sets in $Y^{**}$. Now, a call to Theorem 3.12 tells us that $T^*$ is $p$-summing.

We finish this section by showing that our definition of $k_p$ coincides with that in [13]. An operator $T \in \mathcal{L}(X,Y)$ belongs to $\mathcal{K}_p(X,Y)$ if and only if there exists $\hat{y} = (y_n) \in \ell_p(Y)$ such that $T^* = S_{\hat{y}} \circ \phi_{\hat{y}}^*$, where $\phi_{\hat{y}}^* : y^* \in Y^* \mapsto \langle \langle y_n, y^* \rangle \rangle \in H := \{ \langle \langle y_n, y^* \rangle \rangle : y^* \in Y^* \}$ and $S_{\hat{y}} : \langle \langle y_n, y^* \rangle \rangle \in H \mapsto T^* y^* \in X^*$ [13, Theorem 3.2]. Using this decomposition, we can endow $\mathcal{K}_p(X,Y)$ with the norm $\kappa_p$ defined by

$$\kappa_p(T) = \inf \{ \| S_{\hat{y}} \| \cdot \| \hat{y} \|_p : \hat{y} = (y_n) \in \ell_p(Y), \ T^* = S_{\hat{y}} \circ \phi_{\hat{y}}^* \}.$$ 

**Proposition 3.15.** Let $X$ and $Y$ be Banach spaces and $p \geq 1$. Then $k_p(T) = \kappa_p(T)$ for all $T \in \mathcal{K}_p(X,Y)$.

**Proof.** Given $T \in \mathcal{K}_p(X,Y)$ and $\hat{y} = (y_n) \in \ell_p(Y)$, we know that $\| T^* y^* \| \leq \| \langle \langle y_n, y^* \rangle \rangle \|_p$ for all $y^* \in Y^*$ if and only if $T(B_X) \subset p\text{-co}(y_n)$ (Proposition 3.1). Since $\| S_{\hat{y}}(\langle y_n, y^* \rangle) \| = \| T^* y^* \|$, it follows that $\| S_{\hat{y}} \| \leq 1$ and $\kappa_p(T) \leq k_p(T)$.

Now, given $0 < \varepsilon < 1$, consider $\hat{y} = (y_n) \in \ell_p(Y)$ such that

$$\kappa_p(T) + \varepsilon > \| S_{\hat{y}} \| \cdot \| \hat{y} \|_p.$$

Moreover, $\hat{y}$ can be chosen so that $\| S_{\hat{y}} \| > 1 - \varepsilon$. Otherwise, $\| T^* y^* \| = \| S_{\hat{y}}(\langle y_n, y^* \rangle) \| \leq \| \langle \langle (1 - \varepsilon)y_n, y^* \rangle \rangle \|_p$ for all $y^* \in Y^*$ and this means that $T(B_X) \subset p\text{-co}((1 - \varepsilon)y_n)$ (Proposition 3.1). But then

$$\| S_{(1-\varepsilon)\hat{y}} \| = \sup \{ \| T^* y^* \| : \| \langle \langle (1 - \varepsilon)y_n, y^* \rangle \rangle \|_p \leq 1 \}$$

$$= \sup \left\{ \left\| T^* \left( \frac{1}{1 - \varepsilon} z^* \right) \right\| : \left( \left\langle \left\langle (1 - \varepsilon)y_n, \frac{1}{1 - \varepsilon} z^* \right\rangle \right\rangle \right) \right\|_p \leq 1 \right\}$$

$$= \| S_{\hat{y}} \| / (1 - \varepsilon),$$

which implies $\kappa_p(T) + \varepsilon > \| S_{(1-\varepsilon)\hat{y}} \| \cdot \| (1 - \varepsilon)\hat{y} \|_p$. By induction, we have $T(B_X) \subset p\text{-co}((1 - \varepsilon)^m y_n)$ for all $m \in \mathbb{N}$, which is impossible if $T \neq 0$. So

$$\kappa_p(T) + \varepsilon > (1 - \varepsilon)\| \hat{y} \|_p > (1 - \varepsilon)k_p(T),$$

and since $\varepsilon$ can be chosen arbitrarily, $\kappa_p(T) \geq k_p(T)$. ■

4. The operator ideal $\mathcal{V}_p$. We will denote by $\mathcal{V}_p(X,Y)$ the vector space of all operators from $X$ into $Y$ that map relatively weakly $p$-compact subsets of $X$ to relatively $p$-compacts subsets of $Y$. In [13], the authors proved that $\Pi_p(X,Y) \subset \mathcal{V}_p(X,Y)$. First of all, we give sufficient conditions for which the converse inclusion holds for $p = 1, 2$. We will denote by $\ell_p^u(X)$ the subspace of $\ell_p^w(X)$ consisting of all unconditionally $p$-summable
sequences in $X$, that is, those sequences $(x_n)$ satisfying
\[
\lim_{n \to \infty} \left( \sup_{\|x^*\| \leq 1} \sum_{m \geq n} |\langle x_m, x^* \rangle|^p \right) < \infty.
\]

**Proposition 4.1.** If $Y$ is an $\mathcal{L}_1$-space, then $\Pi_1(X, Y) = \mathcal{V}_1(X, Y)$ for every Banach space $X$.

**Proof.** If $(x_n) \in \ell_1^w(X)$ and $T \in \mathcal{V}_1(X, Y)$, then the set
\[
\left\{ \sum_n \alpha_n T(x_n) : (\alpha_n) \in B_{c_0} \right\}
\]
is relatively 1-compact in $Y$. So, the operator $A : e_n \in c_0 \mapsto T(x_n) \in Y$ is 1-compact. By Corollary 3.4, its adjoint $A^* : Y^* \to \ell_1$ is quasi 1-nuclear, and therefore it is 1-summing. As $Y^*$ is an $\mathcal{L}_\infty$-space, $A^*$ is integral. Actually, $A^*$ is nuclear because $\ell_1$ is a dual space and has the Radon–Nikodym property. According to [3, Theorem VIII.7], $A$ is nuclear. This yields $\sum_n \|T(x_n)\| < \infty$.

**Proposition 4.2.** If $Y$ is a Banach space isomorphic to a Hilbert space, then $\Pi_2(X, Y) = \mathcal{V}_2(X, Y)$ for every Banach space $X$.

**Proof.** Let $T \in \mathcal{V}_2(X, Y)$ and $(x_n) \in \ell_2^w(X)$. By hypothesis, the operator $S : \ell_2 \to Y$ defined by $S(e_n) = T(x_n)$ is 2-compact, and therefore its adjoint $S^* : Y^* \to \ell_2$ is quasi 2-nuclear (Corollary 3.4). According to [2, Theorem 4.19], $S^*$ has a 2-summing adjoint because $Y^*$ is isomorphic to a Hilbert space. In particular, $S$ is 2-summing and this implies that $\sum_n \|T(x_n)\|^2 < \infty$. So, we have proved that $T$ is 2-summing.

However, in general, $\Pi_p(X, Y)$ is strictly contained in $\mathcal{V}_p(X, Y)$ for all $p \in [1, \infty)$. The following relationships are obvious for all $p \geq 1$:
\begin{equation}
\Pi_p(\ell_{p'}, X) \subset \mathcal{K}_p(\ell_{p'}, X) \subset \mathcal{V}_p(\ell_{p'}, X).
\end{equation}
If $p > 1$, the first inclusion is strict whenever $X$ is not a subspace of a quotient of an $L_p$-space [15, Theorem 3.1]. So, only the case $p = 1$ needs to be studied.

Let $1 \leq p < 2$. Let $C_p$ be the ideal of all operators mapping weakly $p$-summable sequences to unconditionally $p$-summable sequences. First of all, we will prove that $\Pi_p \circ C_p \subset \mathcal{V}_p$ for every $p \geq 1$. So, let $T = T_2 \circ T_1$, where $T_1$ belongs to $C_p(X, Y)$ and $T_2 \in \Pi_p(Y, Z)$. If $(x_n)$ is a weakly $p$-summable sequence in $X$ and $A = \{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_{p'}}\}$, notice that $T_1(A)$ is relatively compact in $Y$. Then $T_2(T_1(A))$ is relatively $p$-compact (Theorem 3.12).

Now we are going to show that the inclusion $\Pi_p \subset \mathcal{V}_p$ is, in general, strict for every $1 \leq p < 2$. Denote by $I_{2.0}$ the identity map from $\ell_2$ into $c_0$. According to [1, Lemma 6] the identity map from $\ell_2$ onto $\ell_2$ belongs to $C_p$ for
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On the other hand, $(I_{2,0})^*$ is $p$-summing, so $I_{2,0} \in \Pi_p^d \circ C_p \subset V_p$ for all $p < 2$. Nevertheless, $I_{2,0}$ is not $p$-summing.

Finally, we have obtained the following result about the biadjoint of an operator $T \in V_2$. Here, $I_p$ denotes the Banach ideal of $p$-integral operators.

**Proposition 4.3.** Let $X$ be a Banach space such that $I_{X^{**}} \in C_2$. If $T \in V_2(X,Y)$, then $T^{**} \in V_2(X^{**},Y^{**})$.

**Proof.** Given $T \in V_2(X,Y)$, consider the linear map

$$U : (x_n) \in \ell_2^u(X) \mapsto \sum_n Tx_n \otimes e_n \in QN_2(Y^*,\ell_2).$$

It is easy to prove that $U$ has closed graph, and therefore it is continuous. Its adjoint maps $J_2(\ell_2, Y^{**})$ into $J_1(\ell_2, X^*)$. Put $V = U^*|_{N_2(\ell_2, Y^*)}$. Since $N_1(\ell_2, X^*)$ is isometric to a subspace of $J_1(\ell_2, X^*)$ it follows easily that $V$ maps $N_2(\ell_2, Y^*)$ into $N_1(\ell_2, X^*)$. We also denote by $V$ the operator

$$\sum_n e_n^* \otimes y_n^* \in N_2(\ell_2, Y^*) \mapsto \sum_n e_n^* \otimes T^* y_n^* \in N_1(\ell_2, X^*).$$

Taking adjoints again we obtain the operator

$$V^* : (x_n^{**}) \in L(X^*, \ell_2) \mapsto \sum_n T^{**} x_n^{**} \otimes e_n \in \Pi_2(Y^*, \ell_2).$$

As every 2-summing operator is 2-integral and the 2-summing norm coincides with the 2-integral norm, (5) can be written in the form

$$(x_n^{**}) \in L(X^*, \ell_2) \mapsto \sum_n T^{**} x_n^{**} \otimes e_n \in J_2(Y^*, \ell_2).$$

Now, as in the proof of (b)$\Rightarrow$(a) in Theorem 3.14, we can prove that $V^*$ maps $\ell_2^u(X^{**})$ into $N_2(Y^*, \ell_2)$. This shows that the operator $A : y^* \in Y^* \mapsto (\langle T^{**} x_n^{**}, y^* \rangle) \in \ell_2$ is 2-nuclear whenever $(x_n^{**})$ is unconditionally 2-summable in $X^{**}$, and therefore its adjoint $A^* : e_n \in \ell_2 \mapsto T^{**} x_n^{**} \in Y^{**}$ belongs to $N^2$. So, $A^*$ is 2-compact and this concludes the proof.  

**Acknowledgments.** This research was supported by MTM2009-14483-C02-01 project (Spain).

**References**


J. M. Delgado, C. Piñeiro, E. Serrano
Departamento de Matemáticas
Campus Universitario del Carmen
Universidad de Huelva
Avda. de las Fuerzas Armadas s/n
21071 Huelva, Spain
E-mail: jmdelga@uhu.es
candido@uhu.es
eserrano@uhu.es

Received June 24, 2009