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Perturbations of isometries between $C(K)$-spaces

by

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Abstract. We study the Gromov–Hausdorff and Kadets distances between $C(K)$-spaces and their quotients. We prove that if the Gromov–Hausdorff distance between $C(K)$ and $C(L)$ is less than $1/16$ then K and L are homeomorphic. If the Kadets distance is less than one, and K and L are metrizable, then $C(K)$ and $C(L)$ are linearly isomorphic. For K and L countable, if $C(L)$ has a subquotient which is close enough to $C(K)$ in the Gromov–Hausdorff sense then K is homeomorphic to a clopen subset of L.

1. Introduction. The aim of this paper is to obtain a nonlinear version of the Amir–Cambern theorem [1, 4, 5] which states that if K and L are locally compact spaces such that the Banach–Mazur distance between $C_0(K)$ and $C_0(L)$ is less than 2, then K and L are homeomorphic. For this, we need a definition of nonlinear distances. Before giving it, let us recall some notation.

In this paper, X, Y and E are Banach spaces. The closed unit ball of X is denoted by B_X and its unit sphere by S_X. If K is a Hausdorff compact set, the space of real continuous functions on K is denoted by $C(K)$. It is equipped with the supremum norm. For a Hausdorff locally compact space K we also consider $C_0(K)$, the space of continuous functions on K which vanish at infinity. It coincides with $C(K)$ if K is compact. For $f \in C_0(K)$ and $U \subset K$, we write $\|f\|_U = \sup_{u\in U} |f(u)|$.

We define some nonlinear distances between Banach spaces related to the Banach–Mazur distance $\mathrm{d_{BM}}(\cdot,\cdot)$. We recall a definition from [8]. Let A and B be two bounded subsets of some pseudo-metric space M (we do not assume that the "distance" d separates the points of M). The *Hausdorff distance* between A and B is

$$\max[\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)].$$

The *Gromov–Hausdorff distance* between Banach spaces X and Y is the infimum of all Hausdorff distances between $i(B_X)$ and $j(B_Y)$, where i :

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$B_X \to M$ and $j : B_Y \to M$ are isometric embeddings into a common pseudo-metric space M. We denote this distance by $\mathrm{d_{GH}}(X,Y)$. The *Kadets distance* $\mathrm{d_K}(X,Y)$ is the infimum of all Hausdorff distances between $i(B_X)$ and $j(B_Y)$, where $i : X \to E$ and $j : Y \to E$ are linear isometric embeddings into a common normed space E. These are pseudo-metrics, since two Banach spaces can have Kadets distance 0 without being isomorphic (see [11]).

In Section 2, we study separable $C(K)$-spaces which are close in the Gromov–Hausdorff sense. We prove in particular that if K and L are compact sets with $\mathrm{d_{GH}}(C(K), C(L)) < 1/16$, then K and L are homeomorphic. This result may be regarded as an extension of a theorem of Jarosz on Lipschitz homeomorphisms between $C(K)$-spaces [7]. Then we prove that, in the separable case, if $d_{\mathrm{K}}(C(K), C(L)) < 1$ then $C(K)$ and $C(L)$ are isomorphic (this does not however imply that K and L are homeomorphic). In Section 4, we study similar questions on quotients and subspaces of $C(K)$-spaces with K countable. In particular, we show that if K and L are countable Hausdorff compact sets and if X is a subquotient (that is to say, a subspace of an isometric quotient) of $C(L)$ with $\mathrm{d_{GH}}(C(K), X) < \varepsilon(L)$, where $\varepsilon(L)$ is some positive constant depending only on L, then K is homeomorphic to a subset of L.

2. Remarks on nonlinear distances. Let us also define some alternative nonlinear distances between Banach spaces.

When f is a Lipschitz map between two metric spaces, we denote by $l(f)$ the Lipschitz constant of f. The *Lipschitz distance* between two metric spaces X and Y is $\mathrm{d_L}(X,Y) = \inf l(f) \cdot l(f^{-1})$, where the infimum is taken over all Lipschitz homeomorphisms between X and Y.

If X and Y are two Banach spaces and $u : X \to Y$ is uniformly continuous, we call the number

$$l_\infty(u) = \inf_{\eta>0} \sup_{\|x-x'\|\ge\eta} \frac{\|u(x)-u(x')\|}{\|x-x'\|}$$

the *Lipschitz constant* of u *at infinity*. The *uniform distance* between X and Y is $\mathrm{d_U}(X,Y) = \inf l_\infty(u) \cdot l_\infty(u^{-1})$, where the infimum is taken over all uniform homeomorphisms between X and Y.

Following [2, Chapter 10], an (a,b)-*net* in a Banach space X is a subset $\mathcal{N}$ of X such that, for any $x, x' \in \mathcal{N}$ with $x \neq x'$, we have $\|x - x'\| \ge a$ and, for any $x \in X$, there exists $y \in \mathcal{N}$ with $\|x - y\| \le b$. We say that two Banach spaces are *net-equivalent* when they have Lipschitz homeomorphic nets. The *net distance* between X and Y is the number $\mathrm{d_N}(X,Y) = \inf \mathrm{d_L}(\mathcal{N},\mathcal{M})$, where the infimum is taken over all pairs $(\mathcal{N},\mathcal{M})$ of nets $\mathcal{N} \subset X$ and $\mathcal{M} \subset Y$. The following inequalities are clear: for any couple (X,Y) of Banach spaces, we have

$$\mathrm{d_N}(X,Y) \le \mathrm{d_U}(X,Y) \le \mathrm{d_L}(X,Y) \le \mathrm{d_{BM}}(X,Y).$$

The following fact shows that results on the Gromov–Hausdorff distance automatically yield similar results for the other notions of distance.

PROPOSITION 2.1. *If X and Y are Banach spaces, we have*

$$\mathrm{d_{GH}}(X,Y) \le \mathrm{d_K}(X,Y) \le \log \mathrm{d_{BM}}(X,Y),$$
$$\mathrm{d_{GH}}(X,Y) \le \mathrm{d_N}(X,Y) - 1.$$

Proof. The inequality $\mathrm{d_K}(X,Y) \le \log \mathrm{d_{BM}}(X,Y)$ is due to Ostrovskii [13]. We only show the last inequality.

Let C be a constant greater than $\mathrm{d_N}(X,Y)$ and $\eta > 0$ be arbitrary. There exists a positive number ε, an (ε,η)-net $\mathcal{M} \subset X$ and a one-to-one map $f : \mathcal{M} \to Y$ such that $\mathcal{N} = f(\mathcal{M})$ is an (ε',η)-net in Y for some $\varepsilon' > 0$ and, for any $m, m' \in \mathcal{M}$,

$$\|m - m'\| \le \|f(m) - f(m')\| \le C\|m - m'\|. \tag{1}$$

We can suppose that $0 \in \mathcal{M}$ and $0 = f(0)$. Let $J = \mathcal{M} \cap (1+\eta)B_X$, $A = B_X \cup J$ and $B = B_Y \cup f(J)$. It is easy to check that J is an (ε,η)-net of A and that $f(J)$ is an (ε',η)-net of B. On the disjoint union M of A and B, we define a new metric which coincides with the norm metric on A and B by defining, for every $x \in A$ and $y \in B$,

$$d(x,y) = \inf\{\|x - m\| + \alpha + \|y - f(m)\|\,;\, m \in J\},$$

where α is a positive number that we will choose in order that d is a metric. There are several cases to check in the triangle inequality. We only check the less easy one: for $x \in A$ and $y, y' \in B$, we have $d(x,y) + d(x,y') \ge \|y - y'\|$. It is sufficient to prove that, for any $(m,m') \in \mathcal{M}$, we have $\Delta \ge \|y - y'\|$, where

$$\Delta = \|y - f(m)\| + \alpha + \|m - x\| + \|x - m'\| + \alpha + \|y' - f(m')\|.$$

Using (1), we have

$$\Delta \ge 2\alpha + \|y - f(m)\| + \|y' - f(m')\| + \frac{\|f(m) - f(m')\|}{C} \ge 2\alpha + \frac{\|y - y'\|}{C}.$$

Since $\|y - y'\| \le \|y\| + \|y'\| \le 2C(1+\eta)$, we have $\Delta \ge \|y - y'\|$ provided $\alpha \ge (C-1)(1+\eta)$. Since $\eta > 0$ and $C > \mathrm{d_N}(X,Y)$ are arbitrary, we find $\mathrm{d_{GH}}(X,Y) \le \mathrm{d_N}(X,Y) - 1$. ∎

We conclude this section by taking the opportunity to correct an unfortunate error in the statement and proof of Proposition 3.3 in [8]. We are grateful to Tamara Kucherenko for bringing this error to our attention. Fortunately, the only difference in the new statement is in the size of the constants, and this does not change subsequent results in [8].

Proposition 2.2 (Proposition 3.3 in [8] corrected). *Let X and Y be Banach spaces and suppose $\Phi : X \to Y$ is a homogeneous map satisfying $\frac{1}{2}\|x\|_X \le \|\Phi(x)\|_Y \le \|x\|_X$ such that for a constant $0 \le \sigma < 1$ we have:*

(1) *Given $y \in Y$ there exists $x \in X$ with*

$$\|x\|_X \le \|y\|_Y \quad \text{and} \quad \|y - \Phi(x)\|_Y \le \sigma\|y\|_Y.$$

(2) *If $x_1, x_2, x_3 \in X$ and $\sum_{k=1}^3 x_k = 0$ then*

$$\Big\|\sum_{k=1}^{3}\Phi(x_k)\Big\|_Y \le \sigma\sum_{k=1}^{3}\|x_k\|_X.$$

Then if $\Psi : Y \to X$ is a homogeneous map satisfying $\|\Psi(y)\|_X \le \|y\|_Y$ and $\|y - \Phi(\Psi(y))\|_Y \le \sigma\|y\|_Y$ (whose existence is guaranteed by (1)) we have

$$|\,\|x - \Psi(y)\|_X - \|y - \Phi(x)\|_Y| \le 15\sigma(\|x\|_X + \|y\|_Y).$$

Furthermore for each $0 < r < 1$ there is a universal constant $C = C(r)$ such that $\Delta_r(\Phi, \Psi) \le C\sigma$.

If further we have:

(3) *If $x_1, \ldots, x_n \in X$ and $\sum_{k=1}^n x_k = 0$ then*

$$\Big\|\sum_{k=1}^{n}\Phi(x_k)\Big\|_Y \le \sigma\sum_{k=1}^{n}\|x_k\|_X,$$

then $\Delta(\Phi, \Psi) \le 30\sigma$.

Remark. In the statement we have replaced a constant 6 by 15 and 20 by 30. These constants have no essential importance in the remainder of the paper.

Proof. The error occurs on the sixth line of page 27 of [8]. There it is falsely assumed that $\|x - \Psi(y)\|_X \le \|\Phi(x - \Psi(y))\|_Y$.

From the inequality on line 2 of page 27, by putting $y = \Phi(x)$, it follows that

$$\|\Phi(x - \Psi\Phi(x))\|_Y \le 3\sigma(\|x\|_X + \|\Phi(x)\|_X) \le 6\sigma\|x\|_X$$

and so

$$\|x - \Psi\Phi(x)\|_X \le 12\sigma\|x\|_X.$$

This gives

$$\|x\|_X \le \|\Phi(x)\|_Y + 12\sigma\|x\|_X.$$

Replacing x by $x - \Psi(y)$ gives

$$\|x - \Psi(y)\|_X \le \|\Phi(x - \Psi(y))\|_Y + 12\sigma(\|x\|_X + \|y\|_Y),$$

and this implies

$$\|x - \Psi(y)\|_X \le \|\Phi(x) - y\|_Y + 15\sigma(\|x\|_X + \|y\|_Y).$$

This gives the first part of the proposition with constant 15. We leave it to the reader to make the appropriate adjustments in the latter part of the proof to obtain the constant 30. ■

3. Small nonlinear distances between $C_0(K)$-spaces

THEOREM 3.1. *Let K and L be Hausdorff locally compact spaces. If the Gromov–Hausdorff distance $\mathrm{d_{GH}}(C_0(K), C_0(L))$ is less than $1/16$ then K and L are homeomorphic.*

Proof. Suppose $\mathrm{d_{GH}}(C_0(K), C_0(L)) < \eta < 1/16$. Then there is a pseudometric d on $B_{C_0(K)} \cup B_{C_0(L)}$ such that:

1. If $f, g \in B_{C_0(K)}$ then $d(f, g) = \|f - g\|_K$.
2. If $f, g \in B_{C_0(L)}$ then $d(f, g) = \|f - g\|_L$.
3. If $f \in B_{C_0(K)}$ there exists $g \in B_{C_0(L)}$ with $d(f, g) < \eta$.
4. If $f \in B_{C_0(L)}$ there exists $g \in B_{C_0(K)}$ with $d(f, g) < \eta$.

Now we claim:

CLAIM 1. *Let $f \in B_{C_0(K)}$ and $g \in B_{C_0(L)}$ be such that $d(f, g) < \eta$. Then $|\,\|f\|_K - \|g\|_L| < 4\eta$.*

Proof of Claim 1. First suppose that the function $h \in B_{C_0(L)}$ is so that $d(0_{C_0(K)}, h) < \eta$. Then there exists $h' \in B_{C_0(L)}$ with $\|h - h'\|_L = 1 + \|h\|_L$. Pick $\varphi \in B_{C_0(K)}$ so that $d(\varphi, h') < \eta$. Then $1 \geq \|\varphi\|_K > \|h - h'\|_L - 2\eta$ so that $\|h\|_L < 2\eta$. It then follows that if $f \in B_{C_0(K)}$ and $g \in B_{C_0(L)}$ are such that $d(f, g) < \eta$ we have

$$\|g\|_L \leq \|h\|_L + \|g - h\|_L < 4\eta + \|f\|_K.$$

By symmetry we have

$$|\,\|f\|_K - \|g\|_L| < 4\eta.$$

CLAIM 2. *Given $s \in K$ there exists $t \in L$ and a scalar $\gamma = \pm 1$ such that if $f \in B_{C_0(K)}$ with $\|f\|_K < 1/2$ and $g \in B_{C_0(L)}$ with $d(f, g) < \eta$ then $|f(s) - \gamma g(t)| \leq 4\eta$.*

Proof of Claim 2. Let U be an open neighborhood of s. Pick a function $h = h_U \in B_{C_0(K)}$ with $h(s) = 1$, $0 \leq h \leq 1$ with support contained in U. Then pick $\varphi_1, \varphi_2 \in B_{C_0(L)}$ so that $d(h, \varphi_1) < \eta$ and $d(-h, \varphi_2) < \eta$. Since $d(h, -h) = 2$ we conclude that $\|\varphi_1 - \varphi_2\|_L > 2 - 2\eta$. Now pick a point $t_U \in L$ such that

$$|\varphi_1(t_U) - \varphi_2(t_U)| > 2 - 2\eta.$$

Let γ_U be the sign of $\varphi_1(t_U)$. It is then clear that $\gamma_U \varphi_1(t_U) > 1 - 2\eta$ and $\gamma_U \varphi_2(t_U) < -1 + 2\eta$.

Suppose $f \in B_{C_0(K)}$ and $\|f\| < 1/2$. Let $\delta_U = \sup_{u\in U} |f(u) - f(s)|$. Since $\|f-h\|_K > 1/2 \geq \|f-h\|_{K\setminus U}$, we have

$$\|f-h\|_K \leq \sup_{u\in U}(|f(u)-f(s)| + |f(s)-h(u)|) \leq \delta_U + 1 - f(s).$$

Similarly, we prove $\|f+h\|_K \leq \delta_U + 1 + f(s)$.

Now suppose $g \in B_{C_0(L)}$ and $d(f,g) < \eta$. Then

$$|g(t_U) - \varphi_1(t_U)| \leq d(g,\varphi_1) < \delta_U + 1 - f(s) + 2\eta,$$
$$|g(t_U) - \varphi_2(t_U)| \leq d(g,\varphi_2) < \delta_U + 1 + f(s) + 2\eta.$$

Since the left-hand sides sum to at least $2-2\eta$, it follows that

$$|g(t_U) - \varphi_1(t_U)| > 1 - f(s) - \delta_U - 4\eta,$$
$$|g(t_U) - \varphi_2(t_U)| > 1 + f(s) - \delta_U - 4\eta.$$

Since $\|g\|_L < 1/2 + 4\eta < \gamma_U\varphi_1(t_U)$, we have

$$1 \geq \gamma_U\varphi_1(t_U) > 1 - f(s) + \gamma_U g(t_U) - \delta_U - 4\eta,$$

so $\gamma_U g(t_U) - f(s) < \delta_U + 4\eta$. Working with φ_2 instead of φ_1 gives the symmetric inequality. Finally, we have

$$|\gamma_U g(t_U) - f(s)| < \delta_U + 4\eta.$$

Now we claim that t_U lies in some compact subset of L for sufficiently small U. Indeed, for sufficiently small U, we have $\delta_U < \eta$. Choose f in $C_0(K)$ so that $f(s) = \|f\|_K = 7\eta < 1/2$. Let $g \in C_0(L)$ be such that $d(f,g) < \eta$. There is a compact subset W of L such that $\|g\|_{L\setminus W} < 2\eta$. We have $|\gamma_U g(t_U) - 7\eta| < 5\eta$ so $|g(t_U)| > 2\eta$ and $t_U \in W$. Hence the net (t_U) has a limit point t as U runs through all open neighborhoods of s. Choosing γ a limit point of the net (γ_U) as well, we obtain the claim.

It follows from the second claim that we can define maps $\alpha : K \to L$, $\beta : L \to K$ and functions $u : K \to \{\pm 1\}$, $v : L \to \{\pm 1\}$ so that if $f \in B_{C_0(K)}$ and $g \in B_{C_0(L)}$ with $d(f,g) < \eta$ then

1. If $\|f\|_K < 1/2$ then for $s \in K$, $|f(s) - u(s)g(\alpha(s))| \leq 4\eta$,
2. If $\|g\|_L < 1/2$ then for $t \in L$, $|g(t) - v(t)f(\beta(t))| \leq 4\eta$.

We conclude the proof by showing that α is invertible and $\beta = \alpha^{-1}$, and that α, β are continuous.

Suppose $s \in K$ and let $s' = \beta \circ \alpha(s)$. Assume $s' \neq s$. Then we may find $f \in C_0(K)$ with $\|f\|_K = u(s)f(s) = 1/2 - 4\eta$ and $v(\alpha(s))f(s') = 4\eta - 1/2$. Pick $g \in B_{C_0(L)}$ so that $d(f,g) < \eta$. Then, using Claim 1, we see that $\|g\|_L < 1/2$. Hence

$$|f(s) - u(s)g(\alpha(s))| \leq 4\eta, \qquad |v(\alpha(s))f(s') - g(\alpha(s))| \leq 4\eta.$$

Thus

$$|u(s)f(s) - v(\alpha(s))f(s')| \leq 8\eta.$$

Hence $1-8\eta \leq 8\eta$, which contradicts $\eta < 1/16$. Thus $\beta \circ \alpha(s) = s$ for $s \in K$ and similarly $\alpha \circ \beta(t) = t$ for $t \in L$.

Let us next show that u is continuous. Fix $s \in K$. To prove that u is continuous at s, we first prove that there is an open neighborhood V of s so that $\alpha(V)$ is relatively compact. If not, we can construct a net (s_μ) tending to s and such that $\alpha(s_\mu)$ tends to infinity. Then choose $f \in C_0(K)$ such that $f(s) = \|f\|_K = 1/4$ and any $g \in B_{C_0(L)}$ such that $d(f,g) < \eta$. We have

$$|f(s_\mu) - u(s_\mu)g(\alpha(s_\mu))| \leq 4\eta.$$

Since $g(\alpha(s_\mu))$ tends to 0 and $u(s_\mu)$ is bounded, we obtain $1/4 \leq 4\eta$, which contradicts $\eta < 1/16$. Thus $\alpha(V)$ is relatively compact.

Now choose $g \in C_0(L)$ such that $g = 1/2 - 4\eta$ on $\alpha(V)$ and $\|g\|_L = 1/2 - 4\eta$. We may then pick $f \in C_0(K)$ with $\|f\|_K < 1/2$ and $d(f,g) < \eta$. Then $|f(s') - (1/2-4\eta)u(s')| \leq 4\eta$ for any $s' \in V$. So the sets $\{u = 1\} \cap V = \{f > 0\} \cap V$ and $\{u = -1\} \cap V = \{f < 0\} \cap V$ are both open. Hence u is continuous on V and in particular at s. Similarly v is continuous.

It remains to show that α is continuous. Suppose E is a closed subset of L. Suppose $s' \notin \alpha^{-1}(E)$. Then we may find $g \in C_0(L)$ with $\|g\|_L = 1/2 - 4\eta$ and $g(t) = 4\eta - 1/2$ for $t \in E$ but $g(\alpha(s')) = 1/2 - 4\eta$. Pick $f \in B_{C_0(K)}$ with $d(f,g) < \eta$ so that as in Claim 1, $\|f\|_K < 1/2$. Then

$$|u(s)f(s) + 1/2 - 4\eta| \leq 4\eta \quad \text{ for } s \in \alpha^{-1}(E)$$

but

$$|u(s')f(s') - 1/2 + 4\eta| \leq 4\eta.$$

Set $f' = uf$; this is a continuous function on K with $f'(s') > 0 > \sup_{s\in\alpha^{-1}(E)} f'(s)$ since $\eta < 1/16$. The point s' is not in the closure of $\alpha^{-1}(E)$; it follows that $\alpha^{-1}(E)$ is a closed set. This means that α and similarly β are continuous. ■

We do not know whether the constant $1/16$ can be improved. We notice that almost the same proof gives the same result for the Kadets distance with $1/10$ instead of $1/16$.

Before turning to subquotients of $C(K)$-spaces (for countable K), we add a result on the Kadets distance. It is optimal since the Kadets distance between two Banach spaces cannot exceed 1.

THEOREM 3.2. *Let K and L be two metrizable compact spaces. If the Kadets distance* $\mathrm{d_K}(C(K), C(L))$ *is less than* 1 *then $C(K)$ and $C(L)$ are isomorphic. If K and L are not supposed to be metrizable*, *then this condition implies* $\mathrm{Sz}(C(K)) = \mathrm{Sz}(C(L))$.

Let us recall that the Szlenk index of a Banach space X is defined as follows: let $\varepsilon > 0$ and C be a weak*-closed subset of X^*. The *first Szlenk derivative* of C is the set $C_\varepsilon^{[1]}$ of all points $x^* \in C$ which are weak* limits of sequences (x_n^*) in C such that $\|x_n^* - x^*\| \geq \varepsilon$ for any n. We can iterate this derivation transfinitely taking $C_\varepsilon^{[\alpha+1]} = (C_\varepsilon^{[\alpha]})_\varepsilon^{[1]}$ and, if α is a limit ordinal, $C_\varepsilon^{[\alpha]} = \bigcap_{\beta<\alpha} C_\varepsilon^{[\beta]}$. The *$\varepsilon$-index* of C is the smallest ordinal α such that $C_\varepsilon^{[\alpha]} = \emptyset$ (the ε-index is ω_1 when there is no such ordinal). The *ε-Szlenk index* $\mathrm{Sz}(X;\varepsilon)$ of X is the ε-index of B_{X^*}; the supremum of $\mathrm{Sz}(X;\varepsilon)$ for $\varepsilon > 0$ is denoted by $\mathrm{Sz}(X)$. It is shown in [14] that two separable $C(K)$-spaces are isomorphic if and only if they have the same Szlenk index.

Note that it is *not* true that $d_{\mathrm{K}}(C(K), C(L)) < 1$ implies that K and L are homeomorphic. Indeed, Cohen [6] gave an example of two non-homeomorphic metrizable compact sets K, L with $d_{\mathrm{BM}}(C(K), C(L)) = 2$. By Proposition 2.1 we have $d_{\mathrm{K}}(C(K), C(L)) \leq \log 2 < 1$.

Proof of Theorem 3.2. The proof requires the use of trees. Let us recall some basic facts about them. Let $s = (s_1, \ldots, s_m)$ be a finite sequence of integers. We define $s^+ = \{(s_1, \ldots, s_m, n)\,;\, n \in \mathbb{N}\}$. A *tree* is a nonempty set T of finite sequences of integers such that if $s^+ \cap T \neq \emptyset$ then $s \in T$. The *leaves* of the tree T are the elements $s \in T$ such that $s^+ \cap T = \emptyset$. The subtree T' is made of the elements $s \in T$ which are not leaves. We define a family $(T_\alpha)_{\alpha<\omega_1}$ of trees by taking $T_0 = \{\emptyset\}$, $T_{\alpha+1} = \{\emptyset\} \cup \bigcup_{n=0}^\infty n^\frown T_\alpha$, where $n^\frown T_\alpha = \{(n, s_1, \ldots, s_m)\,;\, (s_1, \ldots, s_m) \in T_\alpha\}$. To complete the definition of $(T_\alpha)_{\alpha<\omega_1}$, we choose for any limit ordinal α a sequence (ξ_n^α) which increases to α and we put $T_\alpha = \{\emptyset\} \cup \bigcup_{n=0}^\infty n^\frown T_{\xi_n^\alpha}$.

Let X be a separable Banach space, $\varepsilon > 0$ and $\alpha < \omega_1$. An *(ε, α)-Szlenk tree map* is a family $(x_s^*)_{s\in T_\alpha}$ in B_{X^*} such that:

1. For any $s \in T'_\alpha$, x_s^* is a weak*-cluster point of $\{x_t^*\,;\, t \in s^+\}$.
2. For any $s \in T'_\alpha$ and $t \in s^+$, we have $\|x_s^* - x_t^*\| \geq \varepsilon$.

The point $x_\emptyset^*$ is called the *root* of the tree map. It is clear that x^* is an element of $(B_{X^*})_\varepsilon^{[\alpha]}$ if and only if x^* is the root of some (ε, α)-Szlenk tree map. We prove two simple claims about Szlenk tree maps from which it is easy to deduce Theorem 3.2.

Claim 1. *Let E be a separable Banach space and X a subspace of E. Any (ε, α)-Szlenk tree map on X can be lifted to an (ε, α)-Szlenk tree map on E.*

Proof of Claim 1. We argue by induction on α. The initial step is the Hahn–Banach theorem. Let $\alpha \geq 1$ be an ordinal. If $\alpha = \beta + 1$, let us write $\xi_n^\alpha = \beta$ for any $n \in \mathbb{N}$. By our induction hypothesis, there exist families $(e_{s,n}^*)_{s\in T_{\xi_n^\alpha}}$ in B_{E^*} extending the families $(x_s^*)_{s\in T_{\xi_n^\alpha}}$. Since E is separable, by passing to subsequences, we can suppose that (x_n^*) weak*-converges to $x_\emptyset^*$ and $(e_{\emptyset,n}^*)$ weak*-converges to some point that we denote by $e_\emptyset^*$. Then we define $e_{n^\frown s}^* = e_{s,n}^*$ for any $n \in \mathbb{N}$ and $s \in T_{\xi_n^\alpha}$. It suffices to consider the family $(e_s^*)_{s\in T_\alpha}$ to establish the claim.

Claim 2. *Let $\eta > 0$ and $\varepsilon > 2\eta$. If X and Y are separable Banach spaces such that* $\mathrm{d_K}(X,Y) < \eta$, *then* $\mathrm{Sz}(Y;\varepsilon-2\eta) \geq \mathrm{Sz}(X;\varepsilon)$.

Proof of Claim 2. Let E be a separable Banach space containing isometric copies of X and Y (that we denote by X and Y) such that the Hausdorff distance between B_X and B_Y is less than η. Let $\alpha < \mathrm{Sz}(X;\varepsilon)$ and let $(x_s^*)_{s\in T_\alpha}$ be an (ε,α)-Szlenk tree map. Define $(e_s^*)_{s\in T_\alpha}$ as in Claim 1 and $(y_s^*)_{s\in T_\alpha}$ as the restriction of the family $(e_s^*)_{s\in T_\alpha}$ to Y. Let s be in T'_α and t in s^+. There exists $x \in S_X$ such that $\langle x_t^* - x_s^*, x\rangle > \varepsilon$. Let $y \in S_Y$ be such that $\|x-y\| < \eta$. Then $\|y_t^* - y_s^*\| \geq \langle e_t^* - e_s^*, y\rangle \geq \varepsilon - 2\eta$. Hence $(y_s^*)_{s\in T_\alpha}$ is an $(\varepsilon-2\eta,\alpha)$-Szlenk tree map. Therefore, $\mathrm{Sz}(Y;\varepsilon-2\eta) > \alpha$. Since $\alpha < \mathrm{Sz}(X;\varepsilon)$ is arbitrary, we are done.

Let us return to the proof of the theorem. Let K and L be two metrizable compact spaces such that $\mathrm{d_K}(C(K),C(L)) < \eta < 1$. If K and L are both uncountable, the classical Milyutin theorem ensures that $C(K)$ and $C(L)$ are isomorphic. Suppose that K is countable. By Claim 2, we have $\mathrm{Sz}(C(L);2) \leq \mathrm{Sz}(C(K);2-2\eta) \leq \mathrm{Sz}(C(K)) < \omega_1$. By Lemma 4.1 of [14], this implies that L is countable and $\mathrm{Sz}(C(L)) \leq \mathrm{Sz}(C(K))$. Symmetry gives us $\mathrm{Sz}(C(L)) = \mathrm{Sz}(C(K))$, which shows that $C(K)$ and $C(L)$ are isomorphic.

Now, suppose that K and L are Hausdorff compact spaces such that $\mathrm{d_K}(C(K),C(L)) < \eta < 1$. We use the following claims which prove that the Kadets distance is separably determined.

Claim 3. *Let X and Y be two subspaces of a Banach space E and let $A \subseteq X$, $B \subseteq Y$ be separable subspaces. For any ε greater than the Hausdorff distance between B_X and B_Y, there exist separable subspaces A_ε and B_ε such that $A \subseteq A_\varepsilon \subseteq X$, $B \subseteq B_\varepsilon \subseteq Y$ and the Hausdorff distance between B_{A_ε} and B_{B_ε} is less than or equal to ε.*

Proof of Claim 3. We put $A_0 = A$, $B_0 = B$ and, by induction, let $(x_{n,p})_{p\in\mathbb{N}}$ be a sequence dense in S_{A_n}. For any integer p, we can find a vector $y_{n,p} \in S_Y$ such that $\|x_{n,p} - y_{n,p}\| < \varepsilon$. We let B_{n+1} be the closed linear span of $B_n \cup \{y_{n,p}\,;\, p \in \mathbb{N}\}$. Since B_{n+1} is separable, we can choose $(z_{n,p})_{p\in\mathbb{N}}$ dense in $S_{B_{n+1}}$. For every $p \in \mathbb{N}$, we get $w_{n,p} \in S_X$ such that $\|w_{n,p} - z_{n,p}\| < \varepsilon$. We define A_{n+1} as the closed linear span of $A_n \cup \{w_{n,p}\,;\, p \in \mathbb{N}\}$. It is separable and we can continue the construction. Finally, we define $A_\varepsilon = \overline{\bigcup_{n=0}^\infty A_n}$ and $B_\varepsilon = \overline{\bigcup_{n=0}^\infty B_n}$.

Claim 4. *Let K and L be two Hausdorff compact spaces and ε greater than* $\mathrm{d_K}(C(K),C(L))$. *Let $A \subseteq C(K)$ and $B \subseteq C(L)$ be separable subspaces. There exist two continuous maps $\varrho_K : K \to \widetilde{K}$ and $\varrho_L : L \to \widetilde{L}$ such that $A \subseteq \{f\circ\varrho_K\,;\, f \in C(\widetilde{K})\}$, $B \subseteq \{g\circ\varrho_L\,;\, g\in C(\widetilde{L})\}$, $\widetilde{K}$ and $\widetilde{L}$ are metrizable and* $\mathrm{d_K}(C(\widetilde{K}),C(\widetilde{L})) \leq \varepsilon$.

Proof of Claim 4. We use a classical lemma: if A is a separable subspace of $C(K)$, then there exists a continuous map $r : K \to K_1$ such that K_1 is metrizable and $A \subseteq \{f \circ r\,;\, f \in C(K_1)\}$.

Let X and Y be two isometric copies of $C(K)$ and $C(L)$ as in Claim 3. We apply the classical lemma to $C(K)$ (which defines $r : K \to K_1$) and to $C(L)$ (which gives us $s : L \to L_1$). Set $A^1 = \{f \circ r\,;\, f \in C(K_1)\}$ and $B^1 = \{g \circ s\,;\, g \in C(L_1)\}$. Then we apply Claim 3 with A^1 and B^1 instead of A and B. We denote by A^2 and B^2 the resulting subspaces. We can iterate this construction inductively. Finally, we put

$$A' = \overline{\bigcup_{n\geq 0} A^{2n+1}} = \overline{\bigcup_{n\geq 0} A^{2n+2}}, \qquad B' = \overline{\bigcup_{n\geq 0} B^{2n+1}} = \overline{\bigcup_{n\geq 0} B^{2n+2}}.$$

The sets A' and B' are unital subalgebras of $C(K)$ and $C(L)$. On K, we define the equivalence relation $a \sim b$ when $f(a) = f(b)$ for any $f \in A'$. The Stone–Weierstrass theorem proves that $A' = \{f \circ \varrho_K\,;\, f \in C(\widetilde{K})\}$, where $\widetilde{K}$ is the quotient space defined by $\sim$ and $\varrho_K : K \to \widetilde{K}$ is the canonical surjection. Similarly, we define $\widetilde{L}$ and ϱ_L. It is easy to check that the Hausdorff distance between $B_{A'}$ and $B_{B'}$ does not exceed ε.

Let us return to the proof of Theorem 3.2. By Claim 4, there exist metrizable compact spaces $\widetilde{K}$ and $\widetilde{L}$ and subspaces A' and B' such that $A \subseteq A' \subseteq C(K)$, $B \subseteq B' \subseteq C(L)$, A' is isometric to $C(\widetilde{K})$, B' is isometric to $C(\widetilde{L})$ and $\mathrm{d_K}(C(\widetilde{K}), C(\widetilde{L})) \leq \eta < 1$. Using the metric case, we infer that $\mathrm{Sz}\,(C(\widetilde{K})) = \mathrm{Sz}\,(C(\widetilde{L}))$. This shows that $\mathrm{Sz}(A) \leq \mathrm{Sz}\,(C(L))$ and $\mathrm{Sz}(B) \leq \mathrm{Sz}\,(C(K))$. Then Proposition 4.12 of [10] ensures that $C(K)$ and $C(L)$ have the same Szlenk index. ■

We conclude by remarking that for any $\varepsilon > 0$ there is a Banach space $X = X_\varepsilon$ so that $\mathrm{d_K}(X, C[0,1]) < \varepsilon$ but X is not isomorphic to a $C(K)$-space. This can be formally deduced from the following fact: if Y is a Banach space and E is a closed subspace of Y and Z is isomorphic to $Y/E \oplus E$ then there is a sequence of spaces Y_n each isomorphic to Y but such that $\lim_{n\to\infty} \mathrm{d_K}(Y_n, Z) = 0$. This idea is used in [12, Lemma 5.9] and [8, Proposition 5.7]. We sketch the idea. Consider the direct sum $Y \oplus Y/E$ with a norm such that the subspace $E \oplus Y/E$ is isometric to Z. Let Y_n be the subspace $\{(n^{-1}y, Qy)\,;\, y \in Y\}$, where $Q : Y \to Y/E$ is the quotient map. Then Y_n converges in Kadets metric to $E \oplus Y/E$ which is an isometric copy of Z.

Thus it suffices to produce a space X which is not a $C(K)$-space but with a subspace E such that $X/E \oplus E$ is isomorphic to $C[0,1]$. See [3, Corollary 2.4] for such an example.

4. Subquotients of $C(K)$-spaces. We begin this section with some new linear results on $C(K)$-spaces when K is countable.

Let us introduce some definition. Suppose K is a countable Hausdorff compact set. The *Cantor–Bendixson derivative* of K is the set $K^{(1)}$ of all cluster points of K. We can iterate transfinitely this derivation taking $K^{(\alpha+1)} = (K^{(\alpha)})^{(1)}$ and, if α is a limit ordinal, $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$. It is obvious that there exists a countable ordinal α such that $K^{(\alpha)}$ is finite. The minimal ordinal with this property is denoted by $\sigma(K)$; notice that $K^{(\sigma(K)+1)} = \emptyset$. We denote the cardinality of $K^{(\sigma(K))}$ by $\nu(K)$. The couple $\chi(K) = (\sigma(K), \nu(K)) \in \omega_1 \times \omega$ is called the *characteristic system* of K. In the Cantor normal form $\sigma(K) = \omega^{\alpha_1} + \cdots + \omega^{\alpha_p}$ (with $\alpha_1 \geq \cdots \geq \alpha_p$), we write $o(K) = \alpha_1$ and $p(K) = p$. If L is another countable Hausdorff compact set, we say that K is *simpler* than L when $\chi(K) \leq \chi(L)$ (with the usual lexicographic order on $\omega_1 \times \omega$). If $\chi(K) < \chi(L)$, then K is said to be *strictly simpler* than L.

It is a classical result (see [15, p. 155]) that two countable Hausdorff compact sets are homeomorphic if and only if they have the same characteristic system. Moreover, K is simpler than L if and only if K is homeomorphic to a clopen subset of L, which is equivalent to saying that there exists a continuous map from L onto K.

Next we introduce some technical terminology.

Let X and Y be two Banach spaces and T be a linear map from X onto Y. We denote by $\gamma(T)$ the lower bound of the constants $r > 0$ such that $T(rB_X) \supset B_Y$. If $r > \gamma(T)$, then $x \in X$ is an (r,T)*-preimage* if $\|x\| \leq r\|Tx\|$.

For $r > 1$, we write $\varrho_0(r) = 1$ and, inductively on j, we define $\varrho_{j+1}(r) = 2\varrho_j(r) - r$. We obtain immediately

$$\varrho_j(r) > 0 \;\Leftrightarrow\; 1 < r < \frac{2^j}{2^j - 1}.$$

Let K, L and R be Hausdorff compact sets, let $j \in \mathbb{N}$ and $r > 1$. We say that (K, L, R) *satisfies the scheme* $\mathcal{H}(j, r)$ if there exists a linear map T from $C(R)$ onto $C(L)$ such that the following conditions hold:

$\mathcal{H}_1$: K is a clopen subset of R.
$\mathcal{H}_2$: $\|T\| \leq 1$ and $\gamma(T) < r$.
$\mathcal{H}_3$: Every (r,T)-preimage $g \in C(R)$ satisfies $\|g\|_K \geq \varrho_j(r)\|Tg\|_L$.

Lemma 4.1. *Let R and L be countable Hausdorff compact spaces, K a finite subset of R, and $j \geq 0$ and $r > 1$ such that (K, L, R) satisfies $\mathcal{H}(j, r)$. If $r < 2^j(2^j - 1)^{-1}$, then $|L| \leq |K|$.*

Proof. We use the following claim:

Claim. *Let X be a Banach space, V a finite-codimensional subspace of X, and U a subspace which has codimension greater than that of V. For every $t > 1$, there exists a vector $v \in V$ such that $\|v\| = 1$ and $d(v, U) \geq 1/t$.*

Proof of Claim. Consider the two subspaces $E = V^\perp$ and $F = U^\perp$ of X^*. We have $\dim E < \infty$ and $\dim E < \dim F$. Applying Lemma 2.c.8 of [2], we find $f \in F$ such that $\|f\| = 1$ and $d(f, E) = 1$. For any $t > 1$, there exists $v \in V$ such that $\|v\| = 1$ and $\langle f, v\rangle \geq 1/t$. This implies $d(v, U) \geq 1/t$ and proves the Claim.

Now, consider $T : C(R) \to C(L)$ as in the definition of $\mathcal{H}(j, r)$ with $r < 2^j(2^j - 1)^{-1}$. Let $t > 1$ be such that $t\gamma(T) < r$. We suppose that $|L| > |K|$. Define $U = \ker T$ and $V = \{g \in C(R) ; g = 0 \text{ on } K\}$. The codimension of V is finite and less than the codimension of U. Applying the Claim, we find $v \in V$ such that $\|v\|_R = 1$ and $d(v, U) \geq 1/t$. We immediately obtain

$$\|Tv\|_L \geq \frac{1}{\gamma(T)}\, d(v, U) \geq \frac{1}{t\gamma(T)} \geq \frac{1}{r}\|v\|_R > 0.$$

Hence v is an (r, T)-preimage. Using $\mathcal{H}_3$, we have $\|v\|_K \geq \varrho_j(r)\|Tv\|_L$. Since $r < 2^j(2^j - 1)^{-1}$, we have $\varrho_j(r) > 0$, which contradicts $v \in V$. This shows Lemma 4.1. ∎

Lemma 4.2. *Let K, L and R be countable Hausdorff compact sets, $j \geq 0$ and $r > 1$. Set $\alpha = \omega^{o(K)}$. If (K, L, R) satisfies the scheme $\mathcal{H}(j, r)$ then $(K^{(\alpha)}, L^{(\alpha)}, R)$ satisfies the scheme $\mathcal{H}(j + 1, r)$.*

Proof. Let T be a linear map from $C(R)$ onto $C(L)$ with properties $\mathcal{H}_1$–$\mathcal{H}_3$. Let $P : C(L) \to C(L^{(\alpha)})$ be the restriction map and $T' = P \circ T$. We have $\|T'\| \leq 1$ and $\gamma(T') < r$. It is enough to prove that T' has properties $\mathcal{H}_2$ and $\mathcal{H}_3$. Let $f_1 \in C(L^{(\alpha)})$ be a norm 1 vector and $g \in C(R)$ be an (r, T')-preimage of f_1. We want to prove $\|g\|_{K^{(\alpha)}} \geq \varrho_{j+1}(r)$. We take $\varepsilon > 0$ and $\eta > \|g\|_{K^{(\alpha)}}$. It is sufficient to show that

$$\eta \geq (2 - 2\varepsilon)\varrho_j(r) - r. \tag{2}$$

Define $f = Tg$ and take s such that $\gamma(T) < s < r$ and $r + s \leq (2 - 2\varepsilon)r$. There exists a clopen neighborhood W of $K^{(\alpha)}$ in K such that $\|g\|_W \leq \eta$. We put $M = K\backslash W$. Then $M^{(\alpha)} = \emptyset$. Moreover, $\|f_1\|_{L^{(\alpha)}} = 1$ and $Pf = f_1$. This implies that $|f|$ attains the value 1 at some point $u \in L^{(\alpha)}$. Of course, we can suppose $f(u) = 1$. Let Λ be a clopen neighborhood of u in L such that $f(\lambda) \geq 1 - \varepsilon$ for every $\lambda \in \Lambda$. Since $\Lambda \cap L^{(\alpha)}$ contains u, we have $\sigma(\Lambda) \geq \alpha$. Hence, there exists a clopen subset Q of Λ such that $\sigma(Q) = \alpha$ and $\nu(Q) = 1$. Let $P' : C(L) \to C(Q)$ be the restriction map and $S = P' \circ T : C(R) \to C(Q)$. We clearly have $\|S\| \leq 1$ and $\gamma(S) < s$.

Claim. *There exists $h \in C(R)$ such that $\|h\|_R \leq s$, $\|Sh\|_Q \geq 1 - \varepsilon$ and $h = 0$ on M.*

Proof of Claim. Suppose there is no such h. Then for any $h \in C(R)$ we have

$$(\|h\|_R \leq s \text{ and } h = 0 \text{ on } M) \;\Rightarrow\; \|Sh\|_Q < 1 - \varepsilon.$$

Let f_0 be any norm 1 vector in $C(Q)$. Let $h'_1 \in C(R)$ be such that $\|h'_1\|_R \le s$ and $Th'_1 = f_0$. We define h_1 as the function which coincides with h'_1 on M and vanishes on $R \setminus M$. Since M is clopen, h_1 is continuous. Moreover, using the preceding implication, we deduce $\|f_0 - Sh_1\|_Q < 1 - \varepsilon$. We can proceed in the standard way and define a sequence (h_n) in $C(R)$ such that $\|h_n\|_R \le r\varepsilon^{n-1}$, $\|f_0 - S(h_1 + \cdots + h_n)\|_Q < (1-\varepsilon)^n$ and $h_n = 0$ on $R \setminus M$. We then put $h = \sum h_n$. Since $h = 0$ on $R \setminus M$ and $Sh = f_0$, we showed that S induced a linear quotient map from $C(M)$ onto $C(Q)$. Since M is strictly simpler than Q, since the index $\sigma(Q)$ is a power of ω and since $\nu(Q) = 1$, the proposition of Section 3 in [14] shows that there is no such linear quotient map. This contradiction proves the Claim.

Let h be the vector given by the Claim. Considering $-h$ if necessary, we can suppose that there exists $t \in Q$ with $Sh(t) \ge 1 - \varepsilon$. Hence, we have

$$\|T(g+h)\|_L \ge f(t) + Sh(t) \ge 2 - 2\varepsilon.$$

Since $\|g+h\|_R \le r + s$, the choice of s ensures that $g + h$ is an (r,T)-preimage. By $\mathcal{H}_3$, we deduce that

$$\|g+h\|_K \ge (2-2\varepsilon)\varrho_j(r).$$

Since also $\|g+h\|_K = \max(\|g+h\|_W, \|g+h\|_M) \le r + \eta$, we obtain inequality (2) and we are done. ∎

Now, we can deduce our first result on linear quotients of $C(K)$-spaces.

Proposition 4.3. *Let K and L be countable Hausdorff compact sets. Suppose that there exists a linear map T from $C(K)$ onto $C(L)$ such that $\|T\| \cdot \gamma(T) < 2^{p(K)}(2^{p(K)} - 1)^{-1}$. Then L is simpler than K.*

Proof. Actually, we prove the following technical fact which immediately implies the proposition (take $R = K$ and $j = 0$).

Claim. *Let K, L, R be countable Hausdorff compact sets, and let $j \ge 0$ and $r > 1$ be such that (K, L, R) satisfies the scheme $\mathcal{H}(j,r)$. If $r < 2^m(2^m - 1)^{-1}$ with $m = j + p(K)$, then L is simpler than K.*

We proceed by induction on $p(K)$. If $p(K) = 0$, then K is finite. Hence Lemma 4.1 starts our induction.

Write $\sigma(K) = \omega^{\alpha_1} + \cdots + \omega^{\alpha_{p(K)}}$ with $\alpha_1 \ge \cdots \ge \alpha_{p(K)}$ and $\alpha = \omega^{\alpha_1}$. It is easy to check that $\sigma(K^{(\alpha)}) = \omega^{\alpha_2} + \cdots + \omega^{\alpha_{p(K)}}$ and $p(K^{(\alpha)}) = p(K) - 1$. By Lemma 4.2, $(K^{(\alpha)}, L^{(\alpha)}, R)$ satisfies $\mathcal{H}(j+1, r)$. By our induction hypothesis, $L^{(\alpha)}$ is simpler than $K^{(\alpha)}$, which implies that L is simpler than K. ∎

Now we prove that if $C(K)$ is a subspace (with K countable) of some separable space X, then $C(K)$ is also a linear quotient of X. For that, we need some lemmas.

Lemma 4.4. *Let X be a Banach space and $K = \{x_n^* \,;\, n \in \mathbb{N}\}$ a countable weak*-closed subset of B_{X^*}. Suppose that the sequence (x_n^*) is r-equivalent to the canonical basis of ℓ_1. Then there exists a linear map T from X onto $C(K)$ such that $\|T\| \le 1$ and $\gamma(T) \le r$.*

Proof. For $x \in X$, define Tx as the restriction of $x \in X^{**}$ to K. It is easy to check that T^* is an isomorphism onto its range and that $\|(T^*)^{-1}\| \le r$. Hence $\gamma(T) \le r$. ∎

Lemma 4.5. *Let E be a separable Banach space and X a subspace of E. Endow B_{E^*} with a metric d^* which defines the weak* topology on B_{E^*}. Denote by $P : E^* \to X^*$ the canonical quotient map. Let K be a countable weak*-closed subset of B_{X^*}. Write $\alpha = \sigma(K)$ and suppose that $K^{(\alpha)} = \{r\}$. For any $\varepsilon > 0$, there exists a subset $\widetilde{K}$ of K and a subset L of B_{E^*} such that*:

(K1) *$\widetilde{K}$ is homeomorphic to K.*
(K2) *The d^*-diameter of $\widetilde{K}$ is less than ε.*
(K3) *$r \in \widetilde{K}$.*
(L1) *L is weak*-closed.*
(L2) *The d^*-diameter of L is less than ε.*
(L3) *$L^{(\alpha)} = \{s\}$ and $Ps = r$.*
(KL) *The map P induces a homeomorphism from L onto $\widetilde{K}$.*

Proof. First we can notice that condition (K2) is obvious: the d^*-closed ball of center r and radius $\varepsilon/3$ in K is homeomorphic to K. We prove the result by transfinite induction on α. If $\alpha = 0$, then $K = \{r\}$, the result comes simply from the Hahn–Banach theorem.

Now, suppose that $\alpha \ge 1$. The compact set K is homeomorphic to the ordinal $\omega^\alpha + 1$. If $\alpha = \beta + 1$, we put $\alpha_n = \beta$ for every n. If α is a limit ordinal, we consider an increasing sequence of ordinals (α_n) which tends to α. In both cases, we have

$$\omega^\alpha + 1 = \Big(\sum_{n=0}^{\infty} (\omega^{\alpha_n} + 1)\Big) + 1.$$

Let $\varphi : \omega^\alpha + 1 \to K$ be a homeomorphism. For every n, we define

$$r_n = \varphi\Big(\sum_{k=0}^{n} (\omega^{\alpha_k} + 1)\Big), \qquad K_n = \{\varphi(t) \,;\, r_{n-1} < t \le r_n\}.$$

The K_n's are clopen subsets of K homeomorphic to $\omega^{\alpha_n}+1$ and (r_n) tends to r. By our induction hypothesis, there exist subsets $\widetilde{K}_n$ of K_n and L_n of B_{E^*} satisfying conditions (K1) to (KL) with $\varepsilon/2^{n+2}$ instead of ε. Writing $\{s_n\} = L_n^{(\sigma(L_n))}$, we can find a subsequence $(s_{n_k})_{k\ge 0}$ such that $d^*(s_{n_k}, s_{n_{k+1}}) <$

$\varepsilon/3^{k+1}$. Its limit s belongs to B_{E^*}. We define

$$\widetilde{K} = \{r\} \cup \bigcup_{k\ge 0} \widetilde{K}_{n_k}, \qquad L = \{s\} \cup \bigcup_{k\ge 0} L_{n_k}.$$

Conditions (K2), (K3), (L3) and (KL) are obvious. To check (L1), we prove that L is compact: this follows directly from the fact that the d^*-diameters of the L_{n_k}'s tend to 0. Similarly, we establish the compactness of $\widetilde{K}$. Since $\widetilde{K} \subset K$ and $r \in \widetilde{K}^{(\alpha)}$, we have $\chi(K) = \chi(\widetilde{K})$, which gives (K1). To prove condition (L2), it is enough to notice that the d^*-diameter of L is upper bounded by $\sup_{p,q}(\varepsilon/2^{n_p+2} + \varepsilon/2^{n_q+2} + d^*(s_{n_p}, s_{n_q})) < \varepsilon$. ∎

The assumption $\nu(K) = 1$ is unnecessary, as the following lemma shows:

LEMMA 4.6. *Let E be a separable Banach space and X a subspace of E. Let K be a countable weak*-closed subset of B_{X^*}. There exists a closed subset $\widetilde{K}$ of K, homeomorphic to K, and a weak*-closed subset L of B_{E^*} such that the restriction map induces a homeomorphism from L onto $\widetilde{K}$.*

Proof. Define $n = \nu(K)$. There is a partition of K into n clopen subsets $K_1, \dots, K_n$ such that $\nu(K_i) = 1$. We can apply Lemma 4.5 to the K_i's and find $\widetilde{K}_1, \dots, \widetilde{K}_n, L_1, \dots, L_n$. The sets $\widetilde{K} = \widetilde{K}_1 \cup \dots \cup \widetilde{K}_n$ and $L = L_1 \cup \dots \cup L_n$ give the result. ∎

These lemmas allow us to prove the second result of this section:

PROPOSITION 4.7. *Let K be a countable Hausdorff compact set and E be a separable Banach space. If R is an isomorphism from $C(K)$ onto a subspace X of E, then there exists a linear map S from E onto $C(K)$ such that $\|S\| \cdot \gamma(S) \le \|R\| \cdot \|R^{-1}\|$.*

Proof. The map $\delta : K \to C(K)^*$ defined by the formula $\delta(k)(f) = f(k)$ for any $k \in K$ and $f \in C(K)$ is a homeomorphism onto its range. Suppose $\|R^{-1}\| = 1$ and put $r = \|R\|$. The set $(R^{-1})^*(\delta(K))$ is r-equivalent to the canonical basis of ℓ_1. Applying Lemma 4.6, we construct $L \subset B_{E^*}$, weak*-closed and homeomorphic to K and r-equivalent to the canonical basis of ℓ_1. Then Lemma 4.4 proves that there exists a linear quotient map $S : E \to C(K)$ such that $\|S\| \cdot \gamma(S) \le r$. ∎

As a direct consequence of Propositions 4.3 and 4.7, we obtain the following corollary:

COROLLARY 4.8. *Let K and L be countable Hausdorff compact spaces. If there is a subquotient X of $C(K)$ with $\mathrm{d_{BM}}(X, C(L)) < 2^{p(K)}(2^{p(K)} - 1)^{-1}$, then L is simpler than K.*

These linear preliminaries enable us to prove our result on subspaces of quotients of $C(K)$-spaces (with K countable).

Theorem 4.9. *Suppose that K and L are countable Hausdorff compact sets. If there exists a subquotient X of $C(L)$ such that* $\mathrm{d_K}(X, C(K)) < 2^{-p(L)-1}$, *then K is simpler than L.*

The similar question for uncountable compact sets is irrelevant since $C(L)$ is universal for separable Banach spaces provided L is metrizable and uncountable.

Proof. We have $\mathrm{d_K}(X, C(K)) < \eta_0 = 2^{-p(L)-1}$. Up to isometries, we can suppose that $C(K)$ and X are subspaces of a common separable Banach space E such that the Hausdorff distance between $B_{C(K)}$ and B_X is less than η_0. Applying Lemma 4.6 to $\delta(K) \subset B_{C(K)^*}$, we can find weak*-compact sets $\widetilde{K} \subset \delta(K)$, homeomorphic to K, and $L_0 \subset B_{E^*}$ such that the restriction map $P : E^* \to C(K)^*$ induces a homeomorphism from L_0 onto $\widetilde{K}$. Let $Q : E^* \to X^*$ be the restriction map and $\widetilde{L} = Q(L_0)$. We make the following two claims:

Claim 1. *The map Q induces a homeomorphism from L_0 onto $\widetilde{L}$. In particular*, *the compact spaces K*, *$\widetilde{K}$ and $\widetilde{L}$ are homeomorphic.*

Proof of Claim 1. It suffices to prove that Q is one-to-one on L_0. Let e_1^* and e_2^* be two distinct elements of L_0. Define $y_i^* = P(e_i^*)$ and $x_i^* = Q(e_i^*)$ for $i = 1, 2$. Since P is one-to-one on L_0, we have $y_1^* \neq y_2^*$, which implies $\|y_1^* - y_2^*\| = 2$. For an arbitrary ε, taking $y \in S_{C(K)}$ such that $\langle y_1^* - y_2^*, y\rangle > 2 - \varepsilon$, we can find $x \in S_X$ such that $\|x - y\| < \eta_0$. Then $\langle x_1^* - x_2^*, x\rangle > 2 - \varepsilon - 2\eta_0 > 0$ for sufficiently small ε.

Claim 2. *The set $\widetilde{L}$ is r-equivalent to the canonical basis of ℓ_1 for some $r < 2^{p(L)}(2^{p(L)} - 1)^{-1}$.*

Proof of Claim 2. We put $L_0 = \{e_n^*\}$ and $y_n^* = P(e_n^*)$, $x_n^* = Q(e_n^*)$. Let $\varepsilon > 0$ and let $\eta < \eta_0$ be greater than the Hausdorff distance between B_X and $B_{C(K)}$. Let (λ_n) be a finitely nonzero sequence of scalars such that $\sum |\lambda_n| = 1$. Since (y_n^*) is 1-equivalent to the canonical basis of ℓ_1, we can find $y \in S_{C(K)}$ such that $\sum \langle \lambda_n y_n^*, y\rangle \geq 1 - \varepsilon$. Choosing $x \in S_X$ such that $\|x - y\| \leq \eta$, we find $\|\sum \lambda_n x_n^*\| \geq \sum \langle \lambda_n x_n^*, x\rangle \geq 1 - \varepsilon - 2\eta$. Finally, we conclude that (x_n^*) is $(1 - 2\eta)^{-1}$-equivalent to the canonical basis of ℓ_1.

Using Proposition 4.7, we see that X is a quotient of $C(L)$. By Proposition 4.3, it is enough to prove that there exists a linear map $T : X \to C(K)$ such that $\|T\| \cdot \gamma(T) < 2^{p(L)}(2^{p(L)} - 1)^{-1}$. Using Claim 1, we just have to prove that there is such a quotient map from $C(L)$ onto $C(\widetilde{L})$, which is direct from Claim 2 and Lemma 4.4. ∎

Corollary 4.10. *There exists a universal function $\mu : \mathbb{N} \to]0, 1]$ such that*, *for any countable Hausdorff compact sets K and L*, *if there exists* *a subquotient X of $C(L)$ such that* $\mathrm{d_{GH}}(X, C(K)) < \mu(p(L))$, *then K is simpler than L.*

Proof. By Theorem 6.3 of [9] and Theorem 3.7 of [8], for any $j \in \mathbb{N}$, there exists a number $\mu(j)$ such that for any Banach space X, we have $\mathrm{d_K}(X, C(K)) < 2^{-j-1}$ provided $\mathrm{d_{GH}}(X, C(K)) < \mu(j)$. ∎

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References

[1] D. Amir, *On isomorphisms of continuous function spaces*, Israel J. Math. 3 (1965), 205–210.

[2] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol. I, Colloq. Publ. 48, Amer. Math. Soc., 2000.

[3] F. Cabello Sánchez, J. M. F. Castillo, N. J. Kalton and D. Yost, *Twisted sums with $C(K)$-spaces*, Trans. Amer. Math. Soc. 355 (2003), 4523–4541.

[4] M. Cambern, *A generalized Banach–Stone theorem*, Proc. Amer. Math. Soc. 17 (1966), 396–400.

[5] —, *On isomorphisms with small bound*, ibid. 18 (1967), 1062–1066.

[6] H. B. Cohen, *A bound-two isomorphism between $C(X)$ Banach spaces*, ibid. 50 (1975), 215–217.

[7] K. Jarosz, *Nonlinear generalization of the Banach–Stone theorem*, Studia Math. 93 (1989), 97–107.

[8] N. J. Kalton and M. I. Ostrovskii, *Distances between Banach spaces*, Forum Math. 11 (1999), 17–48.

[9] N. J. Kalton and J. W. Roberts, *Uniformly exhaustive submeasures and nearly additive set functions*, Trans. Amer. Math. Soc. 278 (1983), 803–816.

[10] G. Lancien, *On the Szlenk index and the weak*-dentability index*, Quart. J. Math. Oxford 47 (1996), 59–71.

[11] M. I. Ostrovskii, *On the properties of the opening and related closeness characterizations of Banach spaces*, Amer. Math. Soc. Transl. 136 (1987), 109–119.

[12] —, *Topologies on the set of all subspaces of a Banach space and related questions of Banach space geometry*, Quaestiones Math. 17 (1994), 259–319.

[13] —, *Paths between Banach spaces*, Glasgow Math. J. 44 (2002), 261–273.

[14] C. Samuel, *Indice de Szlenk des $C(K)$ (K espace topologique compact dénombrable)*, Sém. Géométrie des espaces de Banach, Vol. 1, Publ. Math. Univ. Paris VII 18, 1983, 81–91.

[15] Z. Semadeni, *Banach Spaces of Continuous Functions*, Vol. I, Monograf. Mat. 55 PWN–Polish Sci. Publ., Warszawa, 1971.

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