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Periodic solutions for second order integro-differential equations with infinite delay in Banach spaces

by

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Abstract. We study the maximal regularity on different function spaces of the second order integro-differential equations with infinite delay

$$(P)\qquad u''(t)+\alpha u'(t)+\frac{d}{dt}\Big(\int_{-\infty}^{t} b(t-s)u(s)\,ds\Big)=Au(t)-\int_{-\infty}^{t} a(t-s)Au(s)\,ds+f(t)$$

$(0\le t\le 2\pi)$ with periodic boundary conditions $u(0)=u(2\pi), u'(0)=u'(2\pi)$, where A is a closed operator in a Banach space X, $\alpha\in\mathbb{C}$, and $a,b\in L^1(\mathbb{R}_+)$. We use Fourier multipliers to characterize maximal regularity for (P). Using known results on Fourier multipliers, we find suitable conditions on the kernels a and b under which necessary and sufficient conditions are given for the problem (P) to have maximal regularity on $L^p(\mathbb{T},X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T},X)$ and periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T},X)$.

1. Introduction. In a series of recent publications operator-valued Fourier multipliers on vector-valued function spaces have been studied (see e.g. [2, 3, 1, 6, 14, 15]). They are needed to study the existence and uniqueness of differential equations on Banach spaces. In [2, 3, 1, 6], the authors study the maximal regularity of the classical second order problem (P_1) on L^p spaces, Besov spaces and Triebel–Lizorkin spaces using operator-valued Fourier multipliers, where

$$(P_1)\qquad \begin{cases} u''(t)+Au(t)=f(t) & (0\le t\le 2\pi),\\ u(0)=u(2\pi), & u'(0)=u'(2\pi);\end{cases}$$

here A is a closed linear operator defined in a Banach space X and f is an X-valued function defined on $[0,2\pi]$. If X is a UMD Banach space and

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$1 < p < \infty$, then the problem (P_1) has maximal regularity on $L^p(\mathbb{T}, X)$ if and only if $k^2 \in \varrho(A)$ for all $k \in \mathbb{Z}$ and the sequence $(k^2R(k^2, A))_{k\in\mathbb{Z}}$ is Rademacher bounded [2]. In the setting of Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$, the maximal regularity is equivalent to the condition that $k^2 \in \varrho(A)$ for all $k \in \mathbb{Z}$ and $(k^2R(k^2, A))_{k\in\mathbb{Z}}$ is bounded [3, 6].

In this paper, we consider a more general evolution equation, namely the second order integro-differential equation with infinite delay:

$$(P_2)\quad \begin{cases} u''(t) + Bu'(t) + \dfrac{d}{dt}\Big(\displaystyle\int_{-\infty}^{t} b(t-s)u(s)\,ds\Big) \\ \qquad = Au(t) - \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t) \quad (0 \le t \le 2\pi), \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases}$$

where A and B are closed linear operators in a Banach space X and $a, b \in L^1(\mathbb{R}_+)$. Much literature has been devoted to a similar first order integro-differential equation (P_3):

$$(P_3)\quad \begin{cases} \gamma_0 u'(t) + \dfrac{d}{dt}\Big(\displaystyle\int_{-\infty}^{t} b(t-s)u(s)\,ds\Big) + \gamma_\infty u(t) \\ \qquad = c_0Au(t) - \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t) \quad (0 \le t \le 2\pi), \\ u(0) = u(2\pi), \end{cases}$$

where γ_0, γ_∞, c_0 are constants, A is a closed linear operator in X, and $a, b \in L^1(\mathbb{R}_+)$. The class of equations of type (P_2) and (P_3) arises as models for nonlinear heat conduction in materials of fading memory type, and in population dynamics. In [11], Keyantuo and Lizama obtained the maximal regularity of (P_3) on L^p spaces and Besov spaces. They also studied this equation in the case $\gamma_0 = c_0 = 1, b = \gamma_\infty = 0$ in a previous paper [10]. Clément and Da Prato [8] studied (P_3) on the real line in the case $a = 0$ and obtained maximal regularity results in Sobolev spaces and Hölder spaces as well as in the space of bounded uniformly continuous functions. Da Prato and Lunardi [9] investigated periodic solutions of (P_3) in the case $b = 0$. Hölder continuous solutions of (P_3) have been studied on the real line by Lunardi [12] in the case of A being the Laplacian operator in a bounded domain $\Omega \subset \mathbb{R}^N$ and $X = C(\overline{\Omega})$.

We notice that the problem (P_2) has been studied by several authors in a simpler form and for different boundary conditions. For instance, R. Chill and S. Srivastava [7] have considered the L^p-maximal regularity on a finite interval $[0,T)$ for the abstract second order problem

$$(P_4) \qquad \begin{cases} u''(t) + Bu'(t) + Au(t) = f(t) & (0 \le t < T), \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

The semigroup theory and trace spaces played important roles in that discussion. Under a suitable condition on the operators A and B, they gave a necessary and sufficient condition for the problem (P_4) to have L^p-maximal regularity.

In this paper, we are interested in the second order integro-differential equation (P_2) with periodic boundary conditions. Since A and B are not necessarily generators of semigroups in our situation, semigroup theory is no longer applicable. So our main tool in the study of maximal regularity of (P_2) is operator-valued Fourier multipliers. The presence of two closed linear operators in the operator-valued multiplier functions makes the verification of the sufficient condition for Fourier multipliers particularly complicated. Therefore in this paper, we just consider the simpler case $B = \alpha I$ for some fixed $\alpha \in \mathbb{C}$ (the general case will be studied elsewhere).

We want to obtain maximal regularity of (P_2) with $B = \alpha I$ for some $\alpha \in \mathbb{C}$ on three function spaces: $L^p(\mathbb{T}, X)$ for $1 < p < \infty$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ for $1 \le p, q \le \infty$, $s > 0$, and periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$ for $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$, where $\mathbb{T} = [0, 2\pi]$. The main tools are the operator-valued Fourier multiplier theorems obtained in [2, Theorem 1.3], [3, Theorem 4.5] and [6, Theorem 3.2]. The differences between these multiplier theorems on different function spaces make us impose different conditions on the kernels a and b to obtain the maximal regularity on these spaces. These conditions are satisfied by a class of functions which correspond to the most common kernels encountered in applications. Furthermore, it is easy to see that in the case $\alpha = 0$, $a = b = 0$ our results are in accordance with the well known results for (P_1) [2, 3, 6].

The paper is organized as follows. In Section 2, we establish a general maximal regularity result for a problem (P_2) in the case $B = \alpha I$ for some $\alpha \in \mathbb{C}$, in terms of operator-valued Fourier multipliers. In Section 3, we apply the general result to three concrete function spaces: $L^p(\mathbb{T}, X)$, $B^s_{p,q}(\mathbb{T}, X)$ and $F^s_{p,q}(\mathbb{T}, X)$, still in the case $B = \alpha I$ for some $\alpha \in \mathbb{C}$.

2. Maximal regularity via Fourier multipliers. Let X be a Banach space. We will consider the problem (P_2) in a simpler form

$$(P_5)\qquad \begin{cases} u''(t) + \alpha u'(t) + \dfrac{d}{dt}\Big(\int\limits_{-\infty}^{t} b(t-s)u(s)\,ds\Big) \\ \qquad\qquad = Au(t) - \int\limits_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t) \quad (0 \le t \le 2\pi), \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases}$$

where A is a closed linear operator in X, $a, b \in L^1(\mathbb{R}_+)$, f is an X-valued function defined on $\mathbb{T} := [0, 2\pi]$ and $\alpha \in \mathbb{C}$ is a constant. The solution of (P_5) will be an X-valued function defined on $\mathbb{T}$ (extended to $\mathbb{R}$ by periodicity).

Fourier multipliers will be very useful in our study of maximal regularity of the problem (P_5) on different function spaces. These spaces include $L^p(\mathbb{T}, X)$ for $1 < p < \infty$, $B^s_{p,q}(\mathbb{T}, X)$ for $1 \le p, q \le \infty$, $s > 0$ and $F^s_{p,q}(\mathbb{T}, X)$ for $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$. For detailed information about vector-valued periodic Besov and Triebel–Lizorkin spaces, we refer to [3, Section 2] and [6, Section 2].

If Y is another Banach space, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. For $1 \le p \le \infty$ and $f \in L^p(\mathbb{T}, X)$, we denote by

$$\widehat{f}(k) = \frac{1}{2\pi} \int\limits_0^{2\pi} e_{-k}(t) f(t)\,dt$$

the kth Fourier coefficient of f, where $k \in \mathbb{Z}$ and $e_k(t) = e^{ikt}$ for $t \in \mathbb{R}$. For $x \in X$, we let $e_k \otimes x$ be the X-valued function given by $t \mapsto e_k(t)x$.

DEFINITION 2.1. Let X and Y be Banach spaces and let $\Gamma(\mathbb{T}, X)$ be one of the following X-valued function spaces: $L^p(\mathbb{T}, X)$ $(1 \le p < \infty)$, $B^s_{p,q}(\mathbb{T}, X)$ $(1 \le p, q \le \infty, s \in \mathbb{R})$ or $F^s_{p,q}(\mathbb{T}, X)$ $(1 \le p < \infty, 1 \le q \le \infty, s \in \mathbb{R})$. We say that a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is a *Γ-multiplier* if for each $f \in \Gamma(\mathbb{T}, X)$, there exists a unique $g \in \Gamma(\mathbb{T}, Y)$ such that $\widehat{g}(k) = M_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$ [2, 3, 6].

REMARK 2.2. 1. It follows from the closed graph theorem that if $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is a Γ-multiplier, then there exists a constant $C > 0$ such that for $f \in \Gamma(\mathbb{T}, X)$, we have $\|\sum_{k\in\mathbb{Z}} e_k \otimes M_k \widehat{f}(k)\|_\Gamma \le C\|f\|_\Gamma$. This implies that each Γ-multiplier is a bounded sequence.

2. It is clear from the definition that if $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ and $(N_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(Y, Z)$ are Γ-multipliers, then so is $(N_k M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Z)$.

Let X be a Banach space and let $\Gamma(\mathbb{T}, X)$ be one of the following: $L^p(\mathbb{T}, X)$ $(1 < p < \infty)$, $B^s_{p,q}(\mathbb{T}, X)$ $(1 \le p, q \le \infty, s > 0)$ or $F^s_{p,q}(\mathbb{T}, X)$ $(1 \le p < \infty, 1 \le q \le \infty, s > 0)$. We denote the first order "Sobolev" space by $\Gamma^{[1]}(\mathbb{T}, X)$ and the second order "Sobolev" spaces by $\Gamma^{[2]}(\mathbb{T}, X)$:

if $\Gamma(\mathbb{T},X) = L^p(\mathbb{T},X)$, then

$$\begin{aligned}\Gamma^{[1]}(\mathbb{T},X) &= \{u \in L^p(\mathbb{T},X) : \text{there exists } v \in L^p(\mathbb{T},X) \\ &\qquad \text{such that } \widehat{v}(k) = ik\widehat{u}(k) \text{ for all } k \in \mathbb{Z}\} \\ &= \{u \in L^p(\mathbb{T},X) : u \text{ is differentiable a.e., } u' \in L^p(\mathbb{T},X), \\ &\qquad \text{and } u(0) = u(2\pi)\}.\end{aligned}$$

$$\begin{aligned}\Gamma^{[2]}(\mathbb{T},X) &= \{u \in L^p(\mathbb{T},X) : \text{there exists } v \in L^p(\mathbb{T},X) \\ &\qquad \text{such that } \widehat{v}(k) = -k^2\widehat{u}(k) \text{ for all } k \in \mathbb{Z}\}, \\ &= \{u \in L^p(\mathbb{T},X) : u \text{ is twice differentiable a.e., } u', u'' \in L^p(\mathbb{T},X), \\ &\qquad \text{and } u(0) = u(2\pi), u'(0) = u'(2\pi)\}.\end{aligned}$$

If $\Gamma(\mathbb{T},X) = B^s_{p,q}(\mathbb{T},X)$ $(1 \le p,\ q \le \infty,\ s > 0)$, then

$$\Gamma^{[1]}(\mathbb{T},X) = B^{s+1}_{p,q}(\mathbb{T},X), \qquad \Gamma^{[2]}(\mathbb{T},X) = B^{s+2}_{p,q}(\mathbb{T},X).$$

If $\Gamma(\mathbb{T},X) = F^s_{p,q}(\mathbb{T},X)$ $(1 \le p < \infty,\ 1 \le q \le \infty,\ s > 0)$, then

$$\Gamma^{[1]}(\mathbb{T},X) = F^{s+1}_{p,q}(\mathbb{T},X), \qquad \Gamma^{[2]}(\mathbb{T},X) = F^{s+2}_{p,q}(\mathbb{T},X).$$

We refer to [2, Section 2, 6], [3, Section 2] and [6, Section 2] for more information about these spaces. For $g \in L^1(\mathbb{R}_+)$ and $u \in L^1(\mathbb{T},X)$ (extended to $\mathbb{R}$ by periodicity), we define

$$F(t) = (g \mathbin{\dot{*}} u)(t) := \int_{-\infty}^{t} g(t-s)u(s)\,ds. \tag{2.1}$$

In this notation, (P_5) has the following more compact form:

$$u'' + \alpha u' + \frac{d}{dt}(b \mathbin{\dot{*}} u) = Au - a \mathbin{\dot{*}} u + f$$

with periodic boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$.

Let $\widetilde{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\,dt$ be the Laplace transform of g. An easy computation shows that

$$\widehat{F}(k) = \widetilde{g}(ik)\widehat{u}(k) \qquad (k \in \mathbb{Z}). \tag{2.2}$$

Now we define the Γ-maximal regularity of the problem (P_5).

DEFINITION 2.3. Let X be a Banach space, A be a closed linear operator in X, $\alpha \in \mathbb{C}$ and let $a, b \in L^1(\mathbb{R}_+)$. Let $\Gamma(\mathbb{T},X)$ be one of the following: $L^p(\mathbb{T},X)$ $(1 < p < \infty)$, $B^s_{p,q}(\mathbb{T},X)$ $(1 \le p,q \le \infty,\ s > 0)$ or $F^s_{p,q}(\mathbb{T},X)$ $(1 \le p < \infty,\ 1 \le q \le \infty,\ s > 0)$,

1. Let $f \in \Gamma(\mathbb{T},X)$. A function $u \in \Gamma^{[2]}(\mathbb{T},X)$ is called a *strong Γ-solution* of (P_5) if $u(t) \in D(A)$ and the equation of (P_5) holds for almost all $t \in \mathbb{T}$, and $u'', u', Au, a \mathbin{\dot{*}} Au, \frac{d}{dt}(b \mathbin{\dot{*}} u) \in \Gamma(\mathbb{T},X)$.

2. The problem (P_5) is said to have *Γ-maximal regularity* if for every $f \in \Gamma(\mathbb{T}, X)$, there exists a unique strong Γ-solution of (P_5).

In what follows, we always set $g_k = \widetilde{g}(ik)$ for any $g \in L^1(\mathbb{R}_+)$ and $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \varrho(A)$, where $\varrho(A)$ is the resolvent set of A. If $a \in \mathbb{C}$, we will simply denote the bounded linear operator aI by a, where I is the identity of X. We consider the following two hypotheses for a scalar function g defined on $\mathbb{R}_+$:

(**H0a**) $g \in L^1(\mathbb{R}_+)$ and $(g_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ is a Γ-multiplier.
(**H0b**) $g_k \neq 1$ for all $k \in \mathbb{Z}$ and $((1-g_k)^{-1})_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ is a Γ-multiplier.

We shall write (**H0**) when (**H0a**) and (**H0b**) are both satisfied. For convenience, for $a, b \in L^1(\mathbb{R}_+)$ we adopt the following notations: for $k \in \mathbb{Z}$,

$$\begin{aligned} a_k &:= \widetilde{a}(ik), \qquad b_k := \widetilde{b}(ik), \\ d_k &:= \frac{ik(\alpha + b_k) - k^2}{1 - a_k}, \\ M_k &:= \frac{-k^2}{1 - a_k} R(d_k, A). \end{aligned} \tag{2.3}$$

Now, we are ready to state the main result of this section.

THEOREM 2.4. *Let X be a Banach space, $A : D(A) \subset X \to X$ be a closed linear operator, $\alpha \in \mathbb{C}$ and let $a, b \in L^1(\mathbb{R}_+)$. Let $\Gamma(\mathbb{T}, X)$ be one of the following: $L^p(\mathbb{T}, X)$ $(1 < p < \infty)$, $B^s_{p,q}(\mathbb{T}, X)$ $(1 \le p, q \le \infty,\ s > 0)$ or $F^s_{p,q}(\mathbb{T}, X)$ $(1 \le p < \infty,\ 1 \le q \le \infty,\ s > 0)$. Assume that a satisfies* (**H0**) *and b satisfies* (**H0a**). *Then the following assertions are equivalent*:

(i) *The problem (P_5) has Γ-maximal regularity.*
(ii) *$(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ and $(M_k)_{k\in\mathbb{Z}}$ is a Γ-multiplier.*

Proof. We notice that if $s > 0$, then $B^s_{p,q}(\mathbb{T}, X)$ and $F^s_{p,q}(\mathbb{T}, X)$ embed continuously into $L^p(\mathbb{T}, X)$ [3, 6], thus we will freely use results in $L^p(\mathbb{T}, X)$ for functions in $B^s_{p,q}(\mathbb{T}, X)$ or $F^s_{p,q}(\mathbb{T}, X)$ when $s > 0$.

(i)⇒(ii): Let $k \in \mathbb{Z}$ and $y \in X$ be fixed. We define $f(t) = e^{ikt}y$. Then $\widehat{f}(k) = y$. By assumption, there exists $u \in \Gamma^{[2]}(\mathbb{T}, X)$ such that $u(t) \in D(A)$ and

$$u''(t) + \alpha u'(t) + \frac{d}{dt}(b \dot{*} u(t)) = Au(t) - a \dot{*} Au(t) + f(t)$$

for almost all $t \in \mathbb{T}$, $u'', u', Au, a \dot{*} Au, \frac{d}{dt}(b \dot{*} u) \in \Gamma(\mathbb{T}, X)$ and $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$. Taking Fourier transforms on both sides, using (2.2) and the closedness of A, we find that $\widehat{u}(k) \in D(A)$ and

$$[-k^2 + ik\alpha + ikb_k - (1 - a_k)A]\widehat{u}(k) = y.$$

Thus $-k^2 + ik\alpha + ikb_k - (1 - a_k)A$ is surjective. To show that it is also injective, let $x \in D(A)$ be such that $[-k^2 + ik\alpha + ikb_k - (1 - a_k)A]x = 0$.

Then

$$Ax = \frac{-k^2 + ik(\alpha + b_k)}{1 - a_k}\, x = d_k x,$$

where we have used the assumption that the kernel a satisfies (**H0**) and therefore $a_k - 1 \neq 0$ for $k \in \mathbb{Z}$. Hence $u(t) = e^{ikt}x$ defines a solution of $u''(t) + \alpha u'(t) + \frac{d}{dt}\,(b \dot{*} u(t)) = Au(t) - a \dot{*} Au(t)$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$. Indeed,

$$\begin{aligned} Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)\,ds &= Ae^{ikt}x - \int_{-\infty}^{t} a(t-s)Ae^{iks}x\,ds \\ &= e^{ikt}Ax - e^{ikt}a_k Ax = (1-a_k)e^{ikt}Ax = [-k^2 + ik(\alpha + b_k)]e^{ikt}x \\ &= u''(t) + \alpha u'(t) + \frac{d}{dt}(b \dot{*} u(t)). \end{aligned}$$

By the uniqueness assumption, we have $x = 0$. We have shown that $-k^2 + ik\alpha + ikb_k - (1-a_k)A$ is bijective. Since A is closed, we conclude that

$$d_k = \frac{-k^2 + ik\alpha + ikb_k}{1 - a_k} \in \varrho(A) \quad \text{for each } k \in \mathbb{Z}.$$

Next, we show that $(M_k)_{k\in\mathbb{Z}}$ is a Γ-multiplier where M_k is defined by (2.3). If $f \in \Gamma(\mathbb{T}, X)$, there exists $u \in \Gamma^{[2]}(\mathbb{T}, X)$ solving (P_5) by assumption. Taking Fourier transforms, we obtain

$$[-k^2 + ik\alpha + ikb_k - (1-a_k)A]\widehat{u}(k) = \widehat{f}(k) \quad (k \in \mathbb{Z}).$$

Since $-k^2 + ik\alpha + ikb_k - (1-a_k)A$ is invertible, we have

$$\widehat{u}(k) = \frac{1}{1-a_k}\, R(d_k, A)\widehat{f}(k) \quad \text{and} \quad -k^2\widehat{u}(k) = M_k\widehat{f}(k).$$

Since $u \in \Gamma^{[2]}(\mathbb{T}, X)$, it is twice differentiable a.e. on $\mathbb{T}$, $u', u'' \in \Gamma(\mathbb{T}, X)$ and

$$\widehat{u''}(k) = -k^2\widehat{u}(k) = M_k\widehat{f}(k) \quad (k \in \mathbb{Z}).$$

From this and the definition of Γ-multiplier, we conclude that $(M_k)_{k\in\mathbb{Z}}$ is a Γ-multiplier.

(ii)⇒(i): Let $f \in \Gamma(\mathbb{T}, X)$. We define

$$N_k = \frac{1}{1-a_k}\, R(d_k, A).$$

By Remark 2.2, $N_k = (-1/k^2)M_k$ and $ikN_k = (1/ik)M_k$ are Γ-multipliers as the sequences $(-1/k^2)_{k\in\mathbb{Z}}$ and $(1/ik)_{k\in\mathbb{Z}}$ are Γ-multipliers by [2, Theorem 1.3], [3, Theorem 4.5] and [6, Theorem 3.2]. Since $(N_k)_{k\in\mathbb{Z}}$ is a Γ-multiplier, there exists $u \in \Gamma(\mathbb{T}, X)$ such that $\widehat{u}(k) = N_k\widehat{f}(k)$ for all $k \in \mathbb{Z}$. This implies that $\widehat{u}(k) \in D(A)$ and

$$[-k^2 + ik(\alpha + b_k) - (1-a_k)A]\widehat{u}(k) = \widehat{f}(k) \tag{2.4}$$

for all $k \in \mathbb{Z}$. Since $(ikN_k)_{k\in\mathbb{Z}}$ is also a Γ-multiplier, there exists $v \in \Gamma(\mathbb{T}, X)$ such that

$$\widehat{v}(k) = ikN_k\widehat{f}(k) = ik\widehat{u}(k).$$

By [2, Lemma 2.1], u is differentiable a.e. with $v = u'$ and $u(0) = u(2\pi)$. Therefore $u \in \Gamma^{[1]}(\mathbb{T}, X)$. As $(M_k)_{k\in\mathbb{Z}}$ is a Γ-multiplier by assumption, there exists $w \in \Gamma(\mathbb{T}, X)$ such that

$$\widehat{w}(k) = M_k\widehat{f}(k) = ik\widehat{v}(k) = -k^2\widehat{u}(k).$$

By [2, Lemma 2.1], $v = u'$ is differentiable a.e. with $w = v' = u''$ and $u'(0) = u'(2\pi)$. This implies that $u \in \Gamma^{[2]}(\mathbb{T}, X)$.

Next, we show that $u(t) \in D(A)$ for almost all $t \in \mathbb{T}$. We have remarked that for $k \in \mathbb{Z}$, we have $\widehat{u}(k) \in D(A)$ and

$$\begin{aligned} A\widehat{u}(k) &= \frac{-k^2\widehat{u}(k)}{1-a_k} + \frac{(\alpha + b_k)ik\widehat{u}(k)}{1-a_k} - \frac{\widehat{f}(k)}{1-a_k} \\ &= \frac{\widehat{w}(k)}{1-a_k} + \frac{(\alpha + b_k)\widehat{v}(k)}{1-a_k} - \frac{\widehat{f}(k)}{1-a_k}. \end{aligned}$$

In view of assumptions (**H0**) on a and (**H0a**) on b and the facts that $w, v, f \in \Gamma(\mathbb{T}, X) \subset L^1(\mathbb{T}, X)$, there exists $g \in \Gamma(\mathbb{T}, X)$ such that $A\widehat{u}(k) = \widehat{g}(k)$. Then by [2, Lemma 3.1], $u(t) \in D(A)$ for almost all $t \in \mathbb{T}$ and $Au \in \Gamma(\mathbb{T}, X)$. Clearly,

$$\left(\frac{d}{dt}(b \dot{*} u)\right)^{\wedge}(k) = ikb_k\widehat{u}(k) = b_k(ikN_k)\widehat{f}(k)$$

and

$$(a \dot{*} Au)^{\wedge}(k) = a_kA\widehat{u}(k) = a_k\widehat{g}(k).$$

Since $(a_k)_{k\in\mathbb{Z}}, (b_k)_{k\in\mathbb{Z}}$ and $(ikN_k)_{k\in\mathbb{Z}}$ are Γ-multipliers, we conclude that $\frac{d}{dt}(b \dot{*} u), a \dot{*} Au \in \Gamma(\mathbb{T}, X)$.

Now, from (2.4) and the uniqueness theorem of Fourier coefficients, we conclude that $u(t)$ satisfies (P_5) for a.e. $t \in [0, 2\pi]$. This shows the existence.

To show the uniqueness, let $u \in \Gamma^{[2]}(\mathbb{T}, X)$ be such that

$$u''(t) + \alpha u'(t) + \frac{d}{dt}(b \dot{*} u(t)) - Au(t) + a \dot{*} Au(t) = 0$$

for almost all $t \in \mathbb{T}$ and $u(0) = u(2\pi), u'(0) = u'(2\pi)$. Then taking Fourier transforms we have $\widehat{u}(k) \in D(A)$ and $[-k^2 + ik(\alpha + b_k) - (1 - a_k)A]\widehat{u}(k) = 0$ by [2, Lemma 3.1]. Since $d_k = (-k^2 + ik(\alpha + b_k))/(1 - a_k) \in \varrho(A)$, we must have $\widehat{u}(k) = 0$ for all $k \in \mathbb{Z}$. Thus $u = 0$ and the proof is finished. ∎

We remark that on a Hilbert space X, each bounded sequence is an L^2-multiplier. By the Riemann–Lebesgue lemma, if $a \in L^1(\mathbb{R}_+)$, then $\lim_{k\to\infty} a_k = 0$. Thus on a Hilbert space X the above theorem takes a particularly simple form:

COROLLARY 2.5. *Let X be a Hilbert space, $A : D(A) \subset X \to X$ be a closed linear operator, $\alpha \in \mathbb{C}$ and let $a, b \in L^1(\mathbb{R}_+)$. Assume that $a_k \neq 1$ for all $k \in \mathbb{Z}$. Then the following assertions are equivalent*:

(i) *The problem (P_5) has L^2-maximal regularity.*
(ii) $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$.

3. Maximal regularity on three function spaces. In this section, we apply Theorem 2.4 in three concrete function spaces: $L^p(\mathbb{T}, X)$ ($1 < p < \infty$), $B^s_{p,q}(\mathbb{T}, X)$ ($1 \le p, q \le \infty,\ s > 0$) and $F^s_{p,q}(\mathbb{T}, X)$ ($1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$ by imposing some conditions on the kernels $a, b \in L^1(\mathbb{R}_+)$. The three operator-valued multiplier theorems obtained in [2, 3, 6] on these function spaces are fundamental for our discussion. Versions of the multiplier theorems on the real line can be found in [14, 15].

For results about R-boundedness, we can refer to Bourgain [4], Weis [14, 15] and Arendt–Bu [2]. We merely recall the definition and some basic properties.

We let r_j be the jth Rademacher function on $[0, 1]$ given by $r_j(t) = \mathrm{sgn}(\sin(2^{j-1}t))$. For $x \in X$, we denote by $r_j \otimes x$ the vector-valued function $t \mapsto r_j(t)x$.

DEFINITION 3.1. Let X and Y be Banach spaces. A family $\mathbf{T} \subset \mathcal{L}(X, Y)$ is called *R-bounded* if there exists $C \ge 0$ such that

$$\Big\|\sum_{j=1}^{n} r_j \otimes T_j x_j\Big\|_{L^1(0,1;Y)} \le C\Big\|\sum_{j=1}^{n} r_j \otimes x_j\Big\|_{L^1(0,1;X)} \tag{3.1}$$

for all $T_1, \ldots, T_n \in \mathbf{T}$, $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$.

REMARK 3.2.

(a) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then it is clear from the definition that $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R-bounded.
(b) Each subset $\mathbf{M} \subset \mathcal{L}(X)$ of the form $\mathbf{M} = \{\lambda I : \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded. This follows from Kahane's contraction principle [13, §3.5.4].

In order to state our main results, we will use the following hypotheses for a scalar function $a \in L^1(\mathbb{R}_+)$ (we recall that the sequence $(a_k)_{k\in\mathbb{Z}}$ is defined by (2.3)):

(H1a) $(k(a_{k+1} - a_k))_{k\in\mathbb{Z}}$ is bounded.
(H1b) $a_k \neq 1$ for all $k \in \mathbb{Z}$.
(H2) $(ka_k)_{k\in\mathbb{Z}}$ and $(k^2(a_{k+1} - 2a_k + a_{k-1}))_{k\in\mathbb{Z}}$ are bounded.
(H3) $(ka_k)_{k\in\mathbb{Z}}$, $(k^2(a_{k+1} - 2a_k + a_{k-1}))_{k\in\mathbb{Z}}$ and $(k^3(a_{k+1} - 3a_k + 3a_{k-1} - a_{k-2}))_{k\in\mathbb{Z}}$ are bounded.

REMARK 3.3. From [11, Remarks 3.4 and 3.5], we know that these conditions are satisfied by a large class of functions, which correspond to the most common kernels encountered in applications. When we refer simply to (**H1**), we mean (**H1a**) and (**H1b**).

LEMMA 3.4.

(1) *Let X be a UMD space. Assume that $a \in L^1(\mathbb{R}_+)$ satisfies* (**H1**) *and $b \in L^1(\mathbb{R}_+)$ satisfies* (**H1a**). *Then $(a_k)_{k\in\mathbb{Z}}$, $((1-a_k)^{-1})_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ are L^p-multipliers whenever $1<p<\infty$.*
(2) *Let X be a Banach space. Assume that $a, b \in L^1(\mathbb{R}_+)$ satisfy* (**H2**) *and a satisfies* (**H1b**). *Then $(a_k)_{k\in\mathbb{Z}}$, $((1-a_k)^{-1})_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers whenever $1 \le p,q \le \infty$ and $s \in \mathbb{R}$.*
(3) *Let X be a Banach space. Assume that $a, b \in L^1(\mathbb{R}_+)$ satisfy* (**H3**) *and a satisfies* (**H1b**). *Then $(a_k)_{k\in\mathbb{Z}}$, $((1-a_k)^{-1})_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ are $F^s_{p,q}$-multipliers whenever $1 \le p < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$.*

These assertions follow from [10, Lemmas 2.9 and 3.8] and [5, Proposition 3.4]. We omit the details. The following is one of the main results of this paper.

THEOREM 3.5. *Let X be a UMD space and let A be a closed linear operator in X. Assume that $a, b \in L^1(\mathbb{R}_+)$ satisfy* (**H1**) *and* (**H1a**), *respectively. Then the following statements are equivalent*:

(i) *The problem (P_5) has L^p-maximal regularity for some (equivalently, all) $1<p<\infty$.*
(ii) *$(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ and $(M_k)_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. Since $a \in L^1(\mathbb{R}_+)$ satisfies (**H1**) and $b \in L^1(\mathbb{R}_+)$ satisfies (**H1a**), it follows that a satisfies (**H0**) and b satisfies (**H0a**) by Lemma 3.4. Thus Theorem 2.4 is applicable in the case $\Gamma(\mathbb{T}, X) = L^p(\mathbb{T}, X)$ when $1<p<\infty$.

(i)⇒(ii): Assume that (P_5) has L^p-maximal regularity for some $1<p<\infty$. By Theorem 2.4, $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ and $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier. The R-boundedness of $(M_k)_{k\in\mathbb{Z}}$ follows from [2, Proposition 1.11].

(ii)⇒(i): Fix $1<p<\infty$, and assume that $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ and $(M_k)_{k\in\mathbb{Z}}$ is R-bounded. In view of Theorem 2.4, it suffices to show that $(M_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier. We define

$$\mu_k = k^2 R(d_k, A) = -(1-a_k)M_k.$$

Then μ_k is R-bounded by Remark 3.2. We claim that $(k(\mu_{k+1}-\mu_k))_{k\in\mathbb{Z}}$ is also R-bounded. Indeed,

$$\begin{aligned} k(\mu_{k+1}-\mu_k) &= k[(k+1)^2 R(d_{k+1}, A) - k^2 R(d_k, A)] \\ &= k[(k+1)^2 (R(d_{k+1}, A) - R(d_k, A)) + (2k+1) R(d_k, A)] \end{aligned}$$

$$= k(k+1)^2(d_k - d_{k+1})R(d_{k+1},A)R(d_k,A) + (2k+1)kR(d_k,A)$$
$$= \frac{d_k - d_{k+1}}{k}\,\mu_k\mu_{k+1} + \frac{2k+1}{k}\,\mu_k.$$

We have

$$\frac{d_k - d_{k+1}}{k} = \frac{i(\alpha+b_k)}{1-a_k} - \frac{k+1}{k}\,\frac{i(\alpha+b_{k+1})}{1-a_{k+1}} + \frac{2k+1}{k}\,\frac{1}{1-a_{k+1}} + \frac{k(a_{k+1}-a_k)}{(1-a_k)(1-a_{k+1})},$$

which is clearly bounded. From the assumption on $a, b \in L^1(\mathbb{R}_+)$ and Lemma 3.4, we know that $((d_k - d_{k+1})/k)_{k\in\mathbb{Z}}$ is R-bounded. It follows that $(k(\mu_{k+1} - \mu_k))_{k\in\mathbb{Z}}$ is R-bounded. From [2, Theorem 1.3], we deduce that $(\mu_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier. Then $M_k = \frac{-1}{1-a_k}\mu_k$ is also an L^p-multiplier by Lemma 3.4 and Remark 2.2. The proof is complete. ∎

Now, we consider the maximal regularity for the problem (P_5) on periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$, where $1 \le p, q \le \infty$, $s > 0$. By [3], if X is an arbitrary Banach space, then the *Marcinkiewicz condition of order 2*, that is,

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k(M_{k+1} - M_k)\| + \|k^2(M_{k+1} - 2M_k + M_{k-1})\|) < \infty,$$

is sufficient for the sequence $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier whenever $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. So for the maximal regularity of the problem (P_5) in $B^s_{p,q}(\mathbb{T}, X)$, we must impose the stronger assumption (**H2**) on $a, b \in L^1(\mathbb{R}_+)$.

THEOREM 3.6. *Let X be a Banach space and let A be a closed linear operator in X. Assume that $a, b \in L^1(\mathbb{R}_+)$ satisfy* (**H2**) *and a satisfies* (**H1b**). *Then the following statements are equivalent*:

- (i) *The problem (P_5) has $B^s_{p,q}$-maximal regularity for some* (*equivalently, all*) $1 \le p, q \le \infty$ *and* $s > 0$.
- (ii) $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ *and* $(M_k)_{k\in\mathbb{Z}}$ *is bounded.*

Proof. Since $a, b \in L^1(\mathbb{R}_+)$ satisfy (**H2**) and a satisfies (**H1b**), we see that a satisfies (**H0**) and b satisfies (**H0a**) by Lemma 3.4. Thus Theorem 2.4 is applicable in the case $\Gamma(\mathbb{T}, X) = B^s_{p,q}(\mathbb{T}, X)$ when $1 \le p, q \le \infty$ and $s > 0$.

(i)⇒(ii): Assume that (P_5) has $B^s_{p,q}$-maximal regularity for some $1 \le p, q \le \infty$ and $s > 0$. Then in view of Theorem 2.4, $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ and $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Hence $(M_k)_{k\in\mathbb{Z}}$ must be bounded [3, Theorem 5.1].

(ii)⇒(i): Let $1 \le p, q \le \infty$ and $s > 0$ be fixed. To show that (P_5) has $B^s_{p,q}$-maximal regularity, it suffices to prove that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by Theorem 2.4. We let $\mu_k = k^2R(d_k, A)$ for $k \in \mathbb{Z}$ and we first show that

$(\mu_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. It is clear that (**H2**) implies (**H1a**). From the proof of Theorem 3.5, we know that $(\mu_k)_{k\in\mathbb{Z}}$ and $(k(\mu_{k+1}-\mu_k))_{k\in\mathbb{Z}}$ are bounded. To show that $(\mu_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier, we need only show that $(k^2(\mu_{k+1}-2\mu_k+\mu_{k-1}))_{k\in\mathbb{Z}}$ is bounded, by the Fourier multiplier theorem on periodic Besov spaces [3, Theorem 4.5]. For $k\in\mathbb{Z}$, we have

$$\begin{aligned}k^2(\mu_{k+1}-2\mu_k+\mu_{k-1}) &= k^4[R(d_{k+1},A)-2R(d_k,A)+R(d_{k-1},A)]\\ &\quad+2k^3[R(d_{k+1},A)-R(d_{k-1},A)]\\ &\quad+k^2[R(d_{k+1},A)+R(d_{k-1},A)] =: I_1+I_2+I_3.\end{aligned}$$

For I_1, we have

$$\begin{aligned}I_1 &= k^4R(d_k,A)[(d_k-d_{k+1})R(d_{k+1},A)-(d_{k-1}-d_k)R(d_{k-1},A)]\\ &= \mu_k k^2[(d_k-d_{k+1})(R(d_{k+1},A)-R(d_{k-1},A))\\ &\quad-(d_{k+1}-2d_k+d_{k-1})R(d_{k-1},A)]\\ &= \frac{d_k-d_{k+1}}{k}\,\frac{d_{k-1}-d_{k+1}}{k}\,\mu_k(k^2R(d_{k+1},A))(k^2R(d_{k-1},A))\\ &\quad-\mu_k(d_{k+1}-2d_k+d_{k-1})(k^2R(d_{k-1},A)).\end{aligned}$$

Since $(d_k-d_{k+1})/k$ is bounded, so is

$$\frac{d_{k-1}-d_{k+1}}{k}=\frac{d_{k-1}-d_k}{k}+\frac{d_k-d_{k+1}}{k}.$$

The sequences $k^2R(d_{k-1},A)=\frac{k^2}{(k-1)^2}\mu_{k-1}$ and $k^2R(d_{k+1},A)=\frac{k^2}{(k+1)^2}\mu_{k+1}$ are bounded. To show that I_1 is bounded, it remains to consider $d_{k+1}-2d_k+d_{k-1}$. We have

$$\begin{aligned}&d_{k+1}-2d_k+d_{k-1}\\ &\quad= ik\alpha\left(\frac{1}{1-a_{k+1}}-\frac{2}{1-a_k}+\frac{1}{1-a_{k-1}}\right)+i\alpha\left(\frac{1}{1-a_{k+1}}-\frac{1}{1-a_{k-1}}\right)\\ &\qquad+i\left(\frac{(k+1)b_{k+1}}{1-a_{k+1}}-\frac{2kb_k}{1-a_k}+\frac{(k-1)b_{k-1}}{1-a_{k-1}}\right)\\ &\qquad-k^2\left(\frac{1}{1-a_{k+1}}-\frac{2}{1-a_k}+\frac{1}{1-a_{k-1}}\right)\\ &\qquad-\frac{2+2k(a_{k+1}-a_{k-1})-a_{k+1}-a_{k-1}}{(1-a_{k+1})(1-a_{k-1})}.\end{aligned}$$

Each term in the above expression is bounded by the assumption on $a,b\in L^1(\mathbb{R}_+)$. We have shown that I_1 is bounded.

To estimate I_2 and I_3, we have

$$\begin{aligned}I_2 &= \frac{2(d_{k-1}-d_{k+1})}{k}\,(k^2R(d_{k+1},A))(k^2R(d_{k-1},A)),\\ I_3 &= k^2R(d_{k+1},A)+k^2R(d_{k-1},A).\end{aligned}$$

Thus the boundedness of I_2 and I_3 follows easily from the boundedness of $(d_{k-1} - d_{k+1})/k$, $k^2R(d_{k+1}, A)$ and $k^2R(d_{k-1}, A)$. We have shown that $(\mu_k)_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order 2. Therefore it is a $B^s_{p,q}$-multiplier [3, Theorem 4.5]. By the assumption on a and Lemma 3.4, $((1-a_k)^{-1})_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-multiplier. Therefore $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by Remark 2.2. The proof is complete. ∎

If the underlying Banach space X has a non-trivial Fourier type and $1 \le p, q \le \infty$, $s \in \mathbb{R}$, then the *Marcinkiewicz condition of order 1*, that is,

$$\sup_k (\|M_k\| + \|k(M_{k+1} - M_k)\|) < \infty,$$

is already sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier [3, Theorem 4.5]. From this fact and the proof of Theorem 3.5, we easily deduce the following result on the $B^s_{p,q}$-maximal regularity of (P_5) under a weaker condition on a, b when X has a non-trivial Fourier type.

THEOREM 3.7. *Let X be a Banach space with non-trivial Fourier type. Assume that $a, b \in L^1(\mathbb{R}_+)$ satisfy* (**H1**) *and* (**H1a**), *respectively. Then for $1 \le p, q \le \infty$ and $s > 0$, the following statements are equivalent*:

(i) *The problem (P_5) has $B^s_{p,q}$-maximal regularity for some* (*equivalently, all*) $1 \le p, q \le \infty$, $s > 0$.
(ii) $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ *and $(M_k)_{k\in\mathbb{Z}}$ is bounded.*

Periodic Hölder continuous function spaces are a particular case of $B^s_{p,q}(\mathbb{T}, X)$. From [3, Theorem 3.1], we have

$$B^\alpha_{\infty,\infty}(\mathbb{T}, X) = C^\alpha_{\mathrm{per}}(\mathbb{T}, X) \quad \text{whenever } 0 < \alpha < 1,$$

where $C^\alpha_{\mathrm{per}}(\mathbb{T}, X)$ is the space of all X-valued functions f defined on $\mathbb{T}$ and such that $f(0) = f(2\pi)$ and $\sup_{x\ne y} \|f(x) - f(y)\|/|x-y|^\alpha$ is finite. Moreover, the norm

$$\|u\|_{C^\alpha_{\mathrm{per}}} := \max_{t\in\mathbb{T}} \|u(t)\| + \sup_{x\ne y} \frac{\|f(x) - f(y)\|}{|x-y|^\alpha}$$

on $C^\alpha_{\mathrm{per}}(\mathbb{T}, X)$ is an equivalent norm of $B^\alpha_{\infty,\infty}(\mathbb{T}, X)$. Thus Theorems 3.6 and 3.7 have the following corollary, where for $0 < \alpha < 1$ we say that (P_5) has C^α_{per}-maximal regularity if for every $f \in C^\alpha_{\mathrm{per}}(\mathbb{T}, X)$, there exists a unique $u \in C^{\alpha+2}_{\mathrm{per}}(\mathbb{T}, X)$ such that $u(t) \in D(A)$ and the equation of (P_5) holds for all $t \in [0, 2\pi]$, and $u'', u', Au, a \mathbin{\dot{*}} Au, \frac{d}{dt}(b \mathbin{\dot{*}} u) \in C^\alpha_{\mathrm{per}}(\mathbb{T}, X)$.

COROLLARY 3.8. *Let X be a Banach space and let $a, b \in L^1(\mathbb{R}_+)$. Then*:

1. *If $a, b \in L^1(\mathbb{R}_+)$ satisfy* (**H2**) *and a satisfies* (**H1b**), *then the problem (P_5) has C^α_{per}-maximal regularity for some* (*equivalently, all*) $0<\alpha<1$ *if and only if $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ and $(M_k)_{k\in\mathbb{Z}}$ is bounded.*

2. *If* X *has a non-trivial Fourier type and if* a *satisfies* (**H1**) *and* b *satisfies* (**H1a**), *then the problem* (P_5) *has* $C^{\alpha}_{\mathrm{per}}$*-maximal regularity for some* (*equivalently, all*) $0 < \alpha < 1$ *if and only if* $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ *and* $(M_k)_{k\in\mathbb{Z}}$ *is bounded.*

Let $M = (M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. We say that M satisfies the *Marcinkiewicz condition of order 3* if M satisfies the Marcinkiewicz condition of order 2 and

$$\sup_k \|k^3(M_{k+1} - 3M_k + 3M_{k-1} - M_{k-2})\| < \infty$$

(see [6]). Next, we prove maximal regularity of (P_5) on periodic Triebel spaces $F^s_{p,q}(\mathbb{T},X)$ when $1 \le p < \infty,\ 1 \le q \le \infty,\ s > 0$. We need the stronger condition (**H3**) on $a, b \in L^1(\mathbb{R}_+)$ because the Marcinkiewicz condition of order 3 is needed in the $F^s_{p,q}$-multiplier case [6, Theorem 3.2].

THEOREM 3.9. *Let* X *be a Banach space. Assume that* $a, b \in L^1(\mathbb{R}_+)$ *satisfy* (**H3**) *and* a *satisfies* (**H1b**). *Then the following assertions are equivalent*:

(i) *The problem* (P_5) *has* $F^s_{p,q}$*-maximal regularity for some* (*equivalently, all*) $1 \le p < \infty$, $1 \le q \le \infty$ *and* $s > 0$.
(ii) $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ *and* $(M_k)_{k\in\mathbb{Z}}$ *is bounded.*

Proof. Since $a, b \in L^1(\mathbb{R}_+)$ satisfy (**H3**) and a satisfies (**H1b**), we infer that a satisfies (**H0**) and b satisfies (**H0a**) by Lemma 3.4. Thus Theorem 2.4 is applicable in the case $\Gamma(\mathbb{T},X) = F^s_{p,q}(\mathbb{T},X)$ when $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$.

(i)⇒(ii): Assume that (P_5) has $F^s_{p,q}$-maximal regularity for some $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$. Then $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ and $(M_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier by Theorem 2.4. Hence $(M_k)_{k\in\mathbb{Z}}$ must be bounded [6, Theorem 4.1].

(ii)⇒(i): Let $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$ be fixed. To show that (P_5) has $F^s_{p,q}$-maximal regularity, it suffices to prove that $(M_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier by Theorem 2.4. We let $\mu_k = k^2R(d_k,A)$ for $k \in \mathbb{Z}$ and we first show that $(\mu_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier. It is clear that (**H3**) implies (**H2**). Thus the boundedness of μ_k, $k(\mu_{k+1}-\mu_k)$ and $k^2(\mu_{k+1}-2\mu_k+\mu_{k-1})$ follows from the proofs of Theorems 3.5 and 3.6. It remains to show that $k^3(\mu_{k+1} - 3\mu_k + 3\mu_{k-1} - \mu_{k-2})$ is bounded. We have

$$\begin{aligned}
&k^3(\mu_{k+1} - 3\mu_k + 3\mu_{k-1} - \mu_{k-2}) \\
&\quad = k^5[R(d_{k+1},A) - 3R(d_k,A) + 3R(d_{k-1},A) - R(d_{k-2},A)] \\
&\qquad + 2k^4[R(d_{k+1},A) - 3R(d_{k-1},A) + 2R(d_{k-2},A)] \\
&\qquad + k^3[R(d_{k+1},A) + 3R(d_{k-1},A) - 4R(d_{k-2},A)] =: J_1 + J_2 + J_3.
\end{aligned}$$

Now,

$$J_1 = -k(d_{k+1} - 3d_k + 3d_{k-1} - d_{k-2})\,\frac{k^2\mu_k\mu_{k-1}}{(k-1)^2}$$
$$+ \frac{d_{k+1} - d_{k-2}}{k}(d_{k+1} - 2d_k + d_{k-1})\,\frac{k^4\mu_{k-1}\mu_k\mu_{k+1}}{(k^2-1)^2}$$
$$+ \frac{d_{k+1} - d_{k-2}}{k}(d_k - 2d_{k-1} + d_{k-2})\,\frac{k^4\mu_{k-2}\mu_k\mu_{k+1}}{(k^2-k-2)^2}$$
$$+ 2\,\frac{d_{k+1} - d_{k-2}}{k}\,\frac{d_k - d_{k-1}}{k}\,\frac{d_{k-2} - d_{k-1}}{k}\,\frac{k^6\mu_{k-2}\mu_{k-1}\mu_k\mu_{k+1}}{(k^3-2k^2-k+2)^2},$$

$$J_2 = -2(d_{k+1} - 2d_k + d_{k-1})\,\frac{k^2\mu_{k-1}\mu_k}{(k-1)^2}$$
$$+ 2\,\frac{d_k - d_{k+1}}{k}\,\frac{d_{k-1} - d_{k+1}}{k}\,\frac{k^4\mu_{k-1}\mu_k\mu_{k+1}}{(k^2-1)^2}$$
$$+ 4\,\frac{d_{k-1} - d_k}{k}\,\frac{d_{k-2} - d_k}{k}\,\frac{k^4\mu_{k-2}\mu_{k-1}\mu_k}{(k^2-3k+2)^2}$$
$$- 4(d_k - 2d_{k-1} + d_{k-2})\,\frac{k^4\mu_{k-2}\mu_{k-1}}{(k^2-3k+2)^2},$$

$$J_3 = \frac{d_{k-2} - d_{k+1}}{k}\,\frac{k^4\mu_{k-2}\mu_{k+1}}{(k^2-k-2)^2}$$
$$+ \frac{3(d_{k-2} - d_{k-1})}{k}\,\frac{k^4\mu_{k-2}\mu_{k-1}}{(k^2-3k+2)^2}.$$

The boundedness of $\mu_k, (d_k - d_{k+1})/k$ and $d_{k+1} - 2d_k + d_{k-1}$ follows from the proof of Theorem 3.6. To show that J_1, J_2 and J_3 are bounded, it suffices to show that $k(d_{k+1} - 3d_k + 3d_{k-1} - d_{k-2})$ is bounded. We have

$$k(d_{k+1} - 3d_k + 3d_{k-1} - d_{k-2})$$
$$= ik^2\alpha\left(\frac{1}{1-a_{k+1}} - \frac{3}{1-a_k} + \frac{3}{1-a_{k-1}} - \frac{1}{1-a_{k-2}}\right)$$
$$+ ik\alpha\left(\frac{1}{1-a_{k+1}} - \frac{3}{1-a_{k-1}} + \frac{2}{1-a_{k-2}}\right)$$
$$+ ik\left(\frac{b_{k+1}}{1-a_{k+1}} - \frac{3b_{k-1}}{1-a_{k-1}} + \frac{2b_{k-2}}{1-a_{k-2}}\right)$$
$$+ ik^2\left[\left(\frac{b_{k+1}}{1-a_{k+1}} - \frac{2b_k}{1-a_k} + \frac{b_{k-1}}{1-a_{k-1}}\right)\right.$$

$$- \left(\frac{b_k}{1-a_k} - \frac{2b_{k-1}}{1-a_{k-1}} + \frac{b_{k-2}}{1-a_{k-2}} \right) \Bigg]$$
$$- k^3 \left[\frac{1}{1-a_{k+1}} - \frac{3}{1-a_k} + \frac{3}{1-a_{k-1}} - \frac{1}{1-a_{k-2}} \right]$$
$$- k \left[\frac{2k+1}{1-a_{k+1}} + \frac{3(-2k+1)}{1-a_{k-1}} - \frac{-4k+4}{1-a_{k-2}} \right]$$

The boundedness of the first and fifth brackets follows from the proof of [5, Proposition 3.4]. The fourth bracket is bounded by the proof of [11, Theorem 3.12]. We also have

$$ik\alpha \left(\frac{1}{1-a_{k+1}} - \frac{3}{1-a_{k-1}} + \frac{2}{1-a_{k-2}} \right) = i\alpha \frac{k(a_{k+1} + 2a_{k-2} - 3a_{k-1}) + ka_{k-1}a_{k-2} - 3ka_{k+1}a_{k-2} + 2ka_{k+1}a_{k-1}}{(1-a_{k+1})(1-a_{k-1})(1-a_{k-2})}$$

and

$$k \left[\frac{2k+1}{1-a_{k+1}} + \frac{3(-2k+1)}{1-a_{k-1}} - \frac{-4k+4}{1-a_{k-2}} \right] = \frac{J}{(1-a_{k+1})(1-a_{k-1})(1-a_{k-2})}$$

where

$$\begin{aligned} J = {} & 2k^2(a_{k+1} - 2a_k + a_{k-1}) + 4k^2(a_k - 2a_{k-1} + a_{k-2}) \\ & + (3ka_{k-1} - 4ka_{k-2} + ka_{k+1}) + (2k+1)ka_{k-1}a_{k-2} \\ & + (-6k+3)ka_{k+1}a_{k-2} + (4k-4)ka_{k+1}a_{k-1}. \end{aligned}$$

We have shown that $k(d_{k+1} - 3d_k + 3d_{k-1} - d_{k-2})$ is bounded by the assumption on $a \in L^1(\mathbb{R}_+)$. We deduce that $(\mu_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier by [6, Theorem 3.2]. From the assumptions on $a \in L^1(\mathbb{R}_+)$ and Lemma 3.4, $((1-a_k)^{-1})_{k\in\mathbb{Z}}$ is also an $F^s_{p,q}$-multiplier. From Remark 2.2, we deduce that $M_k = \frac{-1}{1-a_k}\mu_k$ is an $F^s_{p,q}$-multiplier. The proof is finished. ∎

Remark 3.10. When $1 < p < \infty$, $1 < q \le \infty$ and $s \in \mathbb{R}$, the Marcinkiewicz condition of order 2 is already sufficient for a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ to be an $F^s_{p,q}$-multiplier [6, Theorem 3.2]. This fact together with the proof of Theorem 3.6 implies that if $a, b \in L^1(\mathbb{R}_+)$ satisfy (**H2**), and a satisfies (**H1b**), then the problem (P_5) has $F^s_{p,q}$-maximal regularity for some (equivalently, all) $1 < p < \infty, 1 < q \le \infty$ and $s > 0$ if and only if $(d_k)_{k\in\mathbb{Z}} \subset \varrho(A)$ and $(M_k)_{k\in\mathbb{Z}}$ is bounded.

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