

On group decompositions of bounded cosine sequences

by

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Abstract. A two-sided sequence $(c_n)_{n \in \mathbb{Z}}$ with values in a complex unital Banach algebra is a cosine sequence if it satisfies $c_{n+m} + c_{n-m} = 2c_n c_m$ for any $n, m \in \mathbb{Z}$ with c_0 equal to the unity of the algebra. A cosine sequence $(c_n)_{n \in \mathbb{Z}}$ is bounded if $\sup_{n \in \mathbb{Z}} \|c_n\| < \infty$. A (bounded) group decomposition for a cosine sequence $c = (c_n)_{n \in \mathbb{Z}}$ is a representation of c as $c_n = (b^n + b^{-n})/2$ for every $n \in \mathbb{Z}$, where b is an invertible element of the algebra (satisfying $\sup_{n \in \mathbb{Z}} \|b^n\| < \infty$, respectively). It is known that every bounded cosine sequence possesses a universally defined group decomposition, here referred to as a standard group decomposition. The present paper reveals various classes of bounded operator-valued cosine sequences for which the standard group decomposition is bounded. One such class consists of all bounded $\mathcal{L}(X)$ -valued cosine sequences $(c_n)_{n \in \mathbb{Z}}$, with X a complex Banach space and $\mathcal{L}(X)$ the algebra of all bounded linear operators on X , for which c_1 is scalar-type prespectral. Every bounded $\mathcal{L}(H)$ -valued cosine sequence, where H is a complex Hilbert space, falls into this class. A different class of bounded cosine sequences with bounded standard group decomposition is formed by certain $\mathcal{L}(X)$ -valued cosine sequences $(c_n)_{n \in \mathbb{Z}}$, with X a reflexive Banach space, for which c_1 is not scalar-type spectral—in fact, not even spectral. The isolation of this class uncovers a novel family of non-prespectral operators. Examples are also given of bounded $\mathcal{L}(H)$ -valued cosine sequences, with H a complex Hilbert space, that admit an unbounded group decomposition, this being different from the standard group decomposition which in this case is necessarily bounded.

1. Introduction. Let \mathbf{A} be a complex Banach algebra with a unity e and a norm $\|\cdot\|$. A two-sided sequence $c = (c_n)_{n \in \mathbb{Z}}$ with values in \mathbf{A} is called a *cosine sequence*, or a *discrete cosine function*, if

- (i) $c_{n+m} + c_{n-m} = 2c_n c_m$ for any $n, m \in \mathbb{Z}$,
- (ii) $c_0 = e$.

As is easily verified, every cosine sequence c is even: the equality $c_{-n} = c_n$ holds for all $n \in \mathbb{Z}$. Furthermore, every cosine sequence is uniquely deter-

2000 *Mathematics Subject Classification*: Primary 47D09, 47D03, 39B42; Secondary 47B40, 42C05.

Key words and phrases: cosine sequence, cosine function, Chebyshev polynomial, group decomposition, translation, unit shift, scalar-type prespectral operator, generalised scalar operator, prespectral operator, spectral operator, doubly power bounded operator.

mined by its element indexed by 1. More specifically, if c is a cosine sequence, then

$$(1.1) \quad c_n = T_{|n|}(c_1) \quad (n \in \mathbb{Z}),$$

where, for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $T_n(x)$ is the n th Chebyshev polynomial of the first kind

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} (x^2 - 1)^k.$$

The representation (1.1) follows easily from the evenness property of cosine sequences mentioned above and the recursive formulae

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

(cf. [31, Section 1.2.1]). The element c_1 is commonly referred to as the *generator* of c . Every element of \mathbf{A} generates a unique cosine sequence. The cosine sequence generated by $a \in \mathbf{A}$ is given by $c_n(a) = T_{|n|}(a)$ for $n \in \mathbb{Z}$ and is denoted $c(a)$.

An \mathbf{A} -valued cosine sequence $(c_n)_{n \in \mathbb{Z}}$ satisfying $\sup_{n \in \mathbb{Z}} \|c_n\| < \infty$ is termed *bounded*.

Let $\text{Inv}\mathbf{A}$ be the group of invertible elements of \mathbf{A} . It is readily verified that, for each $b \in \text{Inv}\mathbf{A}$, the sequence $\underline{c}(b)$ defined by

$$\underline{c}_n(b) = \frac{1}{2} (b^n + b^{-n}) \quad (n \in \mathbb{Z})$$

is a cosine sequence. A *group decomposition* for an \mathbf{A} -valued cosine sequence $c = (c_n)_{n \in \mathbb{Z}}$ is a representation of c in the form

$$(1.2) \quad c = \underline{c}(b)$$

for some $b \in \text{Inv}\mathbf{A}$. Note that, in view of the uniqueness property of cosine sequences, for (1.2) to hold it is necessary and sufficient that

$$c_1 = \underline{c}_1(b) = \frac{1}{2} (b + b^{-1}).$$

The element b in (1.2) will henceforth be referred to as the generator of the corresponding group decomposition. If b is *doubly power bounded*, i.e., if $\sup_{n \in \mathbb{Z}} \|b^n\| < \infty$, then the group decomposition is termed *bounded* and b is said to generate a bounded group decomposition.

It is known that every bounded cosine sequence with values in a complex unital Banach algebra admits a special group decomposition, here called a *standard* group decomposition [7]. A precise definition will be given later, but for now we informally characterise the standard group decomposition of a bounded cosine sequence as being reminiscent of the formula

$$\cos nt = \frac{1}{2} [(\cos t + i\sqrt{1 - \cos^2 t})^n + (\cos t + i\sqrt{1 - \cos^2 t})^{-n}]$$

$(n \in \mathbb{Z}, t \in [0, 2\pi)).$

In general, the standard group decomposition of a bounded cosine sequence may fail to be bounded. For example, there exist bounded cosine sequences with the property that all their group decompositions, including the standard one, are unbounded [7].

The main purpose of this paper is to investigate under what conditions the standard group decomposition of a bounded cosine sequence is itself bounded. From a broader perspective, the paper can be seen as an addition to a growing number of studies exploring the relationship between cosine functions (including those more general than discrete) and group representations [4, 16, 25, 35]; see also [1, Section 3.16], [17, Section 2.5], [18, Sections III.6 and III.8], [28, Section III.1.1]. While most of the interest in cosine families comes from differential equations, where cosine functions are parametrised by \mathbb{R} rather than \mathbb{Z} , the discrete cosine functions occupy a special position with regard to group decomposability. Unlike bounded cosine sequences, bounded cosine functions on \mathbb{R} fail in general to admit a group decomposition [24, 26] (although for some, a group decomposition always exists; this is the case, for example, with any bounded strongly continuous cosine function taking values in $\mathcal{L}(X)$, where X is a UMD space [8]). Ref. [7] sheds light on why there is a difference between \mathbb{Z} and \mathbb{R} in relation to cosine families, by characterising Abelian groups G with the property that every bounded cosine function on G admits a (bounded) group decomposition.

The rest of the article is laid out as follows. Following Section 2 that contains operator-theoretic prerequisites, Section 3 presents a simplified construction of the standard group decomposition for a bounded cosine sequence. Section 4 establishes that if a bounded cosine sequence with values in $\mathcal{L}(X)$, where X is a complex Banach space, is generated by a scalar-type prespectral operator, then its standard group decomposition is bounded. One consequence of this result is the fact that every bounded $\mathcal{L}(H)$ -valued cosine sequence, where H is a complex Hilbert space, has a bounded standard decomposition. The next three sections aim to show that a bounded cosine sequence with bounded standard group decomposition can be generated by an operator that is *not* scalar-type prespectral. Relevant examples hinge on identification of a novel family of non-prespectral operators. More specifically, following Section 5 which is of technical character, it is first shown in Section 6 that, when $1 < p < \infty$, the operator A_p defined as half the sum of the backward and forward unit shifts in $l^p(\mathbb{Z})$ generates a bounded cosine sequence with bounded standard group decomposition. Next in Section 7 it is shown that A_p is not prespectral when $1 \leq p \leq \infty$, $p \neq 2$. Given that an operator which is not prespectral is much less scalar-type prespectral, it is then concluded that A_p with $1 < p < \infty$, $p \neq 2$ is not scalar-type prespectral and generates a cosine sequence with bounded standard group

decomposition. Interestingly, this result not only has implications for cosine sequences, but also relies upon manipulations with cosine sequences. The final Section 8 reveals that a bounded $\mathcal{L}(H)$ -valued cosine sequence, with H a complex Hilbert space, may admit an unbounded group decomposition, this being different from the standard group decomposition which, by the result on decomposability of bounded cosine sequences in Hilbert space mentioned earlier, is necessarily bounded in this case. Two examples are given, the simpler one involving a cosine sequence generated by a spectral operator, and the more complicated one involving a cosine sequence generated by a non-spectral operator. In neither case can the generator be scalar-type pre-spectral, but in the second example the generator turns out to be generalised scalar.

2. Preliminaries. In this section, we establish all general operator-theoretic definitions and facts that will be needed later on.

Suppose that X is a Banach space. The dual space of X is denoted by X' . The value of a functional $x' \in X'$ at $x \in X$ is written $\langle x, x' \rangle$. $\mathcal{L}(X)$ is the Banach algebra of all bounded linear operators on X . The identity operator on X is denoted I_X .

Recall that a subset $\Gamma \subset X'$ is *total* if, for any $x \in X$, $\langle x, x' \rangle = 0$ for all $x' \in \Gamma$ implies $x = 0$.

Let \mathcal{M} be a σ -algebra of subsets of a set Ω and let Γ be a total subset of X' . A *spectral measure of class Γ* is a map $E: \mathcal{M} \rightarrow \mathcal{L}(X)$ such that

- (i) $E(\emptyset) = 0$ and $E(\Omega) = I_X$,
- (ii) $E(\omega \cap \omega') = E(\omega)E(\omega')$ for any $\omega, \omega' \in \mathcal{M}$,
- (iii) $\omega \mapsto \langle E(\omega)x, x' \rangle$ is σ -additive for any $x \in X$ and $x' \in \Gamma$,
- (iv) $\sup_{\omega \in \Omega} \|E(\omega)\| < \infty$.

It follows from the Orlicz–Pettis theorem that any spectral measure of class X' is strongly σ -additive—that is, the function $\mathcal{M} \ni \omega \mapsto E(\omega)x \in E$ is σ -additive for each $x \in X$.

The spectrum of an operator $T \in \mathcal{L}(X)$ is denoted by $\sigma(T)$. For $T \in \mathcal{L}(X)$ and $Y \subset X$ such that $T(Y) \subset Y$, $T|_Y$ denotes the restriction of T to Y .

The Borel σ -algebra of a topological space Y is designated by $\mathcal{B}(Y)$.

Following Dunford [12] (cf. also [11, 13, 14]), an operator $T \in \mathcal{L}(X)$ is called *prespectral of class Γ* if there exists a spectral measure $E: \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{L}(X)$ of class Γ such that

- (i) $TE(\omega) = E(\omega)T$ for each $\omega \in \mathcal{B}(\mathbb{C})$,
- (ii) $\sigma(T|_{E(\omega)X}) \subset \bar{\omega}$ for each $\omega \in \mathcal{B}(\mathbb{C})$, with the bar denoting the set closure.

The spectral measure $E: \mathcal{M} \rightarrow \mathcal{L}(X)$ of class Γ satisfying (i) and (ii) is uniquely determined by T and is called the *resolution of the identity of class Γ* for T [11, Theorem 5.13]. Any resolution of the identity E for a prespectral operator $T \in \mathcal{L}(X)$, of some class, is supported on $\sigma(T)$ in the sense that $E(\sigma(T)) = I_X$. In general, a prespectral operator of some class can also be a prespectral operator of another class, with a possibly different resolution of the identity [19] (see also [11, Example 5.35]).

If $T \in \mathcal{L}(X)$ has the form

$$T = \int_{\sigma(T)} \lambda dE(\lambda),$$

where $E: \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{L}(X)$ is a spectral measure of class Γ , then T is a prespectral operator of class Γ and E is its resolution of the identity of class Γ . In this case, T is termed a *scalar-type operator of class Γ* .

An operator $Q \in \mathcal{L}(X)$ is called *quasinilpotent* if $\lim_{n \rightarrow \infty} \|Q^n\|^{1/n} = 0$, which is equivalent to $\sigma(Q) = \{0\}$.

If $T \in \mathcal{L}(X)$ is a prespectral operator with resolution of the identity E of class Γ and if

$$(2.1) \quad S = \int_{\sigma(T)} \lambda dE(\lambda), \quad Q = T - S,$$

then S is a scalar-type operator with resolution of the identity E of class Γ and Q is a quasinilpotent operator commuting with $\{E(\omega) \mid \omega \in \mathcal{B}(\mathbb{C})\}$; moreover $\sigma(T) = \sigma(S)$. This characterisation of prespectral operators has a partial converse: If $S \in \mathcal{L}(X)$ is a scalar-type operator with resolution of the identity E of class Γ and Q is a quasinilpotent operator commuting with $\{E(\omega) \mid \omega \in \mathcal{B}(\mathbb{C})\}$, then $S+Q$ is prespectral with resolution of the identity E of class Γ ; moreover, $\sigma(S+Q) = \sigma(S)$ [11, Theorem 5.15].

The decomposition $T = S + Q$ in (2.1) is called the *Jordan decomposition* of T . It does not depend on the spectral measure E used to define S (and, effectively, also Q)—all spectral measures for which T is prespectral yield the same S and Q . This follows from the fact that if an operator $T \in \mathcal{L}(X)$, prespectral or not, can be represented as $T = S + Q = S_0 + Q_0$, where $S, S_0 \in \mathcal{L}(X)$ are scalar-type prespectral, and $Q, Q_0 \in \mathcal{L}(X)$ are quasinilpotent, satisfying $SQ = QS$ and $S_0Q_0 = Q_0S_0$, then $S = S_0$ and $Q = Q_0$ [11, Theorem 5.23]. If $T \in \mathcal{L}(X)$ can be written as $T = S + Q$ with $S \in \mathcal{L}(X)$ of scalar type and $Q \in \mathcal{L}(X)$ quasinilpotent with $SQ = QS$, then S is said to be the *scalar part* of T and Q is its *radical part*.

An operator $T \in \mathcal{L}(X)$ is a *spectral operator* if it is a prespectral operator of class X' . In this case, T has a unique resolution of the identity [11, Theorem 6.7]. An operator $T \in \mathcal{L}(X)$ is spectral if and only if it has the form $T = S + Q$, where $S \in \mathcal{L}(X)$ is a scalar-type spectral operator and $Q \in \mathcal{L}(X)$ is a quasinilpotent operator which commutes with S . Then

T and S have the same spectrum and the same resolution of the identity [11, Theorem 6.8].

Let $C^\infty(\mathbb{C})$ be the algebra of all infinitely differentiable complex-valued functions on the complex plane, endowed with the topology of uniform convergence on compact sets for the functions and all their partial derivatives. Following Foiaş [21] (see also [9, 29, 36]), an operator $T \in \mathcal{L}(X)$ is called *generalised scalar* if it admits a functional calculus on $C^\infty(\mathbb{C})$; the latter means that there exists a continuous \mathbb{C} -algebra homomorphism $\Phi: C^\infty(\mathbb{C}) \rightarrow \mathcal{L}(X)$ for which $\Phi(1) = I_X$ and $\Phi(\text{id}_{\mathbb{C}}) = T$, with $\text{id}_{\mathbb{C}}$ the identity function on \mathbb{C} . Any Φ with the above properties can be viewed as an $\mathcal{L}(X)$ -valued distribution with compact support—a *spectral distribution* for T . A generalised scalar operator, even as simple as the identity operator, may possess several different spectral distributions (cf. [9, p. 94], [21]).

Every scalar-type prespectral operator is generalised scalar: If $T \in \mathcal{L}(X)$ is of scalar type with a resolution of the identity E , then the mapping $\Phi: C^\infty(\mathbb{C}) \rightarrow \mathcal{L}(X)$ given by

$$\Phi(f) = \int_{\sigma(T)} f(\lambda) dE(\lambda) \quad (f \in C^\infty(\mathbb{C}))$$

is a continuous algebra homomorphism such that $\Phi(1) = I_X$ and $\Phi(\text{id}_{\mathbb{C}}) = T$; see [11, Proposition 5.9] for a more general result.

One can also define a *generalised spectral operator*, but this rather involved notion will not intervene in this paper [9, 29, 36].

3. Standard group decomposition. For each x in the interval $[-1, 1]$, define

$$\phi(x) = \sqrt{1 - x^2} = |\sin(\arccos x)|.$$

By considering the Fourier cosine series expansion of the function $[-\pi, \pi) \ni t \mapsto |\sin t| \in \mathbb{R}$, one can easily check that

$$\phi(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} T_{2k}(x) = -\frac{2}{\pi} \sum_{k \in \mathbb{Z}} \frac{1}{4k^2 - 1} T_{2|k|}(x).$$

Here both series are absolutely convergent, this being a consequence of the representation

$$T_n(x) = \cos(n \arccos x) \quad (n \in \mathbb{N}_0, x \in [-1, 1])$$

(cf. [31, Section 1.2.1]) guaranteeing that the polynomials $T_n(x)$ assume values from $[-1, 1]$ whenever $|x| \leq 1$. Set

$$\psi(x) = x + i\phi(x) = x + i\sqrt{1 - x^2}$$

for $|x| \leq 1$. It is clear that the function ψ maps $[-1, 1]$ homeomorphically onto the closed upper unit semicircle $\mathbb{T}^+ = \{z \in \mathbb{C} \mid |z| = 1, \text{Im } z \geq 0\}$.

Let \mathbf{A} be a complex Banach algebra with a unity e . Suppose that an element $a \in \mathbf{A}$ generates a bounded cosine sequence. Then the series $\sum_{k \in \mathbb{Z}} (4k^2 - 1)^{-1} c_{2k}(a)$ is absolutely convergent and, bearing in mind (1.1), one can define actions of ϕ and ψ on a by setting

$$\phi(a) = -\frac{2}{\pi} \sum_{k \in \mathbb{Z}} \frac{1}{4k^2 - 1} c_{2k}(a), \quad \psi(a) = a + i\phi(a).$$

The importance of both these actions lies in the following result.

THEOREM 3.1. *Let \mathbf{A} be a complex unital Banach algebra. If $a \in \mathbf{A}$ generates a bounded cosine sequence c , then $\psi(a)$ is invertible and generates a group decomposition for c .*

Given an \mathbf{A} -valued bounded cosine sequence c generated by $a \in \mathbf{A}$, we define the *standard group decomposition* of c as the group decomposition generated by $\psi(a)$.

Theorem 3.1 was first established in [7]. A streamlined proof of the theorem appeared in [4]. Below we present an even simpler proof.

Proof of Theorem 3.1. Suppose that $a \in \mathbf{A}$ generates a bounded cosine sequence. We clearly have

$$(3.1) \quad \phi^2(x) = 1 - x^2$$

for $|x| \leq 1$. We shall show that $\phi^2(a) = e - a^2$, where e is the unity of \mathbf{A} . The latter identity rewritten as $\psi(a)(a - i\phi(a)) = e$ makes it obvious that $a - i\phi(a)$ is the inverse of $\psi(a)$ and that

$$(3.2) \quad a = \frac{1}{2} (\psi(a) + \psi(a)^{-1}),$$

this being all what is needed to accomplish the proof.

We have

$$\begin{aligned} (3.3) \quad \phi^2(a) &= \frac{4}{\pi^2} \sum_{k, l \in \mathbb{Z}} \frac{1}{(4k^2 - 1)(4l^2 - 1)} c_{2k}(a) c_{2l}(a) \\ &= \frac{4}{\pi^2} \cdot \frac{1}{2} \sum_{k, l \in \mathbb{Z}} \frac{1}{(4k^2 - 1)(4l^2 - 1)} (c_{2(k+l)}(a) + c_{2(k-l)}(a)) \\ &= \frac{4}{\pi^2} \sum_{k, n \in \mathbb{Z}} \frac{1}{(4k^2 - 1)(4(n-k)^2 - 1)} c_{2n}(a) \\ &= \frac{4}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(4k^2 - 1)^2} c_0(a) \\ &\quad + \frac{4}{\pi^2} \sum_{n \in \mathbb{N}} \left[\sum_{k \in \mathbb{Z}} \frac{1}{4k^2 - 1} \left[\frac{1}{4(n-k)^2 - 1} + \frac{1}{4(n+k)^2 - 1} \right] \right] c_{2n}(a). \end{aligned}$$

By the same token,

$$(3.4) \quad \phi^2(x) = \frac{4}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(4k^2 - 1)^2} T_0(x) \\ + \frac{4}{\pi^2} \sum_{n \in \mathbb{N}} \left[\sum_{k \in \mathbb{Z}} \frac{1}{4k^2 - 1} \left[\frac{1}{4(n-k)^2 - 1} + \frac{1}{4(n+k)^2 - 1} \right] \right] T_{2n}(x)$$

for $|x| \leq 1$. As $T_0(x) = 1$ and $T_2(x) = 2x^2 - 1$, (3.1) can be rewritten in the form

$$(3.5) \quad \phi^2(x) = \frac{1}{2} T_0(x) - \frac{1}{2} T_2(x).$$

In view of the orthogonality relations

$$\int_{-1}^1 T_i(x) T_j(x) (1-x^2)^{-1/2} dx = 0 \quad (i \neq j)$$

(see [31, Section 4.2.2]), the polynomial expansions in (3.4) and (3.5) coincide, so

$$\frac{4}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(4k^2 - 1)^2} = \frac{1}{2}, \\ \frac{4}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{1}{(4k^2 - 1)} \left[\frac{1}{4(n-k)^2 - 1} + \frac{1}{4(n+k)^2 - 1} \right] = \begin{cases} -\frac{1}{2} & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

With these identities, (3.3) can now be rewritten as

$$\phi^2(a) = \frac{1}{2} c_0(a) - \frac{1}{2} c_2(a) = e - a^2,$$

as was to be shown. ■

4. A condition for boundedness. This section presents a sufficient condition for the standard group decomposition of a bounded $\mathcal{L}(X)$ -valued cosine sequence, where X is a complex Banach space, to be bounded. It requires that the generator of a cosine sequence should be of scalar type. As will be shown later, the condition is not a necessary one. Amongst its consequences, the most fundamental is that any bounded $\mathcal{L}(H)$ -valued cosine sequence, where H is a complex Hilbert space, has a bounded standard group decomposition.

We begin with two preliminary results.

Let \mathbb{T} denote the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. For an element a of a complex Banach algebra, let $\sigma(a)$ denote the spectrum of a .

PROPOSITION 4.1. *Let \mathbf{A} be a complex unital Banach algebra. If $b \in \text{Inv}\mathbf{A}$ is such that $\underline{c}(b)$ is bounded, then $\sigma(b) \subset \mathbb{T}$.*

Proof. Suppose that for $b \in \text{Inv}\mathbf{A}$ the cosine sequence $\underline{c}(b)$ is bounded. Let \mathbf{A}_0 be the smallest complex Banach algebra containing b and the unity of \mathbf{A} . Clearly, \mathbf{A}_0 is commutative. Since the spectrum of b relative to \mathbf{A} is contained in the spectrum of b relative to \mathbf{A}_0 , we may assume, replacing \mathbf{A} by \mathbf{A}_0 if necessary, that \mathbf{A} is commutative. Let $\Delta(\mathbf{A})$ be the set of all complex-valued homomorphisms on \mathbf{A} . Fix $\tau \in \Delta(\mathbf{A})$ arbitrarily. The invertibility of b implies that $\tau(b) \neq 0$. Furthermore,

$$(4.1) \quad \tau(\underline{c}_n(b)) = \frac{1}{2} (\tau(b)^n + \tau(b)^{-n})$$

for each $n \in \mathbb{N}$. Since all members of $\Delta(\mathbf{A})$ have unit norm, it follows that

$$(4.2) \quad |\tau(\underline{c}_n(b))| \leq \sup_{n \in \mathbb{N}} \|\underline{c}_n(b)\|.$$

From this we deduce that $|\tau(b)| = 1$; indeed, should $|\tau(b)| \neq 1$ hold, the right-hand side of (4.1) would diverge in modulus to infinity as $n \rightarrow \infty$, contradicting (4.2). To complete the proof, it suffices to invoke the identity $\sigma(b) = \{\tau(b) : \tau \in \Delta(\mathbf{A})\}$ (see e.g. [5, Chapter 1, §16, Proposition 9]). ■

PROPOSITION 4.2. *Let \mathbf{A} be a complex unital Banach algebra. If $a \in \mathbf{A}$ generates a bounded cosine sequence, then $\sigma(a) \subset [-1, 1]$.*

Proof. By Theorem 3.1 and Proposition 4.1, $\psi(a)$ has spectrum contained in \mathbb{T} . As the function $z \mapsto (z + z^{-1})/2$ maps \mathbb{T} onto $[-1, 1]$, the result follows immediately from (3.2) and the spectral mapping theorem (see e.g. [5, Chapter 1, §7, Theorem 4]). ■

We can now pass to more substantial results.

THEOREM 4.3. *Let X be a complex Banach space. If $B \in \text{Inv}\mathcal{L}(X)$ is scalar-type prespectral and $\underline{c}(B)$ is bounded, then B is doubly power bounded.*

Proof. By Proposition 4.1, the spectrum of B is contained in \mathbb{T} . Let E be a resolution of the identity for B , and let $K_E = \sup_{\omega \in \mathcal{B}(\sigma(B))} \|E(\omega)\|$. Then, for each $n \in \mathbb{Z}$, $B^n = \int_{\sigma(B)} \lambda^n dE(\lambda)$. Since

$$\left\| \int_{\omega} f(\lambda) dE(\lambda) \right\| \leq 4K_E \sup_{\lambda \in \omega} \|f(\lambda)\|$$

for any $\omega \in \mathcal{B}(\mathbb{C})$ and any complex-valued bounded Borel function f on ω [11, p. 120], it follows that

$$\|B^n\| \leq 4K_E \sup_{\lambda \in \sigma(B)} |\lambda^n| \leq 4K_E \sup_{\lambda \in \mathbb{T}} |\lambda^n| = 4K_E$$

for each $n \in \mathbb{Z}$. ■

THEOREM 4.4. *Let X be a complex Banach space. If $A \in \mathcal{L}(X)$ is a scalar-type operator of class Γ that generates a bounded cosine sequence, then $\psi(A)$ is a scalar-type operator of class Γ that is doubly power bounded.*

Proof. By Proposition 4.2, we have $\sigma(A) \subset [-1, 1]$. Let E be the resolution of the identity of class Γ for A . Then, on account of $\psi([-1, 1]) = \mathbb{T}^+$,

$$\psi(A) = \int_{\sigma(A)} \psi(\lambda) dE(\lambda) = \int_{\mathbb{T}^+} z dF(z),$$

where $F: \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{L}(E)$ is the spectral measure of class Γ defined by

$$F(\omega) = E(\psi^{-1}(\omega)) \quad (\omega \in \mathcal{B}(\mathbb{C})).$$

Thus $\psi(A)$ is a scalar-type operator of class Γ . By Theorem 3.1, $\underline{c}(\psi(A)) = c(A)$ and $c(A)$ is bounded by assumption. Now Theorem 4.3 guarantees that $\psi(A)$ is doubly power bounded. ■

Putting the last two theorems together yields the following fundamental result.

THEOREM 4.5. *Let X be a complex Banach space. If a bounded $\mathcal{L}(X)$ -valued cosine sequence is generated by a scalar-type prespectral operator, then the standard group decomposition for this cosine sequence is bounded.*

It is well known that the generator of a bounded $\mathcal{L}(H)$ -valued cosine sequence, where H is a complex Hilbert space, is similar to a normal (in fact, hermitian) operator [6, Theorem 2.1]. This fact combined with Theorem 4.5 and the elementary result that any operator similar to a normal operator is scalar-type spectral leads to the following assertion.

THEOREM 4.6. *The standard group decomposition of a bounded $\mathcal{L}(H)$ -valued cosine sequence, where H is a complex Hilbert space, is bounded.*

5. A spectrality result. Here we record a result that will be of relevance in what follows. It should be compared with Theorem 4.4.

Given a linear subspace Y of a Banach space X , we denote by Y^\perp the annihilator of Y in X' defined as

$$Y^\perp = \{x' \in X' \mid \langle x, x' \rangle = 0 \text{ for all } x \in Y\}.$$

THEOREM 5.1. *Let X be a complex Banach space. If $A \in \mathcal{L}(X)$ is prespectral of class Γ and generates a bounded cosine sequence, then $\psi(A)$ is prespectral of class Γ .*

Proof. Let E be the spectral resolution of the identity for A of class Γ . Then setting

$$F(\omega) = E(\psi^{-1}(\omega)) \quad (\omega \in \mathcal{B}(\mathbb{C}))$$

defines a spectral measure of class Γ . As the range of ψ coincides with \mathbb{T}^+ , F is supported on \mathbb{T}^+ . Since $\{E(\omega) \mid \omega \in \mathcal{B}(\mathbb{C})\}$ commutes with A , it follows that $\{F(\omega) \mid \omega \in \mathcal{B}(\mathbb{C})\}$ commutes with $B = \psi(A)$. The proof will

be complete once we show that

$$(5.1) \quad \sigma(B|_{F(\omega)X}) \subset \bar{\omega}$$

for each $\omega \in \mathcal{B}(\mathbb{T}^+)$.

Fix $\omega \in \mathcal{B}(\mathbb{T}^+)$ arbitrarily. For each $n \in \mathbb{N}$, let

$$\begin{aligned} \omega_n^{(-1)} &= \omega \cap \psi([-1, -1 + n^{-1}]), \\ \omega_n^{(0)} &= \omega \cap \psi([-1 + n^{-1}, 1 - n^{-1}]), \\ \omega_n^{(1)} &= \omega \cap \psi((1 - n^{-1}, 1]). \end{aligned}$$

We have

$$B|_{F(\omega)X} = B|_{F(\omega_n^{(-1)})X} \oplus B|_{F(\omega_n^{(0)})X} \oplus B|_{F(\omega_n^{(1)})X}$$

and further

$$(5.2) \quad \sigma(B|_{F(\omega)X}) = \sigma(B|_{F(\omega_n^{(-1)})X}) \cup \sigma(B|_{F(\omega_n^{(0)})X}) \cup \sigma(B|_{F(\omega_n^{(1)})X})$$

(cf. [11, Proposition 1.37]). As A is prespectral, we see that

$$(5.3) \quad \sigma(A|_{F(\omega_n^{(0)})X}) \subset \overline{\psi^{-1}(\omega_n^{(0)})} = \psi^{-1}(\overline{\omega_n^{(0)}}),$$

where the last equality results from ψ being a homeomorphism. Furthermore, $A|_{F(\omega_n^{(0)})X}$ is prespectral with resolution of the identity $\{E(\omega)|_{F(\omega_n^{(0)})X} \mid \omega \in \mathcal{B}(\mathbb{C})\}$ of class $\Gamma/(F(\omega_n^{(0)})X)^\perp$; here the quotient space $\Gamma/(F(\omega_n^{(0)})X)^\perp$ is a total subspace of $X'/(F(\omega_n^{(0)})X)^\perp$, the latter being identified with $(F(\omega_n^{(0)})X)'$ (cf. [11, Theorem 14.2]). Since $\psi^{-1}(\overline{\omega_n^{(0)}}) \subset [-1 + n^{-1}, 1 - n^{-1}]$ and since ψ has a holomorphic extension to a neighbourhood of $[-1 + n^{-1}, 1 - n^{-1}]$, namely

$$\psi(z) = z + \sqrt{1 - z^2} \quad (z \in \mathbb{C}, |z| < 1),$$

where $\sqrt{1 - z^2}$ employs the branch of the square root function $w \mapsto \sqrt{w}$ defined for all $w \in \mathbb{C}$ with $\operatorname{Re} w > 0$, we conclude that $\psi(A|_{F(\omega_n^{(0)})X})$ is prespectral of class $\Gamma/(F(\omega_n^{(0)})X)^\perp$ (cf. [11, Theorem 5.16]). By (5.3) and the spectral mapping theorem (see e.g. [5, Chapter 1, §7, Theorem 4]),

$$\psi(A|_{F(\omega_n^{(0)})X}) \subset \overline{\omega_n^{(0)}}.$$

This together with the identity

$$B|_{F(\omega_n^{(0)})X} = \psi(A|_{F(\omega_n^{(0)})X})$$

implies

$$\sigma(B|_{F(\omega_n^{(0)})X}) \subset \overline{\omega_n^{(0)}},$$

whence, in particular,

$$(5.4) \quad \sigma(B|_{F(\omega_n^{(0)})X}) \subset \bar{\omega}.$$

For $x \in \mathbb{C}$ and $r > 0$, let $D(x, r) = \{z \in \mathbb{C} \mid |z - x| \leq r\}$. As we shall see, there exist two sequences $(\delta_n^-)_{n \in \mathbb{N}}$ and $(\delta_n^+)_{n \in \mathbb{N}}$ of positive numbers such that

$$(5.5) \quad \sigma(B|_{F(\omega_n^{(-1)})X}) \subset D(-1, \delta_n^-) \quad \text{and} \quad \lim_{n \rightarrow \infty} \delta_n^- = 0,$$

$$(5.6) \quad \sigma(B|_{F(\omega_n^{(1)})X}) \subset D(1, \delta_n^+) \quad \text{and} \quad \lim_{n \rightarrow \infty} \delta_n^+ = 0.$$

Assuming this for now, let $\Lambda = \{-1, 1\} \setminus \bar{\omega}$. Observe that if Λ is non-void, then, for each $\lambda \in \Lambda$, there exists $n_\lambda \in \mathbb{N}$ such that $\omega_n^{(\lambda)} = \emptyset$ whenever $n \geq n_\lambda$; indeed, otherwise, for some $\lambda \in \Lambda$, there would exist a sequence $(x_{n_k})_{k \in \mathbb{N}}$ with $x_{n_k} \in \omega_{n_k}^{(\lambda)}$ and $\lim_{k \rightarrow \infty} n_k = \infty$, implying that $\lambda \in \bar{\omega}$, which is a contradiction. Let

$$n' = \begin{cases} \max\{n_\lambda \mid \lambda \in \Lambda\} & \text{if } \Lambda \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, if Λ is non-void and $\lambda \in \Lambda$, then $\omega_n^{(\lambda)} = \emptyset$ whenever $n \geq n'$. Hence $\bigcup_{\lambda \in \Lambda} \sigma(B|_{F(\omega_n^{(\lambda)})X}) = \emptyset$ for $n \geq n'$ regardless of whether Λ is empty or not, and (5.2) reduces to

$$\sigma(B|_{F(\omega)X}) = \sigma(B|_{F(\omega_n^{(0)})X}) \cup \bigcup_{\lambda \in \{-1, 1\} \setminus \Lambda} \sigma(B|_{F(\omega_n^{(\lambda)})X})$$

for $n \geq n'$. This in conjunction with (5.4)–(5.6) implies that

$$\sigma(B|_{F(\omega)X}) \subset \bar{\omega} \cup (\{-1, 1\} \setminus \Lambda).$$

As $\{-1, 1\} \setminus \Lambda = \{-1, 1\} \cap \bar{\omega} \subset \bar{\omega}$, we immediately obtain (5.1).

We are left with establishing the existence of $(\delta_n^-)_{n \in \mathbb{N}}$ and $(\delta_n^+)_{n \in \mathbb{N}}$. We shall confine ourselves to indicating how to construct the first sequence, the construction of the other being completely analogous. For each $n \in \mathbb{N}$, let

$$\begin{aligned} X_n &= F(\omega_n^{(-1)})X, & I_n &= I_{X_n}, \\ A_n &= A|_{X_n}, & B_n &= B|_{X_n}. \end{aligned}$$

Given $T \in \mathcal{L}(X)$, denote by $r(T)$ the spectral radius of T . We shall show that

$$(5.7) \quad \lim_{n \rightarrow \infty} r(I_n + B_n) = 0.$$

With this formula, the desired sequence is immediately obtained by setting $\delta_n^- = r(I_n + B_n)$. Since $X_n = E(\psi^{-1}(\omega_n^{(-1)}))X$ and $\psi^{-1}(\omega_n^{(-1)}) \subset [-1, -1 + n^{-1}]$, it follows that $\sigma(A_n) \subset [-1, -1 + n^{-1}]$. Hence

$$(5.8) \quad r(I_n + A_n) \leq \frac{1}{n}.$$

Given that $B_n = A_n + i\phi(A_n)$ and that A_n and $\phi(A_n)$ commute, we have

$$(5.9) \quad r(I_n + B_n) \leq r(I_n + A_n) + r(\phi(A_n))$$

(cf. [5, p. 19]). Now $\phi(A_n)^2 = I_n - A_n^2$ and so

$$\begin{aligned} r(\phi(A_n))^2 &= r(\phi(A_n)^2) = r((I_n - A_n)(I_n + A_n)) \\ &\leq r(I_n - A_n)r(I_n + A_n) \leq (1 + \|A_n\|)r(I_n + A_n) \\ &\leq (1 + \|A\|)r(I_n + A_n), \end{aligned}$$

where the first equality follows from the spectral radius formula and the first inequality follows from the fact that $I_n - A_n$ and $I_n + A_n$ commute (cf. [5, p. 19]). Putting this together with (5.8) and (5.9) yields (5.7) immediately. ■

6. Cosine sequences generated by translations. This section is concerned with certain cosine sequences that are naturally defined in terms of translation operators on the l^p spaces over the additive group of integers. We isolate from among these cosine sequences those that have a bounded standard group decomposition.

For $1 \leq p \leq \infty$, let $l^p(\mathbb{Z})$ be the space of all complex-valued two-sided sequences, p -summable when $p < \infty$ and bounded when $p = \infty$, with the standard $\|\cdot\|_p$ norm. Given a two-sided sequence ξ and $k \in \mathbb{Z}$, the *translate* of ξ by k , denoted $\mathsf{T}_k\xi$, is the sequence

$$(\mathsf{T}_k\xi)_n = \xi_{n+k} \quad (n \in \mathbb{Z}).$$

For $1 \leq p \leq \infty$ and $k \in \mathbb{Z}$, let $\mathsf{T}_k^{(p)}$ be the operator $l^p(\mathbb{Z}) \ni \xi \mapsto \mathsf{T}_k\xi \in l^p(\mathbb{Z})$. If p is understood, $\mathsf{T}_k^{(p)}$ will be abbreviated to T_k . $\mathsf{T}_k^{(p)}$ is a surjective linear isometry and its inverse is $\mathsf{T}_{-k}^{(p)}$. $\mathsf{T}_1^{(p)}$ and $\mathsf{T}_{-1}^{(p)}$ are known as the *backward unit shift* and *forward unit shift* in $l^p(\mathbb{Z})$, respectively. Let

$$\mathsf{A}_p = \frac{1}{2} (\mathsf{T}_1^{(p)} + \mathsf{T}_{-1}^{(p)}).$$

Consider the cosine sequence generated by A_p . Obviously,

$$c_n(\mathsf{A}_p) = \frac{1}{2} (\mathsf{T}_n^{(p)} + \mathsf{T}_{-n}^{(p)})$$

for each $n \in \mathbb{Z}$. By construction, $\mathsf{T}_1^{(p)}$ generates a group decomposition for $c(\mathsf{A}_p)$ and, since $\|\mathsf{T}_1^{(p)}\| = 1$, this group decomposition is bounded. We shall show that the standard group decomposition of $c(\mathsf{A}_p)$ is bounded only if $1 < p < \infty$. We start with the following result.

THEOREM 6.1. *If $1 < p < \infty$, then the standard group decomposition of $c(\mathsf{A}_p)$ is bounded.*

Proof. For simplicity, we relabel $\psi(\mathsf{A}_p)$ as B_p . To prove that B_p is doubly power bounded, observe first that, for each $\xi \in l^2(\mathbb{Z}) \cap l^p(\mathbb{Z})$,

$$\widehat{\mathsf{A}_p\xi}(t) = (\cos t)\widehat{\xi}(t)$$

for almost every (a.e.) $t \in [0, 2\pi)$. Here $\widehat{\xi}$ denotes the Fourier transform of ξ —the element of $L^2([0, 2\pi))$ defined by

$$\widehat{\xi}(t) = \sum_{k \in \mathbb{Z}} \xi(k) e^{-ikt} \quad \text{for a.e. } t \in [0, 2\pi),$$

with the right-hand side understood as the limit in the L^2 -norm of the sequence $(d_n(\xi))_{n \in \mathbb{N}}$ in $L^2([0, 2\pi))$ given by

$$d_n(\xi)(t) = \sum_{k=-n}^n \xi(k) e^{-ikt} \quad (t \in [0, 2\pi)).$$

It is easily verified that

$$(6.1) \quad \widehat{\mathbb{B}_p \xi}(t) = (\cos t + i|\sin t|)\widehat{\xi}(t) \quad \text{for a.e. } t \in [0, 2\pi).$$

Let $f: [0, 2\pi) \rightarrow \mathbb{R}$ be given by

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < \pi, \\ -1 & \text{if } \pi \leq t < 2\pi. \end{cases}$$

Since f is bounded, Plancherel's theorem ensures the existence of a bounded linear operator $M_f: l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$ satisfying

$$\widehat{M_f \xi} = f \widehat{\xi} \quad (\xi \in l^2(\mathbb{Z})).$$

Since, in addition, f is of bounded variation, it follows from a result of Stechkin [34] (cf. also [15, Theorem 6.4.4]) that f is a p -multiplier—there exists a positive number m_p such that

$$(6.2) \quad \|M_f \xi\|_p \leq m_p \|\xi\|_p \quad (\xi \in l^2(\mathbb{Z}) \cap l^p(\mathbb{Z})).$$

Here the assumption $1 < p < \infty$ is critical. The estimate (6.2) together with $l^2(\mathbb{Z}) \cap l^p(\mathbb{Z})$ being dense in $l^p(\mathbb{Z})$ allows M_f to be uniquely extended to a bounded linear operator from $l^p(\mathbb{Z})$ into itself, also denoted M_f , of norm $\leq m_p$.

Define two (projection) operators P^\pm in $\mathcal{L}(l^p(\mathbb{Z}))$ by

$$P^\pm = \frac{1}{2} (I_{l^p(\mathbb{Z})} \pm M_f).$$

Clearly, we have $P^+ + P^- = I_{l^p(\mathbb{Z})}$, and $\widehat{P^+ \xi} = 1_{[0, \pi)} \widehat{\xi}$ and $\widehat{P^- \xi} = 1_{[\pi, 2\pi)} \widehat{\xi}$ for $\xi \in l^2(\mathbb{Z}) \cap l^p(\mathbb{Z})$. In view of (6.1), if $n \in \mathbb{Z}$ and $\xi \in l^2(\mathbb{Z}) \cap l^p(\mathbb{Z})$, then

$$\widehat{\mathbb{B}_p^n P^+ \xi}(t) = (\cos t + i \sin t)^n 1_{[0, \pi)}(t) \widehat{\xi}(t) = e^{int} \widehat{P^+ \xi}(t) = \widehat{\mathbb{T}_n P^+ \xi}(t),$$

$$\widehat{\mathbb{B}_p^n P^- \xi}(t) = (\cos t - i \sin t)^n 1_{[\pi, 2\pi)}(t) \widehat{\xi}(t) = e^{-int} \widehat{P^- \xi}(t) = \widehat{\mathbb{T}_{-n} P^- \xi}(t)$$

for a.e. $t \in [0, 2\pi)$. Hence $\mathbb{B}_p^n P^+ = \mathbb{T}_n P^+$ and $\mathbb{B}_p^n P^- = \mathbb{T}_{-n} P^-$, and further

$$\|\mathbb{B}_p^n\| \leq \|\mathbb{B}_p^n P^+\| + \|\mathbb{B}_p^n P^-\| = \|\mathbb{T}_n P^+\| + \|\mathbb{T}_{-n} P^-\| = \|P^+\| + \|P^-\|.$$

Thus \mathbb{B}_p is doubly power bounded. ■

To treat the cases $p = 1$ and $p = \infty$, we need an auxiliary result. For $1 \leq p \leq \infty$, denote by $l_e^p(\mathbb{Z})$ the space of all even sequences in $l^p(\mathbb{Z})$.

THEOREM 6.2. *Assume that either $p = 1$ or $p = \infty$. If $B \in \mathcal{L}(l^p(\mathbb{Z}))$ generates a group decomposition for $c(A_p)$ and if $l_e^p(\mathbb{Z})$ is invariant for both B and B^{-1} , then B is not doubly power bounded.*

Proof. If B were doubly power bounded, then so too would be $B_e = B|_{l_e^p(\mathbb{Z})}$. Consequently, B_e would generate a bounded group decomposition for the cosine sequence engendered by $A_{p,e} = A_p|_{l_e^p(\mathbb{Z})}$. But this is impossible, since, in view of [7, Theorems 2.2 and 2.4], no group decomposition of $c(A_{p,e})$ is bounded when either $p = 1$ or $p = \infty$. ■

We can now state the final result of this section.

THEOREM 6.3. *The standard group decompositions of $c(A_1)$ and $c(A_\infty)$ fail to be bounded.*

Proof. Suppose that either $p = 1$ or $p = \infty$. Observe that while $l_e^p(\mathbb{Z})$ is not an invariant subspace for T_k whenever $k \in \mathbb{Z} \setminus \{0\}$, it is an invariant subspace for $T_k + T_{-k}$ for each $k \in \mathbb{Z}$. Hence $l_e^p(\mathbb{Z})$ is invariant for A_p and $\phi(A_p)$. Set $B_p = \psi(A_p) = A_p + i\phi(A_p)$. Then $B_p^{-1} = A_p - i\phi(A_p)$ and $l_e^p(\mathbb{Z})$ is invariant for both B_p and B_p^{-1} . Now Theorem 6.2 ensures that B_p is not doubly power bounded. ■

7. Lack of prespectrality. The main goal of this section is to establish the following result.

THEOREM 7.1. *If $1 \leq p \leq \infty$ and $p \neq 2$, then A_p is not prespectral.*

While Theorem 7.1 is of interest in its own right, its primary significance here is that it permits showing that a bounded cosine sequence may have a bounded standard group decomposition without the generator of the cosine sequence being scalar-type prespectral. Indeed, Theorems 6.1 and 7.1, the reflexivity of $l^p(\mathbb{Z})$ for $1 < p < \infty$, and the fact that a prespectral operator on a reflexive Banach space is spectral [11, Theorem 6.11] imply the following result.

THEOREM 7.2. *If $1 < p < \infty$ and $p \neq 2$, then A_p is not spectral and generates a bounded cosine sequence with bounded standard group decomposition.*

One consequence of the above theorem is that, barring the case $p = 2$, Theorem 6.1 cannot be deduced directly from Theorem 4.5. That Theorem 6.1 in the case $p = 2$ reduces indeed to Theorem 4.5 results from A_2 being of scalar type (see comments before Theorem 4.6).

Theorem 7.1 has a predecessor in the result of Fixman [19] and Krabbe [27] stating that, for each $1 \leq p \leq \infty$ with $p \neq 2$, the (backward) unit shift in $l^p(\mathbb{Z})$ fails to be spectral. It is worth noting that as far as spectrality is concerned, the unit shift is typical of translation operators in general locally compact Abelian groups—if G is such a group, then except in trivial cases, translations in $L^p(G)$, $1 \leq p \leq \infty$, $p \neq 2$, are not spectral [22], [11, Theorem 20.30].

Theorem 7.1 makes no direct appeal to cosine sequences, but its proof will make a critical use of the standard group decompositions of certain bounded cosine sequences. Before giving this proof, we present a few results that we shall need.

For each $\lambda \in [-1, 1]$, let X_λ be the eigenspace of A_∞ corresponding to the eigenvalue λ . If $a = (a_n)_{n \in \mathbb{Z}} \in X_\lambda$, then

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2\lambda & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \cdots = \begin{bmatrix} 2\lambda & -1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

for each $n \in \mathbb{Z}$, showing that a depends linearly on a_0 and a_1 , and hence that $\dim X_\lambda \leq 2$. For each $t \in \mathbb{T}$, let $\chi_t = (t^n)_{n \in \mathbb{Z}}$. It is immediately verified that $\chi_1 \in X_1$, $\chi_{-1} \in X_{-1}$, and that if $|\lambda| < 1$, then $\chi_{\psi(\lambda)}$ and $\chi_{\overline{\psi(\lambda)}}$ both belong to X_λ and are linearly independent. In particular, if $|\lambda| < 1$, then $\dim X_\lambda = 2$, and X_λ is spanned by $\chi_{\psi(\lambda)}$ and $\chi_{\overline{\psi(\lambda)}}$. As we shall prove next, X_1 and X_{-1} are one-dimensional, spanned by χ_1 and χ_{-1} , respectively.

If $a \in X_1$, then, for each $n \in \mathbb{Z}$, $a_{n+1} - a_n = a_n - a_{n-1}$, implying that $a_{n+1} - a_n = b$, where $b = a_1 - a_0$. Hence $a_n = a_0 + nb$, and, since a is bounded, we have $b = 0$. Consequently, $a = a_0\chi_1$.

If $a \in X_{-1}$, then, for each $n \in \mathbb{Z}$, $a_{n+1} + a_n = -(a_n + a_{n-1})$, so $a_{n+1} + a_n = (-1)^n c$, where $c = a_0 + a_1$. Hence, for each $k \in \mathbb{Z}$, $a_{2k+1} - a_{2k-1} = a_{2k+1} + a_{2k} - (a_{2k} + a_{2k-1}) = 2c$ and further $a_{2k+1} = 2kc + a_1$. The boundedness of a now implies that $c = 0$. Thus $a_{n+1} = -a_n$ for each $n \in \mathbb{Z}$ and further $a = a_0\chi_{-1}$.

Recall that a complex-valued two-sided sequence $(\xi_n)_{n \in \mathbb{Z}}$ is *almost periodic* if, for each $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that every set of the form $\{k, k+1, \dots, k+K\}$, $k \in \mathbb{Z}$, contains $N \in \mathbb{Z}$ with the property that $|\xi_n - \xi_{n+N}| < \varepsilon$ for all $n \in \mathbb{Z}$. Equivalently, $(\xi_n)_{n \in \mathbb{Z}}$ should belong to the closed linear span of $\{\chi_t \mid t \in \mathbb{T}\}$ in $l^\infty(\mathbb{Z})$. In particular, every almost periodic sequence is bounded. Let $ap(\mathbb{Z})$ denote the space of complex-valued almost periodic two-sided sequences. Obviously, $ap(\mathbb{Z})$ is translation invariant, and hence is also an invariant subspace for A_∞ .

LEMMA 7.3. *If A_∞ is prespectral, then $A_\infty|_{ap(\mathbb{Z})}$ is scalar-type prespectral.*

Proof. First note that if R is a linear operator on $l^\infty(\mathbb{Z})$ commuting with A_∞ , then, for each $|\lambda| \leq 1$, X_λ is invariant for R . Suppose now that A_∞ is prespectral of class Γ . Let S and Q be the scalar and radical parts of A_∞ , and let E be the resolution of the identity for A_∞ of class Γ . Since S , Q , and $\{E(\omega) \mid \omega \in \mathcal{B}(\mathbb{C})\}$ commute with A_∞ , it follows that, for each $|\lambda| \leq 1$, X_λ is invariant for S , Q , and all of the $E(\omega)$. We also have

$$A_\infty|_{X_\lambda} = S|_{X_\lambda} + Q|_{X_\lambda},$$

with $S|_{X_\lambda}$ and $Q|_{X_\lambda}$ commuting. Setting $E|_{X_\lambda}(\omega) = E(\omega)|_{X_\lambda}$ for each $\omega \in \mathcal{B}(\mathbb{C})$ defines a spectral measure $E|_{X_\lambda}$ in X_λ of class Γ/X_λ^\perp , the quotient space Γ/X_λ^\perp being a total subspace of X'/X_λ^\perp identified with X'_λ . Because X_λ is finite-dimensional, all vector topologies on X_λ coincide and, as a result, $E|_{X_\lambda}$ is of class X'_λ . Clearly, $S|_{X_\lambda} = \int_{\mathbb{C}} \lambda dE|_{X_\lambda}(\lambda)$, so $S|_{X_\lambda}$ is of scalar type. Also, we have $\lim_{n \rightarrow \infty} \|(Q|_{X_\lambda})^n\| \leq \lim_{n \rightarrow \infty} \|Q^n\| = 0$, implying that $Q|_{X_\lambda}$ is quasinilpotent. Thus $A_\infty|_{X_\lambda}$ is spectral, and $S|_{X_\lambda}$ and $Q|_{X_\lambda}$ are its scalar and radical parts. But $A_\infty|_{X_\lambda} = \lambda I_{X_\lambda}$, so $A_\infty|_{X_\lambda}$ is in fact of scalar type. Since the Jordan decomposition is unique, it follows that $Q|_{X_\lambda} = 0$.

As every χ_t ($t \in \mathbb{T}$) can be represented as either $\chi_{\psi(\lambda)}$ or $\chi_{\overline{\psi(\lambda)}}$ for some $|\lambda| \leq 1$, the closed linear space spanned by the X_λ in $l^\infty(\mathbb{Z})$ coincides with $ap(\mathbb{Z})$. Consequently, $Q|_{ap(\mathbb{Z})} = 0$ and further $A_\infty|_{ap(\mathbb{Z})} = S|_{ap(\mathbb{Z})}$. Since $S|_{ap(\mathbb{Z})} = \int_{\mathbb{C}} \lambda dE|_{ap(\mathbb{Z})}(\lambda)$, where $E|_{ap(\mathbb{Z})}$ is the spectral measure of class $\Gamma/ap(\mathbb{Z})^\perp$ defined by $E|_{ap(\mathbb{Z})}(\omega) = E(\omega)|_{ap(\mathbb{Z})}$ for $\omega \in \mathcal{B}(\mathbb{C})$, it follows that $A_\infty|_{ap(\mathbb{Z})}$ is a scalar-type operator of class $\Gamma/ap(\mathbb{Z})^\perp$. ■

Proof of Theorem 7.1. Arguing contrapositively, assume that A_p is prespectral of class Γ . Introduce the notation

$$A = \begin{cases} A_p & \text{if } 1 \leq p < \infty \text{ and } p \neq 2, \\ A_\infty|_{ap(\mathbb{Z})} & \text{if } p = \infty. \end{cases}$$

We first show that A is of scalar type. We shall consider three cases.

Assume first that $p = \infty$. Then $A = A_\infty|_{ap(\mathbb{Z})}$ and that A is of scalar type in this case is ensured by Lemma 7.3.

Suppose next that $p = 1$. Then $A = A_1$. Let $A = S + Q$ be the Jordan decomposition of A , with S and Q the respective scalar and radical parts. Then the dual operator A' is prespectral of class $l^1(\mathbb{Z})$, and S' and Q' are the scalar and radical parts of A' [11, Theorem 5.22]. Upon identifying the dual of $l^1(\mathbb{Z})$ with $l^\infty(\mathbb{Z})$, A' becomes identical with A_∞ . Thus A_∞ is prespectral and, in view of Lemma 7.3, $Q'|_{ap(\mathbb{Z})} = 0$. Since $ap(\mathbb{Z})$ is dense in $l^\infty(\mathbb{Z})$ under the weak* topology [3], it follows that $Q = 0$. Consequently, A_1 is of scalar type.

Finally, suppose that $1 < p < \infty$ and $p \neq 2$. Then $A = A_p$. By Theorem 5.1, the assumption that A is prespectral leads to the conclusion that also $B = \psi(A)$ is prespectral. Since $l^p(\mathbb{Z})$ is reflexive for the adopted value of p ,

\mathbf{B} is in fact spectral. Moreover, by Theorem 6.1, \mathbf{B} is doubly power bounded. According to a theorem proved independently by Fixman [19] and Foguel [20] and further extended by Dowson [10] (see also [11, Theorem 10.17]), every doubly power bounded spectral operator is of scalar type. In view of this result, \mathbf{B} is scalar-type spectral, and hence also $\mathbf{A} = \mathbf{B} + \mathbf{B}^{-1}$ is scalar-type spectral.

Having established that \mathbf{A} is of scalar type, we now produce a final contradiction. Our approach will be patterned after [22]; see also [11, Theorem 20.30]. For each $n \in \mathbb{N}$, let p_n be the n th Rudin–Shapiro polynomial. There is then a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of numbers, each with $\varepsilon_n = -1$ or $\varepsilon_n = 1$, such that

$$p_n(z) = \sum_{k=1}^n \varepsilon_k z^k$$

and

$$(7.1) \quad \sup_{z \in \mathbb{T}} |p_n(z)| \leq 5n^{1/2}$$

for each $n \in \mathbb{N}$. The existence of such a sequence of polynomials is proved in [33]. For each $n \in \mathbb{N}$, set

$$q_n(z) = \sum_{k=1}^n \varepsilon_k T_k(z),$$

where—let us recall— $T_k(z)$ stands for the k th first-kind Chebyshev polynomial. By Theorem 4.4, $\mathbf{B} = \psi(\mathbf{A})$ is of scalar type of class Γ . Let E be the resolution of the identity for \mathbf{B} of class Γ , and let $K_E = \sup_{\omega \in \mathcal{B}(\sigma(\mathbf{B}))} \|E(\omega)\|$. Since $\sigma(\mathbf{B}) \subset \mathbb{T}$, it follows that, for each $n \in \mathbb{N}$, $p_n(\mathbf{B}) = \int_{\mathbb{T}} p_n(z) dE(z)$ and $p_n(\mathbf{B}^{-1}) = \int_{\mathbb{T}} p_n(\bar{z}) dE(z)$, and further that

$$\|p_n(\mathbf{B})\| \leq 4K_E \sup_{z \in \mathbb{T}} |p_n(z)|, \quad \|p_n(\mathbf{B}^{-1})\| \leq 4K_E \sup_{z \in \mathbb{T}} |p_n(z)|.$$

But, as a moment’s reflection reveals,

$$q_n(\mathbf{A}) = \frac{1}{2} (p_n(\mathbf{B}) + p_n(\mathbf{B}^{-1}))$$

so, in view of (7.1),

$$(7.2) \quad \|q_n(\mathbf{A})\| \leq \frac{1}{2} (\|p_n(\mathbf{B})\| + \|p_n(\mathbf{B}^{-1})\|) \leq 20K_E n^{1/2}.$$

We shall now look at three cases. Suppose first that $1 \leq p < 2$. For each $k \in \mathbb{N}$, denote by e_k the two-sided sequence with all entries equal to 0 except the k th entry which is equal to 1. Clearly,

$$q_n(\mathbf{A}_p)e_0 = \frac{1}{2} \sum_{k=1}^n \varepsilon_k (e_{-k} + e_k),$$

implying that

$$(7.3) \quad 2^{1/p-1}n^{1/p} = \|q_n(\mathbf{A}_p)e_0\|_p \leq \|q_n(\mathbf{A}_p)\|.$$

Bearing in mind that $\mathbf{A} = \mathbf{A}_p$ when $1 \leq p < 2$, we see that this estimate is incompatible with (7.2) for large n . This establishes the theorem in the case $1 \leq p < 2$.

Assume now that $2 < p < \infty$. First observe that with the dual of $l^p(\mathbb{Z})$ identified with $l^q(\mathbb{Z})$, where q is the conjugate index $q = p/(p - 1)$, \mathbf{A}'_p coincides with \mathbf{A}_q . Given that $1 < q < 2$, we can now invoke (7.3) with p replaced by q to conclude that

$$2^{1/q-1}n^{1/q} \leq \|q_n(\mathbf{A}_q)\| = \|q_n(\mathbf{A}_p)\|.$$

But, remembering that $\mathbf{A} = \mathbf{A}_p$ when $2 < p < \infty$, we obtain a contradiction with (7.2) for large n . This establishes the theorem in the case $2 < p < \infty$.

Finally, assume that $p = \infty$. Recall that a subspace Y of the dual X' to a normed space X is called *norming* (*1-norming*) if the pseudonorm defined by

$$\|x\| = \sup\{|\langle x, x' \rangle| \mid x' \in Y, \|x'\| \leq 1\} \quad (x \in X)$$

is equivalent (equal, respectively) to the original norm on X . Using the equality $\mathbf{A}'_1 = \mathbf{A}_\infty$ and the fact that $ap(\mathbb{Z})$ is a 1-norming subspace of $l^\infty(\mathbb{Z})$ [3], we obtain

$$\begin{aligned} \|q_n(\mathbf{A}_1)e_0\|_1 &= \sup_{\substack{\xi \in ap(\mathbb{Z}), \\ \|\xi\|_\infty=1}} |\langle q_n(\mathbf{A}_1)e_0, \xi \rangle| \\ &= \sup_{\substack{\xi \in ap(\mathbb{Z}), \\ \|\xi\|_\infty=1}} |\langle e_0, q_n(\mathbf{A}_\infty)\xi \rangle| \leq \|q_n(\mathbf{A}_\infty|_{ap(\mathbb{Z})})\| \end{aligned}$$

for each $n \in \mathbb{N}$. On the other hand, the equality in (7.3) specialised to $p = 1$ yields $n = \|q_n(\mathbf{A}_1)e_0\|_1$ for each $n \in \mathbb{N}$. Therefore, $n \leq \|q_n(\mathbf{A}_\infty|_{ap(\mathbb{Z})})\|$ for each $n \in \mathbb{N}$. But, as $\mathbf{A} = \mathbf{A}_\infty|_{ap(\mathbb{Z})}$ when $p = \infty$, this is incompatible with (7.2) for large n , establishing the theorem in the case $p = \infty$ and finishing the proof. ■

8. Unbounded group decompositions. According to Theorem 4.6, any bounded $\mathcal{L}(H)$ -valued cosine sequence, where H is a complex Hilbert space, has a bounded standard group decomposition. Here we show that a bounded $\mathcal{L}(H)$ -valued cosine sequence may also admit an unbounded group decomposition. We present two examples, of which the simpler, given first, involves a finite-dimensional Hilbert space.

EXAMPLE 8.1. Let $H = \mathbb{C}^2$ and let c be the $\mathcal{L}(H)$ -valued cosine sequence generated by $A = I_H$. Clearly, c is bounded, as all of its elements coincide with I_H . Identify $\mathcal{L}(H)$ with the algebra of all 2×2 complex-valued

matrices and let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It is immediately verified that $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is the inverse of B and that $A = (B + B^{-1})/2$. Thus B generates a group decomposition for c . This group decomposition is unbounded because $B^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for each $n \in \mathbb{Z}$.

The generator of the group decomposition in the above example is spectral, as is indeed the case with any operator in a finite-dimensional Hilbert space. The next example will exhibit a bounded $\mathcal{L}(H)$ -valued cosine sequence, with H an infinite-dimensional complex Hilbert space, admitting an unbounded group decomposition generated by a non-spectral operator. While, in view of Theorem 4.3, the generator of an unbounded group decomposition for an operator-valued bounded cosine sequence cannot be scalar-type prespectral, the non-spectral generator in the upcoming example will be generalised scalar.

EXAMPLE 8.2. Let H be an infinite-dimensional complex Hilbert space. Let $(P_k)_{k \in \mathbb{N}}$ and $(Q_k)_{k \in \mathbb{N}}$ be two sequences of projections in H such that

- (a) $P_k P_l = Q_k Q_l = 0$ for all $k, l \in \mathbb{N}$ with $k \neq l$,
- (b) $P_k Q_l = Q_l P_k = 0$ for all $k, l \in \mathbb{N}$,
- (c) $\sum_{k=1}^{\infty} (P_k + Q_k) = I_H$, where the sum converges in the strong operator topology,
- (d) $\sup_{k \in \mathbb{N}} \|P_k\| = \infty$,
- (e) $K_{PQ} = \sup_{\omega \in \mathcal{F}(\mathbb{N})} \|\sum_{k \in \omega} (P_k + Q_k)\| < \infty$, where $\mathcal{F}(\mathbb{N})$ denotes the set of all finite subsets of \mathbb{N} .

Such sequences can be constructed as follows. Exploiting the assumption that H is infinite-dimensional, we first represent H as the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{H}$, where \mathcal{H} is a copy of H . Let $(U_k)_{k \in \mathbb{N}}$ be a sequence of projections in \mathcal{H} with $U_k U_l = 0$ for $k \neq l$ and such that $\sum_{k=1}^{\infty} U_k$ is *unconditionally* convergent to $I_{\mathcal{H}}$ in the strong operator topology. Let $(V_k)_{k \in \mathbb{N}}$ be a sequence of projections with $V_k V_l = 0$ for $k \neq l$ and such that $\sum_{k=1}^{\infty} V_k$ is *conditionally*, but not unconditionally, convergent to $I_{\mathcal{H}}$ in the strong operator topology. The existence of the latter sequence follows from the existence of conditional Schauder bases in a separable, infinite-dimensional Hilbert space, which was first established by K. I. Babenko [2]. Another construction of conditional bases in a separable Hilbert space, due to C. A. McCarthy and J. Schwartz [32], is presented in detail in [30, p. 74]. Given $\omega \in \mathcal{F}(\mathbb{N})$, let $V_{\omega} = \sum_{k \in \omega} V_k$. For each $k \in \mathbb{N}$, choose $\omega_k \in \mathcal{F}(\mathbb{N})$ so that $\|V_{\omega_k}\| \geq k \|U_k\|^{-1}$, this being possible because $\sum_{k=1}^{\infty} V_k$ is not unconditionally convergent. Set

$$P_k = U_k \otimes V_{\omega_k}, \quad Q_k = U_k \otimes (I_{\mathcal{H}} - V_{\omega_k})$$

for each $k \in \mathbb{N}$. It is easily seen that the sequences $(P_k)_{k \in \mathbb{N}}$ and $(Q_k)_{k \in \mathbb{N}}$ satisfy conditions (a)–(e).

Select a sequence $(t_k)_{k \in \mathbb{N}}$ in $\mathbb{T} \setminus \{1\}$ such that $\{t_k, \bar{t}_k\} \cap \{t_l, \bar{t}_l\} = \emptyset$ for $k \neq l$ and

$$(8.1) \quad \sum_{k=1}^{\infty} |t_k - \bar{t}_k| \|Q_k\| < \infty.$$

For example, a sequence $(t_k)_{k \in \mathbb{N}}$ of distinct points in the open upper semi-circle $\{z \in \mathbb{C} \mid |z| = 1, \text{Im } z > 0\}$ satisfying $|t_k - 1| \leq (1 + \|Q_k\|)^{-1} k^{-2}$ for each $k \in \mathbb{N}$ will do. Let

$$B = \sum_{k=1}^{\infty} (t_k P_k + \bar{t}_k Q_k).$$

The series on the right-hand side converges in the strong operator topology, given that it can be represented as

$$\sum_{k=1}^{\infty} t_k (P_k + Q_k) + \sum_{k=1}^{\infty} (\bar{t}_k - t_k) Q_k,$$

where, by (e), the first series converges unconditionally in the strong operator topology, and, by (8.1), the second series converges in the norm operator topology. By (a)–(c), for each $n \in \mathbb{Z}$,

$$B^n = \sum_{k=1}^{\infty} t_k^n (P_k + Q_k) + \sum_{k=1}^{\infty} (\bar{t}_k^n - t_k^n) Q_k$$

and further

$$(8.2) \quad B^n + B^{-n} = \sum_{k=1}^{\infty} (t_k^n + \bar{t}_k^n) (P_k + Q_k).$$

Condition (e) implies that

$$(8.3) \quad \left\| \sum_{k=1}^{\infty} c_k (P_k + Q_k) \right\| \leq 4K_{PQ} \sup_{k \in \mathbb{N}} |c_k|$$

for every bounded sequence $(c_k)_{k \in \mathbb{N}}$. In particular, for each $n \in \mathbb{Z}$,

$$\left\| \sum_{k=1}^{\infty} t_k^n (P_k + Q_k) \right\| \leq 4K_{PQ}.$$

As $|\bar{t}_k^n - t_k^n| \leq |n| |\bar{t}_k - t_k|$, we have

$$\left\| \sum_{k=1}^{\infty} (\bar{t}_k^n - t_k^n) Q_k \right\| \leq |n| \sum_{k=1}^{\infty} |t_k - \bar{t}_k| \|Q_k\|.$$

Consequently, bearing in mind (8.1), we find that

$$\|B^n\| = O(|n|).$$

This estimate immediately implies that B is a generalised scalar operator with $\sigma(B) \subset \mathbb{T}$ (cf. [23], [29, Theorem 1.5.12]).

Let $A = (B + B^{-1})/2$. The cosine sequence generated by A coincides with $\underline{c}(B)$ and is bounded, since, by (8.2) and (8.3),

$$\|B^n + B^{-n}\| = \left\| \sum_{k=1}^{\infty} (t_k^n + \bar{t}_k^n)(P_k + Q_k) \right\| \leq 8K_{PQ}$$

for each $n \in \mathbb{Z}$. Our work will be complete once we show that B is neither power bounded nor spectral.

We first show that B is not power bounded. Assume, on the contrary, that $K_B = \sup_{n \in \mathbb{N}} \|B^n\| < \infty$. Note that

$$(8.4) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (\bar{s}t)^n = \delta_{st}$$

for $s, t \in \mathbb{T}$, where δ_{st} denotes the Kronecker delta. For each $m \in \mathbb{N}$, define a projection R_m by

$$R_m = \sum_{k=1}^m (P_k + Q_k).$$

Clearly, by (e),

$$(8.5) \quad \sup_{m \in \mathbb{N}} \|R_m\| \leq K_{PQ}.$$

Also

$$B^n R_m = \sum_{k=1}^m (t_k^n P_k + \bar{t}_k^n Q_k)$$

for any $n, m \in \mathbb{N}$. Now, in view of (8.4),

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \bar{s}^n B^n R_m = \sum_{k=1}^m (\delta_{st_k} P_k + \delta_{s\bar{t}_k} Q_k)$$

for each $m \in \mathbb{N}$ and each $s \in \mathbb{T}$. Putting $s = t_m$ in the above equality and taking into account that $t_1, \dots, t_m, \bar{t}_1, \dots, \bar{t}_m$ are all different, we find that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \bar{t}_m^n B^n R_m = P_m.$$

Hence, bearing in mind (8.5),

$$\sup_{m \in \mathbb{N}} \|P_m\| \leq \sup_{n \in \mathbb{N}} \|B^n\| \sup_{m \in \mathbb{N}} \|R_m\| \leq K_B K_{PQ},$$

contrary to (d). Thus B is not power bounded.

We finally show that B is not spectral. Assume, on the contrary, that B is spectral and let E be its resolution of the identity. Fix $m \in \mathbb{N}$ arbitrarily.

Since

$$BR_m = R_mB = \sum_{k=1}^m (t_k P_k + \bar{t}_k Q_k),$$

it follows that the range space of R_m , denoted X_m , is invariant for B . Moreover, as

$$P_l R_m = R_m P_l = P_l \quad (1 \leq l \leq m),$$

we see that X_m is invariant for P_l when $1 \leq l \leq m$. Therefore

$$(8.6) \quad B|_{X_m} = \sum_{k=1}^m (t_k P_k|_{X_m} + \bar{t}_k Q_k|_{X_m}).$$

Remembering that $t_1, \dots, t_m, \bar{t}_1, \dots, \bar{t}_m$ are all different, we deduce immediately that the set function $F_m: \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{L}(H)$ defined by

$$F_m(\omega) = \sum_{k=1}^m (\delta_{t_k}(\omega) P_k|_{X_m} + \delta_{\bar{t}_k}(\omega) Q_k|_{X_m}) \quad (\omega \in \mathcal{B}(\mathbb{C}))$$

is a spectral measure. Here, for any given $a \in \mathbb{C}$, δ_a denotes the Dirac measure on \mathbb{C} concentrated at a . Since the right-hand side of (8.6) can be interpreted as $\int_{\mathbb{C}} \lambda dF_m(\lambda)$, we see that $B|_{X_m}$ is a scalar-type spectral operator and F_m is its resolution of the identity. It now follows from a theorem of Fixman [19] (see also [11, Theorem 12.2]) that, for each $\omega \in \mathcal{B}(\mathbb{C})$, X_m is invariant for $E(\omega)$ and $E(\omega)|_{X_m} = F_m(\omega)$. Hence

$$\begin{aligned} E(\omega)R_m &= F_m(\omega)R_m = \left[\sum_{k=1}^m (\delta_{t_k}(\omega) P_k + \delta_{\bar{t}_k}(\omega) Q_k) \right] R_m \\ &= \sum_{k=1}^m (\delta_{t_k}(\omega) P_k + \delta_{\bar{t}_k}(\omega) Q_k). \end{aligned}$$

In particular, $E(\{t_k\})R_m = P_k$ for any $k, m \in \mathbb{N}$ with $k \leq m$. Letting $K_E = \sup_{\omega \in \mathcal{B}(\mathbb{C})} \|E(\omega)\|$ and using (8.5), we conclude that

$$\sup_{k \in \mathbb{N}} \|P_k\| \leq K_E K_{PQ}.$$

But this is incompatible with (d). Thus B is not spectral.

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Received July 19, 2006
Revised version May 16, 2007

(5969)