Extension and lifting of weakly continuous polynomials

by

RAFFAELLA CILIA (Catania) and JOAQUÍN M. GUTIÉRREZ (Madrid)

Abstract. We show that a Banach space \(X\) is an \(L_1\)-space (respectively, an \(L_\infty\)-space) if and only if it has the lifting (respectively, the extension) property for polynomials which are weakly continuous on bounded sets. We also prove that \(X\) is an \(L_1\)-space if and only if the space \(P_{wb}(\mathcal{M}X)\) of \(m\)-homogeneous scalar-valued polynomials on \(X\) which are weakly continuous on bounded sets is an \(L_\infty\)-space.

The problem of lifting holomorphic mappings has attracted attention of various authors (see, for instance, [K, A, AMP, GG4]). The extension of holomorphic mappings from a space to a superspace has been treated in many papers (see, for instance, [AB, A, Z, LRy, GG4]). Here we study the extension and lifting of polynomials which are weakly continuous on bounded sets.

This class of polynomials (the definitions will be recalled below) was introduced in [AP] and has been studied by many authors. In [AHV] it was shown that, if a polynomial is weakly continuous on bounded sets, then it is weakly uniformly continuous on bounded sets. If a dual Banach space \(X^*\) has the approximation property, then these polynomials on \(X\) coincide with the approximable ones [AP, Proposition 2.7]. It was proved in [GG1] that a polynomial \(P\) is weakly continuous on bounded sets if and only if it may be factored in the form \(P = Q \circ T\), where \(T\) is a compact (linear) operator and \(Q\) is a polynomial.

We recall the following well known linear results:

**Theorem 1** ([LR, Theorem 4.1]; see also [L]). Let \(X\) be a Banach space. Then the following facts are equivalent:

---

2000 Mathematics Subject Classification: Primary 46G25; Secondary 46B20, 47H60.

*Key words and phrases:* extension of weakly continuous polynomials, lifting of weakly continuous polynomials, \(L_1\)-space, \(L_\infty\)-space.

The first named author was supported by G.N.A.M.P.A. (Italy).

The second named author was supported in part by Dirección General de Investigación, BFM 2003–06420 (Spain).
(a) $X$ is an $\mathcal{L}_\infty$-space;
(b) for all Banach spaces $Z \supseteq Y$, every compact operator $T : Y \to X$ has an extension to a compact operator $\tilde{T} : Z \to X$;
(c) same as in (b) without the requirement of compactness of $\tilde{T}$;
(d) for all Banach spaces $Z$ and $Y$ with $Z \supseteq X$, every compact operator $T : X \to Y$ has an extension to a compact operator $\tilde{T} : Z \to Y$;
(e) same as in (d) without the requirement of compactness of $\tilde{T}$.

Let $X$, $Y$, and $Z$ be Banach spaces and let $\phi : Z \to Y$ be a surjective operator. A (linear bounded) operator $T : X \to Y$ is said to admit a lifting to $Z$ (with respect to $\phi$) if there is an operator $\tilde{T} : X \to Z$ such that $\phi \circ \tilde{T} = T$. Such a $\tilde{T}$ is called a lifting of $T$ to $Z$.

**Theorem 2** ([LR, Theorem 4.2]). Let $X$ be a Banach space. Then the following facts are equivalent:

(a) $X$ is an $\mathcal{L}_1$-space;
(b) for all Banach spaces $Z$ and $Y$ and any surjective operator $\phi : Z \to Y$, every compact operator $T : X \to Y$ has a compact lifting $\tilde{T} : X \to Z$ with respect to $\phi$;
(c) same as in (b) without the requirement of compactness of $\tilde{T}$;
(d) for all Banach spaces $Z$ and $Y$ and any surjective operator $\phi : Z \to X$, every compact operator $T : Y \to X$ has a compact lifting $\tilde{T} : Y \to Z$ with respect to $\phi$;
(e) same as in (d) without the requirement of compactness of $\tilde{T}$.

Throughout, $X$, $Y$, and $Z$ denote Banach spaces, $X^*$ is the dual of $X$, and $B_X$ stands for its closed unit ball. The closed unit ball $B_X^*$ is a compact space when it is endowed with the weak-star topology, which we denote by $w^*$. By $\mathbb{N}$ we represent the set of all natural numbers while $\mathbb{K}$ denotes the scalar field. By $X \cong Y$ (respectively, $X \equiv Y$), we mean that $X$ and $Y$ are isomorphic (respectively, isometrically isomorphic).

Given $m \in \mathbb{N}$, we denote by $\mathcal{P}^{(m)}(X,Y)$ the space of all $m$-homogeneous (continuous) polynomials from $X$ into $Y$ endowed with the supremum norm. Recall that to each $P \in \mathcal{P}^{(m)}(X,Y)$ we can associate a unique symmetric $m$-linear (continuous) mapping $\hat{P} : X \times \overset{m}{\ldots} \times X \to Y$ so that

$$P(x) = \hat{P}(x,\overset{m}{\ldots},x) \quad (x \in X).$$

For simplicity, we write $\mathcal{P}^{(m)} := \mathcal{P}^{(m)}(X,\mathbb{K})$.

For the general theory of multilinear mappings and polynomials on Banach spaces, we refer to [Di2] and [Mu].

We use the notation $\bigotimes^m X := X \otimes \overset{m}{\ldots} \otimes X$ for the $m$-fold tensor product of $X$, and $\bigotimes_m X$ (respectively, $\bigotimes^m X$) for the $m$-fold tensor product of $X$.
endowed with the projective (respectively, injective) norm. Their completions are denoted by $\bigotimes^m \pi X$ and $\bigotimes^m \varepsilon X$, respectively (see [DF] or [DU] for the theory of tensor products). By $\bigotimes^m \varepsilon X := X \otimes \cdots \otimes X$ we denote the $m$-fold symmetric tensor product of $X$, that is, the set of all elements $u \in \bigotimes^m X$ of the form

$$u = \sum_{j=1}^{n} \lambda_j x_j \otimes \cdots \otimes x_j \quad (n \in \mathbb{N}, \lambda_j \in \mathbb{K}, x_j \in X, 1 \leq j \leq n).$$

The notation $\widetilde{\bigotimes}^m \pi, s X$ stands for the closure of $\bigotimes^m \varepsilon X$ in $\bigotimes^m \pi X$. Given $x_1, \ldots, x_m \in X$, we define

$$x_1 \otimes \cdots \otimes x_m := \frac{1}{m!} \sum_{\sigma \in \mathbb{P}_m} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)},$$

where $\mathbb{P}_m$ is the group of all permutations of $m$ elements. For symmetric tensor products, we refer to [Fl]. For simplicity, we shall use the notation

$$\bigotimes^m x := x \otimes \cdots \otimes x.$$ 

It is well known that $\mathcal{P}(mX) \cong (\widetilde{\bigotimes}^m \pi, s X)^*$ [Fl, Proposition 2.2].

On $\bigotimes^m \varepsilon X$ we define the injective symmetric tensor norm $\varepsilon_s$ by

$$\varepsilon_s(z) = \sup \left\{ \left| \langle z, \bigotimes^m x^* \rangle \right| : x^* \in B_{X^*} \right\} = \sup \left\{ \left| \sum_{k=1}^{n} \lambda_k \langle x_k, x^* \rangle^m \right| : x^* \in B_{X^*} \right\}$$

for $z = \sum_{k=1}^{n} \lambda_k \bigotimes^m x_k$. We denote by $\bigotimes^m \varepsilon_s, s X$ the space $\bigotimes^m \varepsilon X$ endowed with the $\varepsilon_s$-norm, and by $\widetilde{\bigotimes}^m \varepsilon_s, s X$ its completion.

If $P \in \mathcal{P}(mX, Y)$, we define its linearization $\overline{P} : \bigotimes^m \varepsilon_s, s X \rightarrow Y$ by

$$\overline{P} \left( \sum_{i=1}^{n} \lambda_i x_i \otimes \cdots \otimes x_i \right) = \sum_{i=1}^{n} \lambda_i P(x_i)$$

for all $\lambda_i \in \mathbb{K}, x_i \in X$ $(1 \leq i \leq n)$.

If $P \in \mathcal{P}(mX, Y)$, the adjoint of $P$ is the operator $P^* : Y^* \rightarrow \mathcal{P}(mX)$ given by

$$P^*(y^*)(x) = \langle y^*, P(x) \rangle.$$ 

A polynomial $P \in \mathcal{P}(mX, Y)$ is of finite type if it is a finite sum of terms of the form $\gamma^m \otimes y$, with $\gamma \in X^*$ and $y \in Y$, where $(\gamma^m \otimes y)(x) := \gamma(x)^m y$ for all $x \in X$. A polynomial is approximable if it lies in the norm closure of the space of polynomials of finite type. By $\mathcal{P}_A(mX, Y)$ we denote the space of all $m$-homogeneous approximable polynomials from $X$ into $Y$, endowed with the supremum norm.

It is shown in [Fl, Proposition 3.2(6)] that

$$\mathcal{P}_A(mX) \equiv \widetilde{\bigotimes}^m \varepsilon_s, s X^*.$$
We are indebted to the referee for pointing out that
\[ \mathcal{P}_A(mX, Y) \equiv \bigotimes_{\varepsilon, s} X^* \otimes_{\varepsilon} Y. \]
Indeed, define the operator
\[ J : \mathcal{P}_A(mX) \otimes_{\varepsilon} Y \to \mathcal{P}(mX, Y) \]
by
\[ J\left( \sum_{i=1}^{n} P_i \otimes y_i \right) = Q \quad \text{where} \quad Q(x) := \sum_{i=1}^{n} P_i(x)y_i \quad (x \in X). \]
Then
\[ \|Q\| = \sup\{\|Q(x)\| : x \in B_X\} = \sup\{\langle y^*, Q(x) \rangle : x \in B_X, y^* \in B_{Y^*}\} \]
\[ = \sup \left\{ \left\| \sum_{i=1}^{n} P_i(x)y^*(y_i) \right\| : x \in B_X, y^* \in B_{Y^*} \right\}. \]
We can embed \( \bigotimes_{s}^m X \) into \( \mathcal{P}_A(mX)^* \) by the duality
\[ \left\langle \sum_{i=1}^{n} \lambda_i \bigotimes_{s}^m x_i, P \right\rangle = \sum_{i=1}^{n} \lambda_i P(x_i). \]
It follows that the set \( \{ \bigotimes_{s}^m x : x \in B_X\} \) is norming in the unit ball of \( \mathcal{P}_A(mX)^* \), so
\[ \|Q\| = \sup \left\{ \left\| \sum_{i=1}^{n} \Phi(P_i)y^*(y_i) \right\| : \Phi \in \mathcal{P}_A(mX)^*, \|\Phi\| \leq 1, y^* \in B_{Y^*} \right\} \]
\[ = \left\| \sum_{i=1}^{n} P_i \otimes y_i \right\|_{\varepsilon} \]
(see the argument in [Ry, p. 46]) and \( J \) is an into isometry. Clearly, the space \( \mathcal{P}_A(mX, Y) \) is the range of the extension of \( J \) to \( \mathcal{P}_A(mX) \otimes_{\varepsilon} Y \).

A polynomial \( P \in \mathcal{P}(mX, Y) \) is compact if \( P(B_X) \) is relatively compact in \( Y \). A polynomial \( P \in \mathcal{P}(mX, Y) \) is compact if and only if its adjoint \( P^* : Y^* \to \mathcal{P}(mX) \) is a compact operator [AS, Proposition 3.2]. A polynomial \( P \in \mathcal{P}(mX, Y) \) is weakly continuous on bounded subsets if for each bounded net \( (x_\alpha) \subset X \) weakly converging to \( x \), \( (P(x_\alpha)) \) converges to \( P(x) \) in norm. We denote by \( \mathcal{P}_{wb}(mX, Y) \) the space of all polynomials in \( \mathcal{P}(mX, Y) \) which are weakly continuous on bounded sets. Every polynomial in \( \mathcal{P}_{wb}(mX, Y) \) is compact ([AP, Lemma 2.2] and [AHV, Theorem 2.9]). An operator is weakly continuous on bounded sets if and only if it is compact [AP, Proposition 2.5]. With each polynomial \( P \in \mathcal{P}(mX, Y) \) we associate an operator \( T_P : X \to \mathcal{P}(m^{-1}X, Y) \) given by
\[ T_P(x)(y) := \widehat{P}(x, y, (m^{-1})y) \quad (x, y \in X). \]
Then \( P \in \mathcal{P}_{wb}(mX, Y) \) if and only if \( T_P \) is compact [AHV, Theorem 2.9].
The Banach–Mazur distance \( d(X, Y) \) between two isomorphic Banach spaces \( X \) and \( Y \) is defined by \( \inf(||T|| ||T^{-1}||) \) where the infimum is taken over all isomorphisms \( T \) from \( X \) onto \( Y \). Recall that a Banach space \( X \) is an \( \mathcal{L}_p \)-space \( (1 \leq p \leq \infty) \) [LP] if there is \( \lambda \geq 1 \) such that every finite-dimensional subspace of \( X \) is contained in another subspace \( N \) with \( d(N, \ell_p^n) \leq \lambda \) for some integer \( n \).

A Banach space \( X \) is an \( \mathcal{L}_p \)-space \( (1 \leq p \leq \infty) \) if and only if \( X^\ast \) is an \( \mathcal{L}_q \)-space \( (p^{-1} + q^{-1} = 1) \) [LR, Theorem III(a)]. An infinite-dimensional complemented subspace of an \( \mathcal{L}_1 \)-space (respectively, of an \( \mathcal{L}_\infty \)-space) is an \( \mathcal{L}_1 \)-space (respectively, an \( \mathcal{L}_\infty \)-space) [LR, Theorem III(b)].

If \( X \) is an \( \mathcal{L}_1 \)-space, then, for each \( m \), \( \bigotimes_{\pi}^m X \) is an \( \mathcal{L}_1 \)-space [DF, Theorem 34.9] (an easy proof may be found in [GG3, Proposition 2.2]). Since \( \bigotimes_{\pi,s}^m X \) is isomorphic to a complemented subspace of \( \bigotimes_{\pi}^m X \) [Fl, Proposition 2.3], it follows that \( \bigotimes_{\pi,s}^m X \) is an \( \mathcal{L}_1 \)-space. Therefore, \( \mathcal{P}(^m X) \cong (\bigotimes_{\pi,s}^m X)^\ast \) is an \( \mathcal{L}_\infty \)-space.

If \( X \) is an infinite-dimensional \( \mathcal{L}_1 \)-space, then \( X \) contains a complemented copy of \( \ell_1 \) [LP, Proposition 7.3]. Therefore, \( \mathcal{P}_{\text{wb}}(^m X) \) is not complemented in \( \mathcal{P}(^m X) \) [GG2, Lemma 5]. So it is in principle unknown if \( \mathcal{P}_{\text{wb}}(^m X) \) is an \( \mathcal{L}_\infty \)-space. We now prove that the answer to this question is affirmative. This will be useful later on.

**Theorem 3.** Let \( X \) be a Banach space. The following facts are equivalent:

(a) \( X \) is an \( \mathcal{L}_1 \)-space;

(b) for each \( m \in \mathbb{N} \), \( \mathcal{P}_{\text{wb}}(^m X) \) is an \( \mathcal{L}_\infty \)-space;

(c) there exists \( m \in \mathbb{N} \) such that \( \mathcal{P}_{\text{wb}}(^m X) \) is an \( \mathcal{L}_\infty \)-space.

**Proof.** (a) \( \Rightarrow \) (b). Since \( X^\ast \) has the approximation property [DF, p. 306], we have

\[
\mathcal{P}_{\text{wb}}(^m X) = \mathcal{P}_A(^m X) \equiv \bigotimes_{\varepsilon,s}^m X^\ast
\]

(see [AP, Proposition 2.7] and our introduction). Now, \( X^\ast \) is an \( \mathcal{L}_\infty \)-space, so \( \bigotimes_{\varepsilon}^m X^\ast \) is also an \( \mathcal{L}_\infty \)-space (a proof may be found in [GG3, Proposition 2.2]). Then its complemented subspace \( \bigotimes_{\varepsilon,s}^m X^\ast \) [Fl, Proposition 3.1] is again an \( \mathcal{L}_\infty \)-space.

(b) \( \Rightarrow \) (c) is obvious.

(c) \( \Rightarrow \) (a). Since \( X^\ast \) is complemented in \( \mathcal{P}_{\text{wb}}(^m X) \) [AS, Proposition 5.3], \( X^\ast \) is an \( \mathcal{L}_\infty \)-space as well. Thus \( X \) is an \( \mathcal{L}_1 \)-space. ■

We now give two preparatory results.
Proposition 4. Let \( P \in \mathcal{P}(mX, Y) \). Then its adjoint \( P^* : Y^* \to \mathcal{P}(mX) \) is \( w^* - \tau_p \)-continuous, where \( \tau_p \) is the topology of pointwise convergence on \( \mathcal{P}(mX) \). Conversely, let \( S : Y^* \to \mathcal{P}(mX) \) be a \( w^* - \tau_p \)-continuous operator. Then there is a polynomial \( P \in \mathcal{P}(mX, Y) \) such that \( S = P^* \).

Proof. Given \( P \in \mathcal{P}(mX, Y) \), let \( (y_\alpha^*) \) be a weak-star null net in \( Y^* \). Then, for every \( x \in X \), we have

\[
P^*(y_\alpha^*)(x) = \langle y_\alpha^*, P(x) \rangle \to 0,
\]

so \( P^*(y_\alpha^*) \to 0 \) in the \( \tau_p \)-topology on \( \mathcal{P}(mX) \).

Conversely, let \( S : Y^* \to \mathcal{P}(mX) \) be a \( w^* - \tau_p \)-continuous operator. Let

\[
k_m : \mathcal{P}(mX) \to \mathcal{P}(mX)^{**} \approx \mathcal{P}(mX)^*
\]

be the natural embedding and let \( k_Y \) be the natural embedding of \( Y \) into \( Y^{**} \). For \( x \in X \) fixed, consider the functional \( k_m \circ S : Y^* \to \mathbb{K} \).

Clearly,

\[
(k_m \circ S)(y^*) = S(y^*)(x).
\]

By our hypothesis on \( S \), \( k_m \circ S \) is weak-star continuous on \( Y^* \) and so it belongs to \( k_Y(Y) \). Let \( P \in \mathcal{P}(mX, Y) \) be defined by \( P(x) = k_Y^{-1}(k_m \circ S) \). We have

\[
P^*(y^*)(x) = \langle y^*, P(x) \rangle = \langle y^*, k_Y^{-1}(k_m \circ S) \rangle
\]

\[
= k_m \circ S(y^*)(x).
\]

Therefore, \( S = P^* \). □

If \( P \in \mathcal{P}_{wb}(mX, Y) \), then clearly \( P^*(Y^*) \subseteq \mathcal{P}_{wb}(mX) \).

Proposition 5. Let \( P \in \mathcal{P}_{wb}(mX, Y) \). Then its adjoint \( P^* : Y^* \to \mathcal{P}_{wb}(mX) \) is compact and \( w^* - \tau_p \)-continuous. Conversely, let \( S : Y^* \to \mathcal{P}_{wb}(mX) \) be a compact and \( w^* - \tau_p \)-continuous operator. Then there is \( P \in \mathcal{P}_{wb}(mX, Y) \) such that \( S = P^* \).

Proof. If \( P \in \mathcal{P}_{wb}(mX, Y) \), then \( P^* \) is compact and, by Proposition 4, it is also \( w^* - \tau_p \)-continuous.

Conversely, if \( S : Y^* \to \mathcal{P}_{wb}(mX) \) is a compact and \( w^* - \tau_p \)-continuous operator, then by Proposition 4, there is \( P \in \mathcal{P}(mX, Y) \) such that \( S = P^* \). Since \( S \) is compact, \( P \) is compact. Since \( y^* \circ P \in \mathcal{P}_{wb}(mX) \) for each \( y^* \in Y^* \) it follows from [AP, Proposition 2.8] that \( P \in \mathcal{P}_{wb}(mX, Y) \). □

We now extend the equivalences (a) \( \iff \) (d) \( \iff \) (e) of Theorem 2 to the class of polynomials which are weakly continuous on bounded sets.

Theorem 6. Let \( X \) be a Banach space. Then the following facts are equivalent:

1. \( P \) is weakly continuous on bounded sets.
2. \( P \) is \( w^* - \tau_p \)-continuous.
3. \( P \) is compact.
4. \( P \) is \( w^* - \tau_p \)-continuous and \( \mathcal{P}_{wb}(mX, Y) \).
(a) $X$ is an $L_1$-space;
(b) for all $m \in \mathbb{N}$, for every surjective operator $\phi : Z \to X$, every $P \in \mathcal{P}_{wb}^{(m)Y, X}$ has a lifting $\tilde{P} \in \mathcal{P}_{wb}^{(m)Y, Z}$ with respect to $\phi$;
(c) there is $m \in \mathbb{N}$ for which (b) holds;
(d) there is $m \in \mathbb{N}$ for which (b) holds without the requirement that the lifting be weakly continuous on bounded sets.

**Proof.** (a)⇒(b). Let $X$ be an $L_1$-space and $P \in \mathcal{P}_{wb}^{(m)Y, X}$. Then there exist a Banach space $G$, a compact operator $T : Y \to G$ and a polynomial $Q \in \mathcal{P}^{(m)G, X}$ such that $P = Q \circ T$. Since every compact operator factors through two compact operators [F, Corollary 3.3], we can assume that $Q$ is compact. By [GG4, Theorem 2], there exists a lifting $\tilde{Q} \in \mathcal{P}^{(m)G, Z}$ with respect to $\phi$. Then $\tilde{P} := \tilde{Q} \circ T \in \mathcal{P}_{wb}^{(m)Y, Z}$ is a lifting of $P$.
(b)⇒(c) and (c)⇒(d) are obvious.
(d)⇒(a). Let $m \in \mathbb{N}$ ($m \geq 2$). We show that the statement is true for the index $m - 1$. Let $Q \in \mathcal{P}_{wb}^{(m-1)Y, X}$. By Proposition 5, its adjoint $Q^* : X^* \to \mathcal{P}_{wb}^{(m-1)Y}$ is compact and $w^*\tau_p$-continuous. Choose $e \in Y$ and $\gamma \in Y^*$ with $\gamma(e) = 1$. Adapting the proof of [AS, Proposition 5.3], we define the operators

$$j_Y : \mathcal{P}_{wb}^{(m-1)Y} \to \mathcal{P}_{wb}^{(m)Y}$$

by

$$j_Y(R)(y) := \gamma(y)R(y) \quad \text{for } R \in \mathcal{P}_{wb}^{(m-1)Y}, y \in Y,$$

and

$$\pi_Y : \mathcal{P}_{wb}^{(m)Y} \to \mathcal{P}_{wb}^{(m-1)Y}$$

given by

$$\pi_Y(S)(y) := \sum_{i=1}^{m} \binom{m}{i} (-1)^{i+1} \gamma(y)^{i-1} \hat{S}(e^i, y^{m-i}),$$

where

$$\hat{S}(e^i, y^{m-i}) := \hat{S}(e, (i), e, (m-i), y).$$

We show that $\pi_Y \circ j_Y$ is the identity map on $\mathcal{P}_{wb}^{(m-1)Y}$ (and, therefore, $j_Y \circ \pi_Y$ is a projection). Indeed, given $R \in \mathcal{P}_{wb}^{(m-1)Y}$ and $y \in Y$, since

$$j_Y(R)(y_1, \ldots, y_m) = \frac{1}{m} \sum_{i=1}^{m} \gamma(y_i) \hat{R}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m),$$

we have

$$\pi_Y(j_Y(R))(y) = \frac{1}{m} \sum_{i=1}^{m} \binom{m}{i} (-1)^{i+1} \gamma(y)^{i-1} \times [i \gamma(e) \hat{R}(e^{i-1}, y^{m-i}) + (m-i) \gamma(y) \hat{R}(e^i, y^{m-i-1})].$$
For each $i$, we have 

\[
\binom{m}{i-1} (-1)^i (m-i+1) \gamma(y)^{i-1} \hat{R}(e^{i-1}, y^{m-i}) + \binom{m}{i} (-1)^{i+1} i \gamma(y)^{i-1} \hat{R}(e^{i-1}, y^{m-i}) = 0,
\]

so 

\[
\pi_Y(j_Y(R))(y) = R(y).
\]

The operator $j_Y \circ Q^*: X^* \to \mathcal{P}_{wb}(mY)$ is still compact and $w^*-\tau_p$-continuous. Indeed, if $(x^*_\alpha)$ is a net converging to 0 in the weak-star topology of $X^*$ and $y \in Y$, we have 

\[
j_Y(Q^*(x^*_\alpha))(y) = \gamma(y)Q^*(x^*_\alpha)(y) \to 0.
\]

By Proposition 5, there exists a polynomial $P \in \mathcal{P}_{wb}(mY, X)$ such that $P^* = j_Y \circ Q^*$. By the hypothesis, there is a lifting $\tilde{P} \in \mathcal{P}(mY, Z)$ with respect to $\phi$. Using Proposition 4 and considering $\pi_Y$ as an operator $\mathcal{P}(mY) \to \mathcal{P}(m^{-1}Y)$, we see that $\pi_Y \circ (\tilde{P})^*: Z^* \to \mathcal{P}(m^{-1}Y)$ is $w^*-\tau_p$-continuous. So there is $\tilde{Q} \in \mathcal{P}(m^{-1}Y, Z)$ such that $(\tilde{Q})^* = \pi_Y \circ (\tilde{P})^*$. We now prove that $\phi \circ \tilde{Q} = Q$, equivalently, $(\phi \circ \tilde{Q})^* = Q^*$. Indeed, 

\[
(\phi \circ \tilde{Q})^* = (\tilde{Q})^* \circ \phi^* = \pi_Y \circ (\tilde{P})^* \circ \phi^* = \pi_Y \circ P^* = \pi_Y \circ j_Y \circ Q^* = Q^*.
\]

So $\tilde{Q}$ is a lifting of $Q$. Iterating the argument, we deduce that every compact operator from $Y$ into $X$ has a lifting to $Z$. By Theorem 2, $X$ is an $\mathcal{L}_1$-space. 

We now give the equivalences (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) of Theorem 2 for polynomials which are weakly continuous on bounded sets.

**Theorem 7.** Let $X$ be a Banach space. Then the following facts are equivalent:

(a) $X$ is an $\mathcal{L}_1$-space;

(b) for all $m \in \mathbb{N}$, for every surjective operator $\phi: Z \to Y$, every $P \in \mathcal{P}_{wb}(mX, Y)$ admits a lifting $\tilde{P} \in \mathcal{P}_{wb}(mX, Z)$ with respect to $\phi$;

(c) there exists $m \in \mathbb{N}$ for which (b) holds;

(d) there exists $m \in \mathbb{N}$ for which (b) holds without the requirement that the lifting be weakly continuous on bounded sets.

**Proof.** (a)$\Rightarrow$(b). Let $X$ be an $\mathcal{L}_1$-space. Since $X^*$ has the approximation property [DF, p. 306], we have 

\[
\mathcal{P}_{wb}(mX, Y) \equiv \mathcal{P}_A(mX) \hat{\otimes}_\varepsilon Y
\]

(see [AP, Proposition 2.7]). By Theorem 3, $\mathcal{P}_A(mX) = \mathcal{P}_{wb}(mX)$ is an $\mathcal{L}_{\infty}$-space. Then the operators 

\[
I \otimes \phi: \mathcal{P}_A(mX) \otimes_\varepsilon Z \to \mathcal{P}_A(mX) \otimes_\varepsilon Y
\]
and
\[ I \otimes \phi : \mathcal{P}_{A}(mX) \otimes_{\varepsilon} Z \to \mathcal{P}_{A}(mX) \otimes_{\varepsilon} Y \]
are surjective [DF, 23.5, Corollaries 5 and 6], where \( I \) is the identity map on \( \mathcal{P}_{A}(mX) \). A standard argument allows us to conclude that every \( P \in \mathcal{P}_{wb}(mX, Y) \equiv \mathcal{P}_{A}(mX) \otimes_{\varepsilon} Y \) admits a lifting \( \widetilde{P} : \mathcal{P}_{wb}(mX, Z) \equiv \mathcal{P}_{A}(mX) \otimes_{\varepsilon} Z \) with respect to \( \phi \).

(b)\(\Rightarrow\)(c) and (c)\(\Rightarrow\)(d) are obvious.

(d)\(\Rightarrow\)(a). It is enough to show that the statement is true for the index \( m - 1 \). Define the operators
\[ j_{X} : \mathcal{P}_{wb}(m-1X) \to \mathcal{P}_{wb}(mX), \quad \pi_{X} : \mathcal{P}_{wb}(mX) \to \mathcal{P}_{wb}(m-1X) \]
as in the proof of Theorem 6. The proof is analogous to that of (d)\(\Rightarrow\)(a) in Theorem 6. \( \blacksquare \)

Now we give the equivalences (a)\(\Leftrightarrow\)(d)\(\Leftrightarrow\)(e) of Theorem 1 in the setting of polynomials which are weakly continuous on bounded sets.

**Theorem 8.** Let \( X \) be a Banach space. Then the following facts are equivalent:

(a) \( X \) is an \( \mathcal{L}_{\infty} \)-space;
(b) for all \( m \in \mathbb{N} \), for every into isomorphism \( \psi : X \to Z \), every \( P \in \mathcal{P}_{wb}(mX, Y) \) has an extension \( \widetilde{P} \in \mathcal{P}_{wb}(mZ, Y) \) such that \( \widetilde{P} \circ \psi = P \);
(c) there exists \( m \in \mathbb{N} \) for which (b) holds;
(d) there exists \( m \in \mathbb{N} \) for which (b) holds without the requirement that the extension be weakly continuous on bounded sets.

**Proof.** (a)\(\Rightarrow\)(b). Let \( X \) be an \( \mathcal{L}_{\infty} \)-space. Let \( \psi : X \to Z \) be an into isomorphism, and let \( P \in \mathcal{P}_{wb}(mX, Y) \). Then there exist a Banach space \( G \), a compact operator \( T : X \to G \) and a polynomial \( Q \in \mathcal{P}(mG, Y) \) such that \( P = Q \circ T \). By Theorem 1, there is a compact operator \( \widetilde{T} : Z \to G \) such that \( \widetilde{T} \circ \psi = T \). Define \( \widetilde{P} := Q \circ \widetilde{T} \in \mathcal{P}(mZ, Y) \). Then \( \widetilde{P} \) is an extension of \( P \), and \( \widetilde{P} \in \mathcal{P}_{wb}(mZ, Y) \).

(b)\(\Rightarrow\)(c) and (c)\(\Rightarrow\)(d) are obvious.

(d)\(\Rightarrow\)(a). We can assume \( m \geq 2 \). We prove that, for every into isomorphism \( \psi : X \to Z \), every \( Q \in \mathcal{P}_{wb}(m-1X, Y) \) has an extension \( \widetilde{Q} \in \mathcal{P}(m-1Z, Y) \) such that \( \widetilde{Q} \circ \psi = Q \). Let \( \overline{Q} : \bigotimes_{\pi_{s}}^{m-1} X \to Y \) be the linearization of \( Q \). Choose \( x_{0} \in X \) and let \( z_{0} := \psi(x_{0}) \). Select \( z^{*} \in Z^{*} \) with \( z^{*}(z_{0}) = 1 \) and let \( x^{*} = \psi^{*}(z^{*}) \). Then \( x^{*}(x_{0}) = 1 \). Let
\[ \Pi_{X} : \bigotimes_{\pi_{s}}^{m} X \to \bigotimes_{\pi_{s}}^{m-1} X, \quad J_{X} : \bigotimes_{\pi_{s}}^{m-1} X \to \bigotimes_{\pi_{s}}^{m} X \]
be the operators defined in [Bl, p. 168] by
\[ \Pi_{X}(\bigotimes^{m} x) = x^{*}(x) \bigotimes^{m-1} x \]
and
\[ J_X(\bigotimes^{m-1} x) = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} x^{*}(x)^{k-1} x_0 \otimes_s \ldots \otimes_s x_0 \otimes_s x \otimes_s \ldots \otimes_s x. \]

It is shown in [Bl, p. 168] that \( \Pi_X \circ J_X \) is the identity map on \( \tilde{\bigotimes}^{m-1} X \).
Similarly we can define
\[ J_Z : \tilde{\bigotimes}^{m-1} Z \to \tilde{\bigotimes}^{m} Z \]
by
\[ J_Z(\bigotimes^{m-1} z) = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} z^{*}(z)^{k-1} z_0 \otimes_s \ldots \otimes_s z_0 \otimes_s z \otimes_s \ldots \otimes_s z. \]

Consider \( S := \overline{Q} \circ \Pi_X : \tilde{\bigotimes}^{m} X \to Y \) and let \( P \in \mathcal{P}(mX, Y) \) be the polynomial whose linearization is \( S \). Then, for each \( x \in X \),
\[ P(x) = S(\bigotimes^{m} x) = \overline{Q} \circ \Pi_X(\bigotimes^{m} x) = x^{*}(x)\overline{Q}(\bigotimes^{m-1} x) = x^{*}(x)Q(x). \]

Since \( Q \in \mathcal{P}_{wb}(m^{-1}X, Y) \), it follows that \( P \in \mathcal{P}(mX, Y) \). By our hypothesis, there is \( \tilde{P} \in \mathcal{P}(mZ, Y) \) such that \( \tilde{P} \circ \psi = P \). Let \( \tilde{P} \) be the linearization of \( \tilde{P} \), and consider the composition \( \tilde{P} \circ J_Z : \tilde{\bigotimes}^{m-1} Z \to Y \). Let \( \tilde{Q} \in \mathcal{P}(m^{-1}Z, Y) \) be the polynomial whose linearization coincides with \( \tilde{P} \circ J_Z \). We show that \( \tilde{Q} \) extends \( Q \), that is, \( \tilde{Q} \circ \psi = Q \), equivalently, \( \overline{Q} \circ \psi = \overline{Q} \). Indeed,
\[ \overline{Q} \circ \psi(\bigotimes^{m-1} x) = \overline{Q}(\bigotimes^{m-1} \psi(x)) = (\overline{P} \circ J_Z)(\bigotimes^{m-1} \psi(x)) \]
\[ = \overline{P} \left[ \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} z^{*}(\psi(x))^{k-1} \times \psi(x_0) \otimes_s \ldots \otimes_s \psi(x_0) \otimes_s \psi(x) \otimes_s \ldots \otimes_s \psi(x) \right] \]
\[ = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} P(\psi(x_0) \otimes_s \ldots \otimes_s x_0 \otimes_s x \otimes_s \ldots \otimes_s x) \]
\[ = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} z^{*}(\psi(x))^{k-1} \overline{P}(\psi(x_0) \otimes_s \ldots \otimes_s x_0 \otimes_s x \otimes_s \ldots \otimes_s x) \]
\[ = \overline{P} \left[ \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} x^{*}(x)^{k-1} x_0 \otimes_s \ldots \otimes_s x_0 \otimes_s x \otimes_s \ldots \otimes_s x \right] \]
\[ = \overline{P} \circ J_X(\bigotimes^{m-1} x) = \overline{Q} \circ \Pi_X \circ J_X(\bigotimes^{m-1} x) = \overline{Q}(\bigotimes^{m-1} x). \]

Iterating the argument, we conclude that every compact operator \( X \to Y \) has an extension, and so \( X \) is an \( \mathcal{L}_{\infty} \)-space by Theorem 1.
Remark 9. The implications (a)⇒(b) and (a)⇒(c) of Theorem 1 do not extend to the polynomials which are weakly continuous on bounded sets. We shall see that, for every Banach space $X \neq \{0\}$, there are a polynomial $P \in \mathcal{P}_{wb}(2\ell_p, X)$ ($p \geq 2$) and a Banach space $Z$ containing $\ell_p$ such that $P$ does not admit an extension $\tilde{P} : Z \to X$. Indeed, choose $x_0 \in X$ and $x^* \in X^*$ with $x^*(x_0) = 1$; define

$$P(y) = \left(\sum_{n=1}^{\infty} \frac{1}{n^k} y_n^2\right) x_0 \quad \text{for } y = (y_n)_{n=1}^{\infty} \in \ell_p.$$ 

Then $P$ is approximable, so $P \in \mathcal{P}_{wb}(2\ell_p, X)$.

The polynomial $Q \in \mathcal{P}_{wb}(2\ell_p)$ ($p \geq 2$) given by

$$Q(y) = \sum_{n=1}^{\infty} \frac{1}{n^2} y_n^2$$

is not extendible [C, Example 2.6]. Hence there exists a Banach space $Z$ containing $\ell_p$ such that $Q$ does not admit an extension to $Z$. Denote by $\psi$ the embedding of $\ell_p$ into $Z$. Suppose there is a polynomial $\tilde{P} \in \mathcal{P}(2Z, X)$ with $\tilde{P} \circ \psi = P$. Let $N$ be the subspace generated by $x_0$, let $\pi : X \to N$ be the projection given by $\pi(x) = x^*(x) x_0$ ($x \in X$), and denote by $i : N \to \mathbb{K}$ the natural isomorphism. Then

$$i \circ \pi \circ \tilde{P} \circ \psi(y) = i(\pi(P(y))) = i\left(\pi\left(\left(\sum_{n=1}^{\infty} \frac{1}{n} y_n^2\right) x_0\right)\right) = \sum_{n=1}^{\infty} \frac{1}{n} y_n^2 = Q(y),$$

so $i \circ \pi \circ \tilde{P}$ is an extension of $Q$, which is impossible.

We shall now prove that there is no “dual” version of Theorem 3. We recall some preliminary definitions.

A Banach space $X$ is finitely representable in a Banach space $Y$ if, for each $\varepsilon > 0$ and each finite-dimensional subspace $M$ of $X$, there is a finite-dimensional subspace $N$ of $Y$ and a bijective operator $T : M \to N$ such that $\|T\| \|T^{-1}\| \leq 1 + \varepsilon$.

If $\mathcal{P}$ is a property defined for Banach spaces, that is, $\mathcal{P}$ is a subclass of the class of all Banach spaces, then a Banach space $X$ has property “super $\mathcal{P}$” if every Banach space finitely representable in $X$ also has $\mathcal{P}$. A property $\mathcal{P}$ is called a superproperty if $\mathcal{P} = \text{super } \mathcal{P}$ (see [Be, Chapter 4.1]).

Theorem 10. If $X$ is an infinite-dimensional Banach space, then $\mathcal{P}_{wb}(mX)$ and $\mathcal{P}(mX)$ are not $\mathcal{L}_p$-spaces, for $1 \leq p < \infty$ and $m \in \mathbb{N}$ ($m \geq 2$).

Proof. It is shown in [Di1, Corollary 3] that $\ell_\infty$ is finitely representable in $\mathcal{P}(mX)$. In fact, the proof is also valid for $\mathcal{P}_{wb}(mX)$. Suppose that $\mathcal{P}_{wb}(mX)$ (respectively, $\mathcal{P}(mX)$) is an $\mathcal{L}_p$-space for some $1 \leq p < \infty$. Then $\mathcal{P}_{wb}(mX)$
(respectively, $\mathcal{P}(mX)$) has finite cotype [DJT, Corollary 11.7(a)], which implies that $\ell_\infty$ also has finite cotype [DJT, Theorem 11.6], and this contradicts a well known result [DJT, Corollary 11.7(b)].

The above proof shows that $\mathcal{P}_{wb}(mX)$ and $\mathcal{P}(mX)$ do not have finite cotype. Since $\ell_\infty$ does not have type $> 1$ [DJT, Corollary 11.7(b)], neither do $\mathcal{P}_{wb}(mX)$ and $\mathcal{P}(mX)$. These results are mentioned in [Fl, 3.3].

More generally, from the fact that every Banach space is finitely representable in $\ell_\infty$ [FHH, Theorem 9.14], it follows that, if $X$ is an infinite-dimensional Banach space, then $\mathcal{P}_{wb}(mX)$ and $\mathcal{P}(mX)$ do not have any non-trivial superproperty, for $m \in \mathbb{N}$ ($m \geq 2$). This result is mentioned in the introduction to [Dil].

Clearly, all these results (from Theorem 10 on) are also true for the space $\mathcal{L}(mX)$ of $m$-linear forms ($m \geq 2$).

We are grateful to the referee for many suggestions that have improved the paper.

References


[Fl] K. Floret, Natural norms on symmetric tensor products of normed spaces, Note Mat. 17 (1997), 153–188.


