A complete characterization of
R-sets in the theory of differentiation of integrals

by

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Abstract. Let $\mathcal{R}_s$ be the family of open rectangles in the plane $\mathbb{R}^2$ with a side of angle s to the x-axis. We say that a set S of directions is an R-set if there exists a function $f \in L^1(\mathbb{R}^2)$ such that the basis $\mathcal{R}_s$ differentiates the integral of f if $s \notin S$, and $\overline{D}_s f(x) = \limsup_{\mathrm{diam}(R)\to 0,\, x\in R\in\mathcal{R}_s} |R|^{-1} \int_R f = \infty$ almost everywhere if $s \in S$. If the condition $\overline{D}_s f(x) = \infty$ holds on a set of positive measure (instead of a.e.) we say that S is a WR-set. It is proved that S is an R-set (resp. a WR-set) if and only if it is a G_δ (resp. a $G_{\delta\sigma}$).

1. Introduction. For any $s \in [0, \pi/2)$ we define $\mathcal{R}_s$ to be the family of all open rectangles R in $\mathbb{R}^2$ that have a side at angle s to the x-axis. We say that the basis $\mathcal{R}_s$ *differentiates the integral of the function* $f \in L^1(\mathbb{R}^2)$ if

$$(1.1) \qquad \lim_{\mathrm{diam}(R)\to 0,\, x\in R\in\mathcal{R}_s} \frac{1}{|R|} \int_R f = f(x)$$

almost everywhere in $\mathbb{R}^2$, where $d(R)$ is the diameter of R. According to the well-known theorem of Jessen–Marcinkiewicz–Zygmund [3], $\mathcal{R}_s$ differentiates $\int f$ for any $f \in L\log L(\mathbb{R}^2)$. On the other hand, S. Saks [12] constructed an $f \in L^1(\mathbb{R}^2)$ such that

$$\overline{D}_s f(x) = \limsup_{\mathrm{diam}(R)\to 0,\, x\in R\in\mathcal{R}_s} \frac{1}{|R|} \int_R f = \infty \qquad \text{everywhere.}$$

In view of this A. Zygmund posed the following problem (see [2]): for a given $f \in L^1(\mathbb{R}^2)$, is it possible to find a direction s such that $\mathcal{R}_s$ differentiates $\int f$? J. Marstrand in [7] gave a negative answer to this question, proving

THEOREM (J. Marstrand). *There exists a function $f \in L^1(\mathbb{R}^2)$ such that $\overline{D}_s f(x) = \infty$ almost everywhere for any s.*

Different generalizations of this result have been obtained by J. El Helou [1], A. M. Stokolos [13], B. López Melero [6] and G. G. Oniani [9]. A. M. Stokolos [13] extended Marstrand's theorem to higher dimensions. In [6] and [9] the same problem is considered for general translation invariant differentiation bases.

We say that a set $S \subset [0, \pi/2)$ is an *R-set* if there exists a function $f \in L^1(\mathbb{R}^2)$ such that the basis $\mathcal{R}_s$ differentiates $\int f$ whenever $s \in [0, \pi/2) \backslash S$, and $\overline{D}_s f(x) = \infty$ almost everywhere if $s \in S$. If $\overline{D}_s f(x) = \infty$ on a set of positive measure (instead of a.e.), we call S a *WR-set* (weak R-set). In this terminology, Marstrand's theorem asserts that $[0, \pi/2)$ is an R-set. A. M. Stokolos [14] proved the existence of an everywhere dense WR-set which is not the whole $[0, \pi/2)$. G. Lepsveridze [4], [5] proved that any finite set is an R-set and any countable set is contained in some WR-set of measure zero. G. G. Oniani [9], generalizing this result, proved that any countable set is contained in some R-set of measure zero.

The definition of R-sets first appeared in the paper [8] by G. G. Oniani, who posed the problem of characterization of all R-sets. In particular, he asked if there exists an R-set of positive measure and whether any interval is an R-set. In the same paper Oniani shows that any R-set is a G_δ in $[0, \pi/2)$, i.e.

$$G = \Big(\bigcap_{k=1}^{\infty} G_k \Big) \cap [0, \pi/2)$$

where G_n are open sets, and conversely, if a G_δ-set is countable, then it is an R-set. These results characterize the countable R-sets. We note that any countable G_δ-set is nowhere dense. So in [8] Oniani also constructed an R-set of second category. These problems are also stated in Oniani's monograph [9], and the higher dimensional case of the problem is investigated in [10] and [11].

The following theorems give a complete characterization of general R-sets and WR-sets.

THEOREM 1. *A set $S \subset [0, \pi/2)$ is an R-set if and only if it is a G_δ.*

THEOREM 2. *A set $S \subset [0, \pi/2)$ is a WR-set if and only if it is a $G_{\delta\sigma}$.*

The necessity of Theorem 1 is proved by Oniani in [8]. We briefly sketch that proof. If S is an R-set, then there exists $f \in L^1$ such that (1.1) holds if $s \in [0, \pi/2) \setminus S$, and $\overline{D}_s f(x) = \infty$ a.e. if $s \in S$. For any $n \in \mathbb{N}$ define

$$U_n = \{s \in [0, \pi/2) : |\{x \in B(n) : M_s f(x) > n\}| > |B(n)| - 2^{-n}\},$$

where $B(n) = \{x \in \mathbb{R}^2 : \|x\| \le n\}$ and the maximal function $M_s f$ is defined in Section 2. It is easy to check that $U_n = G_n \cap [0, \pi/2)$, where G_n are open

sets and

$$\{s \in [0, \pi/2) : \overline{D}_s f(x) = \infty \text{ a.e.}\} = \bigcap_n U_n = \left(\bigcap_n G_n\right) \cap [0, \pi/2),$$

i.e. it is a G_δ-set in $[0, \pi/2)$, which proves one direction of Theorem 1.

To prove the necessity of Theorem 2 it is enough to prove that for any $f \in L^1(\mathbb{R}^2)$ the set

$$G_f = \{s \in [0, \pi/2) : |\{x \in \mathbb{R}^2 : \overline{D}_s f(x) = \infty\}| > 0\}$$

is a $G_{\delta\sigma}$. Define

$$U_{nm} = \{s \in [0, \pi/2) : |\{x \in B(n) : M_s f(x) > m\}| > 0\}, \qquad n, m = 1, 2, \ldots,$$

where $B(n)$ and $M_s f$ are defined in Section 2. It is clear that the U_{nm} are open sets in $[0, \pi/2)$ and

$$G_f = \bigcup_n \bigcap_m U_{nm}.$$

To show the last equality it suffices to check the following relations:

$$\begin{aligned}
s \in G_f &\Leftrightarrow |\{x \in \mathbb{R}^2 : \overline{D}_s f(x) = \infty\}| > \alpha > 0 \\
&\Leftrightarrow \exists n \text{ such that } |\{x \in B_n : \overline{D}_s f(x) = \infty\}| > \alpha \\
&\Leftrightarrow \exists n \text{ such that } s \in \bigcap_m U_{n,m} \Leftrightarrow s \in \bigcup_n \bigcap_m U_{n,m}.
\end{aligned}$$

Hence the set G_f is a $G_{\delta\sigma}$.

We shall prove the sufficiency of both theorems invoking the probabilistic independence of sets similarly to the original approach of J. Marstrand [7]. This idea is applied in Lemma 1. Of course, we use also Bohr's construction displayed in Saks's classical counterexample. It is important that the function constructed in the proof is not nonnegative, in contrast to all the results stated above. This argument gives more freedom in the construction to ensure differentiability of the integral along some directions. So our method differs from the others, because we essentially use interference of positive and negative values of a function in integrals, which is displayed in Lemmas 2 and 3.

2. Notations and lemmas. The basis $\mathcal{R}_s$ can be defined for any $s \in [0, 2\pi]$. We note that $\mathcal{R}_s = \mathcal{R}_t$ if $s = t \mod \pi/2$. In fact $\bigcup_{s \in [0,\pi/2)} \mathcal{R}_s$ is the family of all rectangles in the plane.

If $n \in \mathbb{N}$ and $c = (c_1, c_2)$, then for any set $A \subset \mathbb{R}^2$ we define

$$\begin{aligned}
\text{dil}_n A &= \{x = (x_1, x_2) \in \mathbb{R}^2 : nx = (nx_1, nx_2) \in A\}, \\
c + A &= \{x \in \mathbb{R}^2 : x = c + a, \, a \in A\}.
\end{aligned}$$

We let $Q_0 = [-1/2, 1/2) \times [-1/2, 1/2)$ and for any $n \in \mathbb{N}$, $k = (k_1, k_2) \in \mathbb{Z}^2$ define $Q_k^n = \text{dil}_n(k + Q_0)$. For a fixed n the family $\{Q_k^n : k \in \mathbb{Z}^2\}$ is a

partition of the plane into squares with side lengths $1/n$. Occasionally we write simply Q_k for Q_k^1.

We denote by $\mathrm{rot}_s A$ the rotation of the set $A \subset \mathbb{R}^2$ about $(0,0)$ by angle s. Let

$$B(\varepsilon) = \{x \in \mathbb{R}^2 : \|x\| = \sqrt{x_1^2 + x_2^2} \le \varepsilon\},$$
$$\Gamma_s(\varepsilon) = \mathrm{rot}_s \{x = (x_1, x_2) : |x_2| < \varepsilon\}.$$

The notation $s^\perp$ stands for the direction $s + \pi/2$. For any direction s define $\mathrm{mes}_s A$ to be the linear Lebesgue measure of the projection of A on the line parallel to $s^\perp$.

For any measurable set $A \subset \mathbb{R}^2$ we define

$$\mathrm{mes}^* A = \sup_{k \in \mathbb{Z}^2} |A \cap Q_k|, \quad \mathrm{mes}_* A = \inf_{k \in \mathbb{Z}^2} |A \cap Q_k|.$$

For $0 < \delta < \mu \le \infty$ we define $\mathcal{R}_s^{[\delta,\mu)}$ to be the family of rectangles $R = R_1 \times R_2 \in \mathcal{R}_s$ with $\delta \le |R_1|, |R_2| < \mu$ and we let $\mathcal{R}_s^\delta$ be the rectangles from $\mathcal{R}_s$ with $|R_1| = |R_2| = \delta$. Define

$$M_s^{[\delta,\mu)} f(x) = \sup_{R \in \mathcal{R}_s^{[\delta,\mu)}} \frac{1}{|R|} \left| \int_R f(x)\, dx \right|.$$

If $\delta = 0$ and $\mu = \infty$ we write simply $M_s f(x)$. We say that $A \subset \mathbb{R}^2$ is a δ-set if it is a union of mutually disjoint rectangles from $\mathcal{R}_s^{[\delta,\infty)}$. The following lemma contains the main idea of the proof of Marstrand's theorem.

LEMMA 1. *Let $0 < \delta_t < 1$, $t = 1, \dots, T$, and let $A_t \subset \mathbb{R}^2$ be δ_t-sets with* $\mathrm{mes}_* A_t > 12\delta_t$, $t = 1, \dots, T$. *Then for any sequence $\{n_t\}$ of integers with* $n_1 = 1$, $n_{t+1} > 4n_t/\delta_t$, *we have*

$$(2.1) \qquad \mathrm{mes}_* \left(\bigcup_{t=1}^T \mathrm{dil}_{n_t} A_t \right) > 1 - \left(1 - \frac{\mathrm{mes}_* A_t}{32} \right)^T.$$

Proof. First we prove that if B is a δ-set with $\mathrm{mes}_* B > 12\delta$, $m, n \in \mathbb{N}$ and $n > (4/\delta)m$, then there exists a set $\widetilde{B}$ such that

  (1) $\widetilde{B} \subset \mathrm{dil}_m B$,
  (2) for any $k \in \mathbb{Z}^2$ the set $\widetilde{B} \cap Q_k^m$ is a union of squares Q_j^n,
  (3) the values $|\widetilde{B} \cap Q_k^m|$ are equal for different $k \in \mathbb{Z}^2$,
  (4) $\mathrm{mes}_* \widetilde{B} > \frac{1}{32} \mathrm{mes}_* B$.

We note that any rectangle $R \in \mathcal{R}_s^{[\delta,\infty)}$ is a union of rectangles from $\mathcal{R}_s^\delta$. So we have $\mathrm{dil}_m B = \bigcup_i R_i$ where $R_i \in \mathcal{R}_s^{\delta/m}$. Set

$$B' = \bigcup_{R_i \subset Q_k^m \text{ for some } k \in \mathbb{Z}^2} R_i \subset \mathrm{dil}_m B.$$

We have diam $R_i = \delta\sqrt{2}/m$. So if $R_i \not\subset Q_k^m$ for some $k \in \mathbb{Z}^2$, then $R_i \cap \widetilde{Q}_k^m$ is empty, where $\widetilde{Q}_k^m$ is the square concentric with Q_k^m with side lengths $(1 - 2\delta\sqrt{2})/m$. Hence we get

$$(2.2) \qquad |B' \cap Q_k^m| > |\mathrm{dil}_m B \cap Q_k^m| - |Q_k^m \setminus \widetilde{Q}_k^m|$$

$$= |\mathrm{dil}_m B \cap Q_k^m| - \frac{1}{m^2}(4\delta\sqrt{2} - 8\delta^2)$$

$$> |\mathrm{dil}_m B \cap Q_k^m| - \frac{6\delta}{m^2} = \frac{1}{m^2}|B \cap Q_k^1| - \frac{6\delta}{m^2}$$

$$\geq \frac{1}{m^2}(\mathrm{mes}_* B - 6\delta) > \frac{\mathrm{mes}_* B}{2m^2}.$$

Using the Besicovitch theorem on covering by squares (see [2, p. 10]), we may choose a subfamily $\{R_i'\}$ of $\{R_i\}$ such that the R_i' are pairwise disjoint and

$$(2.3) \qquad \left| \bigcup_{R_i' \subset Q_k^m} R_i' \right| \geq \frac{1}{4} \left| \bigcup_{R_i \subset Q_k^m} R_i \right| \qquad \text{for any } k \in \mathbb{Z}^2.$$

Therefore, setting

$$B'' = \bigcup R_i' \subset B' \subset \mathrm{dil}_m B,$$

by (2.2) and (2.3) we have

$$(2.4) \qquad |B'' \cap Q_k^m| > \frac{\mathrm{mes}_* B}{8m^2}, \qquad k \in \mathbb{Z}^2.$$

Using simple geometry, one can easily check that if $R \in \mathcal{R}_s^{\delta/m}$ and $n > 4m/\delta$, then

$$\left| \bigcup_{Q_j^n \subset R} Q_j^n \right| > \frac{1}{4}|R|.$$

So, by (2.4), for $n > 4m/\delta$ we have

$$\left| \bigcup_{Q_j^n \subset B'' \cap Q_k^m} Q_j^n \right| > \frac{1}{4}|B'' \cap Q_k^m| > \frac{\mathrm{mes}_* B}{32m^2}.$$

Removing some squares Q_j^n from the left union we can get a set $\widetilde{B} \subset B''$ which is again a union of squares Q_j^n and in addition all the sets $\widetilde{B} \cap Q_k^m$ consist of the same number of squares Q_j^n and

$$|\widetilde{B} \cap Q_k^m| \geq \frac{\mathrm{mes}_* B}{32m^2}, \qquad k \in \mathbb{Z}^2.$$

Clearly, $\widetilde{B}$ satisfies conditions (1)–(4) above.

Taking $n = n_{t+1}$, $m = n_t$, $B = A_{n_t}$, $t = 1, \ldots, T$, we get sets $\widetilde{A}_t$, $t = 1, \ldots, T$, such that

    (1) $\widetilde{A}_t \subset \mathrm{dil}_{n_t} A_t$,

(2) $\widetilde{A}_t \cap Q_k^{n_t}$ is a union of squares $Q_j^{n_{t+1}}$ for any $k \in \mathbb{Z}^2$,

(3) the values $|\widetilde{A}_t \cap Q_k^{n_t}|$ are equal for different $k \in \mathbb{Z}^2$,

(4) $\mathrm{mes}_* \widetilde{A}_t > \mathrm{mes}_* A_t / 32$.

From conditions (2), (3) it follows that for fixed $k \in \mathbb{Z}^2$ the sets $\widetilde{A}_t \cap Q_k$, $t = 1, \ldots, T$, are probabilistically independent. Then by (1) and (4),

$$\mathrm{mes}_* \left(\bigcup_{t=1}^{T} \mathrm{dil}_{n_t} A_t \right) \geq \mathrm{mes}_* \left(\bigcup_{t=1}^{T} \widetilde{A}_t \right) = \left| \bigcup_{t=1}^{T} (\widetilde{A}_t \cap Q_k) \right|$$

$$= 1 - (1 - \mathrm{mes}_* \widetilde{A}_t)^T > 1 - \left(1 - \frac{\mathrm{mes}_* A_t}{32} \right)^T. \quad \blacksquare$$

For any line $l \subset \mathbb{R}^2$ we denote by $\arg l$ the positive value of the minimal angle between l and the x-axis. For two points $\theta, \theta' \in \mathbb{R}^2$ we denote by $\theta\theta'$ the line passing through θ and θ', and by $[\theta, \theta']$ the line segment with endpoints θ and θ'.

LEMMA 2. *Let* $0 < \varepsilon < 1$, $0 < \gamma < \pi/12$ *and set*

(2.5) $\qquad \theta_k = (\varepsilon/2^k, \mathrm{sign}(k) \cdot \tan\gamma \cdot \varepsilon/2^k), \qquad k = \pm1, \pm2, \ldots.$

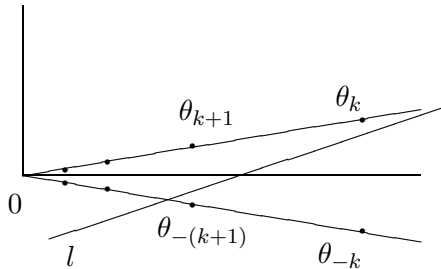*Then for any rectangle* $R \in \mathcal{R}_s$ *with* $3\gamma < |s| < \pi/2 - 3\gamma$, *we have*

(2.6) $\qquad \left| \sum_{0 < |k| \leq m, \, \theta_k \in R} \mathrm{sign}(k) \right| \leq 2, \qquad m = 1, 2, \ldots.$

Proof. First we note that if l is a line on the plane, then

(2.7) $\qquad l \cap [\theta_k, \theta_{-k}] \neq \emptyset, \qquad l \cap [\theta_{k+1}, \theta_{-(k+1)}] \neq \emptyset$

implies

$$\arg l < 3\gamma.$$

Indeed, using simple geometry, one can check that $\arg(\theta_{-k}\theta_{k+1}) < 3\gamma$. Hence $\arg l \leq \arg(\theta_{-k}\theta_{k+1}) < 3\gamma$. Now consider a rectangle

(2.8) $\qquad\qquad R \in \mathcal{R}_s, \qquad 3\gamma < |s| < \pi/2 - 3\gamma.$

Let us show that

(2.9)   if $\theta_n, \theta_{n+1}, \theta_{n+2} \in R$, then $\theta_{-(n+1)} \in R$.

Suppose $\theta_{-(n+1)} \notin R$. Then we find a line l containing a side of R and separating the points $\theta_n, \theta_{n+1}, \theta_{n+2}$ from $\theta_{-(n+1)}$. Obviously,

$$l \cap [\theta_{n+1}, \theta_{-(n+1)}] \neq \emptyset,$$

and either $l \cap [\theta_n, \theta_{-n}] \neq \emptyset$, or $l \cap [\theta_{n+2}, \theta_{-(n+2)}] \neq \emptyset$. So we have (2.7) for $k = n$ or $n + 1$ and therefore $\arg l < 3\gamma$, which contradicts (2.8). Similarly

(2.10)   if $\theta_{-n}, \theta_{-(n+1)}, \theta_{-(n+2)} \in R$, then $\theta_{n+1} \in R$.

Now let p and q be the numbers of elements of the sets $\{1 \leq k \leq m : \theta_k \in R\}$ and $\{-m \leq k \leq -1 : \theta_k \in R\}$ respectively. From (2.9) and (2.10) we conclude $|p - q| \leq 2$, which implies (2.6). $\blacksquare$

LEMMA 3. *For any $0 < \varepsilon < 1$ and $0 < \gamma \leq \pi/12$ there exists a bounded function ϕ on $\mathbb{R}^2$ such that*

(2.11) $\operatorname{supp} \phi \subset B(\varepsilon), \quad \int\limits_{\mathbb{R}^2} \phi(x)\, dx = 0, \quad \int\limits_{\mathbb{R}^2} |\phi(x)|\, dx \leq 1,$

(2.12) $\int\limits_{\operatorname{rot}_s([0,x_1] \times [0,x_2])} \phi(x)\, dx \geq 1/4 \quad \text{if } x_1, x_2 \geq \varepsilon, \ |s| \leq \gamma,$

(2.13) $M_s\phi(x) < \varepsilon \quad \text{if } x \notin \Gamma_s(2\varepsilon) \cup \Gamma_{s\perp}(2\varepsilon), \ 3\gamma < |s| < \pi/2 - 3\gamma.$

Proof. Let $\theta = \theta^+ \cup \theta^-$ where

(2.14)
$$\theta^+ = \{\theta_k : k = 1, 2, \ldots, N\},$$
$$\theta^- = \{\theta_k : k = -1, -2, \ldots, -N\}, \quad N = [10\varepsilon^{-3}] + 1,$$

and the θ_k are defined in (2.5). We have

$$\theta_k \in B(\varepsilon/\sqrt{2}) \subset B(\varepsilon), \quad \theta_k \in \{x : x_2 = \tan\gamma \cdot x_1\}, \quad k = \pm 1, \pm 2, \ldots.$$

Define

$$b_k = \{x \in \mathbb{R}^2 : |x - \theta_k| < r\}, \quad k = \pm 1, \ldots, \pm N.$$

For $r > 0$ small, the following conditions are satisfied:

(1) $b_k \subset B(\varepsilon)$ and they are mutually disjoint,
(2) if $k > 0$, then b_k is in the upper half-plane, and if $k < 0$, in the lower,
(3) any line l with $|\arg l| \geq 3\gamma$ intersects at most two b_k.

We define

$$\phi(x) = \frac{1}{2\pi N r^2} \sum_{k=1}^{N} (\mathbb{I}_{b_k}(x) + \mathbb{I}_{b_{-k}}(x)),$$

where $\mathbb{I}_{b_k}$ is the characteristic function of b_k. The conditions (2.11) are clear. To show (2.12) we shall use (1) and (2). Fix $x_1, x_2 > \varepsilon$. If $0 \le s < \gamma$, then

$$\mathrm{rot}_s([0, x_1] \times [0, x_1]) \cap b_k = \emptyset \quad \text{if } -N \le k < 0,$$

$$|\mathrm{rot}_s([0, x_1] \times [0, x_2]) \cap b_k| > |b_k|/2 = \pi r^2/2 \quad \text{if } 0 < k \le N.$$

Therefore

$$\int\limits_{\mathrm{rot}_s([0,x_1] \times [0,x_2])} \phi(x)\, dx = \frac{1}{2\pi N r^2} \sum_{k=1}^{N} \int\limits_{\mathrm{rot}_s([0,x_1] \times [0,x_2])} \mathbb{I}_{b_k}(x)\, dx \ge \frac{1}{4}.$$

If $-\gamma < s \le 0$, then

$$b_k \subset \mathrm{rot}_s([0, x_1] \times [0, x_2]), \quad k > 0,$$

$$|\mathrm{rot}_s([0, x_1] \times [0, x_2]) \cap b_k| \le |b_k|/2 = \pi r^2/2, \quad k < 0,$$

and then similarly we obtain (2.12). Now we prove that if

(2.15) $$R \in \mathcal{R}_s, \quad 3\gamma < |s| < \pi/2 - 3\gamma,$$

then

(2.16) $$\left| \int\limits_{R} \phi(x)\, dx \right| \le \frac{10}{N} < \varepsilon^3.$$

We have

(2.17) $$\int\limits_{R} \phi(x)\, dx = \frac{1}{2\pi N r^2} \sum_{b_k \cap R \ne \emptyset} \int\limits_{R} \mathbb{I}_{b_k}(x)\, dx$$

$$= \frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int\limits_{R} \mathbb{I}_{b_k}(x)\, dx + \frac{1}{2\pi N r^2} \sum_{\theta_k \notin R,\, b_k \cap R \ne \emptyset} \int\limits_{R} \mathbb{I}_{b_k}(x)\, dx.$$

The conditions $\theta_k \notin R$, $b_k \cap R \ne \emptyset$ mean that b_k intersects a side of R. Also if a line l contains a side of R then $|\arg l| > 3\gamma$. On the other hand, by (3) any line with $|\arg l| > 3\gamma$ can intersect not more than two balls b_k. So the number of terms in the second sum does not exceed 8. Therefore

(2.18) $$\left| \frac{1}{2\pi N r^2} \sum_{\theta_k \notin R,\, b_k \cap R \ne \emptyset} \int\limits_{R} \mathbb{I}_{b_k}(x)\, dx \right| \le \frac{4}{N}.$$

For the same reason the equality

$$\int\limits_{R} \mathbb{I}_{b_k}(x)\, dx = \int\limits_{\mathbb{R}^2} \mathbb{I}_{b_k}(x)\, dx$$

fails for not more than eight different k's. Therefore

$$\left| \frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int\limits_{R} \mathbb{I}_{b_k}(x)\, dx - \frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int\limits_{\mathbb{R}^2} \mathbb{I}_{b_k}(x)\, dx \right| \le \frac{4}{N}.$$

Hence we obtain

$$(2.19) \quad \left| \frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int_R \mathbb{I}_{b_k}(x) \, dx \right| \leq \left| \frac{1}{2\pi N r^2} \sum_{\theta_k \in R} \int_{\mathbb{R}^2} \mathbb{I}_{b_k}(x) \, dx \right| + \frac{4}{N}$$

$$= \left| \frac{1}{2N} \sum_{\theta_k \in R} \mathrm{sign}(k) \right| + \frac{4}{N} \leq \frac{5}{N},$$

where the last inequality follows from Lemma 2. Combining (2.17)–(2.19) we get (2.16). Fix s with $3\gamma < |s| \leq \pi/4$ and take a point $x \in \mathbb{R}^2$ such that

$$x \notin \Gamma_s(2\varepsilon) \cup \Gamma_{s^\perp}(2\varepsilon), \quad x \in R \in \mathcal{R}_s, \quad 3\gamma < |s| < \pi/2 - 3\gamma.$$

We need to prove

$$(2.20) \qquad\qquad \frac{1}{|R|} \int_R \phi(t) \, dt \leq \varepsilon.$$

Assume the lengths of the sides of R are a and b. If R contains no θ_k then (2.20) is trivial. So suppose at least one θ_k is in R. Hence R intersects $B(\varepsilon)$ and $(\Gamma_s(2\varepsilon) \cup \Gamma_{s^\perp}(2\varepsilon))^{\mathrm{c}}$. As $R \in \mathcal{R}_s$, we get $a, b > \varepsilon$. Hence (2.16) yields

$$\frac{1}{|R|} \int_R \phi(t) \, dt \leq \frac{\varepsilon^3}{ab} \leq \varepsilon. \quad \blacksquare$$

LEMMA 4. *For any $0 < \varepsilon, \delta < 1/10$, and any interval $S = [\alpha - \gamma, \alpha + \gamma] \subset [0, \pi/2)$ with $0 < \gamma \leq \pi/12$, there exist a bounded function ϕ and numbers ν, ν' with $0 < \nu < \nu'$ such that*

$$(2.21) \qquad\qquad \sup_{k \in \mathbb{Z}^2} \int_{Q_k} |\phi(x)| \, dx \leq 1,$$

$$(2.22) \qquad \mathrm{mes}^* \{ x \in \mathbb{R}^2 : M_s \phi(x) > \varepsilon \} < \varepsilon, \quad 3\gamma < |s - \alpha| < \pi/2 - 3\gamma,$$

$$(2.23) \qquad \mathrm{mes}^* \{ x \in \mathbb{R}^2 : M_s^{[0,\nu)} \phi(x) > \varepsilon \} < \varepsilon, \quad s \in [0, 2\pi),$$

$$(2.24) \qquad\qquad M_s^{[\nu',\infty)} \phi(x) < \varepsilon, \quad x \in \mathbb{R}^2, \, s \in [0, 2\pi),$$

$$(2.25) \qquad \mathrm{mes}_* \{ M_s^{[\nu,\nu']} \phi(x) > 1/\delta \} > \frac{\delta}{4} \ln \frac{1}{12\delta}, \quad s \in S.$$

Proof. Without loss of generality we may assume $\alpha = 0$, i.e. $S = [-\gamma, \gamma]$. We take $\lambda = \min\{\varepsilon/100, \delta\}$ and consider a double sequence $\varepsilon_k = \varepsilon_{k_1, k_2} = \lambda 2^{-(|k_1| + |k_2|)}$, $k \in \mathbb{Z}^2$. Using Lemma 3 we can find functions ϕ_k satisfying the following conditions:

$$(2.26) \qquad\qquad \mathrm{supp}\,\phi_k \subset B(\varepsilon_k) \subset B(\varepsilon),$$

$$(2.27) \qquad \int_{Q_0} \phi_k(x) \, dx = 0, \quad \int_{Q_0} |\phi_k(x)| \, dx \leq 1,$$

$$(2.28) \qquad \int\limits_{\mathrm{rot}_s R_x} \phi_k(x)\,dx > 1/4, \qquad R_x = [0, x_1] \times [0, x_2],$$

$$x_1, x_2 \geq \delta \geq \varepsilon_k, \ |s| < \gamma,$$

$$(2.29) \quad M_s\phi_k(x) < \varepsilon_k \quad \text{if } x \notin \Gamma_s(2\varepsilon_k) \cup \Gamma_{s^\perp}(2\varepsilon_k), \ 3\gamma < |s| \leq \pi/2 - 3\gamma,$$

where $k = (k_1, k_2)$. Define

$$(2.30) \qquad\qquad \phi(x) = \sum_{k \in \mathbb{Z}^2} \phi_k(x + k),$$

$$(2.31) \qquad\qquad E_s = \bigcup_{k \in \mathbb{Z}^2} (k + (\Gamma_s(2\varepsilon_k) \cup \Gamma_{s^\perp}(2\varepsilon_k))).$$

We obviously have (2.21) and

$$(2.32) \qquad\qquad \mathrm{supp}\,\phi \subset \bigcup_{k \in \mathbb{Z}^2} (k + B(\varepsilon)),$$

$$(2.33) \qquad\qquad \int\limits_{Q_k} \phi(x)\,dx = 0, \qquad k \in \mathbb{Z}^2.$$

Proof of (2.22): For any square Q_j, $j \in \mathbb{Z}^2$, we have

$$|Q_j \cap (k + \Gamma_s(2\varepsilon_k))| \leq \mathrm{diam}\,Q_j \times \mathrm{mes}_s(k + \Gamma_s(2\varepsilon_k)) = 4\varepsilon_k\sqrt{2},$$

$$|Q_j \cap (k + \Gamma_{s^\perp}(2\varepsilon_k))| \leq 4\varepsilon_k\sqrt{2}.$$

Hence we obtain

$$(2.34) \qquad\qquad \mathrm{mes}^* E_s \leq \sum_k 8\sqrt{2}\,\varepsilon_k = 32\sqrt{2}\lambda \leq \varepsilon.$$

From (2.29) it follows that

$$M_s\phi_k(x+k) \leq \varepsilon_k, \quad x \notin E_s \supset k + (\Gamma_s(2\varepsilon_k) \cup \Gamma_{s^\perp}(2\varepsilon_k)), \ 3\gamma < |s| \leq \pi/2 - 3\gamma.$$

Then according to (2.30) and (2.31) we get

$$M_s\phi(x) \leq \sum_k M_s\phi_k(x+k) \leq \sum_k \varepsilon_k \leq \varepsilon, \qquad x \notin E_s, \ 3\gamma < |s| \leq \pi/2 - 3\gamma,$$

and combining this with (2.34) we obtain (2.22).

Proof of (2.23): From (2.32) it follows that

$$\lim_{\nu \to 0} M_s^{[0,\nu)}\phi(x) = 0 \quad \text{if } x \notin \bigcup_{k \in \mathbb{Z}^2} (k + B(\varepsilon)), \ s \in [0, 2\pi),$$

therefore for small $\nu < \delta$ we have (2.23), since

$$\mathrm{mes}^* \Big(\bigcup_{k \in \mathbb{Z}^2} (k + B(\varepsilon)) \Big) = |B(\varepsilon)| = \pi\varepsilon^2 \leq \varepsilon.$$

Proof of (2.24): From (2.33) we obtain

$$\lim_{\nu' \to \infty} \int_R \phi(\nu' x)\, dx = 0$$

for any rectangle R, and the convergence is uniform in $R \in \mathcal{R}_s^{[1,\infty)}$, $s \in [0, 2\pi)$. So for large $\nu' > 1/4$ we have

$$M_s^{[1,\infty)} \phi(\nu' x) < \varepsilon, \quad x \in \mathbb{R}^2, \ s \in [0, 2\pi).$$

By dilation we get

$$M_s^{[\nu',\infty)} \phi(x) = M_s^{[1,\infty)} \phi(\nu' x) < \varepsilon, \quad x \in \mathbb{R}^2, \ s \in [0, 2\pi),$$

which gives (2.24).

Proof of (2.25): Consider the set

$$(2.35) \qquad A = \{x = (x_1, x_2) : x_1 x_2 \le \delta/4, \ \delta \le x_1, x_2 \le 1/4\}.$$

We have

$$(2.36) \qquad \mathrm{rot}_s\, A \subset B(1/2), \quad s \in [-\pi/4, \pi/4),$$

$$(2.37) \qquad |A| = \int_\delta^{1/4} \frac{\delta}{4t}\, dt - \delta(1/4 - \delta) > \frac{\delta}{4} \ln \frac{1}{12\delta}.$$

If $x = (x_1, x_2) \in A$, then

$$(2.38) \qquad 1/4 \ge x_1, x_2 \ge \delta > \varepsilon_k, \quad |R_x| \le \delta/4$$

So by (2.28),

$$\int_{k + \mathrm{rot}_s R_x} \phi_k(k + t)\, dt = \int_{\mathrm{rot}_s R_x} \phi_k(t)\, dt > 1/4 \quad \text{if } x \in A, \ |s| < \gamma,$$

and therefore from (2.26) we get

$$(2.39) \qquad \int_{k + \mathrm{rot}_s R_x} \phi(t)\, dt = \int_{k + \mathrm{rot}_s R_x} \phi_k(k + t)\, dt > 1/4 \quad \text{if } x \in A, \ |s| < \gamma.$$

As $\nu < \delta$, $\nu' > 1/4$ we have $R_x \in \mathcal{R}_0^{[\delta, 1/4]} \subset \mathcal{R}_0^{[\nu, \nu']}$. Since $|R_x| \le \delta/4$, from (2.39) and (2.38) we conclude that

$$(2.40) \qquad M_s^{[\nu, \nu']} \phi(x) > \frac{1}{4|R_x|} > \frac{1}{\delta}, \quad x \in G_s = \bigcup_k (k + \mathrm{rot}_s\, A), \ |s| < \gamma.$$

In addition, by (2.35)–(2.37), for any $m \in \mathbb{Z}^2$ we get

$$|(m + Q_0) \cap G_s| = |m + \mathrm{rot}_s\, A| = |A| > \frac{\delta}{4} \ln \frac{1}{12\delta},$$

which implies

$$\mathrm{mes}^*\, G_s > \frac{\delta}{4} \ln \frac{1}{12\delta}.$$

Combining this with (2.40) we obtain (2.25). ∎

3. Proofs of theorems

Proof of Theorem 1. Let G be an arbitrary G_δ-set in $[0, \pi/2)$. So

$$G = \Big(\bigcap_{k=1}^{\infty} G_k \Big) \cap [0, \pi/2),$$

where $G_k \subset \mathbb{R}$ are open sets and $G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots$. Each G_k is a union $G_k = \bigcup_j I_j^k$ of mutually disjoint intervals.

We note that an arbitrary interval $I = (\alpha, \beta) \subset \mathbb{R}$ can be split into disjoint intervals $I_i = [\alpha_i, \beta_i)$ such that

$$|I_i| \leq \pi/12, \quad 3I_i \subset I, \quad \sum_i \mathbb{I}_{3I_i}(x) \leq 8.$$

For $I = (-1, 1)$ such a partition is

$$[1 - (9/10)^k, 1 - (9/10)^{k+1}), \quad k = 0, 1, 2, \ldots,$$
$$[(9/10)^{k+1} - 1, (9/10)^k - 1), \quad k = 0, 1, 2, \ldots.$$

We do a similar splitting for any I_j^k. Let J_t, $t = 1, 2, \ldots$, be an enumeration of those splitting intervals J for which $J \cap [0, \pi/2) \neq \emptyset$. We set $l_t = J_t \cap [0, \pi/2)$. It is easy to check the following:

(1) if $x \in G$, then x belongs to an infinite number of l_t's,
(2) if $x \notin G$ then x belongs only to a finite number of $3l_t$'s.

We choose integers $0 = m_0 < m_1 < m_2 < \cdots$ satisfying

$$(3.1) \qquad \prod_{k=m_t+1}^{m_{t+1}} \Big(1 - \frac{1}{k \ln k} \Big) < \frac{1}{2^t}, \quad t = 1, 2, \ldots.$$

We define

$$(3.2) \qquad S_k = l_t \quad \text{if } m_t < k \leq m_{t+1}.$$

Using Lemma 4 for $S = S_k$, $\varepsilon = 1/2^k$, $\delta = 1/(k \ln^2 k)$, we may define functions ϕ_k and numbers $0 < \nu_k < \nu_k'$ satisfying (2.21)–(2.25). We set

$$(3.3) \qquad U_{s,k} = \{x \in \mathbb{R}^2 : M_s \phi_k(x) \leq 1/2^k\},$$

$$(3.4) \qquad V_{s,k}' = \{x \in \mathbb{R}^2 : M_s^{[0,\nu_k]} \phi_k(x) \leq 1/2^k\},$$

$$(3.5) \qquad V_{s,k}'' = \{x \in \mathbb{R}^2 : M_s^{[\nu_k, \nu_k']} \phi_k(x) > k \ln^2 k\}.$$

By (2.22), (2.23), (2.25) we have

$$(3.6) \qquad \text{mes}_* U_{s,k} > 1 - 1/2^k, \quad s \in [0, \pi/2) \setminus 3S_k$$

(we may replace the condition $3\gamma < |s - \alpha| < \pi/2 - 3\gamma$ in (2.22) by $s \in$

$[0, \pi/2) \setminus 3S$ because the latter implies the former) and

$$(3.7) \qquad \text{mes}_* V'_{s,k} > 1 - 1/2^k, \qquad s \in [0, \pi/2),$$

$$(3.8) \qquad \text{mes}_* V''_{s,k} > \frac{1}{4k \ln^2 k} \ln \frac{k \ln^2 k}{12} > \frac{c}{k \ln k}, \qquad s \in S_k \ (k \geq 3).$$

From (2.24) we get

$$(3.9) \qquad M_s^{[\nu'_k, \infty)} \phi_k(x) < 1/2^k, \qquad x \in \mathbb{R}^2, \, s \in [0, \pi/2).$$

We define integers $1 = n_0 < n_1 < n_2 < \cdots$ so that

$$(3.10) \qquad \frac{n_k}{n_{k-1}} > \max\left(\frac{4}{\nu_{k-1}}, \frac{\nu'_k}{\nu_{k-1}} \right), \qquad k = 1, 2, \ldots,$$

and set $\mu_k = \nu_k/n_k$. It is clear that

$$\mu_{k-1} > \nu'_k/n_k > \mu_k, \qquad k = 2, 3, \ldots.$$

Consider the functions

$$(3.11) \qquad \psi_k(x) = \phi_k(n_k x), \qquad x \in Q_0.$$

According to (3.3)–(3.5) and (3.11), we obviously have

$$(3.12) \qquad M_s \psi_k(x) \leq 1/2^k, \qquad x \in \text{dil}_{n_k} U_{s,k}, \, s \in [0, \pi/2) \setminus 3S_k,$$

$$(3.13) \qquad M_s^{[0,\mu_k)} \psi_k(x) = M_s^{[0,\nu_k/n_k)} \psi_k(x) \leq 1/2^k,$$

$$x \in \text{dil}_{n_k} V'_{s,k}, \, s \in [0, \pi/2),$$

$$(3.14) \qquad M_s^{[\mu_k, \mu_{k-1}]} \psi_k(x) > M_s^{[\nu_k/n_k, \nu'_k/n_k]} \psi_k(x) > k \ln^2 k,$$

$$x \in \text{dil}_{n_k} V''_{s,k}, \, s \in S_k,$$

$$(3.15) \qquad M_s^{[\mu_{k-1}, \infty)} \psi_k(x) \leq M_s^{[\nu'_k/n_k, \infty)} \psi_k(x) \leq 1/2^k, \qquad x \in \mathbb{R}^2, \, s \in [0, \pi/2).$$

The desired function is

$$(3.16) \qquad f(x) = \sum_{k=1}^{\infty} \frac{\psi_k(x)}{k \ln^{3/2} k}, \qquad x \in Q_0.$$

Set

$$(3.17) \qquad U_s = \limsup_{k \to \infty} ((\text{dil}_{n_k} U_{s,k}) \cap Q_0),$$

where $\limsup_{k \to \infty} A_k$ means $\bigcup_n \bigcap_{k \geq n} A_k$. If $s \notin G$, then (2) implies $s \in [0, \pi/2) \setminus 3S_k$ for $k > k(s)$. Therefore, by (3.6) we have $|\text{dil}_{n_k} U_{s,k} \cap Q_0| \geq \text{mes}_* U_{s,k} > 1 - 1/2^k$, $k > k(s)$, and so we get

$$(3.18) \qquad |U_s| = 1 \quad \text{if } s \notin G.$$

From (3.12) and (3.17) we infer that for any $x \in U_s$,

$$M_s \psi_k(x) \leq 1/2^k, \qquad k > k(x).$$

Hence, if $\varepsilon > 0$, then for an appropriate $N > k(x)$ we get

$$(3.19) \quad M_s \left(\sum_{k=N+1}^{\infty} \frac{\psi_k(x)}{k \ln^{3/2} k} \right) \leq \sum_{k=N+1}^{\infty} \frac{M_s \psi_k(x)}{k \ln^{3/2} k} \leq \sum_{k=N+1}^{\infty} \frac{1}{k 2^k \ln^{3/2} k} < \varepsilon.$$

On the other hand, since

$$\sum_{k=1}^{N} \frac{\psi_k(x)}{k \ln^{3/2} k}$$

is a bounded function, the basis $\mathcal{R}_s$ differentiates its integral. So, by (3.19) and (3.16), $\mathcal{R}_s$ differentiates $\int f$ if $s \in [0, \pi/2) \setminus G$.

Now let $s \in G$. Then $s \in l_{t_i}$, $i = 1, 2, \ldots$. Hence $s \in S_k$ if $m_{t_i} < k \leq m_{t_i+1}$, $i = 1, 2, \ldots$. We notice that each $V''_{s,k}$ defined in (3.5) is a ν_k-set, and by (3.10), $n_{k+1} > 4n_k/\nu_k$. Therefore, using (3.1), from Lemma 1 we obtain

$$(3.20) \quad \left| \bigcup_{k=m_{t_i}+1}^{m_{t_i+1}} \operatorname{dil}_{n_k} V''_{s,k} \cap Q_0 \right| \geq 1 - \prod_{k=m_{t_i}+1}^{m_{t_i+1}} \left(1 - \frac{1}{k \ln k} \right) > 1 - \frac{1}{2^t}.$$

Defining

$$V_s = (\limsup_{k \to \infty} \operatorname{dil}_{n_k} V'_{s,k}) \cap \left(\limsup_{i \to \infty} \bigcup_{k=m_{t_i}+1}^{m_{t_i+1}} \operatorname{dil}_{n_k} V''_{s,k} \right) \cap Q_0,$$

from (3.20) and (3.7) we get

$$(3.21) \qquad\qquad |V_s| = 1, \quad s \in G.$$

On the other hand, if $x \in V_s$, then

$$x \in \operatorname{dil}_{n_{k_i}} V''_{s,k_i}, \quad i = 1, 2, \ldots,$$
$$x \in \operatorname{dil}_{n_k} V'_{s,k}, \quad k > k(x),$$

where $k_i \to \infty$, and therefore, by (3.13) and (3.15) we have

$$M_s^{[\mu_{k_i}, \mu_{k_i}-1]} \psi_j(x) \leq 1/2^{k_i} \quad \text{if } j \neq k_i.$$

The case $j > k_i$ follows from (3.15), and $j < k_i$ from (3.13). From (3.14) we get

$$M_s^{[\mu_{k_i}, \mu_{k_i}-1]} \psi_{k_i}(x) > k_i \ln^2 k_i.$$

So if $k_i > k(x)$, then

$$M_s f(x) \geq M_s^{[\mu_{k_i}, \mu_{k_i}-1]} f(x)$$
$$\geq \frac{M_s^{[\mu_{k_i}, \mu_{k_i}-1]} \psi_{k_i}(x)}{k_i \ln^{3/2} k_i} - \sum_{j \neq k_i} \frac{M_s^{[\mu_{k_i}, \mu_{k_i}-1]} \psi_j(x)}{j \ln^{3/2} j} \geq c\sqrt{\ln k_i} - \sum_{j \neq k_i} \frac{1}{j 2^j \ln^{3/2} j},$$

and so $\overline{D}_s f(x) = \infty$ whenever $x \in V_s$ and $s \in G$. Since $|V_s| = 1$ by (3.21), the theorem is completely proved. ∎

Proof of Theorem 2. The necessity is shown in the introduction. To prove the sufficiency we let $V \subset [0, \pi/2)$ be an arbitrary $G_{\delta\sigma}$-set,

$$V = \bigcup_n V_n$$

where each V_n is a G_δ. According to Theorem 1 for each V_n there exists a function $f_n \in L^1(\mathbb{R}^2)$ such that its integral is differentiated by $\mathcal{R}_s$ if $s \notin V_n$, and $\overline{D}_s f_n(x) = \infty$ a.e. if $s \in V_n$. Set $g_n(x) = \mathbb{I}_{Q_n}(x) f_n(x)$, where Q_n is a family of arbitrary pairwise disjoint unit open squares, and consider the function

$$g(x) = \sum_{n=1}^\infty g_n(x).$$

Since the supports of the functions g_n are disjoint, for any $x \in Q_n$ and any s we have

$$\overline{D}_s g(x) = \overline{D}_s g_n(x) = \overline{D}_s f_n(x).$$

If $s \in V$ then $s \in V_n$ for some n. So we get $\overline{D}_s g(x) = \overline{D}_s f_n(x) = \infty$ almost everywhere on the square Q_n. Using disjointness of the supports of the functions g_n once again we conclude that if $s \notin V$ then

$$\lim_{\mathrm{diam}(R) \to 0, \, x \in R \in \mathcal{R}_s} \frac{1}{|R|} \int_R g = g(x) \qquad \text{a.e.}$$

Finally, we conclude that V is a WR-set and Theorem 2 is proved. ∎

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