STUDIA MATHEMATICA 166 (2) (2005)

Almost-distribution cosine functions and integrated cosine functions

by

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Abstract. We introduce the notion of almost-distribution cosine functions in a setting similar to that of distribution semigroups defined by Lions. We prove general results on equivalence between almost-distribution cosine functions and α-times integrated cosine functions.

Introduction. Integrated cosine functions of operators in Banach spaces have been introduced to study abstract second order "ill-posed" Cauchy problems ([11]). α-Times integrated cosine functions were introduced for $\alpha \in \mathbb{N}$ in [1] and later defined for $\alpha \geq 0$ ([11], [12]). 0-times integrated cosine functions are usual cosine functions. Differential operators in Euclidean spaces are examples of α-times integrated cosine functions (see [1] and [11]).

E. Marschall considered vector-valued cosine transforms defined by cosine functions ([5]) and he applied them to study spectral properties and the spectral mapping theorem for cosine functions. The present author worked with trigonometric convolution products, cosine functions and sine functions (1-times integrated cosine functions) to define vector-valued cosine and sine transforms ([6]). Almost-distribution cosine function is a new related concept, closer to distribution semigroups defined by J.-L. Lions [4].

Every α-times integrated cosine function leads to an almost-distribution cosine function of order α. We apply Banach algebras $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ with respect to cosine convolution product, which are defined using Weyl fractional derivation. Conversely, almost-distribution cosine functions of order α define integrated cosine functions. These ideas also hold in the case of integrated semigroups and distribution semigroups (see [7]). The main facts of fractional calculi are presented in the first section.

2000 *Mathematics Subject Classification*: 47D09, 47D99, 26A33.

Key words and phrases: integrated cosine function, cosine transform, fractional calculus, cosine convolution product.

Partially supported by the Spanish Project BFM2001-1793, MCYT, DGI-FEDER and the DGA project E-12/25.

Notation. $\Re z$ is the real part of a complex number z; Γ is the Gamma function; X, Y are Banach spaces; $X \hookrightarrow Y$ means a continuous embedding; $T : X \to Y$ is a bounded linear map from X to Y and $\ker T$ is the kernel of T; $\mathcal{B}(X)$ is the set of bounded linear operators on X; C_α is a constant which may depend on α.

1. Fractional Banach algebras on $\mathbb{R}^+$. In this section we review some results and also prove new ones about Weyl fractional calculus (see Theorem 3). Let $\tau_0 : [0,\infty) \to [0,\infty)$ be a measurable function on $[0,\infty)$ such that $\tau_0(t+s) \le C_0\tau_0(t)\tau_0(s)$ and $\tau_0(t-s) \le C_0\tau_0(t)\tau_0(s)$ for any $0 < s < t$ and $C_0 > 0$. Then $L^1(\mathbb{R}^+, \tau_0)$ is the Banach space of functions f with $\|f\|_{\tau_0} := \int_0^\infty |f(t)|\tau_0(t)\,dt < \infty$. Take $f, g \in L^1(\mathbb{R}^+, \tau_0)$. Then $f * g, f \circ g \in L^1(\mathbb{R}^+, \tau_0)$, where

$$f * g(t) := \int_0^t f(t-s)g(s)\,ds, \qquad f \circ g(t) := \int_t^\infty f(s-t)g(s)\,ds, \qquad t \ge 0.$$

The *cosine convolution product* $f *_c g$ is defined by $f *_c g := \frac{1}{2}(f * g + f \circ g + g \circ f)$ (see [10]). Let $\mathcal{D}_+$ be the class of $\mathcal{C}^\infty$ functions of compact support on $[0,\infty)$. For $f \in \mathcal{D}_+$ and $\alpha > 0$, the *Weyl fractional integral* $W_+^{-\alpha} f$ of order α is defined by

$$W_+^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s)\,ds, \qquad t \ge 0,$$

and the *Weyl fractional derivative* $W_+^\alpha f$ of order α is given by

$$W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)\,ds, \qquad t \ge 0,$$

with $n = [\alpha] + 1$. It is known that $W_+^{\alpha+\beta} = W_+^\alpha(W_+^\beta)$ for any $\alpha, \beta \in \mathbb{R}$, where $W_+^0 = \mathrm{Id}$ is the identity operator ([8]). The following proposition can be checked directly:

Proposition 1. *Given $f, g \in \mathcal{D}_+$ and $\alpha \in \mathbb{R}$, we have*

(i) $W_+^\alpha(f \circ g) = f \circ W_+^\alpha g$.
(ii) $W_+^\alpha(f *_c g) = \frac{1}{2}(W_+^\alpha(f * g) + f \circ W_+^\alpha g + g \circ W_+^\alpha f)$.

Weyl fractional calculus can also be applied to functions not belonging to $\mathcal{D}_+$ (see [8, p. 248]). For example, let f and g be measurable functions on $[0,\infty)$ such that $W_+^{-\alpha} f$ exists and $g = W_+^{-\alpha} f$ a.e. Then we set $W_+^\alpha g = f$. For example, the *Bochner–Riesz functions* $(R_t^\theta)_{t>0}$ defined by

$$R_t^\theta(s) = \frac{(t-s)^\theta}{\Gamma(\theta+1)} \chi_{(0,t)}(s) \qquad \text{for } t > 0 \text{ and } \theta > -1$$

satisfy $W_+^\alpha R_t^\theta = R_t^{\theta-\alpha}$ for $\theta + 1 > \alpha \ge 0$.

We recall that Ω_α is the set of nondecreasing continuous functions τ_α on $(0,\infty)$ such that $\inf_{u>0} u^{-\alpha}\tau_\alpha(u) > 0$ and there exists a constant $C_\alpha > 0$ with

$$\int_{[0,r]\cup[s,s+r]} u^{\alpha-1}\tau_\alpha(r+s-u)\,du \le C_\alpha \tau_\alpha(r)\tau_\alpha(s), \qquad 0\le r\le s$$

(see [2]). The functions $\tau_\alpha(t)=t^\alpha$; $t^\beta(1+t)^\nu$ with $\beta\in[0,\alpha]$ and $\nu\ge\alpha-\beta$; $t^\beta e^{\tau t}$ with $\tau>0$ and $\beta\in[0,\alpha]$, all belong to Ω_α. If $\tau_\alpha\in\Omega_\alpha$ then $\tau_\nu\in\Omega_\nu$, where $\tau_\nu(t):=t^{\nu-\alpha}\tau_\alpha(t)$ for $t\ge 0$ and $\nu\ge\alpha$. The subset of functions $\tau_\alpha(t)=t^\alpha w_0(t)$, where w_0 is a continuous nondecreasing weight, is denoted by Ω_α^h (see [2] for more details).

Lemma 2. *Let $\alpha>0$ and $\tau_\alpha\in\Omega_\alpha$. If $0<s<t$ then*

(i) $\displaystyle\int_{t-s}^{t} (r-t+s)^{\alpha-1}\tau_\alpha(r)\,dr \le C_\alpha\tau_\alpha(t)\tau_\alpha(s).$

(ii) $\displaystyle\int_0^s (r+t-s)^{\alpha-1}\tau_\alpha(r)\,dr \le C_\alpha\tau_\alpha(t)\tau_\alpha(s).$

Proof. As τ_α is nondecreasing, we get

$$\int_{t-s}^{t} (r-t+s)^{\alpha-1}\tau_\alpha(r)\,dr \le \tau_\alpha(t)\int_{t-s}^{t}(r-t+s)^{\alpha-1}\,dr = \frac{\tau_\alpha(t)}{\alpha}\,s^\alpha \le C_\alpha\tau_\alpha(t)\tau_\alpha(s).$$

(ii) is proven in a similar way. ∎

In [2, Propositions 1.4 and 1.5] the convolution product $*$ is considered, leading to results similar to Theorem 3 below. We denote by $\mathrm{Mul}(\mathcal{A})$ the set of multipliers of a Banach algebra $\mathcal{A}$.

Theorem 3. *Let $\alpha>0$ and $\tau_\alpha\in\Omega_\alpha$. The expression*

$$q_{\tau_\alpha}(f) := \frac{1}{\Gamma(\alpha+1)}\int_0^\infty \tau_\alpha(t)|W_+^\alpha f(t)|\,dt, \qquad f\in\mathcal{D}_+,$$

defines a norm on $\mathcal{D}_+$. Moreover, $q_{\tau_\alpha}(f *_c g)\le C_\alpha q_{\tau_\alpha}(f)q_{\tau_\alpha}(g)$ for $f,g\in\mathcal{D}_+$, and $C_\alpha>0$ is independent of f and g. Denote by $\mathcal{T}_+^{(\alpha)}(\tau_\alpha,*_c)$ the Banach algebra obtained as the completion of $\mathcal{D}_+$ in the norm q_{τ_α}.

(i) $\mathcal{T}_+^{(\alpha)}(\tau_\alpha,*_c)\hookrightarrow\mathcal{T}_+^{(\alpha)}(t^\alpha,*_c)\hookrightarrow L^1(\mathbb{R}^+,*_c)$.

(ii) *If $\beta>\alpha>0$, and $\tau_\beta\in\Omega_\beta$ is such that*

$$\frac{1}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\int_0^t (t-s)^{\beta-\alpha-1}\tau_\alpha(s)\,ds \le \frac{1}{\Gamma(\beta+1)}\,\tau_\beta(t), \qquad t\ge 0,$$

then $\mathcal{T}_+^{(\beta)}(\tau_\beta, *_c) \hookrightarrow \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$; *in particular* $\mathcal{T}_+^{(\beta)}(t^\beta, *_c) \hookrightarrow \mathcal{T}_+^{(\alpha)}(t^\alpha, *_c)$.

(iii) $R_t^{\nu-1} \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ *for* $t>0$ *and* $\nu>\alpha$, *and* $q_{\tau_\alpha}(R_t^{\nu-1}) \le C_{\nu,\alpha} t^{\nu-\alpha}\tau_\alpha(t)$ *for* $t>0$, *where* $C_{\nu,\alpha}>0$ *is independent of* t.

(iv) $R_t^{\alpha-1} \in \mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c))$ *and* $\|R_t^{\alpha-1}\|_{\mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c))} \le C\tau_\alpha(t)$ *for* $t>0$.

Proof. Clearly q_{τ_α} is a norm on $\mathcal{D}_+$ and

$$q_{\tau_\alpha}(f *_c g) \le \frac{1}{2}(q_{\tau_\alpha}(f*g) + q_{\tau_\alpha}(f\circ g) + q_{\tau_\alpha}(g\circ f)).$$

As $q_{\tau_\alpha}(f*g) \le C_\alpha q_{\tau_\alpha}(f) q_{\tau_\alpha}(g)$ (see [2, Proposition 1.4]), it is enough to check $q_{\tau_\alpha}(f\circ g) \le C_\alpha q_{\tau_\alpha}(f) q_{\tau_\alpha}(g)$. We apply Proposition 1(i), the Fubini theorem and Lemma 2 to get

$$\begin{aligned} q_{\tau_\alpha}(f\circ g) &\le \int_0^\infty \tau_\alpha(t) \int_t^\infty \frac{1}{\Gamma(\alpha)} \int_{s-t}^\infty (u-s+t)^{\alpha-1} |W_+^\alpha f(u)|\,du\,|W_+^\alpha g(s)|\,ds\,dt \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^\infty |W_+^\alpha g(s)| \int_0^s |W_+^\alpha f(u)| \int_{s-u}^s (u-s+t)^{\alpha-1}\tau_\alpha(t)\,dt\,du\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^\infty |W_+^\alpha g(s)| \int_s^\infty |W_+^\alpha f(u)| \int_0^s (u-s+t)^{\alpha-1}\tau_\alpha(t)\,dt\,du\,ds \\ &\le C_\alpha q_{\tau_\alpha}(f) q_{\tau_\alpha}(g). \end{aligned}$$

(i) and (ii) are checked directly and (iii) appears in [2].

(iv) Take $f \in \mathcal{D}_+$; we shall prove $R_t^{\alpha-1} * f \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ for any $t>0$. By [2, Proposition 1.5], $R_t^{\alpha-1} \in \mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *))$, and it is enough to prove $R_t^{\alpha-1}\circ f$, $f\circ R_t^{\alpha-1} \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$. Since $W^\alpha(R_t^{\alpha-1}\circ f) = R_t^{\alpha-1}\circ W_+^\alpha f$ and $W_+^\alpha(f\circ R_t^{\alpha-1})(s) = f(s+t)$ for $s,t>0$, we use again Lemma 2 to obtain $R_t^{\alpha-1} \in \mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c))$, and $\|R_t^{\alpha-1}\|_{\mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c))} \le C\tau_\alpha(t)$ for $t>0$. ∎

If $\tau_\alpha \in \Omega_\alpha^h$ with $\alpha \ge 0$, the algebra $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ has bounded approximate identities (take $\phi \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ such that $\int_0^\infty \phi(t)\,dt = 1$ and consider $(\phi_s = (1/s)\phi(\cdot/s))_{0<s<1}$). In general, the algebras $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ do not have any bounded approximate identity.

2. α-Times integrated cosine functions. Given $\alpha>0$, a family $(C_\alpha(t))_{t\ge0} \subset \mathcal{B}(X)$ of strongly continuous operators is an *α-times integrated* *cosine function* if $C_\alpha(0)=0$ and

$$(1)\qquad 2\Gamma(\alpha)C_\alpha(t)C_\alpha(s)x = \Big(\int_t^{t+s} - \int_0^s\Big)(t+s-r)^{\alpha-1}C_\alpha(r)x\,dr + \int_{t-s}^{t}(r-t+s)^{\alpha-1}C_\alpha(r)x\,dr + \int_0^s(r+t-s)^{\alpha-1}C_\alpha(r)x\,dr$$

for all $t>s>0$ and $x\in X$. Every α-times integrated cosine function $(C_\alpha(t))_{t\ge 0}$ yields a ν-times integrated cosine function $(C_\nu(t))_{t\ge0}$ defined by

$$(2)\qquad C_\nu(t)x := \frac{1}{\Gamma(\nu-\alpha)}\int_0^t (t-s)^{\nu-\alpha-1}C_\alpha(s)x\,ds,\qquad t\ge 0,\ x\in X.$$

0-Times integrated cosine functions are usual cosine functions. An α-times integrated cosine function $(C_\alpha(t))_{t\ge0}$ is called *nondegenerate* if $C_\alpha(t)x=0$ for all $t\ge 0$ implies that $x=0$. We only consider nondegenerate integrated cosine functions. We define its *generator* $(A,D(A))$, where $D(A)$ is the set of $x\in X$ such that there exists $y\in X$ satisfying

$$(3)\qquad C_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}\,x = \int_0^t (t-r)C_\alpha(r)y\,dr,\qquad t>0,$$

and $Ax:=y$. It is straightforward to check that $(A,D(A))$ is a closed operator. For every $x\in X$, $\int_0^t(t-s)C_\alpha(s)x\,ds\in D(A)$ and

$$C_\alpha(t)x = A\int_0^t (t-s)C_\alpha(s)x\,ds + \frac{t^\alpha}{\Gamma(\alpha+1)}\,x.$$

If $x\in D(A)$ then $C_\alpha(\,\cdot\,)x$ is differentiable for $t\ge0$ and

$$(4)\qquad \begin{aligned} \frac{d}{dt}C_\alpha(t)x &= \int_0^t C_\alpha(s)Ax\,ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\,x \quad \text{for } \alpha>0,\\ \frac{d}{dt}C_0(t)x &= \int_0^t C_0(s)Ax\,ds \end{aligned}$$

([9], [11]). If $\|C_\alpha(t)\|\le Ce^{\lambda_0 t}$ with $C,\lambda_0\ge 0$, condition (1) is equivalent (via Laplace transform) to

$$R(\lambda^2,A)x := \lambda^{\alpha-1}\int_0^\infty e^{-\lambda t}C_\alpha(t)x\,dt,\qquad x\in X,\ \Re\lambda^2>\lambda_0,$$

being a pseudo-resolvent operator, i.e.,

$$R(\lambda^2,A)-R(\mu^2,A) = (\mu^2-\lambda^2)R(\lambda^2,A)R(\mu^2,A),\qquad \Re\lambda^2,\Re\mu^2>\lambda_0$$

(for $\alpha = n$ see [9, Theorem 1.3]). In the nondegenerate case, $\Re\lambda^2$ belongs to the resolvent set $\varrho(A)$ and $R(\lambda^2, A) = (\lambda^2 - A)^{-1}$.

For $\nu > \alpha$, $(R_t^{\nu-1})_{t>0}$ is a ν-times integrated cosine function in $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$. This may be proved using Laplace transform. This family is the canonical integrated cosine function (see Theorem 4(i)). In the case of cosine functions, this result appears in [5].

THEOREM 4. *Let $(C_\alpha(t))_{t\geq 0}$ be an α-times integrated cosine function on X generated by $(A, D(A))$ such that $\|C_\alpha(t)\| \leq C\tau_\alpha(t)$, $t \geq 0$, where $\tau_\alpha \in \Omega_\alpha$. Then the map $\mathcal{C}_+ : \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c) \to \mathcal{B}(X)$ given by*

$$\mathcal{C}_+(f)x = \int_0^\infty W_+^\alpha f(t) C_\alpha(t)x\,dt, \qquad f \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c),\ x \in X,$$

is a continuous Banach algebra homomorphism. Moreover,

(i) *If $\nu > \alpha$ and $(R_t^{\nu-1})_{t>0}$ are the Bochner–Riesz functions then $C_\nu(t) = \mathcal{C}_+(R_t^{\nu-1})$, where $(C_\nu(t))_{t\geq 0}$ is defined as in* (2) *and*

$$\int_0^\infty W_+^\alpha f(t) C_\alpha(t)x\,dt = \int_0^\infty W_+^\nu f(t) C_\nu(t)x\,dt, \qquad x \in X,$$

for $f \in \mathcal{T}_+^{(\nu)}(\tau_\nu, *_c) \hookrightarrow \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ and $\tau_\nu(t) := t^{\nu-\alpha}\tau_\alpha(t)$ for $t \geq 0$.

(ii) *If $x \in D(A)$ then $C_\alpha(\cdot)x$ is differentiable for any $t \geq 0$, and for $f \in \mathcal{D}_+$,*

$$\int_0^\infty W_+^\alpha f(t) \frac{d}{dt} C_\alpha(t)x\,dt = A\mathcal{C}_+(W_+^{-1}f)x + f(0)x \qquad \text{for } \alpha > 0,$$

$$\int_0^\infty f(t) \frac{d}{dt} C_0(t)x\,dt = A\mathcal{C}_+(W_+^{-1}f)x.$$

Proof. We suppose $\alpha > 0$. As $\|C_\alpha(t)\| \leq C\tau_\alpha(t)$ for any $t \geq 0$, the expression

$$\mathcal{C}_+(f)x := \int_0^\infty W_+^\alpha f(t) C_\alpha(t)x\,dt, \qquad f \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c),\ x \in X,$$

defines a continuous linear homomorphism $\mathcal{C}_+ : \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c) \to \mathcal{B}(X)$. Indeed, for $f, g \in \mathcal{D}_+$ we shall prove $\mathcal{C}_+(f *_c g) = \mathcal{C}_+(f)\mathcal{C}_+(g)$. By Proposition 1(ii), we have

$$\begin{aligned}\Gamma(\alpha)\mathcal{C}_+(f *_c g)x &= \Gamma(\alpha) \int_0^\infty W_+^\alpha (f *_c g)(t) C_\alpha(t)x\,dt \\ &= \frac{\Gamma(\alpha)}{2} \int_0^\infty (W_+^\alpha (f * g) + f \circ W_+^\alpha g + g \circ W_+^\alpha f)(t) C_\alpha(t)x\,dt,\end{aligned}$$

for $x \in X$. Using [7, Proposition 1.1] and Fubini theorem, we get

$$\begin{aligned}
&\Gamma(\alpha)\int_0^\infty W_+^\alpha(f*g)(t)C_\alpha(t)x\,dt\\
&= \int_0^\infty W_+^\alpha g(r)\int_0^r W_+^\alpha f(s)\Big(\Big(\int_r^{s+r}-\int_0^s\Big)(s+r-t)^{\alpha-1}C_\alpha(t)x\,dt\Big)\,ds\,dr\\
&\quad+\int_0^\infty W_+^\alpha g(r)\int_r^\infty W_+^\alpha f(s)\Big(\Big(\int_s^{s+r}-\int_0^r\Big)(s+r-t)^{\alpha-1}C_\alpha(t)x\,dt\Big)\,ds\,dr.
\end{aligned}$$

Again by the Fubini theorem,

$$\int_0^\infty (f\circ W_+^\alpha g)(t)C_\alpha(t)x\,dt=\int_0^\infty W_+^\alpha g(r)\int_0^r f(r-t)C_\alpha(t)x\,dt\,dr$$

and using $f(r-t)=W_+^{-\alpha}(W_+^\alpha f)(r-t)$, we obtain

$$\begin{aligned}
&\Gamma(\alpha)\int_0^\infty (f\circ W_+^\alpha g)(t)C_\alpha(t)x\,dt\\
&\qquad= \int_0^\infty W_+^\alpha g(r)\int_0^r W_+^\alpha f(s)\int_{r-s}^r (s-r+t)^{\alpha-1}C_\alpha(t)x\,dt\,ds\,dr\\
&\qquad\quad+\int_0^\infty W_+^\alpha g(r)\int_r^\infty W_+^\alpha f(s)\int_0^r (t+s-r)^{\alpha-1}C_\alpha(t)x\,dt\,ds\,dr.
\end{aligned}$$

In the same way, we also get

$$\begin{aligned}
&\Gamma(\alpha)\int_0^\infty (g\circ W_+^\alpha f)(t)C_\alpha(t)x\,dt\\
&\qquad= \int_0^\infty W_+^\alpha g(r)\int_r^\infty W_+^\alpha f(s)\int_{s-r}^r (s-r+t)^{\alpha-1}C_\alpha(t)x\,dt\,ds\,dr\\
&\qquad\quad+\int_0^\infty W_+^\alpha g(r)\int_0^r W_+^\alpha f(s)\int_0^r (t+s-r)^{\alpha-1}C_\alpha(t)x\,dt\,ds\,dr.
\end{aligned}$$

We join these six summands to conclude that

$$\begin{aligned}
\Gamma(\alpha)\mathcal{C}_+(f*_c g)x &= \Gamma(\alpha)\int_0^\infty W_+^\alpha g(r)C_\alpha(r)\int_0^r W_+^\alpha f(s)C_\alpha(s)x\,ds\,dr\\
&\quad+\Gamma(\alpha)\int_0^\infty W_+^\alpha g(r)C_\alpha(r)\int_r^\infty W_+^\alpha f(s)C_\alpha(s)x\,ds\,dr\\
&= \Gamma(\alpha)\mathcal{C}_+(g)\mathcal{C}_+(f)x.
\end{aligned}$$

Parts (i) and (ii) are checked using (3) and (4). ■

3. Almost-distribution cosine functions

DEFINITION 5. An *almost-distribution cosine function* on X is a continuous linear map $\mathcal{C}_+ : \mathcal{D}_+ \to \mathcal{B}(X)$ with the following properties:

(i) $\mathcal{C}_+(f *_c g) = \mathcal{C}_+(f)\mathcal{C}_+(g)$ for any $f, g \in \mathcal{D}_+$.

(ii) $\bigcap\{\ker \mathcal{C}_+(f) \mid f \in \mathcal{D}_+\} = \{0\}$.

The *generator* $(A, D(A))$ of an almost-distribution cosine function $\mathcal{C}_+$ is defined by: $x \in D(A)$ when there exists $y \in X$ such that $\mathcal{C}_+(f)y = \mathcal{C}_+(f'')x + f'(0)x$ for any $f \in \mathcal{D}_+$, and $Ax := y$. The generator $(A, D(A))$ is well defined, closed, $\mathcal{C}_+(\mathcal{D}_+)X \subset D(A)$, and $A\mathcal{C}_+(f)x = \mathcal{C}_+(f'')x + f'(0)x$ for any $f \in \mathcal{D}_+$ and $x \in X$. If $x \in D(A)$ and $f \in \mathcal{D}_+$ then $A\mathcal{C}_+(f)x = \mathcal{C}_+(f)Ax$. An almost-distribution cosine function $\mathcal{C}_+$ is said to be of *order* $\alpha > 0$ and *growth* $\tau_\alpha \in \Omega_\alpha$ if $\mathcal{C}_+$, regarded as defined on $\mathcal{D}_+$, can be extended to a continuous linear map $\mathcal{C}_+ : \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c) \to \mathcal{B}(X)$. We connect this kind of almost-distribution cosine functions and integrated cosine functions.

THEOREM 6. *Let $(C_\alpha(t))_{t\ge0}$ be an α-times integrated cosine function on X generated by $(A, D(A))$ such that $\|C_\alpha(t)\| \le C\tau_\alpha(t)$, $(t \ge 0, \tau_\alpha \in \Omega_\alpha)$. Then $(A, D(A))$ generates an almost-distribution cosine function, $\mathcal{C}_+ : \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c) \to \mathcal{B}(X)$,*

$$\mathcal{C}_+(f)x = \int_0^\infty W_+^\alpha f(t) C_\alpha(t)x\, dt, \qquad f \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c),\ x \in X,$$

of order $\alpha > 0$ and growth $\tau_\alpha \in \Omega_\alpha$.

Proof. Condition (i) of Definition 5 is proven in Theorem 4. Take $x \in \bigcap_{f\in\mathcal{D}_+} \ker \mathcal{C}_+(f)$. Then $C_{\alpha+1}(t)x = 0$ and $C_\alpha(t)x = \frac{d}{dt}C_{\alpha+1}(t)x = 0$ for any $t > 0$, so we conclude that $x = 0$ and $\mathcal{C}_+$ is an almost-distribution cosine function.

Let $(B, D(B))$ be the generator of $\mathcal{C}_+$ and $x \in D(A)$ with $y = Ax$. Applying Theorem 4(i) and (3), we see that, for $f \in \mathcal{D}_+$,

$$\begin{aligned}\mathcal{C}_+(f)y &= \int_0^\infty W_+^{\alpha+2} f(t) C_{\alpha+2}(t)y\, dt = \int_0^\infty W_+^{\alpha+1} f(t) \int_0^t (t-s)C_\alpha(s)y\, ds\, dt \\ &= \int_0^\infty W_+^{\alpha+2} f(t)\left(C_\alpha(t)x - \frac{t^\alpha x}{\Gamma(\alpha+1)}\right) dt \\ &= \int_0^\infty W_+^\alpha f''(t) C_\alpha(t)x\, dt - \int_0^\infty W_+^{\alpha+2} f(t)\, \frac{t^\alpha x}{\Gamma(\alpha+1)}\, dt \\ &= \mathcal{C}_+(f'')x + f'(0)x.\end{aligned}$$

Hence $x \in D(B)$ and $Bx = Ax$. Now we take $x \in D(B)$ and $y = Bx$ and since $(R_t^{\alpha+2})_{t>0} \subset \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$, we get

$$C_{\alpha+3}(t)y = C_{\alpha+1}(t)x - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\,x.$$

Differentiating gives $C_{\alpha+2}(t)y = C_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x$. Thus $x \in D(A)$ and $Ax = Bx$. ∎

Now we prove the first converse to Theorem 6.

THEOREM 7. *Given $\alpha \geq 0$, $\tau_\alpha \in \Omega_\alpha$, and an almost-distribution cosine function $\mathcal{C}_+ : \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c) \to \mathcal{B}(X)$ generated by $(A, D(A))$. Then for any $\nu > \alpha$, $(A, D(A))$ generates a ν-times integrated cosine function $(C_\nu(t))_{t\geq 0}$ such that $\|C_\nu(t)\| \leq C_\nu \tau_\nu(t)$ with $\tau_\nu(t) = t^{\nu-\alpha}\tau_\alpha(t)$ for any $t \geq 0$, and*

$$\mathcal{C}_+(f)x = \int_0^\infty W_+^\nu f(t) C_\nu(t)x\,dt, \qquad f \in \mathcal{T}_+^{(\nu)}(\tau_\nu, *_c),\ x \in X.$$

Proof. Take $\nu > \alpha$. By Theorem 3(iii), the Bochner–Riesz functions $(R_t^{\nu-1})_{t\geq 0}$ belong to $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ with $\nu > \alpha$ and $q_{\tau_\alpha}(R_t^{\nu-1}) \leq C_{\nu,\alpha} t^{\nu-\alpha}\tau_\alpha(t)$ for $t \geq 0$. Moreover, $(R_t^{\nu-1})_{t\geq 0}$ is a ν-times integrated cosine function in $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$. We define $C_\nu(t) := \mathcal{C}_+(R_t^{\nu-1})$ for any $t \geq 0$. It is clear that $(C_\nu(t))_{t\geq 0}$ is a ν-times integrated cosine function and by the continuity of $\mathcal{C}_+$, $\|C_\nu(t)\| \leq C_{\nu,\alpha} t^{\nu-\alpha}\tau_\alpha(t)$ for any $t \geq 0$. Take now $\tau_\nu(t) := t^{\nu-\alpha}\tau_\alpha(t)$. Then $\tau_\nu \in \Omega_\nu$ and by Theorem 6, there exists $\mathcal{C}'_+ : \mathcal{T}_+^{(\nu)}(\tau_\nu, *_c) \to \mathcal{B}(X)$ such that

$$\mathcal{C}'_+(f)x = \int_0^\infty W_+^\nu f(t) C_\nu(t)x\,dt, \qquad f \in \mathcal{T}_+^{(\nu)}(\tau_\nu, *_c),\ x \in X.$$

As $\mathcal{T}_+^{(\nu)}(\tau_\nu, *_c) \hookrightarrow \mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$, and $\mathcal{C}_+$ is continuous, we have

$$\mathcal{C}'_+(f)x = \int_0^\infty W_+^\nu f(t)\mathcal{C}_+(R_t^{\nu-1})x\,dt = \mathcal{C}_+\Big(\int_0^\infty W_+^\nu f(t) R_t^{\nu-1}\,dt\Big)x = \mathcal{C}_+(f)x$$

for any $f \in \mathcal{T}_+^{(\nu)}(\tau_\nu, *_c)$ and $x \in X$. If $(B, D(B))$ generates $(C_\nu(t))_{t>0}$ then it generates $\mathcal{C}'_+$ (Theorem 6), and $(A, D(A)) = (B, D(B))$. ∎

By Theorem 3(iv), the Bochner–Riesz functions $(R_t^{\alpha-1})_{t>0}$ are multipliers of the algebra $\mathcal{T}^{(\alpha)}(\tau_\alpha, *_c)$ for any $t > 0$ and $(R_t^{\alpha-1})_{t>0}$ is an α-times integrated cosine function in $\mathrm{Mul}(\mathcal{T}^{(\alpha)}(\tau_\alpha, *_c))$. If $\mathcal{T}_+^{(\alpha)}(\tau_\alpha, *_c)$ has a bounded approximate identity, then Cohen's factorization theorem holds, and we may define $(\mathcal{C}_+(R_t^{\alpha-1}))_{t\geq 0}$. We get the second converse to Theorem 6. The proof is similar to [7, Theorem 4.9].

THEOREM 8. *Let $\alpha \geq 0$, $\tau_\alpha \in \Omega_\alpha^h$ and $(A, D(A))$ a closed and densely defined operator on X. The following conditions are equivalent.*

(i) $(A, D(A))$ *generates an* α*-times integrated cosine function* $(C_\alpha(t))_{t\geq 0}$ *such that* $\|C_\alpha(t)\| \leq C_\alpha \tau_\alpha(t)$ *for any* $t \geq 0$.

(ii) $(A, D(A))$ *generates an almost-distribution cosine function* $\mathcal{C}_+$ *of order* $\alpha > 0$ *and growth* τ_α *such that* $\mathcal{C}_+(\mathcal{D}_+)(X)$ *is dense in* X.

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Received February 23, 2004
Revised version July 13, 2004 (5368)