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On a decomposition for pairs of commuting contractions

by

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Abstract. A new decomposition of a pair of commuting, but not necessarily doubly commuting contractions is proposed. In the case of power partial isometries a more detailed decomposition is given.

1. Introduction. Let H be a complex Hilbert space. Let H_0 be a subspace of H. Then P_{H_0} is the orthogonal projection on H_0. Denote by $L(H)$ the algebra of all bounded linear operators on H. An operator T is called *completely nonunitary* if there is no nontrivial subspace reducing T to a unitary operator. The following decomposition of a contraction with respect to unitarity was given in [7, 11].

Theorem 1.1. *Let $T \in L(H)$ be a contraction. There is a unique decomposition*

$$H = H_u \oplus H_{\neg u},$$

where $H_u, H_{\neg u}$ are maximal subspaces reducing T such that:

- $T|_{H_u}$ *is a unitary operator*,
- $T|_{H_{\neg u}}$ *is a completely nonunitary operator.*

The completely nonunitary part of an operator can be characterized more precisely for certain classes of operators. A well known example is the decomposition of an isometry (Wold [12]), where the completely nonunitary part turns out to be a unilateral shift (of any multiplicity). A larger class of operators with a well characterized completely nonunitary part are power partial isometries. Recall that an operator $T \in L(H)$ is called a *partial isometry* if $T|_{(\ker T)^{\perp}}$ is an isometry. We call T a *power partial isometry* when every power T^n for $n \geq 1$ is a partial isometry. Recall also that a *truncated shift of index k* is an operator T on a Hilbert space $H \oplus \cdots \oplus H$ (k times) given by $T(x_1, \ldots, x_k) = (0, x_1, \ldots, x_{k-1})$ for $k \in \mathbb{Z}_+$. Halmos and Wallen [5] found the decomposition of a power partial isometry.

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Theorem 1.2. *Let $T \in L(H)$ be a power partial isometry. There is a unique decomposition*

$$H = H_u \oplus H_s \oplus H_b \oplus \bigoplus_{k \geq 1} H_k,$$

where H_u, H_s, H_b and $H_k, k \geq 1$, are subspaces reducing the operator T such that:

- $T|_{H_u}$ *is a unitary operator*,
- $T|_{H_s}$ *is a unilateral shift*,
- $T|_{H_b}$ *is a backward shift*,
- $T|_{H_k}$ *is a truncated shift of index k.*

The decompositions given in Theorems 1.1 and 1.2 have a natural generalizations for a pair of doubly commuting operators (i.e., not only do the operators commute, but also each of them commutes with the adjoint of the other). The case of unitary decomposition is given in [9, Proposition 1]:

Theorem 1.3. *Let $T_1, T_2 \in L(H)$ be doubly commuting contractions. There is a unique decomposition*

$$H = H_{uu} \oplus H_{u\neg u} \oplus H_{\neg uu} \oplus H_{\neg u \neg u},$$

where $H_{uu}, H_{u\neg u}, H_{\neg uu}, H_{\neg u\neg u}$ are subspaces reducing T_1, T_2 such that:

- $T_1|_{H_{uu}}, T_2|_{H_{uu}}$ *are unitary*,
- $T_1|_{H_{u\neg u}}$ *is unitary*, $T_2|_{H_{u\neg u}}$ *is completely nonunitary*,
- $T_1|_{H_{\neg uu}}$ *is completely nonunitary operator*, $T_2|_{H_{\neg uu}}$ *is unitary*,
- $T_1|_{H_{\neg u\neg u}}, T_2|_{H_{\neg u\neg u}}$ *are completely nonunitary.*

The case of doubly commuting power partial isometries was described in [3].

Theorem 1.4. *Let $T_1, T_2 \in L(H)$ be doubly commuting power partial isometries. There is a unique decomposition*

$$H = \bigoplus_{\alpha,\beta \in \{u,s,b\} \cup \mathbb{Z}_+} H_{\alpha\beta},$$

where the $H_{\alpha\beta}$ are maximal subspaces reducing T_1, T_2 such that $T_1|_{H_{\alpha\beta}}$ belongs to class α and $T_2|_{H_{\alpha\beta}}$ to class β for $\alpha, \beta \in \{u, s, b\} \cup \mathbb{Z}_+$. The classes are: *class u—unitary operators*, *class s—unilateral shifts*, *class b—backward shifts*, *class k—truncated shifts of index k for $k \in \mathbb{Z}_+$.*

A more universal result generalizing a decomposition of a single operator with respect to any property which is inherited by restrictions to reducing subspaces can be found in [2].

2. Decomposition of a pair of contractions. The decompositions given in Theorems 1.3 and 1.4 are entire, but under a strong double commutativity assumption. A natural example of a commuting, but not doubly commuting pair of isometries is T, T^2, where $T \in L(H)$ is a unilateral shift. For more examples see [1].

For a pair of commuting contractions one can find maximal subspaces H_{uu}, $H_{u\neg u}, H_{\neg uu}$ where the operators are as in Theorem 1.3. However, the orthogonal complement of $H_{uu} \oplus H_{u\neg u} \oplus H_{\neg uu}$ need not reduce the contractions to completely nonunitary operators. Following the definition of a completely nonunitary semigroup of isometries in [10], we say that commuting contractions are a *completely nonunitary pair* if no proper subspace reduces both operators to unitary operators. Any two operators restricted to $H \ominus (H_{uu} \oplus H_{u\neg u} \oplus H_{\neg uu})$ are a completely nonunitary pair. However, the restrictions to $H_{u\neg u}$ or $H_{\neg uu}$ are also completely nonunitary pairs. Therefore we need a more precise definition. We call commuting contractions T_1, T_2 a *strongly completely nonunitary pair* if there is no proper subspace reducing T_1, T_2 and at least one of them to a unitary operator. It turns out that $H \ominus (H_{uu} \oplus H_{u\neg u} \oplus H_{\neg uu})$ is a maximal subspace reducing the contractions T_1, T_2 to a strongly completely nonunitary pair. Precisely, the following decomposition holds.

THEOREM 2.1. *Let $T_1, T_2 \in L(H)$ be commuting contractions. There is a unique decomposition*

$$H = H_{uu} \oplus H_{u\neg u} \oplus H_{\neg uu} \oplus H_{\neg(uu)},$$

where the subspaces $H_{uu}, H_{u\neg u}, H_{\neg uu}, H_{\neg(uu)}$ are maximal with respect to the following:

- *$T_1|_{H_{uu}}, T_2|_{H_{uu}}$ are unitary,*
- *$T_1|_{H_{u\neg u}}$ is unitary, $T_2|_{H_{u\neg u}}$ is completely nonunitary,*
- *$T_1|_{H_{\neg uu}}$ is completely nonunitary, $T_2|_{H_{\neg uu}}$ is unitary,*
- *$T_1|_{H_{\neg(uu)}}, T_2|_{H_{\neg(uu)}}$ is a strongly completely nonunitary pair of contractions.*

Proof. Let $\mathcal{H} := \{H_0 \subset \ker(T_1T_2^* - T_2^*T_1) : P_{H_0}T_i = T_iP_{H_0} \text{ for } i = 1, 2\}$. Recall that a subspace H_0 reduces an operator if and only if the orthogonal projection P_{H_0} commutes with the operator. Thus $H^{dc} = \overline{\mathrm{Span}}\{H_0 \in \mathcal{H}\}$ is a maximal subspace reducing the contractions T_1, T_2 to a doubly commuting pair. By Theorem 1.3 applied to $T_1|_{H^{dc}}, T_2|_{H^{dc}}$, we obtain the decomposition $H^{dc} = H^{dc}_{uu} \oplus H^{dc}_{u\neg u} \oplus H^{dc}_{\neg uu} \oplus H^{dc}_{\neg u\neg u}$. On the other hand, by a simple calculation, or by the Fuglede–Putnam theorem, if one of commuting operators is unitary, they doubly commute. Therefore, H^{dc}_{uu}, $H^{dc}_{u\neg u}$ and $H^{dc}_{\neg uu}$ are maximal subspaces of the entire H reducing T_1, T_2 to operators of the respective types. Thus $H_{uu} = H^{dc}_{uu}$, $H_{u\neg u} = H^{dc}_{u\neg u}$, $H_{\neg uu} = H^{dc}_{\neg uu}$. Set

$H_{\neg(uu)} = H \ominus (H_{uu} \oplus H_{u\neg u} \oplus H_{\neg uu})$. By the maximality of $H_{uu}, H_{u\neg u}, H_{\neg uu}$ the restrictions $T_1|_{H_{\neg(uu)}}, T_2|_{H_{\neg(uu)}}$ are a strongly completely nonunitary pair of contractions.

To show maximality of $H_{\neg(uu)}$, assume that a subspace H_0 reduces the contractions to a strongly completely nonunitary pair. We may decompose $H \ominus H_0 = H'_{uu} \oplus H'_{\neg uu} \oplus H'_{u\neg u} \oplus H'_{\neg(uu)}$, where $T_1|_{H'_{uu}\oplus H'_{u\neg u}}$, $T_2|_{H'_{uu}\oplus H'_{\neg uu}}$ are unitary, $T_1|_{H'_{\neg uu}}, T_2|_{H'_{u\neg u}}$ are completely nonunitary and $T_1|_{H'_{\neg(uu)}}, T_2|_{H'_{\neg(uu)}}$ is a strongly completely nonunitary pair of contractions. The subspace $H_0 \oplus H'_{\neg(uu)}$ also reduces T_1, T_2 to a strongly completely nonunitary pair of contractions. Thus we can assume for convenience that $H \ominus H_0 = H'_{uu} \oplus H'_{\neg uu} \oplus H'_{u\neg u}$. The maximality of $H_{uu}, H_{\neg uu}, H_{u\neg u}$ implies $H'_{uu} \subset H_{uu}$, $H'_{\neg uu} \subset H_{\neg uu}$, $H'_{u\neg u} \subset H_{u\neg u}$. Consequently,

$$(H_{uu} \ominus H'_{uu}) \oplus (H_{u\neg u} \ominus H'_{u\neg u}) \oplus (H_{\neg uu} \ominus H'_{\neg uu}) \subset H \ominus (H'_{uu} \oplus H'_{\neg uu} \oplus H'_{u\neg u}) = H_0.$$

By the definition of a strongly completely nonunitary pair, we deduce that $H_{uu} \ominus H'_{uu} = H_{u\neg u} \ominus H'_{u\neg u} = H_{\neg uu} \ominus H'_{\neg uu} = \{0\}$. The decomposition is unique. ∎

Recall from [8] that a pair of commuting isometries V_1, V_2 is called a *weak bi-shift* when $V_1|_{\bigcap_{i\geq 0} \ker V_2^* V_1^i}, V_2|_{\bigcap_{i\geq 0} \ker V_1^* V_2^i}$ and V_1V_2 are shifts. Theorem 2.1 is a generalization of a result for pairs of isometries given in [8], in particular, a strongly completely nonunitary pair of commuting isometries is a weak bi-shift.

3. Power partial isometries—preliminaries.

The following property of partial isometries can be found in [4].

THEOREM 3.1. *Let $T \in L(H)$. The following conditions are equivalent*:

(1) T *is a partial isometry*, (4) $TT^* = P_{(\ker T^*)^\perp}$,
(2) T^* *is a partial isometry*, (5) $T^*TT^* = T^*$,
(3) $T^*T = P_{(\ker T)^\perp}$, (6) $TT^*T = T$. ∎

Note also a trivial but useful property.

LEMMA 3.2. *Let $T \in L(H)$. If a subspace $H_0 \subset H$ reduces T, then*

$$P_{\ker T}(H_0) = P_{H_0}(\ker T) = \ker T \cap H_0,$$
$$P_{\ker T}(H_0^\perp) = P_{H_0^\perp}(\ker T) = \ker T \cap H_0^\perp.$$

Proof. Since H_0 reduces T, we have $P_{H_0}T = TP_{H_0}$ and $P_{H_0^\perp}T = TP_{H_0^\perp}$. Thus $P_{H_0}(\ker T) \subset \ker T$ and $P_{H_0^\perp}(\ker T) \subset \ker T$. For each $y \in H_0$, since

$P_{H_0^\perp} y = 0$, it follows that $P_{H_0^\perp}(P_{\ker T} y) = -P_{H_0^\perp}(P_{\ker T^\perp} y)$. Therefore

$$\|P_{H_0^\perp}(P_{\ker T} y)\|^2 = (P_{H_0^\perp}(P_{\ker T} y), -P_{\ker T^\perp} y) = 0.$$

Thus $P_{\ker T}(H_0) \subset H_0$ and $P_{\ker T}(H_0^\perp) \subset H_0^\perp$. ∎

The subspaces in Theorem 1.2 may be described by the following formulas.

THEOREM 3.3. *Let $T \in L(H)$ be a power partial isometry and*

$$H = H_u \oplus H_s \oplus H_b \oplus H_t$$

be the decomposition given in Theorem 1.2, *where $H_t = \bigoplus_{k\geq 1} H_k$. Then*

(i) $H_b \oplus H_t = \bigoplus_{n\geq 0} T^{*n}(\ker T)$,
(ii) $H_u \oplus H_b = \bigcap_{n\geq 0} T^n H$,
(iii) $H_s \oplus H_t = \bigoplus_{n\geq 0} T^n(\ker T^*)$,
(iv) $H_u \oplus H_s = \bigcap_{n\geq 0} T^{*n} H$.

Proof. By Theorem 3.1, the operator T^* is a power partial isometry. Note that (iii) for T is equivalent to (i) for T^*. The same relation holds between (iv) and (ii). Therefore it is enough to show (i) and (ii).

For (i) note that since $T|_{H_u \oplus H_s}$ is an isometry, $H_u \oplus H_s$ is orthogonal to $\ker T$. Therefore, by Lemma 3.2, for $H_0 = H_b$ and $H_0^\perp = H_t$, we obtain

$$\ker T = P_{H_b}(\ker T) \oplus P_{H_t}(\ker T) = (\ker T \cap H_b) \oplus (\ker T \cap H_t). \tag{1}$$

By the geometric structure of the unilateral shift $T^*|_{H_b}$ we have

$$H_b = \bigoplus_{n\geq 0} T^{*n}(\ker T \cap H_b). \tag{2}$$

Since $T^*|_{H_k}$ is a truncated shift of index k,

$$H_k = \bigoplus_{n=0}^{k-1} T^{*n}(\ker T \cap H_k) = \bigoplus_{n\geq 0} T^{*n}(\ker T \cap H_k) \tag{3}$$

for $k \geq 1$. By Lemma 3.2, for $H_0 = H_k$ and $H_0^\perp = H_t \ominus H_k$, we obtain $P_{H_k}(\ker T \cap H_t) = \ker T \cap H_k$ for all $k \geq 1$, which implies

$$\ker T \cap H_t = \bigoplus_{k\geq 1} (\ker T \cap H_k). \tag{4}$$

Now from (3) and (4) it follows that

$$\begin{aligned} H_t = \bigoplus_{k\geq 1} H_k &\overset{(3)}{=} \bigoplus_{k\geq 1} \bigoplus_{n\geq 0} T^{*n}(\ker T \cap H_k) \\ &= \bigoplus_{n\geq 0} T^{*n}\Big(\bigoplus_{k\geq 1}(\ker T \cap H_k)\Big) \overset{(4)}{=} \bigoplus_{n\geq 0} T^{*n}(\ker T \cap H_t) \end{aligned}$$

and finally

$$\begin{aligned} H_b \oplus H_t &= \bigoplus_{n\ge 0} T^{*n}(\ker T \cap H_b) \oplus \bigoplus_{n\ge 0} T^{*n}(\ker T \cap H_t) \\ &= \bigoplus_{n\ge 0} T^{*n}\big((\ker T \cap H_b) \oplus (\ker T \cap H_t)\big) \overset{(1)}{=} \bigoplus_{n\ge 0} T^{*n}(\ker T). \end{aligned}$$

For (ii) recall that by [5, Lemmas 4 and 5] the subspace $\bigcap_{n\ge 0} T^n T^{*n} H$ reduces T^* to an isometry. Thus $\bigcap_{n\ge 0} T^n H = \bigcap_{n\ge 0} T^n T^{*n} H \subset H_u \oplus H_b$. Conversely, since $T^{*n}|_{H_u\oplus H_b}$ is an isometry, $H_u \oplus H_b$ is orthogonal to $\ker T^{*n}$ for any $n \ge 1$. Thus $H_u \oplus H_b \subset T^n H$ for any $n \ge 0$. ∎

The theorem implies the following formulas:

$$H_u = \bigcap_{n\ge 0} T^n H \cap \bigcap_{n\ge 0} T^{*n} H, \tag{5}$$

$$H_s = \bigcap_{n\ge 0} T^{*n} H \cap \bigoplus_{n\ge 0} T^n(\ker T^*), \tag{6}$$

$$H_b = \bigcap_{n\ge 0} T^n H \cap \bigoplus_{n\ge 0} T^{*n}(\ker T), \tag{7}$$

$$H_t = \bigoplus_{n\ge 0} T^n(\ker T^*) \cap \bigoplus_{n\ge 0} T^{*n}(\ker T). \tag{8}$$

4. Decomposition of pairs of power partial isometries. The class of power partial isometries is not closed under multiplication.

Example 4.1. Let $\{e_k, f_{k+1}, g_{k+1}\}_{k=0}^{\infty}$ be a set of orthonormal vectors in some complex Hilbert space $\mathbb{H}$. Define a new Hilbert space $H = \bigoplus_{k=0}^{\infty}(\mathbb{C}e_k \oplus \mathbb{C}f_{k+1} \oplus \mathbb{C}g_{k+1})$, and operators (see Fig. 1):

$$\begin{gathered} T_1(e_k) = T_2(e_k) = e_{k+1} \quad \text{for } k \ge 1, \\ T_1(e_0) = (\sqrt{2}/2)e_1 + (\sqrt{2}/2)f_1, \\ T_2(e_0) = (\sqrt{2}/2)e_1 + (\sqrt{2}/2)g_1, \\ T_1(f_k) = f_{k+1}, \quad T_2(f_k) = 0 \qquad \text{for } k \ge 1, \\ T_1(g_k) = 0, \qquad T_2(g_k) = g_{k+1} \quad \text{for } k \ge 1. \end{gathered}$$

One can check that T_1, T_2 commute and are power partial isometries. Since

$$\ker T_2 = \bigoplus_{k=1}^{\infty} \mathbb{C}f_k, \qquad \ker T_1 = \bigoplus_{k=1}^{\infty} \mathbb{C}g_k,$$

and $T_1T_2(\sum_{i=0}^{\infty} \alpha_i e_i) = (\sqrt{2}/2)\alpha_0 e_2 + \sum_{i=1}^{\infty} \alpha_i e_{i+2}$ for all $\{\alpha_i\}_{i\ge 0} \subset \mathbb{C}$, we have $\ker T_1T_2 = \ker T_1 \oplus \ker T_2$. Thus $e_0 \perp \ker T_1T_2$. Since $\|T_1T_2e_0\| = \sqrt{2}/2 \ne \|e_0\|$ the product T_1T_2 is not a partial isometry.

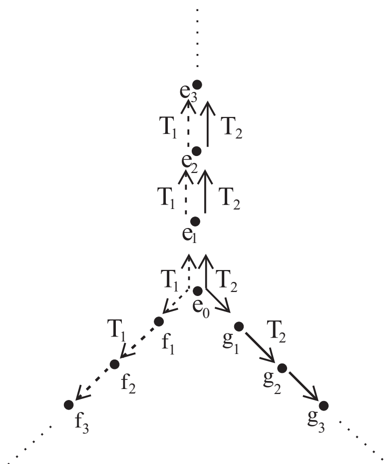


Fig. 1

Recall the result of Halmos and Wallen [5].

LEMMA 4.2. *If T_1 and T_2 are partial isometries, then a necessary and sufficient condition for T_1T_2 to be a partial isometry is that $T_1^*T_1$ and $T_2T_2^*$ commute.*

It follows that the product of doubly commuting power partial isometries is a power partial isometry. When power partial isometries only commute, the following holds:

THEOREM 4.3. *Let $T_1, T_2 \in L(H)$ be commuting power partial isometries. Then*

$$H_p = \overline{\operatorname{Span}}\Big\{H_0 \subset \bigcap_{n\geq 1} \ker(T_2^{n*}T_2^nT_1^nT_1^{*n} - T_1^nT_1^{*n}T_2^{*n}T_2^n) : \\ P_{H_0}T_i = T_iP_{H_0} \text{ for } i = 1,2\Big\}$$

is the maximal subspace reducing T_1, T_2 to a pair such that $T_1T_2|_{H_p}$ is a power partial isometry.

Proof. By Lemma 4.2 the subspace H_p reduces T_1, T_2 to a desired pair, being a closed linear span of such subspaces. The maximality is obvious. ■

Recall that a *co-isometry* is an operator whose adjoint operator is an isometry. Note the following property.

LEMMA 4.4. *Let $L \subset H$ be a subspace reducing commuting power partial isometries $T_1, T_2 \in L(H)$. If at least one of $T_1|_L, T_2|_L$ is an isometry or a co-isometry, then $T_1T_2|_L$ is a power partial isometry.*

Proof. Assume $T_1|_L$ is an isometry. Since L reduces T_2, we have $\|T_1^nT_2^n\| = \|T_2^nx\|$ for any positive integer n and any $x \in L$. On the other hand, $\|T_2^nx\| = \|x\|$ for any $x \in (\ker T_1^nT_2^n)^\perp$. Thus $T_1^nT_2^n|_{(\ker T_1^nT_2^n)^\perp}$ is an isometry for $n \geq 0$. By Theorem 3.1 the case of co-isometry can be deduced from the case of isometry for the adjoint operator. ∎

It follows that any subspace satisfying the conditions of Lemma 4.4 is a subspace of the space H_p of Theorem 4.3.

PROPOSITION 4.5. *Let $T_1, T_2 \in L(H)$ be commuting power partial isometries. Let H_p be the subspace defined in Theorem* 4.3. *The maximal subspace which reduces T_1, T_2 to a pair of isometries is*

$$H_{\mathrm{Iz}^2} = \bigcap_{n\geq 0} T_1^{*n} T_2^{*n} H_p.$$

The maximal subspace which reduces T_1, T_2 to a pair of co-isometries is

$$H_{\mathrm{CoIz}^2} = \bigcap_{n\geq 0} T_1^{n} T_2^{n} H_p.$$

Proof. We will show that H_{CoIz^2} is T_1^*-invariant. The remaining parts of the proof that H_{CoIz^2} and H_{Iz^2} reduce T_1, T_2 are either similar or trivial. Let $y \in H_{\mathrm{CoIz}^2}$ and let $n \geq 0$ be an integer. By Lemma 3.2, $P_{\ker T_1^n T_2^n}(H_p) \subset H_p$ and $P_{(\ker T_1^n T_2^n)^\perp}(H_p) = (I - P_{\ker T_1^n T_2^n})(H_p) \subset H_p$. Thus, there is a vector $x \in H_p \cap (\ker T_1^n T_2^n)^\perp$ such that $y = T_1^n T_2^n x$. Hence $\|y\| = \|T_1^n T_2^n x\| = \|x\|$. Thus,

$$\begin{aligned}\|x\| \geq \|T_1^{n-1} T_2^n x\| \geq \|T_1^n T_2^n x\| \geq \|T_1^* T_1 T_1^{n-1} T_2^n x\| \\ \geq \|T_1 T_1^* T_1 T_1^{n-1} T_2^n x\| \overset{3.1(6)}{=} \|T_1^n T_2^n x\| = \|y\| = \|x\|.\end{aligned}$$

By Theorem 3.1(3), from $\|T_1^* T_1 (T_1^{n-1} T_2^n x)\| = \|T_1^{n-1} T_2^n x\|$, we infer that $T_1^* T_1 (T_1^{n-1} T_2^n x) = T_1^{n-1} T_2^n x$.

It follows that

$$T_1^* y = T_1^* T_1^n T_2^n x = T_1^{n-1} T_2^n x = T_1^{n-1} T_2^{n-1}(T_2 x) \in T_1^{n-1} T_2^{n-1} H_p.$$

We obtain the inclusions $T_1^*(T_1^n T_2^n H_p) \subset T_1^{n-1} T_2^{n-1} H_p$ for $n \geq 1$, while for $n = 0$ we have $T_1^* H_p \subset H_p$. Finally,

$$T_1^* \Big(\bigcap_{n\geq 0} T_1^n T_2^n H_p \Big) \subset \bigcap_{n\geq 0} T_1^n T_2^n H_p.$$

Now, by Theorem 3.3(iii) and the inclusion

$$\bigcap_{n\geq 0} T_1^{*n} T_2^{*n} H_p \subset \bigcap_{n\geq 0} T_1^{*n} H \cap \bigcap_{n\geq 0} T_2^{*n} H$$

the operators $T_1|_{\bigcap_{n\geq 0} T_1^{*n} T_2^{*n} H_p}, T_2|_{\bigcap_{n\geq 0} T_1^{*n} T_2^{*n} H_p}$ are isometries. To show the maximality, assume that a subspace $L \subset H$ reduces T_1, T_2 to isometries. Then $T_1^n T_2^n|_L$ is an isometry for any positive integer n. It follows that $L \subset H_p$

and $L \perp \ker T_1^n T_2^n$ for all n. Thus

$$L \subset \bigcap_{n\geq 0} T_1^{*n} T_2^{*n} H_p.$$

Similarly one can show that $\bigcap_{n\geq 0} T_1^n T_2^n H_p$ is a maximal subspace reducing T_1, T_2 to co-isometries. ■

We needed to know that the product $T_1T_2|_{H_p}$ is a power partial isometry to prove that $\bigcap_{n\geq 0} T_1^{*n}T_2^{*n}H_p$ reduces the power partial isometries T_1, T_2. In Example 4.1, where the product is not a power partial isometry, a similar subspace $\bigcap_{n\geq 0} T_1^{*n}T_2^{*n}H$ does not reduce power partial isometries.

THEOREM 4.6. *Let $T_1, T_2 \in L(H)$ be commuting power partial isometries, and let H_p be the maximal subspace reducing T_1, T_2 such that $T_1T_2|_{H_p}$ is a power partial isometry. Then the maximal subspace reducing T_1, T_2 to a pair of unitary operators is*

$$H_{uu} = \bigcap_{n\geq 0} T_1^n T_2^n H_p \cap \bigcap_{n\geq 0} T_1^{*n} T_2^{*n} H_p.$$

Proof. By Proposition 4.5, H_{uu} reduces T_1, T_2, and the restrictions $T_1|_{H_{uu}}$, $T_2|_{H_{uu}}, T_1^*|_{H_{uu}}, T_2^*|_{H_{uu}}$ are isometries. Therefore $T_1|_{H_{uu}}, T_2|_{H_{uu}}$ are unitary. The maximality of H_{uu} is obvious by Proposition 4.5, because a unitary operator is an isometry and a co-isometry. ■

Recall [6] that a pair of commuting contractions $T_1, T_2 \in L(H)$ is said to belong to $K_{0\cdot}$ when $\lim_{n\to\infty} \|T_1^n T_2^n x\| = 0$ for every $x \in H$. A pair T_1, T_2 is said to belong to $K_{\cdot 0}$ if the pair T_1^*, T_2^* belongs to $K_{0\cdot}$. The intersection of $K_{0\cdot}$ and $K_{\cdot 0}$ is called K_{00}.

PROPOSITION 4.7. *Let $T_1, T_2 \in L(H)$ be commuting power partial isometries and H_p be the subspace defined in Theorem 4.3. Then*

$$H_{pt} = H_p \ominus \overline{\mathrm{Span}}\Big\{ \bigcap_{n\geq 0} T_1^{*n} T_2^{*n} H_p, \bigcap_{n\geq 0} T_1^n T_2^n H_p \Big\}$$

is a maximal subspace reducing T_1, T_2 such that $T_1T_2|_{H_{pt}}$ is an orthogonal sum of truncated shifts of some indices.

Proof. From Theorem 4.3, we infer that $T_1T_2|_{H_p}$ is a power partial isometry. By Theorem 3.1(3), $\{T_1^{*n}T_2^{*n}T_1^nT_2^n|_{H_p}\}_{n\in\mathbb{Z}_+}$ is a decreasing sequence of orthogonal projections onto $\mathcal{R}(T_1^{*n}T_2^{*n}|_{H_p})$. Therefore, $\{T_1^{*n}T_2^{*n}T_1^nT_2^n|_{H_p}\}_{n\in\mathbb{Z}_+}$ converges in the strong operator topology in $L(H_p)$ to the orthogonal projection onto $\bigcap_{n\geq 0} T_1^{*n}T_2^{*n}H_p$. By Proposition 4.5, the latter space is a maximal subspace of H_p reducing T_1, T_2 to isometries. By definition, H_{pt} is orthogonal to $\bigcap_{n\geq 0} T_1^{*n}T_2^{*n}H_p$ and is a subspace of H_p. Thus $T_1^{*n}T_2^{*n}T_1^nT_2^n x$ converges to zero for any $x \in H_{pt}$. Since T_1, T_2 are contractions, we have

$$\text{(9)} \quad \lim_{n\to\infty} \|T_1^n T_2^n x\| \overset{3.1(6)}{=} \lim_{n\to\infty} \|T_1^n T_2^n T_1^{*n} T_2^{*n} T_1^n T_2^n x\| \\ \le \lim_{n\to\infty} \|T_1^{*n} T_2^{*n} T_1^n T_2^n x\| = \|\lim_{n\to\infty} T_1^{*n} T_2^{*n} T_1^n T_2^n x\| = 0.$$

In the same way one can show that

$$\text{(10)} \qquad \lim_{n\to\infty} \|T_1^{*n} T_2^{*n} x\| = 0.$$

Thus the pair $T_1|_{H_{pt}}, T_2|_{H_{pt}}$ belongs to the class K_{00}. Theorem 1.2 can be applied to the product $T_1T_2|_{H_{pt}}$. Note that since $T_1|_{H_{pt}}, T_2|_{H_{pt}}$ is in K_{00}, there is no nontrivial subspace reducing $T_1T_2|_{H_{pt}}$ to an isometry or a co-isometry. Therefore, $T_1T_2|_{H_{pt}}$ is an orthogonal sum of truncated shifts of some indices. To show the maximality, consider a subspace $L \subset H$ reducing T_1, T_2 to a pair such that $T_1T_2|_L$ is an orthogonal sum of truncated shifts of some indices. Then $T_1T_2|_L$ is a power partial isometry and $L \subset H_p$. Since $T_1T_2|_L$ is a sum of truncated shifts, L is orthogonal to the subspaces $\bigcap_{n\ge0} T_1^{*n} T_2^{*n} H_p$ and $\bigcap_{n\ge0} T_1^n T_2^n H_p$ which reduce T_1T_2 to an isometry and a co-isometry, respectively. Thus $L \subset H_{pt}$, which finishes the proof. ■

We can now state the main decomposition theorem.

Theorem 4.8. *Let $T_1, T_2 \in L(H)$ be commuting power partial isometries. There is a unique decomposition*

$$H = H_{uu} \oplus H_{us} \oplus H_{ub} \oplus \bigoplus_{k\ge1} H_{uk} \\ \oplus H_{su} \oplus H_{bu} \oplus \bigoplus_{k\ge1} H_{ku} \oplus H_{\mathrm{iz}^2} \oplus H_{\mathrm{coiz}^2} \oplus H_{K_{00}} \oplus H_{\neg p},$$

where $H_{uu}, H_{us}, H_{ub}, H_{su}, H_{bu}, H_{\mathrm{iz}^2}, H_{\mathrm{coiz}^2}, H_{K_{00}}, H_{\neg p}$ and H_{uk}, H_{ku} for $k \ge 1$ are maximal subspaces reducing, respectively:

- *H_{uu} — T_1, T_2 to unitary operators,*
- *H_{us} — T_1 to a unitary operator, T_2 to a unilateral shift,*
- *H_{ub} — T_1 to a unitary operator, T_2 to a backward shift,*
- *H_{uk} — T_1 to a unitary operator, T_2 to a truncated shift of index k,*
- *H_{su} — T_1 to a unilateral shift, T_2 to a unitary operator,*
- *H_{bu} — T_1 to a backward shift, T_2 to a unitary operator,*
- *H_{ku} — T_1 to a truncated shift of index k, T_2 to a unitary operator,*
- *H_{iz^2} — T_1, T_2 to a strongly completely nonunitary pair of isometries which is a weak bi-shift,*
- *H_{coiz^2} — T_1, T_2 to a strongly completely nonunitary pair of co-isometries,*
- *$H_{K_{00}}$ — T_1, T_2 to a strongly completely nonunitary pair of contractions such that T_1T_2 is an orthogonal sum of truncated shifts of some indices,*

- *$H_{\neg p}$ is a maximal subspace not containing a proper subspace reducing T_1, T_2 to a pair such that the product is a power partial isometry.*

Proof. By Theorem 2.1, we obtain the decomposition $H = H_{uu} \oplus H_{u\neg u} \oplus H_{\neg uu} \oplus H_{\neg(uu)}$. It can be shown as in Theorem 2.1 that if one of the commuting operators is unitary, they doubly commute. Therefore, each of the pairs $T_1|_{H_{\neg uu}}, T_2|_{H_{\neg uu}}$ and $T_1|_{H_{u\neg u}}, T_2|_{H_{u\neg u}}$ doubly commutes and Theorem 1.4 can be applied. This way we obtain a more detailed result: $H = H_{uu} \oplus H_{us} \oplus H_{ub} \oplus \bigoplus_{k\geq 1} H_{uk} \oplus H_{su} \oplus H_{bu} \oplus \bigoplus_{k\geq 1} H_{ku} \oplus H_{\neg(uu)}$. Following Theorem 4.3, denote by H_p a maximal subspace reducing T_1, T_2 such that $T_1T_2|_{H_p}$ is a power partial isometry. Now, set $H_{\neg p} = H \ominus H_p$, which, by Lemma 4.4, is a subspace of $H_{\neg(uu)}$. By Proposition 4.5, there are maximal subspaces reducing T_1, T_2 to a pair of isometries (denoted H_{Iz^2}) and a pair of co-isometries (denoted H_{CoIz^2}). Now set

$$\begin{aligned} H_{\mathrm{iz}^2} &= P_{H_{\neg(uu)}} H_{\mathrm{Iz}^2} = H_{Iz^2} \ominus (H_{uu} \oplus H_{us} \oplus H_{su}), \\ H_{\mathrm{coiz}^2} &= P_{H_{\neg(uu)}} H_{\mathrm{CoIz}^2} = H_{\mathrm{CoIz}^2} \ominus (H_{uu} \oplus H_{ub} \oplus H_{bu}), \\ H_{K_{00}} &= H_{\neg(uu)} \ominus (H_{\neg p} \oplus H_{iz^2} \oplus H_{\mathrm{coiz}^2}) \\ &= H_{\neg(uu)} \cap \Big(H_p \ominus \overline{\mathrm{Span}}\{H_{Iz^2}, H_{\mathrm{CoIz}^2}\}\Big). \end{aligned}$$

By Propositions 4.5, 4.7 and Theorem 2.1, $H_{K_{00}}$ is a maximal subspace reducing T_1, T_2 to a strongly completely nonunitary pair such that $T_1T_2|_{H_{K_{00}}}$ is an orthogonal sum of truncated shifts of some indices. The subspace $H_{K_{00}}$ has been obtained as the orthogonal complement of all previous subspaces. The decomposition is complete. ∎

The decomposition in Theorem 4.8 can easily be made more detailed to yield, in the case of doubly commuting power partial isometries, the same decomposition as in Theorem 1.4. Instead of doing this, which would unreasonably increase the number of decomposing subspaces, we compare these two decompositions.

Remark 4.9. Let $T_1, T_2 \in L(H)$ be power partial isometries. Consider the decomposition defined in Theorem 4.8, using the same notation. Denote by H^{dc} a maximal subspace reducing T_1, T_2 to a doubly commuting pair. Such a subspace can be found as in Theorem 2.1. The pair $T_1|_{H^{dc}}, T_2|_{H^{dc}}$ can be decomposed using Theorem 1.4. We use the same notation for subspaces as in Theorem 1.4 with an additional upper index ^{dc} (e.g. H^{dc}_{us}). Recall that if one of the operators T_1, T_2 is unitary, they doubly commute. It follows that $H^{dc}_{u\alpha} = H_{u\alpha}$, $H^{dc}_{\alpha u} = H_{\alpha u}$ for $\alpha = u, s, b, 1, 2, \ldots$. The subspaces $H^{dc}_{bs}, H^{dc}_{sb}, H^{dc}_{ks}, H^{dc}_{kb}, H^{dc}_{bl}, H^{dc}_{sl}, H^{dc}_{kl}$ are contained in $H_{K_{00}}$ for any $k, l \in \mathbb{Z}_+$. To the operators $T_1|_{H_{\mathrm{iz}^2}}, T_2|_{H_{\mathrm{iz}^2}}$ and adjoint operators $T_1^*|_{H_{\mathrm{coiz}^2}}, T_2^*|_{H_{\mathrm{coiz}^2}}$, we can apply the decomposition for a pair of commuting isometries

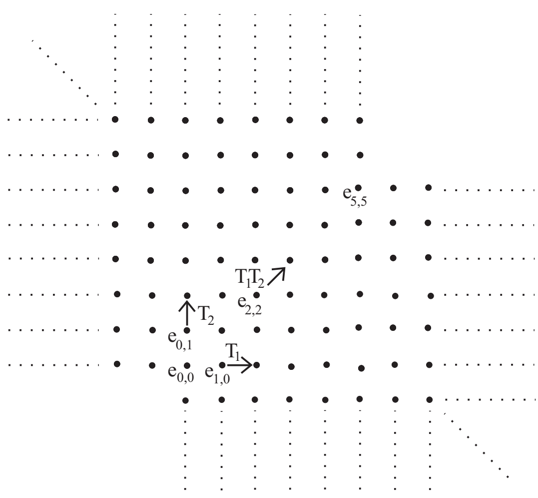
([8, Theorem 3.10] or more detailed [1, Theorem 3.13]). We obtain $H_{ss}^{dc} = H_{ss}$ and $H_{bb}^{dc} = H_{bb}$, where H_{ss} is a subspace of H_{iz^2} reducing T_1, T_2 to doubly commuting unilateral shifts. Similarly, H_{bb} is a subspace of H_{coiz^2} reducing T_1^*, T_2^* to doubly commuting unilateral shifts.

An example of an $H_{K_{00}}$-type pair of power partial isometries can be easily obtained by taking a truncated shift of some index and any other suitable operator. More interesting is the following example, where none of the operators has a truncated shift part, but their product is a sum of truncated shifts of some indices.

Example 4.10. Let $H = \bigoplus_{(i,j)\in J} \mathbb{C}e_{i,j}$ be a Hilbert space where $J = \{(i,j) \in \mathbb{Z}^2 : (i \geq 0 \text{ or } j \geq 0) \text{ and } (i \leq 5 \text{ or } j \leq 5)\}$ and $\{e_{(i,j)}\}_{(i,j)\in J}$ are orthonormal vectors. Define operators $T_1, T_2 \in L(H)$ as follows:

$$\begin{aligned} T_1(e_{5,j}) &= 0 && \text{for } j > 5, \\ T_1(e_{i,j}) &= e_{i+1,j} && \text{for the remaining } (i,j) \in J, \\ T_2(e_{i,5}) &= 0 && \text{for } i > 5, \\ T_2(e_{i,j}) &= e_{i,j+1} && \text{for the remaining } (i,j) \in J. \end{aligned}$$

One can check that the operators commute and both are orthogonal sums of a unilateral shift, a backward shift and a bilateral shift.



Thus, they are power partial isometries. We have $T_1T_2(e_{5,j}) = 0$ for every $j \geq 5$, $T_1T_2(e_{i,5}) = 0$ for every $i \geq 5$, and $T_1T_2(e_{i,j}) = e_{i+1,j+1}$ if $i \leq 4$ or $j \leq 4$. Fix $k \in \mathbb{Z}_- \cup \{0\}$. Then

$$E_k = \langle e_{k,0}; e_{k+1,1}; \ldots; e_{5,5-k}; e_{0,k}; e_{1,k+1}; \ldots; e_{5-k,5} \rangle$$

is a maximal subspace reducing T_1T_2 to a truncated shift of index $5 - k$.

Since $H = \bigoplus_{k\leq 0} E_k$, the product T_1T_2 is an orthogonal sum of truncated shifts with all indices at least 5.

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