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Extension and lifting of weakly continuous polynomials

by

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Abstract. We show that a Banach space X is an $\mathcal{L}_1$-space (respectively, an $\mathcal{L}_\infty$-space) if and only if it has the lifting (respectively, the extension) property for polynomials which are weakly continuous on bounded sets. We also prove that X is an $\mathcal{L}_1$-space if and only if the space $\mathcal{P}_{\rm wb}(^mX)$ of m-homogeneous scalar-valued polynomials on X which are weakly continuous on bounded sets is an $\mathcal{L}_\infty$-space.

The problem of lifting holomorphic mappings has attracted attention of various authors (see, for instance, [K, A, AMP, GG4]). The extension of holomorphic mappings from a space to a superspace has been treated in many papers (see, for instance, [AB, A, Z, LRy, GG4]). Here we study the extension and lifting of polynomials which are weakly continuous on bounded sets.

This class of polynomials (the definitions will be recalled below) was introduced in [AP] and has been studied by many authors. In [AHV] it was shown that, if a polynomial is weakly continuous on bounded sets, then it is weakly *uniformly* continuous on bounded sets. If a dual Banach space X^* has the approximation property, then these polynomials on X coincide with the approximable ones [AP, Proposition 2.7]. It was proved in [GG1] that a polynomial P is weakly continuous on bounded sets if and only if it may be factored in the form $P = Q \circ T$, where T is a compact (linear) operator and Q is a polynomial.

We recall the following well known linear results:

THEOREM 1 ([LR, Theorem 4.1]; see also [L]). *Let X be a Banach space. Then the following facts are equivalent*:

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(a) *X is an $\mathcal{L}_\infty$-space;*
(b) *for all Banach spaces $Z \supseteq Y$, every compact operator $T : Y \to X$ has an extension to a compact operator $\widetilde{T} : Z \to X$;*
(c) *same as in* (b) *without the requirement of compactness of $\widetilde{T}$;*
(d) *for all Banach spaces Z and Y with $Z \supseteq X$, every compact operator $T : X \to Y$ has an extension to a compact operator $\widetilde{T} : Z \to Y$;*
(e) *same as in* (d) *without the requirement of compactness of $\widetilde{T}$.*

Let X, Y, and Z be Banach spaces and let $\phi : Z \to Y$ be a surjective operator. A (linear bounded) operator $T : X \to Y$ is said to *admit a lifting* to Z (with respect to ϕ) if there is an operator $\widetilde{T} : X \to Z$ such that $\phi \circ \widetilde{T} = T$. Such a $\widetilde{T}$ is called a *lifting* of T to Z.

Theorem 2 ([LR, Theorem 4.2]). *Let X be a Banach space. Then the following facts are equivalent*:

(a) *X is an $\mathcal{L}_1$-space;*
(b) *for all Banach spaces Z and Y and any surjective operator $\phi : Z \to Y$, every compact operator $T : X \to Y$ has a compact lifting $\widetilde{T} : X \to Z$ with respect to ϕ;*
(c) *same as in* (b) *without the requirement of compactness of $\widetilde{T}$;*
(d) *for all Banach spaces Z and Y and any surjective operator $\phi : Z \to X$, every compact operator $T : Y \to X$ has a compact lifting $\widetilde{T} : Y \to Z$ with respect to ϕ;*
(e) *same as in* (d) *without the requirement of compactness of $\widetilde{T}$.*

Throughout, X, Y, and Z denote Banach spaces, X^* is the dual of X, and B_X stands for its closed unit ball. The closed unit ball B_{X^*} is a compact space when it is endowed with the weak-star topology, which we denote by w^*. By $\mathbb{N}$ we represent the set of all natural numbers while $\mathbb{K}$ denotes the scalar field. By $X \cong Y$ (respectively, $X \equiv Y$), we mean that X and Y are isomorphic (respectively, isometrically isomorphic).

Given $m \in \mathbb{N}$, we denote by $\mathcal{P}(^m X, Y)$ the space of all m-homogeneous (continuous) polynomials from X into Y endowed with the supremum norm. Recall that to each $P \in \mathcal{P}(^m X, Y)$ we can associate a unique symmetric m-linear (continuous) mapping $\widehat{P} : X \times \overset{(m)}{\cdots} \times X \to Y$ so that

$$P(x) = \widehat{P}(x, \overset{(m)}{\ldots}, x) \quad (x \in X).$$

For simplicity, we write $\mathcal{P}(^m X) := \mathcal{P}(^m X, \mathbb{K})$.

For the general theory of multilinear mappings and polynomials on Banach spaces, we refer to [Di2] and [Mu].

We use the notation $\bigotimes^m X := X \otimes \overset{(m)}{\cdots} \otimes X$ for the m-fold tensor product of X, and $\bigotimes_\pi^m X$ (respectively, $\bigotimes_\varepsilon^m X$) for the m-fold tensor product of X

endowed with the projective (respectively, injective) norm. Their completions are denoted by $\widetilde{\bigotimes}_{\pi}^{m} X$ and $\widetilde{\bigotimes}_{\varepsilon}^{m} X$, respectively (see [DF] or [DU] for the theory of tensor products). By $\bigotimes_{s}^{m} X := X \otimes_s \overset{(m)}{\cdots} \otimes_s X$ we denote the m-fold symmetric tensor product of X, that is, the set of all elements $u \in \bigotimes^m X$ of the form

$$u = \sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \quad (n \in \mathbb{N},\, \lambda_j \in \mathbb{K},\, x_j \in X,\, 1 \le j \le n).$$

The notation $\widetilde{\bigotimes}_{\pi,s}^{m} X$ stands for the closure of $\bigotimes_{s}^{m} X$ in $\widetilde{\bigotimes}_{\pi}^{m} X$. Given $x_1, \ldots, x_m \in X$, we define

$$x_1 \otimes_s \cdots \otimes_s x_m := \frac{1}{m!} \sum_{\sigma \in \mathbb{P}_m} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)},$$

where $\mathbb{P}_m$ is the group of all permutations of m elements. For symmetric tensor products, we refer to [Fl]. For simplicity, we shall use the notation

$$\bigotimes{}^m x := x \otimes \overset{(m)}{\cdots} \otimes x.$$

It is well known that $\mathcal{P}(^m X) \cong (\widetilde{\bigotimes}_{\pi,s}^{m} X)^*$ [Fl, Proposition 2.2].

On $\bigotimes_{s}^{m} X$ we define the *injective symmetric tensor norm* ε_s by

$$\varepsilon_s(z) = \sup\{|\langle z, \bigotimes{}^m x^* \rangle| : x^* \in B_{X^*}\} = \sup\left\{\Big|\sum_{k=1}^{n} \lambda_k \langle x_k, x^* \rangle^m \Big| : x^* \in B_{X^*}\right\}$$

for $z = \sum_{k=1}^{n} \lambda_k \bigotimes^m x_k$. We denote by $\bigotimes_{\varepsilon_s,s}^{m} X$ the space $\bigotimes_{s}^{m} X$ endowed with the ε_s-norm, and by $\widetilde{\bigotimes}_{\varepsilon_s,s}^{m} X$ its completion.

If $P \in \mathcal{P}(^m X, Y)$, we define its *linearization* $\overline{P} : \widetilde{\bigotimes}_{\pi,s}^{m} X \to Y$ by

$$\overline{P}\Big(\sum_{i=1}^{n} \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i\Big) = \sum_{i=1}^{n} \lambda_i P(x_i)$$

for all $\lambda_i \in \mathbb{K}$, $x_i \in X$ $(1 \le i \le n)$.

If $P \in \mathcal{P}(^m X, Y)$, the *adjoint* of P is the operator $P^* : Y^* \to \mathcal{P}(^m X)$ given by

$$P^*(y^*)(x) = \langle y^*, P(x) \rangle.$$

A polynomial $P \in \mathcal{P}(^m X, Y)$ is of *finite type* if it is a finite sum of terms of the form $\gamma^m \otimes y$, with $\gamma \in X^*$ and $y \in Y$, where $(\gamma^m \otimes y)(x) := \gamma(x)^m y$ for all $x \in X$. A polynomial is *approximable* if it lies in the norm closure of the space of polynomials of finite type. By $\mathcal{P}_{\mathrm{A}}(^m X, Y)$ we denote the space of all m-homogeneous approximable polynomials from X into Y, endowed with the supremum norm.

It is shown in [Fl, Proposition 3.2(6)] that

$$\mathcal{P}_{\mathrm{A}}(^m X) \equiv \widetilde{\bigotimes}_{\varepsilon_s,s}^{m} X^*.$$

We are indebted to the referee for pointing out that

$$\mathcal{P}_{\mathrm{A}}({}^m X, Y) \equiv \widetilde{\bigotimes}{}^m_{\varepsilon_s, s} X^* \,\widetilde{\otimes}_\varepsilon\, Y.$$

Indeed, define the operator

$$J : \mathcal{P}_{\mathrm{A}}({}^m X) \otimes_\varepsilon Y \to \mathcal{P}({}^m X, Y)$$

by

$$J\Big(\sum_{i=1}^n P_i \otimes y_i\Big) = Q \quad \text{where} \quad Q(x) := \sum_{i=1}^n P_i(x) y_i \quad (x \in X).$$

Then

$$\begin{aligned}\|Q\| &= \sup\{\|Q(x)\| : x \in B_X\} = \sup\{|\langle y^*, Q(x)\rangle| : x \in B_X,\ y^* \in B_{Y^*}\}\\ &= \sup\Big\{\Big|\sum_{i=1}^n P_i(x) y^*(y_i)\Big| : x \in B_X,\ y^* \in B_{Y^*}\Big\}.\end{aligned}$$

We can embed $\bigotimes^m X$ into $\mathcal{P}_{\mathrm{A}}({}^m X)^*$ by the duality

$$\Big\langle \sum_{i=1}^n \lambda_i \bigotimes{}^m x_i, P\Big\rangle = \sum_{i=1}^n \lambda_i P(x_i).$$

It follows that the set $\{\bigotimes^m x : x \in B_X\}$ is norming in the unit ball of $\mathcal{P}_{\mathrm{A}}({}^m X)^*$, so

$$\begin{aligned}\|Q\| &= \sup\Big\{\Big|\sum_{i=1}^n \Phi(P_i) y^*(y_i)\Big| : \Phi \in \mathcal{P}_{\mathrm{A}}({}^m X)^*,\ \|\Phi\| \le 1,\ y^* \in B_{Y^*}\Big\}\\ &= \Big\|\sum_{i=1}^n P_i \otimes y_i\Big\|_\varepsilon\end{aligned}$$

(see the argument in [Ry, p. 46]) and J is an into isometry. Clearly, the space $\mathcal{P}_{\mathrm{A}}({}^m X, Y)$ is the range of the extension of J to $\mathcal{P}_{\mathrm{A}}({}^m X) \,\widetilde{\otimes}_\varepsilon\, Y$.

A polynomial $P \in \mathcal{P}({}^m X, Y)$ is *compact* if $P(B_X)$ is relatively compact in Y. A polynomial $P \in \mathcal{P}({}^m X, Y)$ is compact if and only if its adjoint $P^* : Y^* \to \mathcal{P}({}^m X)$ is a compact operator [AS, Proposition 3.2]. A polynomial $P \in \mathcal{P}({}^m X, Y)$ is *weakly continuous on bounded subsets* if for each bounded net $(x_\alpha) \subset X$ weakly converging to x, $(P(x_\alpha))$ converges to $P(x)$ in norm. We denote by $\mathcal{P}_{\mathrm{wb}}({}^m X, Y)$ the space of all polynomials in $\mathcal{P}({}^m X, Y)$ which are weakly continuous on bounded sets. Every polynomial in $\mathcal{P}_{\mathrm{wb}}({}^m X, Y)$ is compact ([AP, Lemma 2.2] and [AHV, Theorem 2.9]). An operator is weakly continuous on bounded sets if and only if it is compact [AP, Proposition 2.5]. With each polynomial $P \in \mathcal{P}({}^m X, Y)$ we associate an operator $T_P : X \to \mathcal{P}({}^{m-1} X, Y)$ given by

$$T_P(x)(y) := \widehat{P}(x, y, \overset{(m-1)}{\ldots}, y) \quad (x, y \in X).$$

Then $P \in \mathcal{P}_{\mathrm{wb}}({}^m X, Y)$ if and only if T_P is compact [AHV, Theorem 2.9].

The *Banach–Mazur distance* $d(X,Y)$ between two isomorphic Banach spaces X and Y is defined by $\inf(\|T\|\,\|T^{-1}\|)$ where the infimum is taken over all isomorphisms T from X onto Y. Recall that a Banach space X is an $\mathcal{L}_p$-*space* $(1 \le p \le \infty)$ [LP] if there is $\lambda \ge 1$ such that every finite-dimensional subspace of X is contained in another subspace N with $d(N, \ell_p^n) \le \lambda$ for some integer n.

A Banach space X is an $\mathcal{L}_p$-space $(1 \le p \le \infty)$ if and only if X^* is an $\mathcal{L}_q$-space $(p^{-1} + q^{-1} = 1)$ [LR, Theorem III(a)]. An infinite-dimensional complemented subspace of an $\mathcal{L}_1$-space (respectively, of an $\mathcal{L}_\infty$-space) is an $\mathcal{L}_1$-space (respectively, an $\mathcal{L}_\infty$-space) [LR, Theorem III(b)].

If X is an $\mathcal{L}_1$-space, then, for each m, $\widetilde{\bigotimes}_\pi^m X$ is an $\mathcal{L}_1$-space [DF, Theorem 34.9] (an easy proof may be found in [GG3, Proposition 2.2]). Since $\widetilde{\bigotimes}_{\pi,s}^m X$ is isomorphic to a complemented subspace of $\widetilde{\bigotimes}_\pi^m X$ [Fl, Proposition 2.3], it follows that $\widetilde{\bigotimes}_{\pi,s}^m X$ is an $\mathcal{L}_1$-space. Therefore, $\mathcal{P}(^mX) \cong (\widetilde{\bigotimes}_{\pi,s}^m X)^*$ is an $\mathcal{L}_\infty$-space.

If X is an infinite-dimensional $\mathcal{L}_1$-space, then X contains a complemented copy of ℓ_1 [LP, Proposition 7.3]. Therefore, $\mathcal{P}_{\mathrm{wb}}(^mX)$ is not complemented in $\mathcal{P}(^mX)$ [GG2, Lemma 5]. So it is in principle unknown if $\mathcal{P}_{\mathrm{wb}}(^mX)$ is an $\mathcal{L}_\infty$-space. We now prove that the answer to this question is affirmative. This will be useful later on.

THEOREM 3. *Let X be a Banach space. The following facts are equivalent*:

(a) *X is an $\mathcal{L}_1$-space*;
(b) *for each $m \in \mathbb{N}$, $\mathcal{P}_{\mathrm{wb}}(^mX)$ is an $\mathcal{L}_\infty$-space*;
(c) *there exists $m \in \mathbb{N}$ such that $\mathcal{P}_{\mathrm{wb}}(^mX)$ is an $\mathcal{L}_\infty$-space.*

Proof. (a)⇒(b). Since X^* has the approximation property [DF, p. 306], we have

$$\mathcal{P}_{\mathrm{wb}}(^mX) = \mathcal{P}_{\mathrm{A}}(^mX) \equiv \widetilde{\bigotimes}_{\varepsilon_s,s}^m X^*$$

(see [AP, Proposition 2.7] and our introduction). Now, X^* is an $\mathcal{L}_\infty$-space, so $\widetilde{\bigotimes}_\varepsilon^m X^*$ is also an $\mathcal{L}_\infty$-space (a proof may be found in [GG3, Proposition 2.2]). Then its complemented subspace $\widetilde{\bigotimes}_{\varepsilon_s,s}^m X^*$ [Fl, Proposition 3.1] is again an $\mathcal{L}_\infty$-space.

(b)⇒(c) is obvious.

(c)⇒(a). Since X^* is complemented in $\mathcal{P}_{\mathrm{wb}}(^mX)$ [AS, Proposition 5.3], X^* is an $\mathcal{L}_\infty$-space as well. Thus X is an $\mathcal{L}_1$-space. ■

We now give two preparatory results.

PROPOSITION 4. *Let $P \in \mathcal{P}({}^mX, Y)$. Then its adjoint $P^* : Y^* \to \mathcal{P}({}^mX)$ is w^*-τ_{p}-continuous, where τ_{p} is the topology of pointwise convergence on $\mathcal{P}({}^mX)$. Conversely, let $S : Y^* \to \mathcal{P}({}^mX)$ be a w^*-τ_{p}-continuous operator. Then there is a polynomial $P \in \mathcal{P}({}^mX, Y)$ such that $S = P^*$.*

Proof. Given $P \in \mathcal{P}({}^mX, Y)$, let (y_α^*) be a weak-star null net in Y^*. Then, for every $x \in X$, we have

$$P^*(y_\alpha^*)(x) = \langle y_\alpha^*, P(x)\rangle \to 0,$$

so $P^*(y_\alpha^*) \to 0$ in the τ_{p}-topology on $\mathcal{P}({}^mX)$.

Conversely, let $S : Y^* \to \mathcal{P}({}^mX)$ be a w^*-τ_{p}-continuous operator. Let

$$k_{\widetilde{\bigotimes}^m_{\pi,s} X} : \widetilde{\bigotimes}^m_{\pi,s} X \to (\widetilde{\bigotimes}^m_{\pi,s} X)^{**} \cong \mathcal{P}({}^mX)^*$$

be the natural embedding and let k_Y be the natural embedding of Y into Y^{**}. For $x \in X$ fixed, consider the functional $k_{\widetilde{\bigotimes}^m_{\pi,s} X}(\bigotimes^m x) \circ S : Y^* \to \mathbb{K}$. Clearly,

$$(k_{\widetilde{\bigotimes}^m_{\pi,s} X}(\bigotimes^m x) \circ S)(y^*) = S(y^*)(x).$$

By our hypothesis on S, $k_{\widetilde{\bigotimes}^m_{\pi,s} X}(\bigotimes^m x) \circ S$ is weak-star continuous on Y^* and so it belongs to $k_Y(Y)$. Let $P \in \mathcal{P}({}^mX, Y)$ be defined by $P(x) = k_Y^{-1}(k_{\widetilde{\bigotimes}^m_{\pi,s} X}(\bigotimes^m x) \circ S)$. We have

$$\begin{aligned} P^*(y^*)(x) &= \langle y^*, P(x)\rangle = \langle y^*, k_Y^{-1}(k_{\widetilde{\bigotimes}^m_{\pi,s} X}(\bigotimes^m x) \circ S)\rangle \\ &= k_{\widetilde{\bigotimes}^m_{\pi,s} X}(\bigotimes^m x)(S(y^*)) = S(y^*)(x). \end{aligned}$$

Therefore, $S = P^*$. ∎

If $P \in \mathcal{P}_{\mathrm{wb}}({}^mX, Y)$, then clearly $P^*(Y^*) \subseteq \mathcal{P}_{\mathrm{wb}}({}^mX)$.

PROPOSITION 5. *Let $P \in \mathcal{P}_{\mathrm{wb}}({}^mX, Y)$. Then its adjoint $P^* : Y^* \to \mathcal{P}_{\mathrm{wb}}({}^mX)$ is compact and w^*-τ_{p}-continuous. Conversely, let $S : Y^* \to \mathcal{P}_{\mathrm{wb}}({}^mX)$ be a compact and w^*-τ_{p}-continuous operator. Then there is $P \in \mathcal{P}_{\mathrm{wb}}({}^mX, Y)$ such that $S = P^*$.*

Proof. If $P \in \mathcal{P}_{\mathrm{wb}}({}^mX, Y)$, then P^* is compact and, by Proposition 4, it is also w^*-τ_{p}-continuous.

Conversely, if $S : Y^* \to \mathcal{P}_{\mathrm{wb}}({}^mX)$ is a compact and w^*-τ_{p}-continuous operator, then by Proposition 4, there is $P \in \mathcal{P}({}^mX, Y)$ such that $S = P^*$. Since S is compact, P is compact. Since $y^* \circ P \in \mathcal{P}_{\mathrm{wb}}({}^mX)$ for each $y^* \in Y^*$ it follows from [AP, Proposition 2.8] that $P \in \mathcal{P}_{\mathrm{wb}}({}^mX, Y)$. ∎

We now extend the equivalences (a)⇔(d)⇔(e) of Theorem 2 to the class of polynomials which are weakly continuous on bounded sets.

THEOREM 6. *Let X be a Banach space. Then the following facts are equivalent*:

(a) *X is an $\mathcal{L}_1$-space;*
(b) *for all $m \in \mathbb{N}$, for every surjective operator $\phi : Z \to X$, every $P \in \mathcal{P}_{\mathrm{wb}}(^mY, X)$ has a lifting $\widetilde{P} \in \mathcal{P}_{\mathrm{wb}}(^mY, Z)$ with respect to ϕ;*
(c) *there is $m \in \mathbb{N}$ for which* (b) *holds;*
(d) *there is $m \in \mathbb{N}$ for which* (b) *holds without the requirement that the lifting be weakly continuous on bounded sets.*

Proof. (a)⇒(b). Let X be an $\mathcal{L}_1$-space and $P \in \mathcal{P}_{\mathrm{wb}}(^mY, X)$. Then there exist a Banach space G, a compact operator $T : Y \to G$ and a polynomial $Q \in \mathcal{P}(^mG, X)$ such that $P = Q \circ T$. Since every compact operator factors through two compact operators [F, Corollary 3.3], we can assume that Q is compact. By [GG4, Theorem 2], there exists a lifting $\widetilde{Q} \in \mathcal{P}(^mG, Z)$ with respect to ϕ. Then $\widetilde{P} := \widetilde{Q} \circ T \in \mathcal{P}_{\mathrm{wb}}(^mY, Z)$ is a lifting of P.

(b)⇒(c) and (c)⇒(d) are obvious.

(d)⇒(a). Let $m \in \mathbb{N}$ ($m \geq 2$). We show that the statement is true for the index $m-1$. Let $Q \in \mathcal{P}_{\mathrm{wb}}(^{m-1}Y, X)$. By Proposition 5, its adjoint $Q^* : X^* \to \mathcal{P}_{\mathrm{wb}}(^{m-1}Y)$ is compact and w^*-τ_{p}-continuous. Choose $e \in Y$ and $\gamma \in Y^*$ with $\gamma(e) = 1$. Adapting the proof of [AS, Proposition 5.3], we define the operators

$$j_Y : \mathcal{P}_{\mathrm{wb}}(^{m-1}Y) \to \mathcal{P}_{\mathrm{wb}}(^mY)$$

by

$$j_Y(R)(y) := \gamma(y)R(y) \quad \text{for } R \in \mathcal{P}_{\mathrm{wb}}(^{m-1}Y),\ y \in Y,$$

and

$$\pi_Y : \mathcal{P}_{\mathrm{wb}}(^mY) \to \mathcal{P}_{\mathrm{wb}}(^{m-1}Y)$$

given by

$$\pi_Y(S)(y) := \sum_{i=1}^{m} \binom{m}{i} (-1)^{i+1} \gamma(y)^{i-1} \widehat{S}(e^i, y^{m-i}),$$

where

$$\widehat{S}(e^i, y^{m-i}) := \widehat{S}(e, \overset{(i)}{\ldots}, e, y, \overset{(m-i)}{\ldots}, y).$$

We show that $\pi_Y \circ j_Y$ is the identity map on $\mathcal{P}_{\mathrm{wb}}(^{m-1}Y)$ (and, therefore, $j_Y \circ \pi_Y$ is a projection). Indeed, given $R \in \mathcal{P}_{\mathrm{wb}}(^{m-1}Y)$ and $y \in Y$, since

$$\widehat{j_Y(R)}(y_1, \ldots, y_m) = \frac{1}{m} \sum_{i=1}^{m} \gamma(y_i) \widehat{R}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m),$$

we have

$$\pi_Y(j_Y(R))(y) = \frac{1}{m} \sum_{i=1}^{m} \binom{m}{i} (-1)^{i+1} \gamma(y)^{i-1} \times [i\gamma(e)\widehat{R}(e^{i-1}, y^{m-i}) + (m-i)\gamma(y)\widehat{R}(e^i, y^{m-i-1})].$$

For each i, we have

$$\binom{m}{i-1}(-1)^i(m-i+1)\gamma(y)^{i-1}\widehat{R}(e^{i-1},y^{m-i}) \\ +\binom{m}{i}(-1)^{i+1}i\gamma(y)^{i-1}\widehat{R}(e^{i-1},y^{m-i})=0,$$

so

$$\pi_Y(j_Y(R))(y)=R(y).$$

The operator $j_Y\circ Q^*:X^*\to\mathcal{P}_{\mathrm{wb}}(^mY)$ is still compact and w^*-τ_{p}-continuous. Indeed, if (x^*_α) is a net converging to 0 in the weak-star topology of X^* and $y\in Y$, we have

$$j_Y(Q^*(x^*_\alpha))(y)=\gamma(y)Q^*(x^*_\alpha)(y)\to 0.$$

By Proposition 5, there exists a polynomial $P\in\mathcal{P}_{\mathrm{wb}}(^mY,X)$ such that $P^*=j_Y\circ Q^*$. By the hypothesis, there is a lifting $\widetilde{P}\in\mathcal{P}(^mY,Z)$ with respect to ϕ. Using Proposition 4 and considering π_Y as an operator $\mathcal{P}(^mY)\to\mathcal{P}(^{m-1}Y)$, we see that $\pi_Y\circ(\widetilde{P})^*:Z^*\to\mathcal{P}(^{m-1}Y)$ is w^*-τ_{p}-continuous. So there is $\widetilde{Q}\in\mathcal{P}(^{m-1}Y,Z)$ such that $(\widetilde{Q})^*=\pi_Y\circ(\widetilde{P})^*$. We now prove that $\phi\circ\widetilde{Q}=Q$, equivalently, $(\phi\circ\widetilde{Q})^*=Q^*$. Indeed,

$$(\phi\circ\widetilde{Q})^*=(\widetilde{Q})^*\circ\phi^*=\pi_Y\circ(\widetilde{P})^*\circ\phi^*=\pi_Y\circ P^*=\pi_Y\circ j_Y\circ Q^*=Q^*.$$

So $\widetilde{Q}$ is a lifting of Q. Iterating the argument, we deduce that every compact operator from Y into X has a lifting to Z. By Theorem 2, X is an $\mathcal{L}_1$-space. ∎

We now give the equivalences (a)⇔(b)⇔(c) of Theorem 2 for polynomials which are weakly continuous on bounded sets.

THEOREM 7. *Let X be a Banach space. Then the following facts are equivalent*:

(a) *X is an $\mathcal{L}_1$-space*;
(b) *for all $m\in\mathbb{N}$, for every surjective operator $\phi:Z\to Y$, every $P\in\mathcal{P}_{\mathrm{wb}}(^mX,Y)$ admits a lifting $\widetilde{P}\in\mathcal{P}_{\mathrm{wb}}(^mX,Z)$ with respect to ϕ*;
(c) *there exists $m\in\mathbb{N}$ for which* (b) *holds*;
(d) *there exists $m\in\mathbb{N}$ for which* (b) *holds without the requirement that the lifting be weakly continuous on bounded sets.*

Proof. (a)⇒(b). Let X be an $\mathcal{L}_1$-space. Since X^* has the approximation property [DF, p. 306], we have

$$\mathcal{P}_{\mathrm{wb}}(^mX,Y)\equiv\mathcal{P}_{\mathrm{A}}(^mX)\,\widetilde{\otimes}_\varepsilon\,Y$$

(see [AP, Proposition 2.7]). By Theorem 3, $\mathcal{P}_{\mathrm{A}}(^mX)=\mathcal{P}_{\mathrm{wb}}(^mX)$ is an $\mathcal{L}_\infty$-space. Then the operators

$$I\otimes\phi:\mathcal{P}_{\mathrm{A}}(^mX)\otimes_\varepsilon Z\to\mathcal{P}_{\mathrm{A}}(^mX)\otimes_\varepsilon Y$$

and

$$I \otimes \phi : \mathcal{P}_{\mathrm{A}}({}^m X)\,\widetilde{\otimes}_\varepsilon\, Z \to \mathcal{P}_{\mathrm{A}}({}^m X)\,\widetilde{\otimes}_\varepsilon\, Y$$

are surjective [DF, 23.5, Corollaries 5 and 6], where I is the identity map on $\mathcal{P}_{\mathrm{A}}({}^m X)$. A standard argument allows us to conclude that every $P \in \mathcal{P}_{\mathrm{wb}}({}^m X, Y) \equiv \mathcal{P}_{\mathrm{A}}({}^m X)\widetilde{\otimes}_\varepsilon Y$ admits a lifting $\widetilde{P} : \mathcal{P}_{\mathrm{wb}}({}^m X, Z) \equiv \mathcal{P}_{\mathrm{A}}({}^m X)\widetilde{\otimes}_\varepsilon Z$ with respect to ϕ.

(b)⇒(c) and (c)⇒(d) are obvious.

(d)⇒(a). It is enough to show that the statement is true for the index $m-1$. Define the operators

$$j_X : \mathcal{P}_{\mathrm{wb}}({}^{m-1}X) \to \mathcal{P}_{\mathrm{wb}}({}^m X), \qquad \pi_X : \mathcal{P}_{\mathrm{wb}}({}^m X) \to \mathcal{P}_{\mathrm{wb}}({}^{m-1}X)$$

as in the proof of Theorem 6. The proof is analogous to that of (d)⇒(a) in Theorem 6. ■

Now we give the equivalences (a)⇔(d)⇔(e) of Theorem 1 in the setting of polynomials which are weakly continuous on bounded sets.

THEOREM 8. *Let X be a Banach space. Then the following facts are equivalent*:

(a) *X is an $\mathcal{L}_\infty$-space*;

(b) *for all $m \in \mathbb{N}$, for every into isomorphism $\psi : X \to Z$, every $P \in \mathcal{P}_{\mathrm{wb}}({}^m X, Y)$ has an extension $\widetilde{P} \in \mathcal{P}_{\mathrm{wb}}({}^m Z, Y)$ such that $\widetilde{P} \circ \psi = P$*;

(c) *there exists $m \in \mathbb{N}$ for which* (b) *holds*;

(d) *there exists $m \in \mathbb{N}$ for which* (b) *holds without the requirement that the extension be weakly continuous on bounded sets.*

Proof. (a)⇒(b). Let X be an $\mathcal{L}_\infty$-space. Let $\psi : X \to Z$ be an into isomorphism, and let $P \in \mathcal{P}_{\mathrm{wb}}({}^m X, Y)$. Then there exist a Banach space G, a compact operator $T : X \to G$ and a polynomial $Q \in \mathcal{P}({}^m G, Y)$ such that $P = Q \circ T$. By Theorem 1, there is a compact operator $\widetilde{T} : Z \to G$ such that $\widetilde{T} \circ \psi = T$. Define $\widetilde{P} := Q \circ \widetilde{T} \in \mathcal{P}({}^m Z, Y)$. Then $\widetilde{P}$ is an extension of P, and $\widetilde{P} \in \mathcal{P}_{\mathrm{wb}}({}^m Z, Y)$.

(b)⇒(c) and (c)⇒(d) are obvious.

(d)⇒(a). We can assume $m \geq 2$. We prove that, for every into isomorphism $\psi : X \to Z$, every $Q \in \mathcal{P}_{\mathrm{wb}}({}^{m-1}X, Y)$ has an extension $\widetilde{Q} \in \mathcal{P}({}^{m-1}Z, Y)$ such that $\widetilde{Q} \circ \psi = Q$. Let $\overline{Q} : \widetilde{\bigotimes}{}^{m-1}_{\pi,s} X \to Y$ be the linearization of Q. Choose $x_0 \in X$ and let $z_0 := \psi(x_0)$. Select $z^* \in Z^*$ with $z^*(z_0) = 1$ and let $x^* = \psi^*(z^*)$. Then $x^*(x_0) = 1$. Let

$$\Pi_X : \widetilde{\bigotimes}{}^m_{\pi,s} X \to \widetilde{\bigotimes}{}^{m-1}_{\pi,s} X, \qquad J_X : \widetilde{\bigotimes}{}^{m-1}_{\pi,s} X \to \widetilde{\bigotimes}{}^m_{\pi,s} X$$

be the operators defined in [Bl, p. 168] by

$$\Pi_X\Big(\bigotimes{}^m x\Big) = x^*(x) \bigotimes{}^{m-1} x$$

and

$$J_X(\bigotimes^{m-1} x) = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} x^*(x)^{k-1} x_0 \otimes_s \overset{(k)}{\ldots} \otimes_s x_0 \otimes_s x \otimes_s \overset{(m-k)}{\ldots} \otimes_s x.$$

It is shown in [Bl, p. 168] that $\mathit{\Pi}_X \circ J_X$ is the identity map on $\widetilde{\bigotimes}^{m-1}_{\pi,s} X$. Similarly we can define

$$J_Z : \widetilde{\bigotimes}^{m-1}_{\pi,s} Z \to \widetilde{\bigotimes}^{m}_{\pi,s} Z$$

by

$$J_Z(\bigotimes^{m-1} z) = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} z^*(z)^{k-1} z_0 \otimes_s \overset{(k)}{\ldots} \otimes_s z_0 \otimes_s z \otimes_s \overset{(m-k)}{\ldots} \otimes_s z.$$

Consider $S := \overline{Q} \circ \mathit{\Pi}_X : \widetilde{\bigotimes}^{m}_{\pi,s} X \to Y$ and let $P \in \mathcal{P}(^m X, Y)$ be the polynomial whose linearization is S. Then, for each $x \in X$,

$$P(x) = S(\bigotimes^m x) = \overline{Q} \circ \mathit{\Pi}_X(\bigotimes^m x) = x^*(x)\overline{Q}(\bigotimes^{m-1} x) = x^*(x)Q(x).$$

Since $Q \in \mathcal{P}_{\mathrm{wb}}(^{m-1}X, Y)$, it follows that $P \in \mathcal{P}_{\mathrm{wb}}(^m X, Y)$. By our hypothesis, there is $\widetilde{P} \in \mathcal{P}(^m Z, Y)$ such that $\widetilde{P} \circ \psi = P$. Let $\overline{\widetilde{P}}$ be the linearization of $\widetilde{P}$, and consider the composition $\overline{\widetilde{P}} \circ J_Z : \widetilde{\bigotimes}^{m-1}_{\pi,s} Z \to Y$. Let $\widetilde{Q} \in \mathcal{P}(^{m-1}Z, Y)$ be the polynomial whose linearization coincides with $\overline{\widetilde{P}} \circ J_Z$. We show that $\widetilde{Q}$ extends Q, that is, $\widetilde{Q} \circ \psi = Q$, equivalently, $\overline{\widetilde{Q} \circ \psi} = \overline{Q}$. Indeed,

$$\begin{aligned}
\overline{\widetilde{Q} \circ \psi}(\bigotimes^{m-1} x) &= \overline{\widetilde{Q}}(\bigotimes^{m-1} \psi(x)) = (\overline{\widetilde{P}} \circ J_Z)(\bigotimes^{m-1} \psi(x)) \\
&= \overline{\widetilde{P}}\Big[\sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} z^*(\psi(x))^{k-1} \\
&\qquad \times \psi(x_0) \otimes_s \overset{(k)}{\ldots} \otimes_s \psi(x_0) \otimes_s \psi(x) \otimes_s \overset{(m-k)}{\ldots} \otimes_s \psi(x)\Big] \\
&= \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} z^*(\psi(x))^{k-1} \overline{\widetilde{P} \circ \psi}(x_0 \otimes_s \overset{(k)}{\ldots} \otimes_s x_0 \otimes_s x \otimes_s \overset{(m-k)}{\ldots} \otimes_s x) \\
&= \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} z^*(\psi(x))^{k-1} \overline{P}(x_0 \otimes_s \overset{(k)}{\ldots} \otimes_s x_0 \otimes_s x \otimes_s \overset{(m-k)}{\ldots} \otimes_s x) \\
&= \overline{P}\Big[\sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} x^*(x)^{k-1} x_0 \otimes_s \overset{(k)}{\ldots} \otimes_s x_0 \otimes_s x \otimes_s \overset{(m-k)}{\ldots} \otimes_s x\Big] \\
&= \overline{P} \circ J_X(\bigotimes^{m-1} x) = \overline{Q} \circ \mathit{\Pi}_X \circ J_X(\bigotimes^{m-1} x) = \overline{Q}(\bigotimes^{m-1} x).
\end{aligned}$$

Iterating the argument, we conclude that every compact operator $X \to Y$ has an extension, and so X is an $\mathcal{L}_\infty$-space by Theorem 1. ∎

REMARK 9. The implications (a)⇒(b) and (a)⇒(c) of Theorem 1 do not extend to the polynomials which are weakly continuous on bounded sets. We shall see that, for every Banach space $X \neq \{0\}$, there are a polynomial $P \in \mathcal{P}_{\mathrm{wb}}({}^2\ell_p, X)$ ($p \geq 2$) and a Banach space Z containing ℓ_p such that P does not admit an extension $\widetilde{P} : Z \to X$. Indeed, choose $x_0 \in X$ and $x^* \in X^*$ with $x^*(x_0) = 1$; define

$$P(y) = \Big(\sum_{n=1}^{\infty} \frac{1}{n}\, y_n^2\Big) x_0 \quad \text{for } y = (y_n)_{n=1}^{\infty} \in \ell_p.$$

Then P is approximable, so $P \in \mathcal{P}_{\mathrm{wb}}({}^2\ell_p, X)$.

The polynomial $Q \in \mathcal{P}_{\mathrm{wb}}({}^2\ell_p)$ ($p \geq 2$) given by

$$Q(y) = \sum_{n=1}^{\infty} \frac{1}{n}\, y_n^2$$

is not extendible [C, Example 2.6]. Hence there exists a Banach space Z containing ℓ_p such that Q does not admit an extension to Z. Denote by ψ the embedding of ℓ_p into Z. Suppose there is a polynomial $\widetilde{P} \in \mathcal{P}({}^2Z, X)$ with $\widetilde{P} \circ \psi = P$. Let N be the subspace generated by x_0, let $\pi : X \to N$ be the projection given by $\pi(x) = x^*(x)x_0$ ($x \in X$), and denote by $i : N \to \mathbb{K}$ the natural isomorphism. Then

$$i \circ \pi \circ \widetilde{P} \circ \psi(y) = i(\pi(P(y))) = i\bigg(\pi\bigg(\Big(\sum_{n=1}^{\infty} \frac{1}{n}\, y_n^2\Big) x_0\bigg)\bigg) = \sum_{n=1}^{\infty} \frac{1}{n}\, y_n^2 = Q(y),$$

so $i \circ \pi \circ \widetilde{P}$ is an extension of Q, which is impossible.

We shall now prove that there is no "dual" version of Theorem 3. We recall some preliminary definitions.

A Banach space X is *finitely representable* in a Banach space Y if, for each $\varepsilon > 0$ and each finite-dimensional subspace M of X, there is a finite-dimensional subspace N of Y and a bijective operator $T : M \to N$ such that $\|T\|\,\|T^{-1}\| \leq 1 + \varepsilon$.

If $\mathcal{P}$ is a property defined for Banach spaces, that is, $\mathcal{P}$ is a subclass of the class of all Banach spaces, then a Banach space X has property "super $\mathcal{P}$" if every Banach space finitely representable in X also has $\mathcal{P}$. A property $\mathcal{P}$ is called a *superproperty* if $\mathcal{P} = \text{super}\,\mathcal{P}$ (see [Be, Chapter 4.I]).

THEOREM 10. *If X is an infinite-dimensional Banach space, then $\mathcal{P}_{\mathrm{wb}}({}^mX)$ and $\mathcal{P}({}^mX)$ are not $\mathcal{L}_p$-spaces, for $1 \leq p < \infty$ and $m \in \mathbb{N}$ ($m \geq 2$).*

Proof. It is shown in [Di1, Corollary 3] that ℓ_∞ is finitely representable in $\mathcal{P}({}^mX)$. In fact, the proof is also valid for $\mathcal{P}_{\mathrm{wb}}({}^mX)$. Suppose that $\mathcal{P}_{\mathrm{wb}}({}^mX)$ (respectively, $\mathcal{P}({}^mX)$) is an $\mathcal{L}_p$-space for some $1 \leq p < \infty$. Then $\mathcal{P}_{\mathrm{wb}}({}^mX)$

(respectively, $\mathcal{P}({}^mX)$) has finite cotype [DJT, Corollary 11.7(a)], which implies that ℓ_∞ also has finite cotype [DJT, Theorem 11.6], and this contradicts a well known result [DJT, Corollary 11.7(b)]. ∎

The above proof shows that $\mathcal{P}_{\mathrm{wb}}({}^mX)$ and $\mathcal{P}({}^mX)$ do not have finite cotype. Since ℓ_∞ does not have type > 1 [DJT, Corollary 11.7(b)], neither do $\mathcal{P}_{\mathrm{wb}}({}^mX)$ and $\mathcal{P}({}^mX)$. These results are mentioned in [Fl, 3.3].

More generally, from the fact that every Banach space is finitely representable in ℓ_∞ [FHH, Theorem 9.14], it follows that, if X is an infinite-dimensional Banach space, then $\mathcal{P}_{\mathrm{wb}}({}^mX)$ and $\mathcal{P}({}^mX)$ do not have any non-trivial superproperty, for $m \in \mathbb{N}$ ($m \geq 2$). This result is mentioned in the introduction to [Di1].

Clearly, all these results (from Theorem 10 on) are also true for the space $\mathcal{L}({}^mX)$ of m-linear forms ($m \geq 2$).

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