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Sobolev-type spaces from generalized Poincaré inequalities

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Abstract. We define a Sobolev space by means of a generalized Poincaré inequality and relate it to a corresponding space based on upper gradients.

1. Introduction. This paper addresses the properties of a Sobolev-type space obtained by means of a generalized Poincaré inequality. Unless otherwise stated, $X = (X, \mathrm{d}, \mu)$ is a metric measure space with μ doubling.

Recall that the classical Poincaré inequality states that the estimate

$$\fint_B |u - u_B|\, dx \le C r_B \Big(\fint_B g^p\, dx \Big)^{1/p}, \tag{1}$$

where $\fint_B v = |B|^{-1} \int_B v$, $u_B = \fint_B u$, r_B is the radius of the ball B and $g = |\nabla u|$, holds for each ball B and all functions $u \in W^{1,p}(B)$, $1 \le p < \infty$. In a sense, this single inequality captures the essentials of the theory of the first order Sobolev spaces $W^{1,p}$ consisting of those p-integrable functions that have a p-integrable weak gradient ∇u. Indeed, $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and there is a non-negative function $g \in L^p(\mathbb{R}^n)$ so that (1) holds. A suitable form of this statement extends to many general settings, including those of Heisenberg groups, $\mathbb{R}^n$ equipped with an A_p-weight and general doubling metric measure spaces that support a Poincaré inequality for Lipschitz functions and their pointwise Lipschitz constants. For all this see [14], [7], [11], and [12]. Moreover, (1) is known to yield versions of the usual Sobolev–Poincaré and Trudinger inequalities and other inequalities of this kind [15], [14].

It is then natural to inquire if (1) could be replaced with some other, more general inequality. Such an inequality is given by

$$\fint_B |u - u_B|\, d\mu \le a(\tau B), \tag{2}$$

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where $\tau \geq 1$ is fixed, and $a : \mathcal{B} \to [0, \infty)$ is a functional that satisfies a certain discrete summability condition. Here $\mathcal{B}$ is a collection of balls and $\tau B(x, r) = B(x, \tau r)$. Inequalities of this type were introduced in [8] and further studied in [21], [22], and [9]. In these papers, versions of the Sobolev–Poincaré and Trudinger inequalities were established relying on (2), generalizing the earlier work in [15], [14]. One of the points here is that (2) could well hold for, say, all Lipschitz functions even if there is no usual Poincaré inequality (1) for Lipschitz functions and their pointwise Lipschitz constants. It seems to us that our examples in Section 6 are the first of this kind.

We are then led to consider a Banach space of functions satisfying (2) and to relate this space to some natural Sobolev space. Notice that then the Sobolev–Poincaré and Trudinger inequalities should be automatically satisfied for the functions in our space. To this end, let $\Omega \subset X$ be open, $0 \leq \alpha < \infty$, and $0 < p \leq \infty$. Write $\mathcal{B}_\Omega$ for the collection of all balls in Ω. Denote by $A^{\alpha,p}_\tau(\Omega)$ the set of all locally integrable functions u that satisfy (2) in all balls B for which $\tau B \subset \Omega$ with a functional a of the form

$$a(B) = r^\alpha \left(\frac{\nu(B)}{\mu(B)} \right)^{1/p}, \tag{3}$$

where $\nu : \mathcal{B} \to [0, \infty)$ satisfies

$$\sum_i \nu(B_i) < \infty$$

whenever the balls $B_i \in \mathcal{B}_\Omega$ are disjoint. Then $u \in A^{\alpha,p}_\tau(\Omega)$ if and only if

$$\|u\|_{A^{\alpha,p}_\tau(\Omega)} = \sup_{\mathcal{B} \in \mathcal{B}_\tau(\Omega)} \Big\| \sum_{B \in \mathcal{B}} \Big(r_B^{-\alpha} \fint_B |u - u_B| \, d\mu \Big) \chi_B \Big\|_{L^p(\Omega)}, \tag{4}$$

where

$$\mathcal{B}_\tau(\Omega) = \{\{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } \Omega\},$$

is finite. If $p \geq 1$, then the space $L^p(\Omega) \cap A^{\alpha,p}_\tau(\Omega)$ equipped with the norm $\|\cdot\|_{L^p(\Omega)} + \|\cdot\|_{A^{\alpha,p}_\tau(\Omega)}$ becomes a Banach space. Notice that (3) is the canonical example of a functional a considered in [8], [9], [21], and [22]. In the borderline case $\alpha = 0$, $p = \infty, \tau = 1$ our space reduces to BMO, the space of functions of bounded mean oscillation.

Our substitute for the usual Sobolev class $W^{1,p}$ will be given in terms of a Sobolev space based on upper gradients (cf. [17], [27]). For the connection with the spaces based on pointwise inequalities (cf. [10]), see Section 3 below. In the metric setting, we cannot talk about partial derivatives but the concept of an upper gradient has turned out to be a nice substitute for the length of the gradient. We call a Borel function $g : X \to [0, \infty]$ an *upper gradient* of a function $u : X \to \overline{\mathbb{R}}$ if

$$|u(\gamma(0)) - u(\gamma(l))| \leq \int_\gamma g \, ds \tag{5}$$

for all rectifiable curves $\gamma : [0,l] \to X$. Further, g as above is called a *p-weak upper gradient* if (5) holds for all curves γ except for a family of p-modulus zero. For the definition of the p-modulus in metric spaces, see [17]. This weaker definition is convenient for technical reasons; each p-integrable p-weak upper gradient can be approximated from above in L^p by upper gradients (cf. [19]). We use the Sobolev space $N^{1,p}(X)$ that consists of all functions in $L^p(X)$ having a (p-weak) upper gradient that belongs to $L^p(X)$ (see Section 2.2).

Our first result shows that if $p > 1$, then the functions in $L^p(\Omega) \cap A^{1,p}_\tau(\Omega)$ have p-integrable p-weak upper gradients, and that functions in $A^{1,1}_\tau(\Omega)$ belong to the space of functions of bounded variation as defined in Section 2.4 below. Define

$$\mathcal{B}_{\tau,r}(\Omega) = \{\{B_i\} \in \mathcal{B}_\tau(\Omega) : r_{B_i} \le r \text{ for all } i\}$$

and

$$\|u\|_{A^{\alpha,p}_{\tau,0}(\Omega)} = \lim_{r\to 0} \sup_{\mathcal{B}\in\mathcal{B}_{\tau,r}(\Omega)} \Big\| \sum_{B\in\mathcal{B}} \Big(r_B^{-\alpha} \fint_B |u - u_B| \, d\mu \Big) \chi_B \Big\|_{L^p(\Omega)}.$$

1.1. Theorem. *Let $\Omega \subset X$ be an open set.*

(1) *If $u \in A^{1,1}_\tau(\Omega)$, then $u \in \mathrm{BV}(\Omega)$ and*

$$\|Du\|(\Omega) \le C(C_d,\tau)\|u\|_{A^{1,1}_{\tau,0}(\Omega)}.$$

(2) *If $p > 1$ and $u \in A^{1,p}_\tau(\Omega) \cap L^p(\Omega)$, then a representative of u has a p-weak upper gradient g with*

$$\|g\|_{L^p(\Omega)} \le C(C_d,\tau)\|u\|_{A^{1,p}_{\tau,0}(\Omega)}.$$

The first part of Theorem 1.1 is proven by Miranda in [25, Theorem 3.8]. We added it for the sake of completeness because our proof applies for all $p \ge 1$. One way to view the second part is that $A^{1,p}_\tau(\Omega) \cap L^p(\Omega)$ is the subspace of $N^{1,p}(\Omega)$ consisting of functions that satisfy an abstract form of a Poincaré inequality (and consequently Sobolev–Poincaré and Trudinger inequalities).

If $\alpha < \beta$ and $\|u\|_{A^{\beta,p}_{\tau,0}(\Omega)} < \infty$, then clearly $\|u\|_{A^{\alpha,p}_{\tau,0}(\Omega)} = 0$. Therefore we have the following corollary.

1.2. Corollary. *Let $u \in A^{\alpha,p}_\tau(\Omega) \cap L^p(\Omega)$.*

(1) *If $p \ge 1$ and $\alpha > 1$, then $\|Du\|(\Omega) = 0$.*
(2) *If $p > 1$ and $\alpha > 1$, then the function $g \equiv 0$ is a p-weak upper gradient of a representative of u.*

Our next result shows that $A^{1,p}_\tau(\Omega) \cap L^p(\Omega)$ coincides with the Sobolev class $N^{1,p}(\Omega)$ or with $\mathrm{BV}(\Omega)$ under a Poincaré inequality assumption. Notice that, in general, $A^{1,p}_\tau(\Omega) \cap L^p(\Omega)$ can be larger than the class of functions that allow for a Poincaré inequality of the type (1). For this, see Section 6.

1.3. Corollary. *Assume that X supports a $(1,p)$-Poincaré inequality with constants C_P and τ.*

(1) *If $p = 1$, then $A^{1,1}_\tau(X) = \mathrm{BV}(X)$.*
(2) *If $1 < p < \infty$, then $A^{1,p}_\tau(X) \cap L^p(X) = N^{1,p}(X)$.*
(3) *If $1 \le p < \infty$ and $\alpha > 1$, then $A^{\alpha,p}_\tau(X) = \{constants\}$.*

Recall from the beginning of the introduction that, in the Euclidean setting, the Poincaré inequality (1) with $p = 1$ characterizes $W^{1,1}(\mathbb{R}^n)$, not $\mathrm{BV}(\mathbb{R}^n)$. Questions relating to (1) and (2) in $\mathbb{R}^n$ are studied in [4] and [3]. For integral conditions under which a function is constant in $\mathbb{R}^n$, see [4], and in Ahlfors regular spaces [2]. If $p > 1$, then in the Euclidean case (3) follows from [4].

We close this introduction by briefly commenting on the missing values of the exponent p above. We have only considered the case $p \ge 1$. For the remaining values of p we have the following result.

1.4. Corollary. *Let $u \in A^{\alpha,p}_\tau(\Omega)$, $0 < p < 1$.*

(1) *If $\alpha = 1/p$ and u is bounded, then $u \in \mathrm{BV}(\Omega)$.*
(2) *If $\alpha = 1/p$ and u is uniformly continuous, then $\|Du\|(\Omega) = 0$.*
(3) *If $\alpha > 1/p$, then $\|Du\|(\Omega) = 0$.*

Note that the function $\chi_{[0,\infty)}$ belongs to $A^{1/p,p}_\tau(\mathbb{R})$ for all $0 < p < 1$ and $\|Du\|(\mathbb{R}) = \delta_0(\mathbb{R}) = 1$. Corollary 1.4 still leaves open the case $0 < p < 1$ and $\alpha < 1/p$. In this case, one can construct examples of nontrivial Hölder continuous functions even when Ω is the interval $[0,1]$, equipped with the Lebesgue measure.

The paper is organized as follows. We introduce the necessary notation and terminology in Section 2. Section 3 deals with pointwise inequalities. In Section 4, we give Sobolev–Poincaré and Trudinger type inequalities for functions in $A^{\alpha,p}_\tau(\Omega)$. Section 5 is devoted to the proofs of Theorem 1.1, Corollary 1.3, and Corollary 1.4. Finally, in Section 6, we present examples that illustrate the previous results.

2. Notation and preliminaries

2.1. *Metric measure spaces.* Throughout this paper $X = (X, \mathrm{d}, \mu)$ is a metric space equipped with a doubling measure μ. By a *measure* we mean a Borel regular outer measure satisfying $0 < \mu(U) < \infty$ whenever U is open and bounded. An open ball of radius r centered at x will be denoted by $B(x,r)$. Sometimes we denote the radius of a ball B by r_B. For $\lambda > 0$, we define $\lambda B(x,r) := B(x,\lambda r)$ and $\lambda\{B_i\} := \{\lambda B_i\}$.

A measure μ is *doubling* if there is a constant $C_d \geq 1$ such that

$$\mu(2B) \leq C_d \mu(B)$$

for all balls $B \subset X$. An iteration of the above inequality shows that there are constants C and s depending only on C_d such that

$$\mu(B(x,r)) \geq C \left(\frac{r}{R}\right)^s \mu(B(y,R))$$

whenever $x \in B(y,R)$ and $0 < r \leq R \leq 2\operatorname{diam}(X)$.

In general, C will denote a positive constant whose value is not necessarily the same at each occurrence. By writing $C = C(K,\lambda)$ we indicate that the constant depends only on K and λ.

2.2. *Sobolev spaces on metric measure spaces.* As usual, if $A \subset X$ is μ-measurable, then $L^p(A)$ is the space of μ-measurable functions u for which $\|u\|_{L^p(A)} = (\int_A |u|^p \, d\mu)^{1/p} < \infty$ for $0 < p < \infty$ and $\|u\|_{L^\infty(A)} = \operatorname{ess\,sup}_A |u| < \infty$. A measurable function u is in the *weak L^p-space* if

$$\|u\|_{L^p_w(A)} = \sup_{\lambda > 0} \lambda \, \mu(\{x \in A : |u(x)| > \lambda\})^{1/p}$$

is finite. If $\mu(A) < \infty$ and $1 \leq q < p$, then $L^q_w(A) \subset L^p(A)$ (cf. [20, Theorem 2.18.8]).

The Sobolev space $N^{1,p}(X)$, defined by Shanmugalingam in [27], consists of the functions $u \in L^p(X)$ having a p-weak upper gradient $g \in L^p(X)$. The space $N^{1,p}(X)$ is a Banach space with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over p-weak upper gradients $g \in L^p(X)$ of u.

2.3. *Lipschitz functions and Poincaré inequalities.* A function $u : X \to \mathbb{R}$ is *L-Lipschitz* if $|u(x) - u(y)| \leq L \, \mathrm{d}(x,y)$ for all $x, y \in X$. The lower and upper pointwise Lipschitz constants of a locally Lipschitz function u are

$$\operatorname{lip} u(x) = \liminf_{r \to 0} \frac{L(u,x,r)}{r} \quad \text{and} \quad \operatorname{Lip} u(x) = \limsup_{r \to 0} \frac{L(u,x,r)}{r},$$

where

$$L(u,x,r) = \sup_{\mathrm{d}(x,y) \leq r} |u(x) - u(y)|.$$

The lower Lipschitz constant $\operatorname{lip} u$, and hence also $\operatorname{Lip} u$, is an upper gradient of a locally Lipschitz function u (cf. [5]).

A pair of $u \in L^1_{\mathrm{loc}}(X)$ and a measurable function $g \geq 0$ satisfies a $(1,p)$-*Poincaré inequality* if there are constants $C_P > 0$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \Big(\fint_{\tau B} g^p \, d\mu \Big)^{1/p} \tag{6}$$

for all balls $B \subset X$. If inequality (6) holds for all measurable functions and their upper gradients with fixed constants, then X *supports a* $(1,p)$*-Poincaré inequality.*

2.4. *Functions of bounded variation.* Following [1], [25] we define BV-functions on a doubling metric measure space X by a relaxation procedure starting from Lipschitz functions. The total variation of a locally integrable function u on an open set Ω is

$$\|Du\|(\Omega) = \inf\Big\{ \liminf_{i\to\infty} \int_\Omega \operatorname{lip} u_i \, d\mu \Big\}, \tag{7}$$

where the infimum is taken over all sequences (u_i) of locally Lipschitz functions that converge to u in $L^1_{\mathrm{loc}}(\Omega)$. The set function $\|Du\|$ extends to a measure on X ([25, Theorem 3.4]). A function $u \in L^1_{\mathrm{loc}}(\Omega)$ is in $\mathrm{BV}(\Omega)$ if $\|Du\|(\Omega)$ is finite, and in the local space $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ if $\|Du\|(A)$ is finite for every bounded open set $A \subset \Omega$. The space $\mathrm{BV}(\Omega)$ equipped with (7) is a seminormed space. If X supports a $(1,1)$-Poincaré inequality with constants C_P and τ, then

$$\fint_B |u - u_B| \, d\mu \le C_P r \, \frac{\|Du\|(\tau B)}{\mu(\tau B)} \tag{8}$$

for each $u \in \mathrm{BV}_{\mathrm{loc}}(X)$ and for all balls $B \subset X$ (cf. [9]).

3. Pointwise estimates. Let $\alpha > 0$ and $\Omega \subset X$ be an open set. The *noncentered fractional sharp maximal function* of a function $u \in L^1_{\mathrm{loc}}(\Omega)$ is defined by

$$M^{\#}_{\alpha,\Omega} u(x) = \sup_{x\in B\subset\Omega} r^{-\alpha} \fint_B |u - u_B| \, d\mu. \tag{9}$$

We begin with the following pointwise estimate; the corresponding result for the centered version of (9) is proved in [13]. For the convenience of the reader, we include a proof.

3.1. PROPOSITION. *Let B be a ball, $u \in L^1(2B)$ and $\alpha > 0$. Then*

$$|u(x) - u(y)| \le C(C_d, \alpha)\, \mathrm{d}(x,y)^\alpha (M^{\#}_{\alpha,2B} u(x) + M^{\#}_{\alpha,2B} u(y)) \tag{10}$$

for almost all $x, y \in B$.

Proof. Since u is integrable in B and μ is doubling, almost all points of B are Lebesgue points of u (see [15, Theorem 14.15]). Let $x, y \in B$ be Lebesgue points of u, and let $r = \mathrm{d}(x,y)/2$. For each $i \in \mathbb{N}$, set $B_i = B(x, 2^{-i} r)$. Then $u_{B_i} \to u(x)$ as $i \to \infty$. Since μ is doubling, we have

$$|u(x)-u_{B(x,r))}| \le \sum_{i=0}^{\infty} |u_{B_i}-u_{B_{i+1}}| \le C(C_d)\sum_{i=0}^{\infty} \fint_{B_i} |u-u_{B_i}|\,d\mu$$
$$\le C(C_d)\sum_{i=0}^{\infty} r_{B_i}^{\alpha} M^{\#}_{\alpha,2B}u(x) \le C(C_d,\alpha)\,\mathrm{d}(x,y)^{\alpha} M^{\#}_{\alpha,2B}u(x).$$

If $\mathrm{d}(x,y)\le r_B/2$, then $B(z,r)\subset B(x,2\,\mathrm{d}(x,y))\subset 2B$, and

$$|u_{B(z,r)}-u_{B(x,2\,\mathrm{d}(x,y))}| \le C(C_d,\alpha)\,\mathrm{d}(x,y)^{\alpha} M^{\#}_{\alpha,2B}u(x)$$

for $z=x,y$. Otherwise

$$|u_{B(z,r)}-u_{2B}| \le C(C_d,\alpha)\,\mathrm{d}(x,y)^{\alpha} M^{\#}_{\alpha,2B}u(x).$$

The triangle inequality gives the claim. ∎

3.2. Remark. For $\alpha>0$, denote by $C^{0,\alpha}(\Omega)$ the set of α-Hölder continuous functions on Ω. Then clearly $C^{0,\alpha}(\Omega)\subset A^{\alpha,\infty}_{\tau}(\Omega)$. On the other hand, since $\|M^{\#}_{\alpha,B}u\|_{L^{\infty}(B)} \le \|u\|_{A^{\alpha,\infty}_{\tau}(\tau B)}$, the inequality (10) implies that $A^{\alpha,\infty}_{\tau}(2\tau B)\subset C^{0,\alpha}(B)$.

We continue by showing that $M^{\#}_{\alpha,B}u$ can be controlled in terms of $\|u\|_{A^{\alpha,p}_{\tau}}$.

3.3. Proposition. *Let* $0<p<\infty$. *Then*

$$\|M^{\#}_{\alpha,B}u\|_{L^p_w(B)} \le C(C_d,\tau,p)\|u\|_{A^{\alpha,p}_{\tau}(\tau B)}.$$

Proof. Let $x\in B$ be such that $M^{\#}_{\alpha,B}u(x)>\lambda$. By the definition of $M^{\#}_{\alpha,B}u$, there is a ball $B_x\subset B$ containing x such that

$$r_x^{-\alpha}\fint_{B_x} |u-u_{B_x}|\,d\mu > \lambda.$$

This implies that

$$\mu(B_x) \le \lambda^{-p}\Big(r_x^{-\alpha}\fint_{B_x} |u-u_{B_x}|\,d\mu\Big)^{p}\mu(B_x). \tag{11}$$

By the standard $5r$-covering lemma (cf. [16]), we can cover the set $\{x\in B: M^{\#}_{\alpha,B}(x)>\lambda\}$ by balls $5\tau B_i$ such that the balls τB_i are disjoint and that each B_i is contained in B and satisfies (11). Since μ is doubling, and the balls τB_i are pairwise disjoint, (11) and definition (4) imply that

$$\mu(\{x\in B: M^{\#}_{\alpha,B}u(x)>\lambda\}) \le \sum_i \mu(5\tau B_i) \le C\sum_i \mu(B_i)$$
$$\le C\lambda^{-p}\sum_i \Big(r_i^{-\alpha}\fint_{B_i} |u-u_{B_i}|\,d\mu\Big)^{p}\mu(B_i)$$

$$= C\lambda^{-p} \int_{\tau B} \Big(\sum_i r_i^{-\alpha} \fint_{B_i} |u - u_{B_i}| \, d\mu \, \chi_{B_i} \Big)^p$$
$$\le C\lambda^{-p} \|u\|^p_{A^{\alpha,p}_\tau(\tau B)},$$

where $C = C(C_d, \tau)$. The claim follows by the definition of $\|\cdot\|_{L^p_w}$. ■

For a measurable function u, denote by $D^\alpha(u)$ the set of measurable functions $g \ge 0$ that satisfy

$$|u(x) - u(y)| \le \mathrm{d}(x,y)^\alpha (g(x) + g(y)) \tag{12}$$

for almost every $x, y \in \Omega$. For $0 < \alpha$, $p < \infty$, define, following Hajłasz [10],

$$M^{\alpha,p}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : D^\alpha(u) \cap L^p(\Omega) \ne \emptyset\},$$
$$M^{\alpha,p}_w(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : D^\alpha(u) \cap L^p_w(\Omega) \ne \emptyset\}.$$

From Propositions 3.1 and 3.3 we obtain the following corollary.

3.4. Corollary. *Let $B \subset X$ be a ball. Then $A^{\alpha,p}_\tau(2\tau B) \subset M^{\alpha,p}_w(B)$.*

Denote by $P^{\alpha,p}_\tau(\Omega)$ the set of functions $u \in L^1_{\mathrm{loc}}(\Omega)$ for which there exists a function $0 \le g \in L^p(\Omega)$ such that

$$\fint_B |u - u_B| \, d\mu \le r^\alpha \Big(\fint_{\tau B} g^p \, d\mu \Big)^{1/p}$$

for all balls $\tau B \subset \Omega$. Notice that, trivially, $P^{\alpha,p}_\tau(\Omega) \subset A^{\alpha,p}_\tau(\Omega)$. Thus the previous corollary and the following result almost identify the spaces $A^{\alpha,p}_\tau(\Omega)$ and $M^{\alpha,p}(\Omega)$. However, $A^{\alpha,p}_\tau(\Omega)$ may be strictly larger than $P^{\alpha,p}_\tau(\Omega)$ (see Example 6.2).

3.5. Theorem. *Let $\Omega \subset X$ be an open set with $\mu(\Omega) < \infty$, and let $1 \le q < p$. Then $M^{\alpha,p}_w(\Omega) \subset P^{\alpha,q}_1(\Omega)$. Moreover, $A^{\alpha,p}_\tau(2\tau B) \subset P^{\alpha,q}_1(B)$ whenever $2\tau B \subset \Omega$.*

Proof. By the previous corollary, it suffices to prove the first claim. Let $u \in M^{\alpha,p}_w(\Omega)$ and $g \in L^p_w(\Omega) \cap D^\alpha(u)$. Since $\mu(\Omega) < \infty$ and $q < p$, the function g is in $L^q(\Omega)$. For each ball $B \subset \Omega$ we see by integrating (12) and using Jensen's inequality that

$$\fint_B |u(x) - u_B| \, d\mu(x) \le \fint_B \fint_B |u(x) - u(y)| \, d\mu(y) \, d\mu(y)$$
$$\le C(\alpha) r^\alpha \fint_B g \, d\mu \le C(\alpha) r^\alpha \Big(\fint_B g^q \, d\mu \Big)^{1/q}. \ ■$$

Notice the following consequence of the previous result. Our abstract version of the Poincaré inequality results in a usual inequality provided we relax the integrability requirement on the right-hand side. This relaxation is indeed crucial by an example in Section 6. In fact, one cannot even require that ν in (3) be an absolute continuous measure. We close this section by pointing out that certain choices of α and p only allow for constant functions. For integral conditions with $p \geq 1$ implying that the function is constant, see [4].

3.6. THEOREM. *If* $0<p<1$ *and* $\alpha>1/p$, *then* $M^{\alpha,p}(\mathbb{R}^n)=\{constants\}$.

Proof. If $u \in M^{\alpha,p}(\mathbb{R}^n)$, it follows from Fubini's theorem that $u \in M^{\alpha,p}(l)$ for almost every line l parallel to coordinate axes. Therefore it suffices to prove the theorem in the case $n=1$. Let $u \in M^{\alpha,p}(\mathbb{R})$, and let $\varepsilon>0$. By definition there is $g \in L^p(\mathbb{R})$ and a set $E \subset \mathbb{R}$ of measure zero such that

$$|u(x)-u(y)| \leq |x-y|^\alpha (g(x)+g(y)) < \infty$$

whenever $x, y \in \mathbb{R} \setminus E$. Fix $x, y \in \mathbb{R} \setminus E$ and divide the interval $[x,y]$ into disjoint intervals $I_1, \ldots, I_k$ with $\varepsilon/2 \leq l(I_i) \leq \varepsilon$ for all i. For each i, let $x_i \in I_i \setminus E$ be such that

$$g(x_i) \leq 2 \operatorname{ess\,inf}\{g(z) : z \in I_i\}.$$

Set $x_0 = x$ and $x_{k+1} = y$. Then, by the assumptions on p and α,

$$\begin{aligned}
|u(x)-u(y)| &\leq \sum_{i=0}^{k} |u(x_i)-u(x_{i+1})| \leq C\varepsilon^\alpha \sum_{i=0}^{k} (g(x_i)+g(x_{i+1})) \\
&= C\varepsilon^\alpha \sum_{i=1}^{k} g(x_i) + \varepsilon^\alpha (g(x)+g(y)) \\
&\leq C\varepsilon^{\alpha-1/p} \Big(\sum_{i=1}^{k} \varepsilon g(x_i)^p\Big)^{1/p} + \varepsilon^\alpha (g(x)+g(y)) \\
&\leq C\varepsilon^{\alpha-1/p} \|g\|_{L^p(\mathbb{R})} + \varepsilon^\alpha (g(x)+g(y)),
\end{aligned}$$

and the claim follows by letting $\varepsilon \to 0$. ∎

4. Imbeddings into Lebesgue and Hölder spaces. In [21] MacManus and Pérez showed that if the functional a satisfies a discrete summability condition

$$\sum_i a(B_i)^r \mu(B_i) \leq C a(B)^r \mu(B) \tag{13}$$

whenever the balls B_i are disjoint and contained in the ball B, then the Poincaré type inequality (2) improves to

$$\sup_{\lambda>0} \lambda \left(\frac{\mu(\{x \in B : |u(x)-u_B| > \lambda\})}{\mu(B)} \right)^{1/r} \leq C' a(2\tau B). \tag{14}$$

In [22], they proved that if X is connected and a satisfies a stronger condition

$$\sum a(B_i)^r \leq C a(B)^r, \tag{15}$$

then each function u which satisfies inequality (2), is in the Orlicz space $L^\Phi(B)$, where $\Phi(t) = \exp(t^{r'}) - 1$, and $1/r + 1/r' = 1$. Moreover,

$$\|u - u_B\|_{L^\Phi(B)} \le Ca(2\tau B), \tag{16}$$

where $\|\cdot\|_{L^\Phi(B)}$ is the Luxemburg norm in $L^\Phi(B)$. For Orlicz spaces on metric spaces, see [26]. Without the connectedness assumption, one only obtains (16) with $\Phi(t) = \exp(t)$.

The following result collects the known Sobolev-type imbeddings of $A^{\alpha,p}_\tau$.

4.1. THEOREM. *Let $B \subset X$ be a ball, $\tau \ge 1$, $0 < p < \infty$, and assume that there is $s \ge 1$ such that $\mu(B(x,r)) \ge C_\mu r^s$ whenever $B(x,r) \subset 2\tau B$.*

(a) *If $\alpha p < s$, then $A^{\alpha,p}_\tau(2\tau B) \subset L^q_w(B)$, where $q = sp/(s-\alpha p)$.*
(b) *If $\alpha p = s$, then $A^{\alpha,p}_\tau(2\tau B) \subset L^\Phi(B)$, where $\Phi(t) = \exp(t)$. If X is connected, the above holds with $\Phi(t) = \exp(t^{p/(p-1)})$.*
(c) *If $\alpha p > s$, then $A^{\alpha,p}_\tau(2\tau B) \subset C^{0,\alpha-s/p}(B)$.*

Proof. (a) Recall that $A^{0,\tau}_\tau(2\tau B)$ consists of functions that satisfy (2) with $a(B) = (\nu(B)/\mu(B))^{1/p}$. Such an a satisfies the condition (13), and hence, by (14), it suffices to show that

$$A^{\alpha,p}_\tau(2\tau B) \subset A^{0,\frac{sp}{s-\alpha p}}_\tau(2\tau B).$$

If $u \in A^{\alpha,p}_\tau(2\tau B)$ and $\mathcal{B} \in \mathcal{B}_\tau(2\tau B)$, the assumption $0 < (s-\alpha p)/s < 1$ implies

$$\begin{aligned}
&\sum_{\mathcal{B}} \Big(\fint_B |u - u_B| \, d\mu \Big)^{sp/(s-\alpha p)} \mu(B) \\
&\quad \le \Big(\sum_{\mathcal{B}} \Big(\fint_B |u - u_B| \, d\mu \Big)^p \mu(B)^{(s-\alpha p)/s} \Big)^{s/(s-\alpha p)} \\
&\quad \le C \Big(\sum_{\mathcal{B}} \Big(r_B^{-\alpha} \fint_B |u - u_B| \, d\mu \Big)^p \mu(B) \Big)^{s/(s-\alpha p)} \le C \|u\|^{sp/(s-\alpha p)}_{A^{\alpha,p}_\tau(2\tau B)},
\end{aligned}$$

where $C = C(C_\mu, s, \alpha, p)$, and the claim follows.

(b) If $\alpha p = s$, then $u \in A^{\alpha,p}_\tau(2\tau B)$ satisfies

$$\fint_{B'} |u - u_{B'}| \, d\mu \le r^\alpha_{B'} \mu(B')^{-1/p} \|u\|_{A^{\alpha,p}_\tau(\tau B')} \le C(C_\mu, p) \|u\|_{A^{\alpha,p}_\tau(\tau B')}$$

for $\tau B' \subset 2\tau B$. Since $b(B) = \|u\|_{A^{\alpha,p}_\tau(B)}$ satisfies (15), both claims follow from [22].

(c) If $\alpha p > s$, then $A^{\alpha,p}_\tau(2\tau B) \subset A^{\alpha-s/p,\infty}_\tau(2\tau B)$ and the claim follows from Remark 3.2. ∎

5. Proof of Theorem 1.1. For the proof of Theorem 1.1, which is based on approximation by discrete convolutions, we need a couple of lemmas. Lemma 5.1 follows from a Whitney-type covering result for doubling metric measure spaces (see [6, Theorem III.1.3], [23, Lemma 2.9]). For the proof of Lemma 5.2, we refer to [23, Lemma 2.16].

5.1. Lemma. *Let $\Omega \subset X$ be open. Given $\varepsilon > 0$, $\lambda \geq 1$, there is a cover $\{B_i = B(x_i, r_i)\}$ of Ω with the following properties*:

(1) *$r_i \leq \varepsilon$ for all i,*
(2) *$\lambda B_i \subset \Omega$ for all i,*
(3) *if λB_i meets λB_j, then $r_i \leq 2r_j$,*
(4) *each ball λB_i meets at most $C = C(C_d, \lambda)$ balls λB_j.*

A collection $\{B_i\}$ as above is called an (ε, λ)-*cover* of Ω. Note that an (ε, λ)-cover is an (ε', λ')-cover provided $\varepsilon' \geq \varepsilon$ and $\lambda' \leq \lambda$.

5.2. Lemma. *Let $\Omega \subset X$ be open, and let $\mathcal{B} = \{B_i = B(x_i, r_i)\}$ be an $(\infty, 2)$-cover of Ω. Then there is a collection $\{\varphi_i\}$ of functions $\Omega \to \mathbb{R}$ such that*

(1) *each φ_i is $C(C_d)r_i^{-1}$-Lipschitz,*
(2) *$0 \leq \varphi_i \leq 1$ for all i,*
(3) *$\varphi_i(x) = 0$ for $x \in X \setminus 2B_i$ for all i,*
(4) *$\sum_i \varphi_i(x) = 1$ for all $x \in \Omega$.*

A collection $\{\varphi_i\}$ as above is called a *partition of unity with respect to $\mathcal{B}$*.

Let $\mathcal{B} = \{B_i\}$ be as in the lemma above, and let $\{\varphi_i\}$ be a partition of unity with respect to $\mathcal{B}$. For a locally integrable function u on Ω, define

$$u_{\mathcal{B}}(x) = \sum_i u_{B_i} \varphi_i(x). \tag{17}$$

The following lemma describes the most important properties of $u_{\mathcal{B}}$.

5.3. Lemma.

(1) *The function $u_{\mathcal{B}}$ is locally Lipschitz. Moreover, for each $x \in B_i$,*

$$\operatorname{Lip} u_{\mathcal{B}}(x) \leq C(C_d) r_{B_i}^{-1} \fint_{5B_i} |u - u_{5B_i}| \, d\mu.$$

(2) *Let $u \in L^p(\Omega)$, $p \geq 1$. If $\mathcal{B}_k$ is an $(\varepsilon_k, 2)$-cover of Ω and $\varepsilon_k \to 0$ as $k \to \infty$, then $u_{\mathcal{B}_k} \to u$ in $L^p(\Omega)$.*

Proof. (1) Let $x, y \in B_i$, and let $J = \{j : 2B_j \cap 2B_i \neq \emptyset\}$. Then $\#J \leq C(C_d)$ and $B_j \subset 5B_i$ for each $j \in J$. Using the properties of the functions φ_i, we infer that

$$\begin{aligned} |u_{\mathcal{B}}(x) - u_{\mathcal{B}}(y)| &= \Big| \sum_{j \in J} (u_{B_j} - u_{B_i})(\varphi_j(x) - \varphi_j(y)) \Big| \\ &\leq C(C_d) r_{B_i}^{-1} \, \mathrm{d}(x, y) \max_{j \in J} |u_{B_j} - u_{B_i}| \\ &\leq C(C_d) r_{B_i}^{-1} \, \mathrm{d}(x, y) \fint_{5B_i} |u - u_{5B_i}| \, d\mu, \end{aligned}$$

and the first claim follows.

(2) First we need an estimate for the L^p-norm of $u_{\mathcal{B}}$ on Ω. By Jensen's inequality, $|u_{\mathcal{B}}|^p \le (|u|^p)_{\mathcal{B}}$. Hence, by the properties of the functions φ_i,

$$\int_\Omega |u_{\mathcal{B}}|^p \, d\mu \le \int_\Omega (|u|^p)_{\mathcal{B}} \, d\mu \le \sum_i \int_\Omega (|u|^p)_{B_i} \varphi_i \, d\mu \le C(C_d) \sum_i \int_{2B_i} |u|^p \, d\mu \le C(C_d) \int_\Omega |u|^p \, d\mu. \tag{18}$$

Let $u \in L^p(\Omega)$ and $\varepsilon > 0$. Choose a bounded continuous function v with bounded support such that $\|u - v\|_{L^p(\Omega)} < \varepsilon$ (cf. [15, Theorem 14.2]). Then, estimating as in (18), we obtain

$$\|u_{\mathcal{B}} - v_{\mathcal{B}}\|_{L^p(\Omega)} = \|(u - v)_{\mathcal{B}}\|_{L^p(\Omega)} \le C(C_d, p)\|u - v\|_{L^p(\Omega)} < C(C_d, p)\varepsilon,$$

and so

$$\|u_{\mathcal{B}} - u\|_{L^p(\Omega)} \le \|u_{\mathcal{B}} - v_{\mathcal{B}}\|_{L^p(\Omega)} + \|v_{\mathcal{B}} - v\|_{L^p(\Omega)} + \|v - u\|_{L^p(\Omega)} < \|v_{\mathcal{B}} - v\|_{L^p(\Omega)} + C(C_d, p)\varepsilon.$$

Therefore it suffices to show that $\|v_{\mathcal{B}} - v\|_{L^p(\Omega)} \to 0$ as $\varepsilon_k \to 0$. Now $|v_{\mathcal{B}} - v| \le 2 \sup |v|$, and for all x we have

$$|v_{\mathcal{B}}(x) - v(x)| \le \sum_{2B_i \ni x} \fint_{B_i} |v(y) - v(x)| \, d\mu(y) \le C(C_d) \fint_{B(x, 5\varepsilon_k)} |v(y) - v(x)| \, d\mu(y),$$

which converges to 0 as $\varepsilon_k \to 0$ by the continuity of v. The claim follows from the dominated convergence theorem. ■

Proof of Theorem 1.1. Let $u \in A^{1,p}_\tau(\Omega)$. For $j \in \mathbb{N}$, let $\mathcal{B}_j$ be a $(j^{-1}, 5\tau)$-cover (and hence also a $(j^{-1}, 2)$-cover) of Ω. Then, by Lemma 5.3(2), $u_j := u_{\mathcal{B}_j} \to u$ in $L^p(\Omega)$. Let us show that

$$\limsup_{j \to \infty} \|\operatorname{Lip} u_j\|_{L^p(\Omega)} \le C(C_d, \tau) \|u\|_{A^{1,p}_{\tau,0}(\Omega)}.$$

By Lemma 5.3(1),

$$\operatorname{Lip} u_j \le C(C_d) \sum_{B \in \mathcal{B}_j} r_B^{-1} \fint_{5B} |u - u_{5B}| \, d\mu \, \chi_B.$$

We leave it to the reader to show that since $\mathcal{B}_j$ is a $(j^{-1}, 5\tau)$-cover and μ is doubling, the cover can be divided into $k = C(C_d, \tau)$ subfamilies $\mathcal{B}_{j,1}, \ldots, \mathcal{B}_{j,k}$ so that each of the families $5\tau\mathcal{B}_{j,l}$ consists of disjoint balls. Since the families $5\mathcal{B}_{j,1}, \ldots, 5\mathcal{B}_{j,k}$ belong to $\mathcal{B}_{\tau, 5j^{-1}}(\Omega)$, we have

$$(19)\qquad \|\operatorname{Lip} u_j\|_{L^p(\Omega)} \le C(C_d) \sum_{l=1}^{k} \Big\| \sum_{B \in \mathcal{B}_{j,l}} r_B^{-1} \fint_{5B} |u - u_{5B}|\, d\mu\, \chi_B \Big\|_{L^p(\Omega)}$$
$$\le C(C_d, \tau) \sup \Big\| \sum_{B \in \mathcal{B}} r_B^{-1} \fint_{B} |u - u_B|\, d\mu\, \chi_B \Big\|_{L^p(\Omega)},$$

where the supremum is taken over balls $\mathcal{B} \in \mathcal{B}_{\tau,5j^{-1}}(\Omega)$. Since $\operatorname{lip} u(x) \le \operatorname{Lip} u(x)$, the above estimate for $p = 1$ implies that $u \in \mathrm{BV}(\Omega)$, and that $\|Du\|(\Omega) \le C(C_d, \tau)\|u\|_{A^{1,1}_{\tau,0}(\Omega)}$.

If $p > 1$, then $L^p(\Omega)$ is reflexive. Thus the sequence $(\operatorname{Lip} u_j)$ of upper gradients, which by (19) is bounded in $L^p(\Omega)$, has a subsequence, also denoted by $(\operatorname{Lip} u_j)$, that converges weakly to some $g \in L^p(\Omega)$. By [18, Lemma 3.1], g is a p-weak upper gradient of a representative of u. The second part of the theorem follows because the weak limit g satisfies

$$\|g\|_{L^p(\Omega)} \le \liminf_{j\to\infty} \|\operatorname{Lip} u_j\|_{L^p(\Omega)} \le C(C_d, \tau)\|u\|_{A^{1,p}_{\tau,0}(\Omega)}.$$

Proof of Corollary 1.3. Assume that X supports a $(1, p)$-Poincaré inequality, $1 \le p < \infty$. By the validity of a $(1, p)$-Poincaré inequality, X is connected.

If $p = 1$, then the claim follows from Theorem 1.1 and the Poincaré inequality (8) for BV-functions.

If $1 < p < \infty$, then each function of $A^{1,p}_\tau(X) \cap L^p(X)$ is in $N^{1,p}(X)$ by Theorem 1.1. The assumption that X supports the $(1, p)$-Poincaré inequality gives the inclusion $N^{1,p}(X) \subset A^{1,p}_\tau(X) \cap L^p(X)$.

Let then $1 \le p < \infty$, $\alpha > 1$, and $u \in A^{\alpha,p}_\tau(X)$. If $p = 1$, then the Poincaré inequality (8) for BV-functions together with Corollary 1.2(1) shows that u is constant in each ball of X. For $p > 1$, we notice from Theorem 4.1 that $u \in L^p(B)$ for each ball B. Then Corollary 1.2(2) and the $(1, p)$-Poincaré inequality imply that $u|_B$ is a constant for each ball B. In both cases above, the claim follows by the connectedness of X. All constant functions are trivially in $A^{\alpha,p}_\tau(X)$.

Proof of Corollary 1.4. For the first claim, let $u \in A^{1/p,p}_\tau(\Omega)$, $0 < p < 1$, and $\mathcal{B} \in \mathcal{B}_{\tau,r}(\Omega)$. If there is a constant $M \ge 0$ such that $|u| \le M$ in Ω, then

$$\sum_{B \in \mathcal{B}} \Big(r_B^{-1} \fint_B |u - u_B|\, d\mu \Big) \mu(B) \le (2M)^{1-p} \sum_{B \in \mathcal{B}} \Big(r_B^{-1/p} \fint_B |u - u_B|\, d\mu \Big)^p \mu(B).$$

If u is uniformly continuous, and ω is the modulus of continuity of u, then

$$\sum_{B \in \mathcal{B}} \Big(r_B^{-1} \fint_B |u - u_B|\, d\mu \Big) \mu(B) \le \omega(2r)^{1-p} \sum_{B \in \mathcal{B}} \Big(r_B^{-1/p} \fint_B |u - u_B|\, d\mu \Big)^p \mu(B).$$

By taking supremum over $\mathcal{B}_{\tau,r}(\Omega)$ and letting r tend to zero,we conclude that $\|u\|_{A^{1,1}_{\tau,0}}(\Omega) \le (2M)^{1-p}\|u\|^p_{A^{1/p,p}_{\tau,0}}(\Omega)$ in the former, and $\|u\|_{A^{1,1}_{\tau,0}}(\Omega) = 0$ in the latter case. In both cases, the claim follows from Theorem 1.1.

Let then $u \in A^{\alpha,p}_\tau(\Omega)$, where $\alpha > 1/p$. By the remark after Theorem 1.1, we know that $\|u\|_{A^{1/p,p}_{\tau,0}(\Omega)} = 0$. For $k \in \mathbb{N}$, define $u_k = \min\{k, \max\{u, -k\}\}$. Then each u_k is bounded and, by the first part of the proof,

$$\|u_k\|_{A^{1,1}_{\tau,0}(\Omega)} \le C_k\|u_k\|_{A^{1/p,p}_{\tau,0}(\Omega)} \le 2C_k\|u\|_{A^{1/p,p}_{\tau,0}(\Omega)} = 0,$$

which implies that $\|u\|_{A^{1,1}_{\tau,0}(\Omega)} \le \liminf_{k\to\infty} \|u_k\|_{A^{1,1}_{\tau,0}(\Omega)} = 0$.

6. Examples. In our first example we exhibit a space X that does not support any $(1,q)$-Poincaré inequality, but in which every $u \in N^{1,p}(X)$ satisfies (6) with a certain $g \in L^p(X)$.

6.1. Example. Let B_1 and B_2 be balls in $\mathbb{R}^n$ such that $\mathrm{d}(B_1, B_2) > 0$. Equip $X = B_1 \cup B_2$ with the Euclidean metric of $\mathbb{R}^n$, and let μ be the restriction of the Lebesgue measure to X. By considering the function $u = \chi_{B_1}$, which has $g \equiv 0$ as a weak upper gradient, we see that X cannot support any $(1,q)$-Poincaré inequality.

Let $1 \le p < \infty$, and let $u \in N^{1,p}(X)$ with an upper gradient $g \in L^p(X)$. We will show that the inequality

$$\fint_B |u - u_B|\,d\mu \le C r_B \Big(\fint_B (g + |u|)^p\,d\mu\Big)^{1/p} \tag{20}$$

holds for each ball $B \subset X$.

Fix a ball $B \subset X$. If B intersects only one of the balls B_1, B_2, then (20) holds by the equivalence $N^{1,p}(\Omega) = W^{1,p}(\Omega)$ for domains in $\mathbb{R}^n$ ([27, Theorem 4.5]), and the usual $(1,p)$-Poincaré inequality. Assume that the intersection of B with both B_1 and B_2 is nonempty. Then $2r_B \ge \mathrm{d}(B_1, B_2)$, and by the Hölder inequality, we have

$$\begin{aligned}\fint_B |u - u_B|\,d\mu &\le 2\fint_B |u|\,d\mu \le 2\Big(\fint_B |u|^p\,d\mu\Big)^{1/p}\\ &\le \frac{4r_B}{\mathrm{d}(B_1, B_2)}\Big(\fint_B |u|^p\,d\mu\Big)^{1/p},\end{aligned}$$

which is at most a constant times the right-hand side of (20).

In the next example, the space $A^{1,p}_\tau(X)$ is strictly larger than $P^{1,p}_\tau(X)$.

6.2. Example. Let $X = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| \le |x_2|\}$ be equipped with the Euclidean metric of $\mathbb{R}^2$, and let μ be the restriction of the Lebesgue measure to X. The function $u = \chi_{X_+}$, where $X_+ = \{x \in X : x_1 \ge 0\}$, cannot satisfy the inequality

$$\fint_B |u-u_B|\,d\mu \le Cr\left(\frac{\nu(\tau B)}{\mu(B)}\right)^{1/2}$$

with any measure ν absolutely continuous with respect to μ because we have $\fint_B |u-u_B|\,d\mu = 1/2$ for each ball $B(0,r)$. However, the above inequality holds with

$$\nu(B)=\delta_0(B)=\begin{cases}1 & \text{if } 0\in B,\\ 0 & \text{otherwise.}\end{cases}$$

Our final example shows that, given $0<p<1$ and $1\le\alpha<1/p$, there are Hölder-continuous nonconstant functions in $M^{\alpha,p}([0,1])$.

6.3. Example. Fix $0<p<1$ and $1\le\alpha<1/p$. Let $0<s<1$ and let $C\subset[0,1]$ be the standard Cantor set with $\mathcal{H}^s(C)=1$ (see for example [24, p. 60]). Then the Cantor function $u(x)=\mathcal{H}^s(C\cap[0,x])$ is Hölder-continuous with exponent s. A calculation shows that

$$|u(x)-u(y)|\le \mathrm{d}(x,y)^s\le \mathrm{d}(x,y)^\alpha(\mathrm{d}(x,C)^{s-\alpha}+\mathrm{d}(y,C)^{s-\alpha})$$

for all $x,y\in[0,1]$, and that $g(x)=\mathrm{d}(x,C)^{s-\alpha}$ is in $L^p([0,1])$ provided $0<s<(1-\alpha p)/(1-p)$.

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