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On the characterization of scalar type spectral operators

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Abstract. The paper is concerned with conditions guaranteeing that a bounded operator in a reflexive Banach space is a scalar type spectral operator. The cases where the spectrum of the operator lies on the real axis and on the unit circle are studied separately.

1. Introduction. In [1], [11], [12] (see also [2]) necessary and sufficient conditions were established for an operator defined on a Hilbert space to be similar to a unitary or self-adjoint operator. These conditions were formulated in terms of the resolvent of the operator. The main purpose of this paper is to extend these results to the case of reflexive Banach spaces. In this case unitarity or self-adjointness are replaced by the property that the operator is scalar and its spectrum lies on the unit circle or on the real axis, respectively. Our results strengthen the corresponding results from [5], [8] (see also [4, p. 304]). Moreover, integral estimates for the resolvent ensuring that the corresponding operator is power-bounded are described. In the case of Hilbert spaces these estimates become necessary conditions as well.

The following notation will be used throughout the paper. B will denote a Banach space with the norm $\|\cdot\|$, and B^* its dual space. The norm in B^* will be denoted by $\|\cdot\|_*$. We shall denote by (x, y) the value of the functional y at the vector x. All operators are assumed to be linear and defined on B; I is the identity operator on B. For an operator T on B the resolvent set and the spectrum are denoted by $\varrho(T)$ and $\sigma(T)$, respectively. The resolvent operator $(T - \lambda I)^{-1}$, $\lambda \in \varrho(T)$, will be denoted by $R(T; \lambda)$; $r(T)$ denotes the spectral radius of T. Throughout the paper, c (with or without subscripts) will denote various positive constants, the exact values of which are not essential.

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2. Operators whose spectra lie on the unit circle. In the case of a reflexive Banach space the following result is known [5], [8]: a bounded operator T whose spectrum lies on the unit circle is a scalar type operator if and only if there exists a $c > 0$ such that for every collection $a_n \in \mathbb{C}$ ($n = 0, \pm 1, \ldots, \pm N$) one has

$$\|Q(T)\| \le c \sup_{|\lambda|=1} |Q(\lambda)|, \qquad Q(\lambda) = \sum_{n=-N}^{N} a_n \lambda^n. \tag{2.1}$$

We recall that an operator T is said to be *power-bounded* if

$$\sup_{n\in\mathbb{N}} \|T^n\| < \infty, \tag{2.2}$$

and *polynomially bounded* if there exists a $c > 0$ such that

$$\|P(T)\| \le c \sup_{|\lambda|=1} |P(\lambda)| \quad \text{for all } P(\lambda) = \sum_{n=0}^{N} a_n \lambda^n. \tag{2.3}$$

We note that (2.1) is equivalent to T being polynomially bounded with

$$\sup_{m\in\{0\}\cup\mathbb{N}} \frac{1}{m+1} \sum_{k=0}^{m} \|T^{-k}x\| \le c\|x\| \tag{2.4}$$

for some $c > 0$ and all $x \in B$. In fact, we observe that

$$Q(T) = \frac{1}{N} \sum_{l=N+1}^{2N} \sum_{n=-N}^{N} a_n T^{l+n} T^{-l}$$

and so, by (2.3) for T^* and (2.4), we have

$$\begin{aligned}
|(Q(T)x, y)| &\le \frac{1}{N} \sum_{l=N+1}^{2N} \Big| \Big(\sum_{n=-N}^{N} a_n T^{l+n} T^{-l} x, y \Big) \Big| \\
&= \frac{1}{N} \sum_{l=N+1}^{2N} \Big| \Big(T^{-l} x, \sum_{n=-N}^{N} \overline{a}_n T^{*l+n} y \Big) \Big| \\
&\le \frac{1}{N} \sum_{l=N+1}^{2N} \|T^{-l} x\| \Big\| \sum_{n=-N}^{N} \overline{a}_n T^{*l+n} y \Big\|_* \\
&\le \frac{1}{N} \sum_{l=N+1}^{2N} \|T^{-l} x\| \sup_{|\lambda|=1} \Big| \sum_{n=-N}^{N} \overline{a}_n \lambda^{l+n} \Big| \|y\|_* \\
&\le c \sup_{|\lambda|=1} |Q(\lambda)| \, \|x\| \, \|y\|_*
\end{aligned}$$

for all $x \in B$ and $y \in B^*$. By the uniform boundedness principle (see, for instance, [3, II.1]) the desired estimate (2.1) follows.

The following statement will be needed.

Lemma 2.1. *Let S be a bounded operator on the Banach space B with $r(S)\le 1$ and*

$$\sup_{r>1}\frac{r^2-1}{r}\int_0^{2\pi}|(R^2(S;re^{i\theta})x,y)|\,d\theta\le c\|x\|\,\|y\|_* \tag{2.5}$$

for all $x\in B$ and $y\in B^*$, where c is a positive constant. Then S is power-bounded.

Proof. We use the following identity [2]:

$$S^n=\frac{1}{2\pi i(n+1)}\int_{|\lambda|=r}\lambda^{n+1}R^2(S;\lambda)\,d\lambda,\qquad r>1,$$

which can be obtained from the Riesz integral by repeated integration by parts. For all $x\in B,\ y\in B^*,\ n=1,2,\ldots$ and $r>1$, according to (2.5), we have

$$\begin{aligned}2\pi|(S^nx,y)|&\le\frac{r^{n+2}}{n+1}\int_0^{2\pi}|(R^2(S;re^{i\theta})x,y)|\,d\theta\\&\le\frac{cr^{n+2}}{(r^2-1)(n+1)}\|x\|\,\|y\|_*.\end{aligned}$$

Letting $r=1+1/n$, we obtain

$$\sup_{n\in N}|(S^nx,y)|\le c\|x\|\,\|y\|_*,\qquad x\in B,\ y\in B^*,$$

and the assertion follows. ∎

The main result of this section is the following

Theorem 2.2. *Let T be a bounded operator on a reflexive Banach space B whose spectrum lies on the unit circle. A necessary and sufficient condition for T to be a scalar type spectral operator is that one of the following conditions holds*:

(1) *T and T^{-1} satsfy* (2.5), *i.e. there is a constant $c>0$ such that*

$$\sup_{r>1}(r^2-1)\int_0^{2\pi}[|(R^2(T^{-1};re^{i\theta})x,y)|+|(R^2(T;re^{i\theta})x,y)|]\,d\theta\le c\|x\|\|y\|_*.$$

(2) *T satisfies* (2.5) *and T^{-1} is power-bounded.*

(3) *T satisfies* (2.5) *and T^{-1} satisfies* (2.4).

(4) *There is a constant $c>0$ such that for all $x\in B$ and $y\in B^*$,*

$$\sup_{r>1}\frac{r^2-1}{r}\int_0^{2\pi}|(R(T;re^{i\theta})R(T;r^{-1}e^{i\theta})x,y)|\,d\theta\le c\|x\|\,\|y\|_*. \tag{2.6}$$

Theorem 2.2 is an immediate consequence of the following auxiliary statement, valid for an arbitrary Banach space and of independent interest.

Lemma 2.3. *Let T be a bounded operator on a Banach space with spectrum on the unit circle. Then each of conditions* (1)–(4) *of Theorem* 2.2 *is equivalent to the estimate* (2.1).

Proof. First we prove that (2.1) implies (1). By arguments used in [10], for each $x \in B$ and $y \in B^*$ there exists a function $g(t)$ with bounded variation, depending on x, y, such that

$$(T^n x, y) = \int_0^{2\pi} e^{int}\, dg(t), \qquad n \in \mathbb{Z}.$$

From this it is easy to deduce that

$$(R^2(T;\lambda)x, y) = \int_0^{2\pi} \frac{dg(t)}{(e^{it}-\lambda)^2}, \qquad |\lambda| > 1.$$

Then, putting $\lambda = re^{i\theta},\ r > 1$, one can obtain

$$\begin{aligned}\int_0^{2\pi} |(R^2(T;\lambda)x, y)|\, d\theta &\le \int_0^{2\pi}\int_0^{2\pi} \frac{|dg(t)|}{|e^{it}-\lambda|^2}\, d\theta \\ &\le \operatorname{var}(|g|) \sup_{t\in(0,2\pi)} \int_0^{2\pi} \frac{d\theta}{|e^{it}-\lambda|^2} \le \frac{c(g)}{r^2-1}.\end{aligned}$$

By applying the uniform boundedness principle twice, we obtain (2.5) for T. Similarly, (2.5) holds for the resolvent of T^{-1}.

The implication (1)⇒(2) follows by applying Lemma 2.1 to $S = T^{-1}$. The implication (2)⇒(3) is evident.

Now, we show that (3) implies (2.6). Note that

$$\begin{aligned}R(T;\lambda)R(T;\overline{\lambda}^{-1}) &= R(T;\lambda)[R(T;\overline{\lambda}^{-1}) - R(T;\lambda)] + R^2(T;\lambda) \\ &= (|\lambda|^2-1)R^2(T;\lambda)T^{-1}R(T^{-1};\overline{\lambda}) + R^2(T;\lambda)\end{aligned} \tag{2.7}$$

for $|\lambda| > 1$. On the other hand, since $r(T^{-1}) = 1$, we have

$$R(T^{-1};\lambda) = -\sum_{m=0}^{\infty} \frac{T^{-m}}{\lambda^{m+1}}, \qquad |\lambda| > 1.$$

From this and (2.7) it follows that

$$R(T;\lambda)R(T;\overline{\lambda}^{-1}) = -(|\lambda|^2-1)R^2(T;\lambda)\sum_{m=1}^{\infty} \frac{T^{-m}}{\overline{\lambda}^m} + R^2(T;\lambda),$$

and, by applying (2.5) for $S = T$, we obtain

$$\sup_{r>1} \frac{r^2-1}{r} \int_0^{2\pi} |(R(T;re^{i\theta})R(T;r^{-1}e^{i\theta})x,y)|\,d\theta \le c\|y\|_*\Big[(r^2-1)r^{-2}\sum_{m=0}^{\infty} r^{-m}\|T^{-m}x\| + \|x\|\Big], \qquad r>1.$$

Now, by using the inequality [1]

$$\frac{r^2-1}{r^2}\sum_{k=0}^{\infty}\frac{\alpha_k}{r^k} \le 2\sup_{m\in\mathbb{N}}\frac{1}{m+1}\sum_{k=0}^{m}\alpha_k, \qquad r>1,\ \alpha_k\ge 0,$$

for the sequence $\alpha_k = \|T^{-k}x\|$, and taking into account (2.4), it is easy to obtain (2.6).

To complete the proof it remains to show that (2.6) implies (2.1). From the equality [1]

$$2\pi r^{-|n|}T^n = (r^2-1)\int_0^{2\pi} e^{in\theta}R(T;re^{i\theta})R(T^{-1};re^{-i\theta})\,d\theta, \qquad r>1,\ n=0,\pm1,\ldots,$$

it follows that

$$2\pi\sum_{n=-N}^{N} a_n r^{-|n|}(T^n x,y) = (r^2-1)\int_0^{2\pi} Q(e^{i\theta})(R(T;re^{i\theta})R(T^{-1},re^{-i\theta})x,y)\,d\theta$$

for all $Q(\lambda)$ as in (2.1) and all $x\in B$, $y\in B^*$. Hence, by (2.6),

$$\Big|\sum_{n=-N}^{N} a_n r^{-|n|}(T^n x,y)\Big| \le c\sup_{|\lambda|=1}|Q(\lambda)|\,\|x\|\,\|y\|_*,$$

where $c>0$ does not depend on $r>1$. Letting $r\to 1$, and then using the uniform boundedness principle, we obtain (2.1). ∎

REMARK 2.4. By the uniform boundedness principle, condition (2.5) is equivalent to

$$\sup_{r>1}(r^2-1)\int_0^{2\pi}|(R^2(S;re^{i\theta})x,y)|\,d\theta < \infty, \qquad x\in B,\ y\in B^*.$$

A similar remark can be made with respect to (2.6).

By using Lemma 2.1 and methods from [1], [2] (see also [11], [12]) one can prove the following result concerning power-bounded operators on Hilbert spaces. This result was announced in [6].

COROLLARY 2.5. *Let* T *be a bounded operator on a Hilbert space* $\mathcal{H}$ *and* $r(T)\le 1$. *Then the following statements are equivalent*:

(1) *T is power-bounded.*

(2) *For each $x \in \mathcal{H}$,*

$$\sup_{r>1} (r^2-1) \int_0^{2\pi} [\|R(T;re^{i\theta})x\|^2 + \|R(T^*;re^{i\theta})x\|^2]\, d\theta < \infty. \tag{2.8}$$

(3) *For $x, y \in H$,*

$$\sup_{r>1} (r^2-1) \int_0^{2\pi} |(R^2(T;re^{i\theta})x, y)| d\theta < \infty. \tag{2.9}$$

Proof. It is easy to see that (2) implies (3). Hence, by Lemma 2.1, it is sufficient to prove that (2.8) holds for S power-bounded. But this follows immediately from the relations (cf. [12])

$$\begin{aligned} 2\pi \sum_{n=0}^{\infty} \|S^n x\|^2 r^{2n} &= \int_0^{2\pi} \Big\| \sum_{n=0}^{\infty} (re^{i\theta} S)^n x \Big\|^2 d\theta \\ &= \frac{1}{r} \int_{|\mu|=1/r} \|R(S;\mu)x\|^2\, |d\mu|. \ \blacksquare \end{aligned} \tag{2.10}$$

It follows from some results of van Casteren [1], [2] (see also [11] and [12]) that in a Hilbert space $\mathcal{H}$ an operator S is power-bounded if and only if

$$\sup_{m\in\mathbb{N}} \frac{1}{m} \sum_{k=1}^{m} (\|T^k x\|^2 + \|T^{*k} x\|^2) < \infty$$

for every $x \in \mathcal{H}$. Note that for general Banach spaces condition (2.5) is not necessary for uniform boundedness with respect to the natural powers. This can be seen even for a unitary operator acting on a reflexive space. In [5] (see also [9]) it was shown that the shift operator U defined in the space $l_p(\mathbb{Z})$ $(1 < p < \infty)$ of two-sided sequences $(x_n)_{n\in\mathbb{Z}}$ by

$$U(x_n) = (x_{n+1}) \quad ((x_n) \in l_p(\mathbb{Z}))$$

is not spectral for $p \neq 2$. Therefore, by Theorem 2.2, condition (2.5) cannot be valid. Note also that if condition (2.4) holds for $S = T^{-1}$, then, as is easy to observe, $\|S^n\| \le cn$, $n = 1, 2, \ldots$. This estimate is exact in the power scale, i.e. it is not possible to replace it by $\|S^n\| \le cn^{\alpha}$, $n = 1, 2, \ldots$, with $\alpha < 1$. The corresponding examples can be derived from [2].

3. Operators whose spectra lie on the real axis. Similar considerations to those of Section 2 can be carried out for a bounded operator with real spectrum. For this case it is known [8] that an operator A on a reflexive space B is of scalar type if and only if there exists a constant $c > 0$ such that for each $x \in B$ and $y \in B^*$, we have

$$\sup_{\sigma>0} \int_{-\infty}^{\infty} |((R(A; s+i\sigma) - R(A; s-i\sigma))x, y)|\, ds \le c\|x\|\,\|y\|_*. \tag{3.1}$$

Further, we need the following result [7]: let G be a closed densely defined operator on a Banach space B, with spectrum contained in the half-plane $\operatorname{Im}\lambda \ge 0$ and with

$$\sup_{\sigma>0} \sigma \int_{-\infty}^{\infty} |(R^2(G; s-i\sigma)x, y)|\, ds \le c\|x\|\|y\|_* \tag{3.2}$$

for $x \in B$ and $y \in B^*$. Then the operator iG generates an exponentially bounded C_0-semigroup e^{itG}, $t \ge 0$, i.e. $\|e^{itG}\| \le M$ for $0 \le t < \infty$ and constant $M \ge 1$. We note that in the case of Hilbert spaces the estimate (3.2) is necessary for iG to be the generator of a uniformly bounded C_0-semigroup. For general Banach spaces, the last assertion is not true (see, for instance, [7]).

We will use the inequality (see [1, Lemma 1.1])

$$\sup_{\sigma>0} \sigma \int_0^{\infty} e^{-\sigma t} f(t)\, dt \le \sup_{t>0} \frac{1}{t} \int_0^t f(s)\, ds,$$

holding for every nonnegative measurable function $f(t)$ on $t > 0$. We will also use the following simple auxiliary statement.

LEMMA 3.1. *Let $f(t)$ be a nonnegative continuous function on $t \ge 0$ satisfying the condition $f(t+s) \le f(t)f(s)$ for each $t, s \ge 0$. If*

$$\sup_{t>0} \frac{1}{t} \int_0^t f(s)\, ds = c < \infty$$

then

$$f(t) \le 2act, \qquad t \ge 2,$$

where $a = \max_{0\le t\le 2} f(t)$.

Proof. Since f is continuous on the interval $[n, n+1]$ $(n = 1, 2, \ldots)$, there is a point t_n such that

$$f(t_n) = \int_n^{n+1} f(t)\, dt \le \int_0^{n+1} f(t)\, dt \le c(n+1).$$

Further, for each $t \in [t_n, t_{n+1}]$, we have $t \ge n$, $0 < t - t_n \le 2$, $t_1 \le 2$, and also

$$f(t) = f(t - t_n + t_n) \le f(t-t_n)f(t_n) \le ac(n+1) \le 2acn \le 2act$$

for $t \ge 2$. ■

Corollary 3.2. *Let* $T(t)$, $t \geq 0$, *be a* C_0*-semigroup in a Banach space* B. *Suppose that*, *for all* $x \in B$,

$$\sup_{t>0} \frac{1}{t} \int_0^t \|T(s)x\|\, ds \leq c\|x\|.$$

Then

$$\sup_{\sigma>0} \sigma \int_0^\infty e^{-\sigma t} \|T(t)x\|\, dt \leq c\|x\|, \tag{3.3}$$

and there exists a constant $c_0 > 0$ *such that*

$$\|T(t)\| \leq c_0(t+1), \qquad t \geq 0. \tag{3.4}$$

Note that the estimate (3.4) is exact in the power scale on t (the corresponding examples can be obtained by arguments of [1, cf. Example 2, p. 254], for instance).

Theorem 3.3. *Let* A *be a bounded operator on a reflexive Banach space* B *with spectrum lying on the real axis. Then* A *is a scalar type spectral operator if and only if one of the following conditions holds*:

(1) *For all* $x \in B$ *and* $y \in B^*$,

$$\sup_{\sigma>0} \sigma \int_{-\infty}^\infty |(R^2(A; s \pm i\sigma)x, y)|\, ds \leq c\|x\|\, \|y\|_*. \tag{3.5}$$

(2) *For all* $x \in B$ *and* $y \in B^*$,

$$\sup_{\sigma>0} \sigma \int_{-\infty}^\infty |(R^2(A; s + i\sigma)x, y)|\, ds \leq c\|x\|\, \|y\|_*, \tag{3.6}$$

and for all $x \in B$,

$$\sup_{t>0} \frac{1}{t} \int_0^t \|e^{itA}x\|\, dt \leq c\|x\|. \tag{3.7}$$

Theorem 3.3 follows from the criterion (3.1) and the following lemma, in which A is allowed to be unbounded.

Lemma 3.4. *Let* A *be a densely defined closed linear operator on a Banach space* B, *with spectrum lying on the real axis. Then each of conditions* (1) *and* (2) *of Theorem* 3.3 *is equivalent to the estimate* (3.1).

Proof. In order to prove (1)⇒(2) we observe that by arguments from [7] the estimate (3.2) implies in particular that iA is the generator of a uniformly bounded C_0-semigroup, and therefore (3.7) holds as well.

Next, we prove that (2) implies (3.1). To this end, we observe that from (3.7), by Corollary 3.2, it follows that

$$R(A;s-i\sigma)=iR(iA;\sigma+is)=-i\int_0^\infty e^{-\sigma t}e^{-ist}e^{itA}\,dt,\qquad \sigma>0. \tag{3.8}$$

Also,

$$R(A;\lambda)-R(A;\overline{\lambda})=2i\sigma R(A;\lambda)[R(A;\lambda)-2i\sigma R(A;\lambda)R(A;\overline{\lambda})] \tag{3.9}$$

for $\lambda=s+i\sigma,\ \sigma\neq 0$. By (3.8) and (3.9),

$$\begin{aligned}(R(A;\lambda)x,y)&-(R(A;\overline{\lambda})x,y)\\ &=2i\sigma(R^2(A;\lambda)x,y)-4i\sigma^2\int_0^\infty e^{-\sigma t}e^{-ist}(R^2(A;\lambda)e^{itA}x,y)\,dt\end{aligned} \tag{3.10}$$

for all $x\in B$, $y\in B^*$ and $\lambda=s+i\sigma$, $\sigma>0$.

On the other hand, by applying (3.6) and (3.3) to the semigroup $T(t)=e^{itA}$, we obtain

$$\begin{aligned}\sigma^2\int_{-\infty}^\infty&\Big|\int_0^\infty e^{-\sigma t}e^{-ist}(R^2(A;s+i\sigma)e^{itA}x,y)\,dt\Big|\,ds\\ &\le\sigma^2\int_0^\infty e^{-\sigma t}\int_{-\infty}^\infty|(R^2(A;s+i\sigma)e^{itA}x,y)|\,ds\,dt\\ &\le c\sigma\|y\|_*\int_0^\infty e^{-\sigma t}\|e^{itA}x\|\,dt\le c\|x\|\,\|y\|_*.\end{aligned} \tag{3.11}$$

Now (3.10), (3.11) and (3.6) yield (3.1).

It remains to show that (3.1) implies (3.5). We apply similar arguments to those used in [1] (in particular, see [1, Lemma 1.2 and Theorem 3.1]). For fixed $x\in B$ and $y\in B^*$ we consider the following harmonic function of $s\in\mathbb{R}$ and $\sigma>0$:

$$G(s,\sigma)=(R(A;s+i\sigma)x,y)-(R(A;s-i\sigma)x,y).$$

By (3.1), we have

$$\sup_{\sigma>0}\int_{-\infty}^\infty|G(s,\sigma)|\,ds<\infty$$

and thus (cf. [13])

$$G(s,\sigma)=\frac{\sigma}{\pi}\int_{-\infty}^\infty\frac{d\mu(\eta)}{\sigma^2+(s-\eta)^2},$$

where $\mu(\eta)$ is a finite (complex) measure on $\mathbb{R}$. Since

$$\frac{2\sigma}{\sigma^2+(s-\eta)^2}=\frac{1}{\sigma+i(s-\eta)}+\frac{1}{\sigma-i(s-\eta)},$$

one has

$$(3.12)\quad (R(A;\lambda)x,y)-(R(A;\overline{\lambda})x,y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu(\eta)}{\overline{\lambda}-\eta}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu(\eta)}{\lambda-\eta},\qquad \lambda=s+i\sigma,\ \sigma>0.$$

By differentiating (3.12) with respect to λ and $\overline{\lambda}$, it follows that

$$(R^2(A;\lambda)x,y)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu(\eta)}{(\lambda-\eta)^2},$$

$$(R^2(A;\overline{\lambda})x,y)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu(\eta)}{(\overline{\lambda}-\eta)^2},$$

where $\lambda=s+i\sigma$, $\sigma>0$. Thus

$$\int_{-\infty}^{\infty}|(R^2(s+i\sigma)x,y)|\,ds=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{d\mu(\eta)}{(s+i\sigma-\eta)^2}\right|ds$$
$$\le\frac{M}{2\pi}\sup_{\eta\in\mathbb{R}}\int_{-\infty}^{\infty}\frac{ds}{(s-\eta)^2+\sigma^2}=\frac{M}{2\sigma},\qquad \sigma>0,$$

where

$$M=\int_{-\infty}^{\infty}|d\mu(\eta)|<\infty.$$

Hence,

$$\sup_{\sigma>0}\sigma\int_{-\infty}^{\infty}|(R^2(A;s+i\sigma)x,y)|\,ds<\infty,$$

and, analogously,

$$\sup_{\sigma>0}\sigma\int_{-\infty}^{\infty}|(R^2(A;s-i\sigma)x,y)|\,ds<\infty.$$

By the uniform boundedness principle, (3.5) follows. ∎

4. Comments. In the case of Hilbert spaces a bounded operator whose spectrum lies on the unit circle is of scalar type if and only if it is similar to a unitary operator. Therefore in the case of Hilbert spaces it is natural to compare conditions (1)–(4) of Theorem 2.1 with known criteria for similarity to unitaries. In view of Corollary 2.5 conditions (1) and (2) of Theorem 2.1 for the Hilbert space case amount to the condition that T and T^{-1} are power-bounded, a classical criterion of B. Sz.-Nagy [14]. In turn, condition (4) coincides with the criterion for similarity to a unitary operator obtained in [1] (see also [11], [12]).

Let us turn to condition (3) of Theorem 2.2. For the Hilbert space case, by Corollary 2.5, this condition is equivalent to the condition that T is power-bounded and satisfies (2.4). On the other hand, in [1] it is established that a power-bounded operator T on a Hilbert space is similar to a unitary operator if and only if

$$\sup_{|\lambda|\leq 1}(1-|\lambda|)\|R(T;\lambda)\|<\infty. \tag{4.1}$$

Moreover, the implication (2.4)⇒(4.1) is valid, but the converse is not. Thus, in the case of Hilbert spaces condition (3) of Theorem 2.2 is weaker than the criterion of similarity to a unitary operator ((2.2) and (4.1)). In this connection there remains an open question: is it true that in the case of reflexive Banach spaces conditions (2.2) and (4.1) give a criterion for T to be a scalar operator? A more general question is the following (cf. Lemma 2.3): is it true that in an arbitrary Banach space for a power-bounded operator T conditions (2.4) and (4.1) are equivalent? Analogous remarks can be made with reference to Theorem 3.3. In particular, the following question arises: is it true that in the case of reflexive Banach spaces condition (3.6) and the estimate

$$\sup_{\operatorname{Im}\lambda<0}(\|R(A;\lambda)\|\,|\operatorname{Im}\lambda|)<\infty$$

give a criterion for A to be a scalar type spectral operator?

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