STUDIA MATHEMATICA 188 (1) (2008)

Rosenthal operator spaces

by

M. Junge (Urbana, IL), N. J. Nielsen (Odense)
and T. Oikhberg (Irvine, CA)

Abstract. In 1969 Lindenstrauss and Rosenthal showed that if a Banach space is isomorphic to a complemented subspace of an L_p-space, then it is either an $\mathcal{L}_p$-space or isomorphic to a Hilbert space. This is the motivation of this paper where we study non-Hilbertian complemented operator subspaces of non-commutative L_p-spaces and show that this class is much richer than in the commutative case. We investigate the local properties of some new classes of operator spaces for every $2 < p < \infty$ which can be considered as operator space analogues of the Rosenthal sequence spaces from Banach space theory, constructed in 1970. Under the usual conditions on the defining sequence σ we prove that most of these spaces are operator $\mathcal{L}_p$-spaces, not completely isomorphic to previously known such spaces. However, it turns out that some column and row versions of our spaces are not operator $\mathcal{L}_p$-spaces and have a rather complicated local structure which implies that the Lindenstrauss–Rosenthal alternative does not carry over to the non-commutative case.

Introduction. In 1970 Rosenthal [26] constructed new examples of $\mathcal{L}_p$-spaces for every $2 \leq p < \infty$ using probabilistic methods now famous as the Rosenthal inequalities. These methods were later used by Bourgain, Rosenthal and Schechtman [3] to construct an uncountable family of mutually non-isomorphic $\mathcal{L}_p$-spaces.

In the framework of operator spaces a theory of operator $\mathcal{L}_p$-spaces, called $\mathcal{OL}_p$-spaces, is now being developed (see e.g. [4] and [11]). These are spaces where the operator space structure of the finite-dimensional subspaces is determined by a system of finite-dimensional non-commutative L_p-spaces. If in a given space these L_p-spaces can be chosen to be completely complemented, the space is called a $\mathcal{COL}_p$-space. If they can be chosen to be S_p^n's (S_p denotes the Schatten p-class), then the space is called an $\mathcal{OS}_p$-space,

2000 *Mathematics Subject Classification*: 46B20, 46L07, 46L52.
Key words and phrases: non-commutative L_p-spaces, $\mathcal{OL}_p$-spaces.
Research of M. Junge supported by NSF grant DMS-0301116 and DMS 05-56120.
Research of N. J. Nielsen supported by the Danish Natural Science Research Council, grant 21020436.
Research of T. Oikhberg supported by NSF grant DMS-0500957.

and a $\mathcal{COS}_p$-space if the S_p^n's can be chosen completely complemented. In the present paper we consider some operator space analogues of the Rosenthal spaces, sequence spaces as well as matricial analogues.

For a given $2 < p < \infty$ and a given strictly positive sequence $\sigma = (\sigma_n)$ we construct three families of operator spaces, a sequence space family consisting of spaces called $X_p(\sigma)$, $X_{p,r_p}(\sigma)$, and $X_{p,c_p}(\sigma)$, and two families of matricial operator spaces. All the spaces are mutually non-completely isomorphic as operator spaces, but the spaces in each family are isomorphic to each other as Banach spaces; the three sequence spaces are actually Banach space isomorphic to the original Rosenthal sequence space. One of our main results states that if $2 < p < \infty$, $\sigma_n \to 0$, and $\sum_{n=1}^{\infty} \sigma_n^{2p/(2p-2)} = \infty$, then $X_{p,c_p}(\sigma)$ is completely complemented in a non-commutative L_p-space and contains ℓ_p cb-complemented. However $X_{p,c_p}(\sigma)$ is not an $\mathcal{OL}_p$-space. Similarly for $X_{p,r_p}(\sigma)$. This shows that the Lindenstrauss–Rosenthal alternative [17] does not carry over to the non-commutative case.

We now wish to discuss the arrangement of this paper in greater detail. In Section 1 we construct our spaces, investigate their basic properties and prove among other things that under the above conditions on σ the three sequence spaces are unique up to complete isomorphisms (in analogy with Rosenthal's result). In Section 2 we make a detailed investigation of the local structure of the spaces $X_p(\sigma)$, $X_{p,c_p}(\sigma)$, and $X_{p,r_p}(\sigma)$ and prove that $X_p(\sigma)$ is an $\mathcal{OL}_p$-space while $X_{p,r_p}(\sigma)$ and $X_{p,c_p}(\sigma)$ are not. We also show that some combinations of the different spaces cannot be paved with local pieces of each other. This implies that a general structure theory for completely complemented non-Hilbertian subspaces of non-commutative L_p-spaces is out of reach for the moment (see e.g. Proposition 2.19 and Remark 2.20). Section 3 is devoted to the study of the matricial spaces; we show that they are all $\mathcal{OS}_p$-spaces and that $Y_p(\sigma)$ is cb-complemented in $L_p(\mathcal{R})$ ($\mathcal{R}$ the hyperfinite type II_1 factor) while $Z_p(\sigma)$ does not cb-embed into $L_p(\mathcal{R})$. In Section 4 we prove that certain $\mathcal{OL}_p$-spaces contain cb-uncomplemented copies of themselves.

0. Notation and preliminaries. In this paper we shall use the notation and terminology commonly used in the theory of operator algebras, operator spaces and Banach space theory as it appears in [5], [11], [18], [19], [23] and [28].

If H is a Hilbert space, we let $B(H)$ denote the space of all bounded operators on H and for every $n \in \mathbb{N}$ we let M_n denote the space of all $n \times n$ matrices of complex numbers, i.e. $M_n = B(\ell_2^n)$. If X is a subspace of some $B(H)$ and $n \in \mathbb{N}$, then $M_n(X)$ denotes the space of all $n \times n$ matrices with X-valued entries which we in the natural manner consider as a subspace of

$B(\ell_2^n(X))$. An *operator space* X is a norm closed subspace of some $B(H)$ equipped with the distinguised matrix norm inherited by the spaces $M_n(X)$, $n \in \mathbb{N}$. An abstract matrix norm characterization of operator spaces was given by Ruan (see e.g. [5]).

If X and Y are operator spaces, then a linear operator $T : X \to Y$ is called *completely bounded* (for short, cb-bounded) if the corresponding linear maps $T_n : M_n(X) \to M_n(Y)$ are uniformly bounded in n, i.e.

$$\|T\|_{\mathrm{cb}} = \sup \|T_n\| < \infty.$$

The space of all completely bounded operators from X to Y will be denoted by $CB(X, Y)$.

It follows from [5] that a linear functional on an operator space X is bounded if and only if it is cb-bounded, and its cb-norm and operator norm coincide. This defines an operator structure on X^* so that isometrically we have $M_n(X^*) = CB(X, M_n)$ for all $n \in \mathbb{N}$.

An operator is a *complete contraction*, respectively a *complete isometry*, or a *complete quotient* if $\|T\|_{\mathrm{cb}} \leq 1$, respectively if each T_n is an isometry, or a quotient map. An operator T is called a *complete isomorphism* (for short, a *cb-isomorphism*) if it is a completely bounded linear isomorphism with a completely bounded linear inverse. If X and Y are cb-isomorphic operator spaces we put

$$d_{\mathrm{cb}}(X, Y) = \inf\{\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}} \mid T \text{ is a cb-isomorphism from } X \text{ to } Y\},$$

which is called the *completely bounded Banach–Mazur distance* (for short, the *cb-distance*) between X and Y.

We let $S_\infty \subseteq B(\ell_2)$ denote the subspace of all compact operators on ℓ_2 (hence an operator space in a natural manner). If $1 \leq p < \infty$, then the *Schatten class* S_p is defined to be the space of all compact operators T on ℓ_2 for which $\mathrm{tr}(|T|)^p < \infty$ equipped with the norm

$$\|T\|_{S_p} = (\mathrm{tr}(|T|^p))^{1/p} \quad \text{for all } T \in S_p. \tag{0.1}$$

If $n \in \mathbb{N}$ and p is as above, S_p^n denotes the space of all operators on ℓ_2^n equipped with the norm defined in (0.1). If also $m \in \mathbb{N}$, then $S_p^{n,m}$ denotes the subspace of S_p consisting of those elements which correspond to matrices (a_{ij}) where $a_{ij} = 0$ unless $i \leq n$ and $j \leq m$.

From trace duality it easily follows that $S_\infty^* = S_1$ and hence as a dual space S_1 has a natural operator structure as defined above. It is well known that S_p can be obtained by complex interpolation,

$$S_p = [S_\infty, S_1]_{1/p}.$$

Pisier proved in [23] that

$$M_n(S_p) = [M_n(S_\infty), M_n(S_1)]_{1/p}$$

defines matrix norms on S_p which satisfy Ruan's matrix norm characterization of operator spaces, and this is called the *natural operator space structure* of S_p which we shall always use in the following.

Let e_{ij} denote the element of $B(\ell_2)$ corresponding to the matrix with coefficients equal to one at the i,j entry and zero elsewhere. If $1 \le p \le \infty$, we define the operator subspaces C_p and R_p of S_p by

$$C_p = \overline{\operatorname{span}}\{e_{i1} \mid i \in \mathbb{N}\}, \qquad R_p = \overline{\operatorname{span}}\{e_{1j} \mid j \in \mathbb{N}\}.$$

As Banach spaces, they are both isometric to ℓ_2, but it follows from Pisier [23] that they are not cb-isomorphic as operator spaces.

If $1 \le p \le \infty$, then we put $\mathcal{K}_p = (\sum_{n=1}^{\infty} S_p^n)_p$; $\mathcal{K}_p$ is clearly an operator space in a canonical manner.

If H is an operator Hilbert space, i.e. an operator space which as a Banach space is isometric to a Hilbert space, then we put $H^c = CB(\mathbb{C}, H)$ and $H^r = CB(H, \mathbb{C})$ and if $1 < p < \infty$, then we let $H^{c_p} = [H^c, H^r]_{1/p}$ and $H^{r_p} = [H^r, H^c]_{1/p}$.

If E is an operator space and $1 \le p \le \infty$, it is possible to define $S_p[E]$ (S_p with values in E) as the completion of $S_p \otimes E$ under a certain operator space norm; we refer to [23, Chapter 1] for the details. In particular, we shall often use the following proposition proved by Pisier [23, Lemma 1.7, see also Propositions 2.3, 2.4 and Remark 2.5].

PROPOSITION 0.1. *Let E and F be operator spaces. A linear map $T : E \to F$ is cb-bounded if and only if $\sup_{n\in\mathbb{N}} \|\mathrm{id}_{S_p^n} \otimes T : S_p^n[E] \to S_p^n[F]\| < \infty$. In that case we have $\|T\|_{\mathrm{cb}} = \sup_{n\in\mathbb{N}} \|\mathrm{id}_{S_p^n} \otimes T\|$.*

The norms in $S_p[R_p]$ and $S_p[C_p]$ were computed by Pisier in [23, p. 108], and since we are going to use this frequently, we state it in a proposition.

PROPOSITION 0.2. *If $(x_k)_{k=1}^n \subseteq S_p$, then*

$$\Big\| \sum_{k=1}^n x_k \otimes e_{1k} \Big\|_{S_p[R_p]} = \Big\| \Big(\sum_{k=1}^n x_k x_k^* \Big)^{1/2} \Big\|_{S_p} \tag{0.2}$$

and

$$\Big\| \sum_{k=1}^n x_k \otimes e_{k1} \Big\|_{S_p[C_p]} = \Big\| \Big(\sum_{k=1}^n x_k^* x_k \Big)^{1/2} \Big\|_{S_p}. \tag{0.3}$$

If X is a subspace of S_p and E is an operator space, then we let $X[E]$ denote the closure of $E \otimes X$ in $S_p[E]$.

Let A be a von Neumann algebra with a normal semifinite faithful trace τ (i.e. A is semifinite). The ideal

$$m(\tau) = \Big\{ \sum_{k=1}^n x_k y_k \;\Big|\; n \in \mathbb{N}, \sum_{k=1}^n [\tau(y_k^* y_k) + \tau(x_k^* x_k)] < \infty \Big\}$$

is called the *definition ideal* of τ on which there is a unique linear extension $\tau : m(\tau) \to \mathbb{C}$ so that $\tau(xy) = \tau(yx)$ for all $x, y \in m(\tau)$ (see e.g. [28]). If $1 \le p < \infty$, then we put

$$\|x\| = \tau((x^*x)^{p/2})^{1/p} \quad \text{for all } x \in m(\tau),$$

which is readily seen to be a norm on $m(\tau)$. We define $L_p(A, \tau)$ to be the completion of $m(\tau)$ under this norm. Conventionally we put $L_\infty(A, \tau) = A$. It follows easily that $L_1(A, \tau)^* = A^{\mathrm{op}}$, where A^{op} denotes A equipped with the reverse (or opposite) multiplication, and hence $L_1(A, \tau)$ has a natural operator space structure. It can be shown that the complex interpolation method yields

$$L_p(A, \tau) = [A, L_1(A, \tau)]_{1/p}.$$

Pisier [23] proved that

$$M_n(L_p(A, \tau)) = [M_n(A), M_n(L_1(A, \tau))]_{1/p}$$

defines a natural operator space structure on $L_p(A, \tau)$, which we shall use in what follows. If τ_1 is another normal semifinite faithful trace on A, then it can easily be shown that $L_p(A, \tau)$ is cb-isometric to $L_p(A, \tau_1)$, and therefore we shall often write $L_p(A)$ instead of $L_p(A, \tau)$.

If B is a von Neumann subalgebra of A so that the restriction of τ to B is semifinite again, then it follows from [28, Proposition 2.36] that there exists a faithful normal projection E_B of A onto B such that $\tau = \tau \circ E_B$. The projection E_B is called the *conditional expectation* of A onto B.

An operator space X is called an *operator $\mathcal{L}_p$-space* (for short, *$\mathcal{OL}_p$-space*), $1 \le p \le \infty$, if there exist a $\lambda \ge 1$ and a cofinal family $(F_j)_{j \in I}$ of finite-dimensional subspaces such that $\bigcup_{j \in I} F_j$ is dense in X and for every j there exists a finite-dimensional C^*-algebra A_j with

$$d_{\mathrm{cb}}(L_p(A_j), F_j) \le \lambda. \tag{0.4}$$

In this case we shall also say that X is an *$\mathcal{OL}_{p,\lambda}$-space.* X is called an *$\mathcal{OS}_{p,\lambda}$-space* if we can replace the $L_p(A_j)$'s in (0.4) by $S_p^{n_j}$'s, and a *completely complemented $\mathcal{OL}_{p,\lambda}$-space* (for short, *$\mathcal{COL}_{p,\lambda}$-space*) if in addition the F_j's can be chosen to be cb-complemented in X by projections with cb-norms less than or equal to λ. *$\mathcal{COS}_{p,\lambda}$-spaces* are defined similarly.

If the $L_p(A_j)$'s in (0.4) are of the form $(\bigoplus_{i=1}^k S_p^{n(i),m(i)})_p$, then X is called a *rectangular $\mathcal{OL}_p$-space.*

Let $1 \le p \le \infty$. An operator space X is said to have the *γ_p-approximation property* (for short, γ_p-AP) if there exists a $\lambda > 0$ and nets (U_i) and (V_i) of finite rank operators, $U_i \colon X \to S_p$, $V_i \colon S_p \to X$, so that $\|U_i\|_{\mathrm{cb}} \|V_i\|_{\mathrm{cb}} \le \lambda$ and (V_iU_i) converges pointwise to the identity of X.

Finally, if (x_n) is a finite or infinite sequence in a Banach space X, we let $[x_n]$ denote the closed linear span of the sequence (x_n). If A is a set, $|A|$

denotes its cardinality, and if X and Y are Banach spaces, $X \oplus_p Y$ stands for the direct sum of X and Y equipped with the norm $(\|\cdot\|_X^p + \|\cdot\|_Y^p)^{1/p}$.

1. The Rosenthal operator spaces and their basic properties. In this section we shall investigate some operator spaces which correspond to the $\mathcal{L}_p$-spaces in Banach space theory constructed by Rosenthal in [26].

We let $2 < p < \infty$, $1/p + 1/p' = 1$, $1/2 = 1/p + 1/r$ (i.e. $r = 2p/(p-2)$) and let $\sigma = (\sigma_n)$ be a sequence of real numbers with $\sigma_n > 0$ for all $n \in \mathbb{N}$. We denote the unit vector basis of ℓ_2 by (ξ_n) and let D_σ be the diagonal operator on ℓ_2 defined by $D_\sigma \xi_n = \sigma_n \xi_n$ for all $n \in \mathbb{N}$.

Our first space $\widetilde{X}_p(\sigma)$ is defined to be the space of all sequences $a = (a_n)$ which satisfy

$$\sum_{n=1}^{\infty} |a_n|^p < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |a_n|^2 \sigma_n^2 < \infty, \tag{1.1}$$

equipped with the norm

$$\|a\| = \Big(\sum_{n=1}^{\infty} |a_n|^p + \Big(\sum_{n=1}^{\infty} |a_n|^2 \sigma_n^2\Big)^{p/2}\Big)^{1/p}. \tag{1.2}$$

$\widetilde{X}_p(\sigma)$ is the classical Rosenthal sequence space (except that he used an equivalent norm) and we can clearly identify it with the closed linear span in $S_p \oplus_p S_2$ of the sequence $\{(e_{nn}, \sigma_n e_{nn}) \mid n \in \mathbb{N}\}$. As an operator space we can however represent $\widetilde{X}_p(\sigma)$ in three different ways. We define $X_{p,c_p}(\sigma)$ to be the closed linear span of $\{(e_{nn}, \sigma_n e_{n1}) \mid n \in \mathbb{N}\}$ in $S_p \oplus C_p$. Similarly we let $X_{p,r_p}(\sigma)$ denote the closed linear span of $\{(e_{nn}, \sigma_n e_{1n}) \mid n \in \mathbb{N}\}$ in $S_p \oplus R_p$, and finally we let $X_p(\sigma)$ denote the closed linear span of $\{(e_{nn}, \sigma_n e_{n1}, \sigma_n e_{1n})\}$ in $S_p \oplus C_p \oplus R_p$.

Since $S_p \oplus C_p \oplus R_p$ is cb-isomorphic to S_p, each of the above three spaces is cb-isomorphic to a subspace of S_p. We shall often let $X_{p*}(\sigma)$ denote any of them.

Since we shall often use Proposition 0.1 to check cb-boundedness, it is worth mentioning how the norms in $S_p[X_{p,r_p}(\sigma)]$, $S_p[X_{p,c_p}(\sigma)]$, and $S_p[X_p(\sigma)]$ can be computed. It follows immediately from Proposition 0.2 that if $(x_k)_{k=1}^n \subseteq S_p$, then

$$\Big\| \sum_{k=1}^{n} x_k \otimes (e_{kk} \oplus \sigma_k e_{1k}) \Big\|_{S_p[X_{p,r_p}(\sigma)]} = \Big(\sum_{k=1}^{n} \|x_k\|^p + \Big\| \Big(\sum_{k=1}^{n} \sigma_k^2 x_k x_k^*\Big)^{1/2} \Big\|_{S_p}^p\Big)^{1/p}, \tag{1.3}$$

$$(1.4)\qquad \Big\| \sum_{k=1}^{n} x_k \otimes (e_{kk} \oplus \sigma_k e_{k1}) \Big\|_{S_p[X_{p,c_p}(\sigma)]} = \Big(\sum_{k=1}^{n} \|x_k\|^p + \Big\| \Big(\sum_{k=1}^{n} \sigma_k^2 x_k^* x_k \Big)^{1/2} \Big\|_{S_p}^p \Big)^{1/p},$$

and similarly for $S_p[X_p(\sigma)]$.

It follows easily from these formulas and Proposition 0.1 that though isometric as Banach spaces, these three spaces are not mutually cb-isomorphic as operator spaces.

Throughout the paper we shall often impose at least one of the following two conditions on σ:

$$(1.5)\qquad \liminf_{n\to\infty} \sigma_n = 0,$$

$$(1.6)\qquad \sum_{\sigma_n \le \varepsilon} \sigma_n^r = \infty \quad \text{for all } \varepsilon > 0.$$

It is immediate that if $\sigma_n \to 0$ and $\sigma \notin \ell_r$, then (1.5) and (1.6) are satisfied. (1.6) ensures that the operator $x \mapsto x D_\sigma$ does not act as a bounded operator from S_p to S_2.

It follows from [26] that $\widetilde{X}_p(\sigma)$ is an $\mathcal{L}_p$-space if and only if (1.5) is satisfied, and if both (1.5) and (1.6) hold, then $\widetilde{X}_p(\sigma)$ is the classical Rosenthal $\mathcal{L}_p$-space which is unique up to a Banach space isomorphism. Later in this section we shall prove a similar uniqueness result for the operator space versions.

Our first result states:

THEOREM 1.1. *If σ satisfies* (1.5) *and* (1.6), *then $\widetilde{X}_p(\sigma)^*$ is not Banach space isomorphic to a subspace of $S_{p'}$. Consequently, $\widetilde{X}_p(\sigma)$ is not Banach space isomorphic to a complemented subspace of S_p.*

Proof. Assume that $\widetilde{X}_p(\sigma)^*$ is isomorphic to a subspace of $S_{p'}$ and fix $n \in \mathbb{N}$. By [26, Corollary 8], $\widetilde{X}_p(\sigma)^*$ contains a basic sequence (h_k) equivalent to the unit vector basis of ℓ_2 such that the set of any n elements of that sequence is isometrically equivalent to the unit vector basis of $\ell_{p'}^n$. From [2, Proposition 4 and Lemma 1] it follows that (h_k) has a subsequence which is 4-equivalent to the unit vector basis of ℓ_2. This is a contradiction for large $n \in \mathbb{N}$. ∎

The next theorem is the operator space version of Rosenthal's Lemma 7 in [26].

PROPOSITION 1.2. *Let (g_n) be the natural basis of $X_{p*}(\sigma)$ and let (E_j) be a sequence of mutually disjoint finite subsets of $\mathbb{N}$. For each $j \in \mathbb{N}$ put*

$$f_j = \sum_{n\in E_j} \sigma_n^{r/p} g_n, \tag{1.7}$$

$$\beta_j = \Big(\sum_{n\in E_j} \sigma_n^r\Big)^{1/r}, \tag{1.8}$$

$$\widetilde{f}_j = \beta_j^{-r/p} f_j. \tag{1.9}$$

Then $(\widetilde{f}_j)$ *is a cb-unconditional basic sequence, cb-isometrically equivalent to the natural basis of* $X_{p*}(\beta)$, *and there is a cb-contractive projection of* X_{p*} *onto* $[f_j]$.

Proof. We shall prove the theorem for $X_{p,c_p}(\sigma)$; the other cases can be proved in a similar manner.

If $(x_j)_{j=1}^k \subseteq S_p$, then we get

$$\Big\|\sum_{j=1}^k x_j\otimes f_j\Big\|_{S_p[X_{p,c_p}(\sigma)]} = \Big\|\sum_{j=1}^k \sum_{n\in E_j} \sigma_n^{r/p} x_j \otimes [e_{nn}\oplus \sigma_n e_{n1}]\Big\|_{S_p[X_{p,c_p}(\sigma)]}.$$

It easily follows that

$$\Big\|\sum_{j=1}^k \sum_{n\in E_j} \sigma_n^{r/p} x_j \otimes e_{nn}\Big\|_{S_p[S_p]} = \Big(\sum_{j=1}^k \|x_j\|^p \sum_{n\in E_j}\sigma_n^r\Big)^{1/p} = \Big(\sum_{j=1}^k \|x_k\|^p\beta_j^r\Big)^{1/p}.$$

From (0.3) we get

$$\begin{aligned}\Big\|\sum_{j=1}^k \sum_{n\in E_j} \sigma_n^{r/p} x_j \otimes \sigma_n e_{n1}\Big\|_{S_p[C_p]} &= \Big\|\Big(\sum_{j=1}^k \sum_{n\in E_j} \sigma_n^{2r/p+2} x_j^* x_j\Big)^{1/2}\|_{S_p} \\ &= \Big\|\Big(\sum_{j=1}^k \beta_j^r x_j^* x_j\Big)^{1/2}\Big\|_{S_p}\end{aligned}$$

and therefore

$$\Big\|\sum_{j=1}^k x_j\otimes \widetilde{f}_j\Big\|_{S_p[X_{p,c_p}(\sigma)]} = \Big\|\sum_{j=1}^k x_j\otimes [e_{jj}\oplus\beta_j e_{j1}]\Big\|_{S_p[X_{p,c_p}(\beta)]}. \tag{1.10}$$

Together with Proposition 0.1 this shows that $(\widetilde{f}_j)$ is cb-isometrically equivalent to the natural basis (g_j) of $X_{p,c_p}(\beta)$.

For all $x,y\in X_{p,c_p}(\sigma)$ we put $\langle x,y\rangle = \sum_{j=1}^\infty x(j)\overline{y(j)}\sigma_j^2$ (where $x(j)$, respectively $y(j)$, denotes the jth coordinate of x, respectively y, in the basis (g_j)) and define

$$Px = \sum_{j=1}^\infty \langle x, f_j\rangle \beta^{-r} f_j \quad \text{for all } x\in X_{p,c_p}(\sigma). \tag{1.11}$$

It follows immediately from Rosenthal's argument in [26, Lemma 7] that in the Banach space sense P is a contractive projection of $X_{p,c_p}(\sigma)$ onto $[f_j]$. In addition we need to prove that P is completely bounded with $\|P\|_{\mathrm{cb}} = 1$.

For every $n \in \mathbb{N}$ we get

$$
\begin{aligned}
Pg_n &= \sum_{j=1}^{\infty} \langle g_n, f_j \rangle \beta_j^{-r} f_j = \sigma_n^{r/p+2} \beta_{j_n} f_{j_n} \\
&= \sigma_n^{r/p+2} \beta_{j_n}^{r/p-r} \widetilde{f}_{j_n} = \beta_{j_n}^{-r/p'} \sigma_n^{r/p'} \widetilde{f}_{j_n},
\end{aligned}
\tag{1.12}
$$

where j_n is chosen such that $n \in E_{j_n}$.

Let now $(x_n) \subseteq S_p$ be a finite sequence. From (1.12) and the first part of the proof we obtain

$$
\begin{aligned}
&\Big\| \sum_n x_n \otimes Pg_n \Big\|_{S_p[X_{p,c_p}]} \\
&\quad = \Big\| \sum_j \beta_j^{-r/p'} \Big(\sum_{n \in E_j} \sigma_n^{r/p'} x_n \Big) \otimes \widetilde{f}_j \Big\|_{S_p[X_{p,c_p}(\sigma)]} \\
&\quad = \Big\| \sum_j \beta_j^{-r/p'} \Big(\sum_{n \in E_j} \sigma_n^{r/p'} x_n \Big) \otimes [e_{jj} \oplus \beta_j e_{j1}] \Big\|_{S_p[X_{p,c_p}(\beta)]}.
\end{aligned}
\tag{1.13}
$$

We estimate the two coordinates separately and start with

$$
\begin{aligned}
&\Big\| \sum_j \beta_j^{-r/p'} \Big(\sum_{n \in E_j} \sigma_n^{r/p'} x_n \Big) \otimes e_{jj} \Big\|_{S_p[S_p]} \\
&\qquad = \Big(\sum_j \beta_j^{-rp/p'} \Big\| \sum_{n \in E_j} \sigma_n^{r/p'} x_n \Big\|_{S_p}^p \Big)^{1/p} \\
&\qquad \le \Big(\sum_j \beta_j^{-rp/p'} \Big(\sum_{n \in E_j} \sigma_n^r \Big)^{p/p'} \sum_{n \in E_j} \|x_n\|_{S_p}^p \Big)^{1/p} \\
&\qquad = \Big(\sum_j \sum_{n \in E_j} \|x_n\|_{S_p}^p \Big)^{1/p} = \Big(\sum_n \|x_n\|_{S_p}^p \Big)^{1/p}.
\end{aligned}
\tag{1.14}
$$

The estimate of the other coordinate is slightly more involved. For every $\xi \in \ell_2$ and every j we get

$$
\begin{aligned}
&\Big(\Big(\sum_{n \in E_j} \sigma_n^{r/p'} x_n^* \Big) \Big(\sum_{n \in E_j} \sigma_n^{r/p'} x_n \Big) \xi, \xi \Big) \\
&\qquad = \Big\| \sum_{n \in E_j} \sigma_n^{r/p'} x_n \xi \Big\|^2 \le \Big(\sum_{n \in E_j} \sigma_n^{2r/p'-2} \Big) \Big(\sum_{n \in E_j} \|\sigma_n x_n \xi\|^2 \Big) \\
&\qquad = \sum_{n \in E_j} \sigma_n^r \sum_{n \in E_j} (\sigma_n^2 x_n^* x_n \xi, \xi) = \beta_j^r \sum_{n \in E_j} \sigma_n^2 (x_n^* x_n \xi, \xi),
\end{aligned}
$$

which shows that in the sense of operators on ℓ_2 we have

$$0 \le \sum_j \beta_j^{-r} \Big(\sum_{n\in E_j} \sigma_n^{r/p'} x_n^* \Big) \Big(\sum_{n\in E_j} \sigma_n^{r/p'} x_n \Big) \le \sum_j \sigma_j^2 x_j^* x_j .$$

Together with (0.3) and [7, Theorem 2.3] this gives

$$\begin{aligned}(1.15)\qquad \Big\| \sum_j \beta_j^{-r/p'} \sum_{n\in E_j} \sigma_n^{r/p'} x_n \otimes \beta_j e_{j1} \Big\|_{S_p[C_p]} &= \Big\| \Big(\sum_j \beta_j^{-r} \Big(\sum_{n\in E_j} \sigma_n^{r/p'} x_n^* \Big) \Big(\sum_{n\in E_j} \sigma_n^{r/p'} x_n \Big)\Big)^{1/2} \Big\|_{S_p} \\ &= \Big(\mathrm{tr}\Big(\Big[\sum_j \beta_j^{-r} \sum_{n\in E_j} \sigma_n^{r/p'} x_n^* \sum_{n\in E_j} \sigma_n^{r/p} x_n \Big]^{p/2} \Big)\Big)^{1/p} \\ &\le \Big(\mathrm{tr}\Big(\Big[\sum_j \sigma_j^2 x_j^* x_j \Big]^{p/2} \Big)\Big)^{1/p} = \Big\| \sum_j x_j \otimes \sigma_j e_{j1} \Big\|_{S_p[C_p]} .\end{aligned}$$

(1.13)–(1.15) show that P is completely bounded with $\|P\|_{\mathrm{cb}} = 1$. ■

An application of Theorem 1.1 shows, as in the Banach space case, that if σ in addition satisfies (1.6), then $X_{p*}(\sigma)$ is uniquely determined up to a cb-isomorphism. This is the contents of the next theorem.

THEOREM 1.3. *If* $2 < p < \infty$, *and* σ *and* γ *are two sequences both satisfying* (1.5) *and* (1.6), *then* $X_{p*}(\sigma)$ *is cb-isomorphic to* $X_{p*}(\gamma)$.

Proof. The proof follows the lines of the proofs of [26, Proposition 12 and Theorem 13] and is based on Pełczyński's decomposition method (see e.g. [18, Theorem 2.a.3]). We will therefore first prove that $X_{p*}(\gamma)$ is cb-isomorphic to a cb-complemented subspace of $X_{p*}(\sigma)$ and vice versa.

Since σ satisfies (1.5) and (1.6), we can find a sequence (E_j) of mutually disjoint, finite subsets of $\mathbb{N}$ so that

$$(1.16)\qquad \gamma_j \le \beta_j = \Big(\sum_{n\in E_j} \sigma_n^r \Big)^{1/r} \le 2\gamma_j \qquad \text{for all } j \in \mathbb{N}.$$

From Proposition 1.2 it follows that $X_{p*}(\beta)$ is cb-isometric to a subspace of $X_{p*}(\sigma)$ onto which there is a cb-contractive projection. (1.16) shows that $X_{p*}(\gamma)$ is 2-cb-isomorphic to $X_{p*}(\beta)$. By interchanging the roles of γ and σ we find that also $X_{p*}(\sigma)$ is cb-isomorphic to a cb-complemented subspace of $X_{p*}(\gamma)$.

The next step is to show that $X_{p*}(\sigma)$ is cb-isomorphic to $X_{p*}(\sigma) \oplus X_{p*}(\sigma)$, but we shall only prove it for $X_{p,c_p}(\sigma)$ since the other cases can be obtained in a similar manner.

(1.5) and (1.6) imply that we can find a sequence $\{E_{j,k} \mid j \in \mathbb{N},\ k \in \mathbb{N}\}$ of mutually disjoint finite subsets of $\mathbb{N}$ so that

$$\sigma_j \le \beta_{j,k} = \Big(\sum_{n\in E_{j,k}} \sigma_n^r\Big)^{1/r} \le 2\sigma_j \quad \text{for all } j,k\in\mathbb{N}. \tag{1.17}$$

Put $\beta_k = (\beta_{j,k})_{j=1}^\infty$, let $\widetilde{f}_{j,k} = \beta_{j,k}^{-r/p}\sum_{n\in E_{j,k}} \sigma_n^{r/p} e_{nn} \oplus \sigma_n e_{n1}$ and define $Z = [\widetilde{f}_{j,k} \mid j,k\in\mathbb{N}]$, $Z_1 = [\widetilde{f}_{j,k} \mid j\in\mathbb{N},\ k\ge 2]$. It follows from Proposition 1.2 that Z is cb-contractively complemented in $X_{p,c_p}(\sigma)$ and that for all $k\in\mathbb{N}$, $[\widetilde{f}_{j,k}]$ is cb-contractively complemented and cb-isometric to $X_{p,c_p}((\beta_k))$, which in turn is 2-cb-isomorphic to $X_{p,c_p}(\sigma)$. Hence Z can be viewed as an infinite direct sum of copies of $X_{p,c_p}(\sigma)$. Let $T : \operatorname{span}\{\widetilde{f}_{j,k} \mid j,k\in\mathbb{N}\} \to Z_1$ be defined by $T\widetilde{f}_{j,k} = \widetilde{f}_{j,k+1}$. We shall show that T extends to a cb-isomorphism of Z onto Z_1. If $(x_{j,k}) \subseteq S_p$ is a finite sequence, then we deduce from (1.17) and [7, Theorem 2.3] that

$$\begin{aligned}\Big\|\Big(\sum_k\sum_j \beta_{j,k+1}^2 x_{j,k}^* x_{j,k}\Big)^{1/2}\Big\|_{S_p} &\le 2\Big\|\Big(\sum_k\sum_j \sigma_j^2 x_{j,k}^* x_{j,k}\Big)^{1/2}\Big\|_{S_p}\\ &\le 2\Big\|\Big(\sum_k\sum_j \beta_{j,k}^2 x_{j,k}^* x_{j,k}\Big)^{1/2}\Big\|_{S_p}.\end{aligned} \tag{1.18}$$

In the same manner we get

$$\Big\|\Big(\sum_k\sum_j \beta_{j,k}^2 x_{j,k}^* x_{j,k}\Big)^{1/2}\Big\|_{S_p} \le 2\Big\|\Big(\sum_k\sum_j \beta_{j,k+1} x_{j,k}^* x_{j,k}\Big)^{1/2}\Big\|. \tag{1.19}$$

Similar estimates can easily be obtained for the corresponding p-norms, which implies that

$$\begin{aligned}\frac{1}{2}\Big\|\sum_k\sum_j x_{j,k}\otimes \widetilde{f}_{j,k}\Big\|_{S_p[X_{p,c_p}(\sigma)]} &\le \Big\|\sum_k\sum_j x_{j,k}\otimes \widetilde{f}_{j,k+1}\Big\|_{S_p[X_{p,c_p}(\sigma)]}\\ &\le 2\Big\|\sum_k\sum_j x_{j,k}\otimes \widetilde{f}_{j,k}\Big\|_{S_p[X_{p,c_p}(\sigma)]},\end{aligned}$$

showing that T can be extended to a cb-isomorphism of Z onto Z_1.

Letting $\sim_{\mathrm{cb}}$ denote "cb-isomorphic to", we infer from the above that $Z \sim_{\mathrm{cb}} X_{p,c_p}(\sigma) \oplus Z$. Since Z is cb-complemented in $X_{p,c_p}(\sigma)$, we can find a closed subspace $U \subseteq X_{p,c_p}(\sigma)$ such that

$$X_{p,c_p}(\sigma) = Z\oplus U \sim_{\mathrm{cb}} X_{p,c_p}(\sigma)\oplus Z\oplus U \sim_{\mathrm{cb}} X_{p,c_p}(\sigma)\oplus X_{p,c_p}(\sigma).$$

We are now ready to show that $X_{p,c_p}(\gamma)$ is cb-isomorphic to $X_{p,c_p}(\sigma)$. Indeed, since by the above $X_{p,c_p}(\gamma)$ is cb-isomorphic to a cb-complemented subspace of $X_{p,c_p}(\sigma)$, we can find a closed subspace $G \subseteq X_{p,c_p}(\sigma)$ such that

$$X_{p,c_p}(\sigma) \sim_{\rm cb} X_{p,c_p}(\gamma) \oplus G \sim_{\rm cb} X_{p,c_p}(\gamma) \oplus X_{p,c_p}(\gamma) \oplus G$$
$$\sim_{\rm cb} X_{p,c_p}(\gamma) \oplus X_{p,c_p}(\sigma) \sim_{\rm cb} X_{p,c_p}(\gamma)$$

where the last $\sim_{\rm cb}$ follows by interchanging the roles of σ and γ. ■

Exploiting the decomposition method a bit more we can actually conclude that also the space Z in the above proof is cb-isomorphic to $X_{p,c_p}(\sigma)$.

We are now going to define some operator spaces which we shall call matricial Rosenthal spaces.

Define $\widetilde{Y}_p(\sigma)$ to be the subspace of $\mathcal{K}_p \oplus_p (\sum_{n=1}^\infty S_2^n)_2$ consisting of all elements of the form $((x_n, \sigma_n x_n))$ where $x_n \in S_p^n$ for all $n \in \mathbb{N}$, i.e. we require

$$\sum_{n=1}^\infty \|x_n\|_{S_p^n}^p < \infty \quad \text{and} \quad \sum_{n=1}^\infty \sigma_n^2 \|x_n\|_{S_2^n}^2 < \infty. \tag{1.20}$$

We can view $(\sum_{n=1}^\infty S_2^n)_2$ isometrically as a subspace of $C_p[C_p]$ in the following way: Choose a sequence (m_n) of integers so that $m_1 = 0$ and $m_{n+1} - m_n = n$ for all $n \in \mathbb{N}$. If $x = (x_n) \in (\sum_{n=1}^\infty S_2^n)_2$ with $x_n = (t_{ij}^n)_{i,j=1}^n$, we can identify x with $\sum_{n=1}^\infty \sum_{i,j=m_n+1}^{m_{n+1}} t_{ij}^n e_{ij} \in C_p[C_p]$. Similarly we can consider $(\sum_{n=1}^\infty S_2^n)_2$ as a subspace of $R_p[R_p]$, respectively of $C_p[C_p] \oplus_p R_p[R_p]$.

Hence there is a canonical Banach space isometry w_σ of $\widetilde{Y}_p(\sigma)$ into the operator space $\mathcal{K}_p \oplus_p C_p[C_p]$ and we put $Y_{p,c_p} = w_\sigma(\widetilde{Y}_p(\sigma))$. Similarly we define the spaces $Y_{p,r_p}(\sigma)$ and $Y_{p,c_p\cap r_p}(\sigma)$. In the rest of this paper we shall put $Y_p(\sigma) = Y_{p,c_p\cap r_p}(\sigma)$.

Since we often consider cb-maps to or from these spaces, it is worth mentioning how the norm in $S_p[Y_{p,c_p}(\sigma)]$ is computed (the other cases follow similarly). Let us just compute the "column part" of $S_p[Y_{p,c_p}(\sigma)]$. To this end let $X_n \in S_p \otimes S_p^n$ for all $n \in \mathbb{N}$. We can then find $(x_{jk}^n) \in S_p^n$ so that

$$X_n = \sum_{j,k=m_n+1}^{m_{n+1}} x_{jk}^n \otimes e_{jk}$$

for every $n \in \mathbb{N}$. Note that

$$X_n^* X_n = \sum_{k,l=m_n+1}^{m_{n+1}} \Big(\sum_{j=m_n+1}^{m_{n+1}} x_{jk}^{n*} x_{jl}^n \Big) e_{kl}. \tag{1.21}$$

Using Proposition 0.2 we get

$$\Big\| \sum_n \sigma_n X_n \Big\|_{S_p[C_p[C_p]]} = \Big\| \sum_n \sum_{j,k=m_n+1}^{m_{n+1}} x_{jk}^n \otimes e_{jk} \Big\|_{S_p[C_p[C_p]]} \tag{1.22}$$
$$= \Big\| \Big(\sum_n \sigma_n^2 \sum_{j,k=m_n+1}^{m_{n+1}} x_{jk}^{n*} x_{jk}^n \Big)^{1/2} \Big\|_{S_p} = \Big\| \Big(\sum_n \sigma_n^2 (\mathrm{id} \otimes \mathrm{tr})(X_n^* X_n) \Big)^{1/2} \Big\|_{S_p},$$

where we have used (1.21) to get the last equality. Comparing this with the similar calculations for the other cases it is readily verified that $Y_p(\sigma)$, $Y_{p,c_p}(\sigma)$, and $Y_{p,r_p}(\sigma)$ are mutually non-cb-isomorphic.

Since $\mathcal{K}_p \oplus_p C_p[C_p]$ is cb-isomorphic to a subspace of S_p, the same holds for $Y_{p,c_p}(\sigma)$ as well. In a similar manner we show that $Y_{p,r_p}(\sigma)$ and $Y_p(\sigma)$ are cb-isomorphic to subspaces of S_p. We have the following result on these spaces.

THEOREM 1.4. *Both $\mathcal{K}_p$ and $X_{p,c_p}(\sigma)$ (respectively $X_{p,r_p}(\sigma)$) are cb-isomorphic to complemented subspaces of $Y_{p,c_p}(\sigma)$ (respectively $Y_{p,r_p}(\sigma)$). Consequently, $\widetilde{Y}_p(\sigma)$ is not Banach space isomorphic to a complemented subspace of S_p if σ satisfies* (1.5) *and* (1.6).

Proof. Let $U = X_{p,c_p}(\sigma)$ (respectively $U = X_{p,r_p}(\sigma)$) and $W = Y_{p,c_p}(\sigma)$ (respectively $W = Y_{p,r_p}(\sigma)$). If $(n_k) \subseteq \mathbb{N}$ is a sequence with $\sum_{k=1}^{\infty} \sigma_{n_k}^{2p/(p-2)} < \infty$, then the subspace V consisting of those $((x_n, \sigma_n x_n)) \in W$ for which $x_n = 0$ for all $n \neq n_k$ is readily seen to be completely complemented by a projection of cb-norm 1 and completely isomorphic to $\mathcal{K}_p$.

It is obvious that U can be identified cb-isometrically with the subspace of W consisting of those $((x_n, \sigma_n x_n)) \in \widetilde{Y}_p(\sigma)$ for which x_n is a one-dimensional operator on ℓ_2 for all $n \in \mathbb{N}$. This space is clearly the range of a cb-contractive projection.

It now follows directly from Theorem 1.1 that $\widetilde{Y}_p(\sigma)$ cannot be Banach space isomorphic to a complemented subspace of S_p if σ satisfies (1.5) and (1.6). ■

The last spaces we are going to investigate are defined as follows:

$$Z_{p,c_p}(\sigma) = \{(x, xD_\sigma) \mid x \in A_\sigma\} \subseteq S_p \oplus_p C_p[C_p], \tag{1.23}$$

$$Z_{p,r_p}(\sigma) = \{(x, D_\sigma x) \mid x \in A_\sigma\} \subseteq S_p \oplus_p R_p[R_p], \tag{1.24}$$

$$Z_p(\sigma) = \{(x, xD_\sigma, D_\sigma x) \mid x \in A_\sigma\} \subseteq S_p \oplus_p C_p[C_p] \oplus_p R_p[R_p], \tag{1.25}$$

where

$$A_\sigma = \{x \in S_p \mid xD_\sigma \in S_2\}.$$

In (1.23) we consider xD_σ as an element of $C_p[C_p]$, and similarly in (1.24) and (1.25).

In what follows we let $Z_{p,*}(\sigma)$ denote any of these spaces. Clearly they are isomorphic as Banach spaces, mutually non-cb-isomorphic, and cb-embedded into S_p.

The next theorem gives the basic properties of the spaces $Z_{p*}(\sigma)$.

THEOREM 1.5. *The space $Z_{p,*}(\sigma)$ has the following properties*:

(i) *If σ satisfies* (1.5), *then S_p is cb-isomorphic to a cb-complemented subspace of $Z_{p,*}(\sigma)$.*

(ii) *If σ satisfies both* (1.5) *and* (1.6), *then $Z_{p,*}(\sigma)$ is not isomorphic to a complemented subspace of S_p.*

Proof. (i): We shall only give the argument for $Z_{p,c_p}(\sigma)$. The proof for the other spaces is similar. Let $(n_k) \subseteq \mathbb{N}$ with $\sum_{k=1}^{\infty} \sigma_{n_k}^{2p/(p-2)} < \infty$ and let V consist of those $(x, xD_\sigma) \in Z_p(\sigma)$ for which $x_{ij} = 0$ unless $j = n_k$ for some $k \in \mathbb{N}$. It is readily verified that V is cb-isomorphic to S_p. From Arazy [1, Theorem 1.1] it follows that V contains another subspace U cb-isomorphic to S_p and which is complemented in $Z_p(\sigma)$. This shows (i).

(ii): $X_p(\sigma)$ can easily be identified with those $(x, xD_\sigma) \in Z_p(\sigma)$ for which x is a diagonal matrix. This subspace is clearly contractively complemented in $Z_p(\sigma)$. It now follows from Theorem 1.1 that $Z_p(\sigma)$ is not isomorphic to a complemented subspace of S_p. ∎

Before we go on we need the following lemma on non-commutative L_p-spaces.

Lemma 1.6. *Let $1 < p < \infty$ and let $\mathcal{N}$ be a von Neumann algebra so that $L_p(\mathcal{N})$ is separable and $L_p(0,1)$ does not embed isomorphically into $L_p(\mathcal{N})$. Then there exist sequences (I_k) of countable sets and $(n_k) \subseteq \mathbb{N}$ so that*

$$L_p(\mathcal{N}) = \Big(\sum_{k=1}^{\infty} \ell_p(I_k, S_p^{n_k})\Big)_p. \tag{1.26}$$

Proof. Since $L_p(0,1)$ does not embed into $L_p(\mathcal{N})$, it follows from a result of Marcolino [21] that $\mathcal{N}$ is a type I factor and therefore the separability of $L_p(\mathcal{N})$ and [28] imply that there exist measure spaces $(\Omega_k, \Sigma_k, \mu_k)$ for all $k \in \mathbb{N}$ and $(n_k) \subseteq \mathbb{N}$ so that

$$L_p(\mathcal{N}) = \Big(\sum_{n=1}^{\infty} L_p(\Omega_k, \Sigma_k, \mu_k, S_p^{n_k})\Big)_p. \tag{1.27}$$

Again, since $L_p(0,1)$ does not embed into $L_p(\mathcal{N})$, it follows that all the measure spaces on the right side of (1.27) are purely atomic. ∎

We are now able to prove:

Theorem 1.7. *If σ satisfies* (1.5) *and* (1.6), *then none of the spaces $X_p(\sigma)$, $Y_p(\sigma)$ or $Z_p(\sigma)$ is isomorphic to an $L_p(\mathcal{N})$-space where $\mathcal{N}$ is a von Neumann algebra.*

Proof. Let V be one of the spaces above and assume that there exists a von Neumann algebra $\mathcal{N}$ so that V is isomorphic to $L_p(\mathcal{N})$. Since it follows from [2, Theorem 6] that $L_p(0,1)$ does not embed into S_p, $L_p(\mathcal{N})$ has the form of (1.26) by Lemma 1.6 and therefore it is isomorphic to a complemented subspace of S_p. This contradicts Theorems 1.1, 1.4, and 1.5 above. ∎

2. The operator space structure of the classical Rosenthal sequence spaces. In this section we wish to discuss the operator space structure of the Rosenthal sequence spaces defined in Section 1, and it turns out that the local structure of these spaces behaves quite differently. However, due to the non-commutative Burkholder–Rosenthal inequalities [13], [14] the probabilistic viewpoint from the commutative case is still adequate to determine this structure.

Let $(\sigma_i) \subset [0,1]$ and let $A_i \subset [0,1]$, $i \in \mathbb{N}$, be intervals of measure $\mu(A_i) = \sigma_i^r$, where $1/2 = 1/p + 1/r$. We define $f_i((t_j)) = \mu(A_i)^{-1/p} 1_{A_i}(t_i)$ for all sequences $(t_j) \subseteq [0,1]$. The sequence $(f_i)_{i\in\mathbb{N}}$ is a sequence of independent random variables on $[0,1]^{\mathbb{N}}$. For sequences (s_i) with finite support we define

$$u((s_i)) = \sum_{i=1}^{\infty} s_i \varepsilon_i f_i,$$

$$u_c((s_i)) = \sum_{i=1}^{\infty} s_i e_{i,1} \varepsilon_i f_i, \qquad u_r((s_i)) = \sum_{i=1}^{\infty} s_i e_{1,i} \varepsilon_i f_i,$$

where (ε_i) denotes the sequence of Rademacher functions on $[0,1]$.

Following Rosenthal's argument from [26] using [14] we can now obtain

PROPOSITION 2.1. *Let $2 \le p < \infty$. Then u, u_c, and u_r are cb-isomorphisms between $X_p(\sigma)$, $X_{p,c_p}(\sigma)$, and $X_{p,r_p}(\sigma)$ and their respective images in $L_p([0,1]^{\mathbb{N}})$, $L_p([0,1]^{\mathbb{N}}; C_p)$, and $L_p([0,1]^{\mathbb{N}}; R_p)$. The images are cb-complemented in the respective spaces.*

Proof. We shall only prove the proposition for u_c since the other cases go similarly. Let $(x_i)_{i=1}^n \subseteq S_p$ be arbitrary. From [14, Corollary 1.5] and Proposition 0.2 we get, letting $\sim$ denote two-sided inequalities with constants only depending on p,

$$(2.1)\qquad \Big\| \sum_{i=1}^{n} x_i \otimes \varepsilon_i f_i e_{i1} \Big\|_{S_p[L_p((0,1);C_p)]}$$
$$\sim \max\Big\{ \Big(\sum_{i=1}^{n} \|x_i\|_{S_p}^p \|f_i\|_p^p \Big)^{1/p}, \Big\| \Big(\sum_{i=1}^{n} x_i^* x_i \mathbb{E}(f_i^2) \Big)^{1/2} \Big\|_{S_p},$$
$$\Big(\sum_{i=1}^{n} \|x_i\|_{S_p}^p \mathbb{E}(f_i^2)^{p/2} \Big)^{1/p} \Big\}$$
$$\sim \Big\| \sum_{i=1}^{n} x_i \otimes (e_{ii} \oplus \sigma_i e_{i1}) \Big\|_{S_p[X_{p,c_p}(\sigma)]}.$$

where in the last equivalence we have used the fact that for all $1 \le i \le n$ we have $\|f_i\|_p = 1$, $\mathbb{E}(f_i^2) = \sigma_i^2$, and $\mathbb{E}(f_i^2)^{p/2} = \mu(A_i)^{p/2-1} \le 1$. By Lemma 0.1, u_c is a cb-isomorphism.

For every $1 \le i \le n$ we put $f_i' = \mu(A_i)^{1/p'} 1_{A_i}$ and $u_{p'}((s_i)) = \sum s_i \varepsilon_i f_i'$. Using the second part of [14, Theorem 0.1] in much the same way as above we deduce that $u_{p'}$ acts as a cb-bounded operator from X_{p,c_p}^* to $L_{p'}(0,1)$. It is readily verified that $u_c u_{p'}^*$ is a cb-projection of $L_p(0,1)$ onto the range of u_c. ∎

Corollary 2.2. *The spaces $X_p(\sigma)$, $X_{p,c_p}(\sigma)$, and X_{p,r_p} have the γ_p-AP. More precisely, we have an approximate diagram*

$$\begin{array}{ccc} X_p & \xrightarrow{\ \mathrm{id}\ } & X_p \\ & {\scriptstyle v_n}\searrow \quad \nearrow{\scriptstyle w_n} & \\ & \ell_p^{n_k} & \end{array}$$

For $X_{p,c_p}(\sigma)$ and $X_{p,r_p}(\sigma)$ we replace $\ell_p^{n_k}$ by $\ell_p^{n_k}(C_p^{n_k})$ and $\ell_p^{n_k}(R_p^{n_k})$, respectively.

Corollary 2.3. *If σ satisfies* (1.5), *then the Rosenthal spaces $X_p(\sigma)$ are $\mathcal{COL}_p$-spaces.*

Proof. Follow the proof of [11, Proposition 2.4], using Corollary 2.2 and the fact that $X_p(\sigma)$ contains completely complemented copies of ℓ_p^n's far out. ∎

In the following we want to show that the Rosenthal spaces $X_{p,c_p}(\sigma)$ and $X_{p,r_p}(\sigma)$ are no longer $\mathcal{OL}_p$. Indeed, the mixture between the Hilbert space structure and the ℓ_p structure forms the crucial obstacle.

Lemma 2.4. *If $1 \le p < \infty$ and $\mathcal{N}$ is a finite von Neumann algebra, then C_p is not cb-isomorphic to a subspace of $R_p(L_p(\mathcal{N}))$. Similarly, R_p is not cb-isomorphic to a subspace of $C_p(L_p(\mathcal{N}))$.*

Proof. Suppose to the contrary that C_p is isomorphic to a subspace of $R_p(L_p(\mathcal{N}))$. Using the natural isomorphism between $R_p(R_p)$ and R_p, we deduce that $S_p = R_p(C_p)$ is a Banach space isomorphic to a subspace of $R_p(L_p(\mathcal{N})) \subset L_p(B(\ell_2) \otimes \mathcal{N})$. However, for $x \in R_p(L_p(\mathcal{N}))$ and $p \ge 2$,

$$\|x\|_2 = \|xx^*\|_{L_1(\mathcal{N})}^{1/2} \le \|xx^*\|_{p/2}^{1/2} \le \|x\|_p.$$

So $R_p(L_p(\mathcal{N}))$ is isomorphic to a subspace of $L_p(B(\ell_2)\otimes\mathcal{N}) \cap L_2(B(\ell_2)\otimes\mathcal{N})$ for $2 \le p < \infty$. For $1 \le p \le 2$ a similar argument shows that $R_p(L_p(\mathcal{N}))$ is isomorphic to a subspace of $L_p(\mathcal{N} \otimes B(\ell_2)) + L_2(\mathcal{N} \otimes B(\ell_2))$. According to [9] these spaces are isomorphic to complemented subspaces of $L_p(\mathcal{M})$ for some finite von Neumann algebra $\mathcal{M}$. Hence, S_p is isomorphic to a subspace of $L_p(\mathcal{M})$. This contradicts Sukochev's result for $p \ge 2$, [27], or [6] for $1 \le p \le 2$. By symmetry the same holds for R_p and C_p interchanged. ∎

Corollary 2.5. *Let $2 < p, r < \infty$ and $1/2 = 1/p + 1/r$. If $\sigma \notin \ell_r$, then the spaces $X_{p,c_p}(\sigma)$ and $X_{p,r_p}(\sigma)$ are not cb-isomorphic to subspaces of $L_p(\mathcal{N})$ with $\mathcal{N}$ finite.*

Proof. Assume first that there is an infinite set $A \subset \mathbb{N}$ so that $\inf_{k\in A} \sigma_k > 0$. By interpolation we deduce that for the bounded sequence $(\sigma_k^{-1})_{k\in A}$ the diagonal map $D_{\sigma^{-1}} : C_p \to \ell_p$ is completely bounded. Hence, the subspace of $X_{p,c_p}(\sigma)$ consisting of the sequences having their support in A is cb-isomorphic to C_p. In particular, it cannot embed into $L_p(\mathcal{N})$ cb-isomorphically. Thus $X_{p,c_p}(\sigma)$ cannot embed either in this case.

Since $\sum_j \sigma_j^r = \infty$, in the general case we can find disjoint finite subsets A_j such that if

$$\beta_j = \Big(\sum_{i\in A_j} \sigma_i^r\Big)^{1/r},$$

then $\inf \beta_j > 0$. Proposition 1.2 shows that $X_{p,c_p}(\beta)$ is cb-isomorphic to a subspace of $X_{p,c_p}(\sigma)$ and by the above cb-isomorphic to C_p, and hence the assertion follows. A similar argument applies for the row spaces. ∎

Lemma 2.6. *If $1 \le p \le \infty$ and $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, then $\prod_{\mathcal{U}} \ell_p$ is completely isometrically isomorphic to $L_p(\mathcal{N})$ for a commutative von Neumann algebra $\mathcal{N}$.*

Proof. Let $\mathcal{N} = (\prod_{\mathcal{U}} \ell_1)^*$. From Raynaud's theorem [24] we deduce that for all $n \in \mathbb{N}$, $(S_1^n(\prod_{\mathcal{U}} \ell_1))^* = M_n(\mathcal{N})$, where $\mathcal{N}$ is a commutative von Neumann algebra obtained as the weak closure of $\prod \ell_\infty$. Together with [23, Lemma 5.4] this implies that

$$L_p(M_n \otimes \mathcal{N}) = \prod S_p^n(\ell_p) = S_p^n\Big(\prod\nolimits_{\mathcal{U}} \ell_p\Big) = S_p^n(L_p(\mathcal{N})).$$

Thus $\prod_{\mathcal{U}} L_p$ is completely isometrically isomorphic to $\ell_p(\mathcal{N})$. ∎

Our aim is now to show that $X_{p,c_p}(\sigma)$ is not a rectangular $\mathcal{OL}_p$-space.

Lemma 2.7. *If $2 \le p \le \infty$, then for all $n \in \mathbb{N}$,*

$$n^{1/2-1/p} \le \inf_{E\subset C_p(L_p(0,1))} d_{\rm cb}(R_p^n, E) \le c_p n^{1/2-1/p}.$$

The same estimates hold if R_p and C_p are interchanged.

Proof. By interpolation,

$$d_{\rm cb}(R_p^n, R_p^n \cap C_p^n) \le \|{\rm id} : R_p^n \to C_p^n\|_{\rm cb} \|{\rm id} : R_p^n \cap C_p^n \to R_p^n\|_{\rm cb} \le n^{1/2-1/p}.$$

By the non-commutative Khinchin inequality [20],

$$d_{\rm cb}(R_p^n \cap C_p^n, {\rm span}\{g_j \mid j = 1, \dots, n\}) \le c_p,$$

where the g_j's are independent Gaussian variables. To prove the lower estimate, we consider $E \subset L_p(C_p)$ and a completely bounded contractive isomorphism $\phi : R_p^n \to E$. Let $x_i = \phi(e_{1i})$. Then

$$\Big(\int\Big(\sum_{i=1}^n \|x_i(s)\|_2^2\Big)^{p/2} d\mu(s)\Big)^{1/p} = \Big\|\sum_{i=1}^n e_{i,1}\otimes x_i\Big\|_{L_p(C_p^n(C_p))}$$
$$\le \|\phi\|_{\mathrm{cb}}\Big\|\sum_{i=1}^n e_{i,1}\otimes e_{1,i}\Big\|_{C_p^n[R_p^n]} = \|\mathrm{id}\|_{S_p^n} = n^{1/p}.$$

However, this implies

$$\begin{aligned}\sqrt{n} &= \Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i e_{1,i}\Big\|_2^2\Big)^{1/2} = \Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i \phi^{-1}(x_i)\Big\|_2^2\Big)^{1/2}\\
&\le \|\phi^{-1}\|\Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i(x_i)\Big\|_{L_p(\ell_2)}^2\Big)^{1/2}\\
&\le \|\phi^{-1}\|\Big(\int\Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i(s)\Big\|_2^2\Big)^{p/2}\mu(s)\Big)^{1/p}\\
&= \|\phi^{-1}\|\Big(\int\Big(\sum_{i=1}^n \|x_i(s)\|_2^2\Big)^{p/2}\mu(s)\Big)^{1/p} \le \|\phi^{-1}\|n^{1/p}.\end{aligned}$$

The assertion is proved. ■

Using a similar idea we can even prove a slighly stronger statement.

LEMMA 2.8. *If* $2 \le p \le \infty$, *then for all* $n \in \mathbb{N}$,

$$\frac{1}{c_p} n^{1/2-1/p} \le \inf_{E\in QS(\prod_{\mathcal{U}} L_p(C_p))} d_{\mathrm{cb}}(R_p^n, E) \le c_p n^{1/2-1/p}.$$

Here c_p *is an absolute constant and* $QS(\prod_{\mathcal{U}} L_p(C_p))$ *stands for the class of quotients of subspace of ultraproducts of* $C_p(L_p(0,1))$. *The same estimate holds on exchanging* R_p *with* C_p.

Proof. Let $T : C_p^n \to L_p(0,1)$ be defined by $T(e_{i1}) = \varepsilon_i$, where $(\varepsilon_i)_{i=1}^n$ are the Bernoulli random variables. We will use a sequence (g_j) of independent normalized complex gaussian random variables on (Ω', μ'). Let $h_1, \ldots, h_n \in L_p(\Omega, \mu; \ell_2)$. Then we deduce from the Khinchin/Kahane's inequality [16] that

$$\Big\|\sum_{i=1}^n \varepsilon_i h_i\Big\|_{L_p(\ell_2)}$$
$$= \|g_1\|_p^{-1}\Big(\int_{\Omega\times\Omega'}\int_0^1\Big|\sum_{i=1}^n\sum_{=1}^\infty \varepsilon_i(s) g_j(\omega') h_i(j,\omega)\Big|^p ds\, d\mu'(\omega')\, d\mu(\omega)\Big)^{1/p}$$

$$\leq \|g_1\|_p^{-1} c_0 \sqrt{p} \Big(\int\limits_{\Omega\times\Omega'} \Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{\infty} g_j(\omega') h_i(j,\omega)) |^2 \Big|^{p/2} d\mu'(\omega')\, d\mu(\omega) \Big)^{1/p}$$

$$\leq \|g_1\|_p^{-1} c_0^2 p \Big(\int\limits_{\Omega} \Big(\sum_{i=1}^{n} \sum_{j=1}^{\infty} |h_i(j,\omega)|^2 \Big)^{p/2} d\mu(\omega) \Big)^{1/p}.$$

Since for $p \geq 2$, we have $\|g_1\|_p \sim \sqrt{p}$, we deduce

$$\|T \otimes \mathrm{id}_{C_p(L_p(\Omega))} : C_p^n(C_p(L_p(\Omega))) \to C_p(L_p([0,1]\times\Omega))\| \leq c_0^3\sqrt{p}.$$

This remains true if we pass to an ultraproduct and then to a quotient of a subspace. On the other hand, we have seen in Lemma 2.7 that

$$\|T \otimes \mathrm{id}_{R_p^n}\| \geq n^{1/2-1/p}.$$

Therefore the distance is greater that $n^{1/2-1/p}/c_0^3\sqrt{p}$. ■

The next lemma is a kind of "folklore" but for the convenience of the reader we give a proof.

LEMMA 2.9. *Let $\mathcal{M}$ be a von Neumann algebra and $2 < p \leq \infty$, $2 \leq r < \infty$ with $1/2 = 1/p + 1/r$. Let $F \subset L_p(\mathcal{M})$ be a subspace and $T : F \to R_p$ be a linear map. Then T is a complete contraction if and only if there exists a norm one element $a \in L_r(\mathcal{M})$ and a contraction $W : L_2(\mathcal{M}) \to \ell_2$ such that*

$$T(x) = W(ax) \quad \textit{for all } x \in L_p(\mathcal{N}).$$

In particular, T admits a completely contractive extension $\hat{T} : L_p(\mathcal{M}) \to R_p$. Similarly, every complete contraction $T : F \to C_p$ has a completely contractive extension of the form $T(x) = W(xa)$.

Proof. Let (x_j) be a finite sequence in F. Then

$$\Big(\sum_j \|T(x_j)\|_2^2\Big)^{1/2} = \Big\|\sum_j e_{j,1}\otimes T(x_j)\Big\|_{R_p(R_p)} \leq \Big\|\sum_j e_{j,1}\otimes x_j\Big\|_{R_p(L_p(\mathcal{M}))}$$
$$= \Big\|\sum_j x_j x_j^*\Big\|_{p/2} = \sup_{a\geq 0,\, \|a\|_{r/2}\leq 1} \Big(\sum_j \mathrm{tr}(a x_j x_j^*)\Big)^{1/2}.$$

Let B be the positive part of the unit ball of $L_{r/2}(\mathcal{M})$. The function $f_x(a) \mapsto \mathrm{tr}(ax^*x)$ is continuous with respect to the weak* topology. Hence, the standard separation yields a probability measure μ on B such that

$$\|T(x)\|_2^2 \leq \int\limits_B \mathrm{tr}(ax^*x)\, d\mu(a) = \mathrm{tr}\Big(\Big(\int\limits_B a\, d\mu(a)\Big)x^*x\Big).$$

By convexity, $b = (\int_B a\, d\mu(a)) \in B$ and therefore

$$\|T(x)\|_2 \leq \|b^{1/2}x\|_2.$$

Let $H = \{b^{1/2}x \mid x \in F\} \subset L_2(\mathcal{M})$. Thus there is a linear contraction $W_1 : H \to \ell_2$ such that $W_1(b^{1/2}x) = T(x)$. If P denotes the orthogonal projection onto H, then $W = W_1P$ satisfies the assertion. To prove the converse, we assume $T(x) = W(ax)$ for some $a \in L_r(\mathcal{M})$ of norm less than one. Let $L_a : L_p(\mathcal{M}) \to L_2(\mathcal{M})^{r_p}$ be the left multiplication $L_a(x) = ax$. Let $\phi : L_{p/2} \to \mathbb{C}$ be the induced linear functional $\phi(y) = \mathrm{tr}(ya^*a)$ of norm less than one. If $x \in L_p(B(\ell_2) \otimes \mathcal{M})$, we deduce that for every functional the cb-norm coincides with the norm

$$\begin{aligned}\|(\mathrm{id} \otimes L_a)(x)\|_{S_p(L_2(\mathcal{M})^{r_p})} &= \|(\mathrm{id} \otimes \mathrm{tr})((a \otimes \mathrm{id})xx^*(a^* \otimes \mathrm{id}))\|_{S_{p/2}}^{1/2}\\ &= \|(\mathrm{id} \otimes \mathrm{tr})(xx^*(a^*a \otimes \mathrm{id}))\|_{S_{p/2}}^{1/2} = \|(\mathrm{id} \otimes \phi)(xx^*)\|_{S_{p/2}}^{1/2}\\ &\le \|xx^*\|_{S_{p/2}}^{1/2} = \|x\|_p.\end{aligned}$$

By homogeneity of L_{2,r_p}, this implies $\|WL_a\|_{\mathrm{cb}} \le \|W\|\,\|a\|_r$. ∎

Corollary 2.10. *If $T : X_{p,c_p}(\sigma) \to C_p$ is completely bounded, then T admits a cb-extension to $\ell_p \oplus_p C_p$.*

Proposition 2.11. *If $2 < p < \infty$ and $\mathcal{N}$ is a finite von Neumann algebra, then $\ell_p(C_p)$ is not cb-isomorphic to a subspace of $C_p \oplus_p R_p(L_p(\mathcal{N}))$.*

Proof. Let $2 < r \le \infty$ be such that $1/2 = 1/p+1/r$. Let $T = (T^{(1)}, T^{(2)}) : \ell_p(C_p) \to C_p \oplus_p L_p(\mathcal{N}) \oplus_p R_p(L_p(\mathcal{N}))$ be a complete contraction and $T^{-1} : rg(T) \to \ell_p(C_p)$ be a completely bounded inverse with $\|T^{-1}\|_{\mathrm{cb}} \le C$. We consider the complete contraction $T_1 : \ell_p(S_p) \to C_p$ defined by $T_1(x) = T^{(1)}(P(x))$, P the projection onto the column space. According to Lemma 2.9, we can find $a \in \ell_r(S_r)$ and $W : \ell_2(S_2) \to \ell_2$ such that $T_1(x) = W(xa)$. Let $\varrho = (\|a(i)\|_r)$ and consider the operator $D_\varrho : \ell_p \to \ell_2$. We define the bounded map $W' : \ell_2(\ell_2) \to \ell_2$ by $W'((x_i)) = W((\varrho_i^{-1}x_ia_i))$. In particular, we can find an n such that

$$\Big(\sum_{k\ge n} \varrho_k^r\Big)^{1/r} \le \frac{1}{2C}.$$

In the following, we use the spaces $Y_n = \mathrm{span}\{\sum_k e_k \otimes x_k \mid k > n,\ x_k \in C_p\}$ and deduce

$$\|T^{(1)}|_{Y_n}\|_{\mathrm{cb}} \le \Big(\sum_{k\ge n_1} \varrho_k^r\Big)^{1/r} \|W' : \ell_2(\ell_2) \to \ell_2\| \le \frac{1}{2C}.$$

If $x \in S_p(Y_n)$, we deduce

$$\begin{aligned}\frac{1}{C}\,\|x\|_{S_p(Y_n)} &\le \|(\mathrm{id} \otimes T)(x)\|_p \le \|(\mathrm{id} \otimes T^{(1)}|_{Y_n})(x)\|_{C_p} + \|(\mathrm{id} \otimes T^{(2)}(x)\|_p\\ &\le \frac{1}{2C}\|x\|_{S_p(\ell_p(C_p))} + \|(\mathrm{id} \otimes T^{(1)})(x)\|_p.\end{aligned}$$

Thus

$$\frac{1}{2C}\|x\|_{S_p(Y_n)} \le \|(\mathrm{id}\otimes T^{(1)})(x)\|_{S_p(R_p(L_p(\mathcal{N})))} \le \|x\|_{S_p(\ell_p(C_p))}.$$

In particular, C_p is cb-isomorphic to a subspace of $R_p(L_p(\mathcal{N}))$, which contradicts Lemma 2.4. ∎

For the convenience of the reader we quote the following lemma which is used both in the next proposition and in the next section. The lemma is proved in [9] and [13].

LEMMA 2.12. *Let $\mathcal{M} \subset \mathcal{N}$ be von Neumann algebras, ϕ a faithful normal state on $\mathcal{N}$, and $\mathcal{E} : \mathcal{N} \to \mathcal{M}$ a faithful conditional expectation such that $\phi|_{\mathcal{M}} \circ \mathcal{E} = \phi$. Let $D \in L_1(\mathcal{M})$ be the density of ϕ.*

(i) *If $1/r + 1/s = 1/p \ge 1$, then $\mathcal{E}$ induces a contractive map $\mathcal{E}_p : L_p(\mathcal{N}) \to L_p(\mathcal{M})$ such that*

$$\mathcal{E}_p(axy) = a\mathcal{E}(x)b$$

for all $a \in L_r(\mathcal{M})$, $b \in L_s(\mathcal{M})$ and $x \in \mathcal{N}$.

(ii) *Let $1 \le p, p' \le \infty$ with $1/p+1/p' = 1$ and $L_p(\mathcal{N},\mathcal{E})$ be the completion of $\{aD^{1/p} \mid a\ \phi\text{-analytic}\}$ with respect to the norm*

$$\|aD^{1/p}\|_{L_p(\mathcal{N},\mathcal{E})} = \|D^{1/p}E(a^*a)D^{1/p}\|_{p/2}^{1/2}.$$

For $p = \infty$, we take the closure with respect to the strong topology. Then

$$L_p(\mathcal{N},\mathcal{E})^* = L_{p'}(\mathcal{N},\mathcal{E})$$

and the duality is given by the trace on $\mathcal{M}$.

(iii) *Let $1 \le p' \le 2 \le p \le \infty$ with $1/p + 1/p' = 1$. Then*

$$\|x\|_{L_p(\mathcal{N},\mathcal{E})} \le \|x\|_p \qquad \text{for all } x \in L_p(\mathcal{M}),$$

$$\|x\|_{p'} \le \|x\|_{L_{p'}(\mathcal{N},\mathcal{E})} \qquad \text{for all } x \in L_{p'}(\mathcal{N},\mathcal{E}).$$

PROPOSITION 2.13. *For every separable subspace W of $\prod_{\mathcal{U}} C_p(L_p(0,1))$ there is a commutative von Neumann algebra $\mathcal{N}$ such that W is completely isometrically isomorphic to a subspace of $C_p(L_p(\mathcal{N}))$. If in addition W is cb-complemented, then W can be assumed to be cb-complemented in $C_p(L_p(\mathcal{N}))$. The same holds with R_p replaced by C_p.*

Proof. Consider the commutative von Neumann algebra $\mathcal{N} = (\prod_{\mathcal{U}} L_1)^*$. Let $\iota : \prod_{\mathcal{U}} L_1 \to \prod_{\mathcal{U}} L_1(S_1)$ be the canonical inclusion map, given coordinatewise by $\iota((f(i))) = (e_{00}\otimes f(i))$. Let $q_0 = (e_{00}\otimes 1)$ be the projection onto the first corner. Obviously $q \le q_0$ and $\mathcal{E} = \iota^* : \prod_{\mathcal{U}} L_1(S_1)^* \to \mathcal{N}$ defines a conditional expectation. Let $\mathcal{M} = \prod_{\mathcal{U}} L_1(S_1)^*$ and consider the norm

$$\|x\|_{S_q^n(L_q(\mathcal{M},\mathcal{E}))} = \|(\mathrm{id}\otimes\mathcal{E})(x^*x)^{1/2}\|_{S_q(L_q(\mathcal{N}))}$$

defined on the space of elements $yd^{1/q}$, $d \in L_1(\mathcal{N})$, $y \in L_q(N)$. According to Lemma 2.12, we have

$$L_{p'}(\mathcal{M},\mathcal{E})^* = L_p(\mathcal{M},\mathcal{E}) \quad \text{completely isometrically.}$$

Obviously, the inclusion map $T : \prod_{\mathcal{U}} C_{p'}(L_{p'}(0,1)) \to L_{p'}(\mathcal{M},\mathcal{E})$ is completely isometric and therefore by duality $\prod_{\mathcal{U}} C_p(L_p(0,1))$ is completely contractively complemented in $L_p(\mathcal{M},\mathcal{E})$. Given $x \in S_p^m(W)$, we see that

$$\|x\|_p^2 = \|x^*x\|_{p/2} = \|x^*x\|_{S^m_{p/2}[L_p(\mathcal{N})]}.$$

Since $\bigcup_m S_p^m[W]$ is separable, we can find a density $D \in L_1(\mathcal{N})$ such that

$$x_{ij}^* x_{ij} \le C(x) D^{1/p}$$

for all $x = (x_{ij})_{ij=1}^m$ in a countable dense subset Δ of $\bigcup_m S_p^m[W]$. Multiplying with the support projection q of D, we can work in $\mathcal{N}q$. For every coordinate $y = x_{ij}$, $x = (x_{ij}) \in \Delta$, we consider the polar decomposition

$$y = ub.$$

Using Raynaud's isomorphism [24], we see that $b \in L_p(q\mathcal{N}q)$. Let $\mathcal{N}_1$ be a separable subalgebra generated by the elements $b = b_{ij}(x)$, $x \in \Delta$. Let $\mathcal{M}_1$ be a separable subalgebra containing by the polar decompositions $u = u_{ij}(x)$, $x \in \Delta$, such that there exists a conditional expectation $\mathcal{E}_1 : \mathrm{wcl}(\mathcal{M}_1) \to \mathcal{N}_1$ leaving ϕ invariant. Clearly, W is still a (cb-complemented) subspace of $L_p(\mathcal{M}_1,\mathcal{E})$ and we can consider the right $\mathcal{N}_1$-module F generated by M_1 and $\mathcal{N}_1$. According to [13], $L_p(\mathcal{M}_1,\mathcal{N}_1)$ is completely contractively complemented in $C_p(L_p(\mathcal{N}_1))$, and therefore the assertion is proved. ∎

COROLLARY 2.14. *If $2 < p < \infty$ and F is a quotient of $R_p(L_p(0,1))$, then $\ell_p^n(C_p^n)$ does not embed uniformly into $C_p \oplus_p F$.*

Proof. Supposing the contrary, we can find $T_n = (T_n^{(1)}, T_n^{(2)}) : \ell_p^n(C_p^n) \to C_p \oplus_p F$ such that

$$\|T_n\|_{\mathrm{cb}} \le 1 \quad \text{and} \quad \|T_n^{-1}\|_{\mathrm{cb}} \le C.$$

Let $\mathcal{U}$ be a free ultrafilter on the natural numbers and define

$$T : \ell_p(C_p) \to \prod\nolimits_{\mathcal{U}} C_p \oplus_p \prod\nolimits_{\mathcal{U}} F,$$

by $T(x) = ((T_n^{(1)}(x))_{n\in\mathbb{N}}, (T_n^{(2)}(x))_{n\in\mathbb{N}})$. This is well defined because the union $\bigcup_n \ell_p^n(C_p^n)$ is norm dense in $\ell_p(C_p)$. Moreover, for $x \in S_p^m(\ell_p^n(C_p^n))$, we have

$$\|(\mathrm{id}\otimes T)(x)\| = \lim_{n'>n} \|\mathrm{id}\otimes T_{n'}(x)\|_{S_p(\ell_p)\oplus_p S_p(C_p)} \sim_C \|x\|_{S_p^m(\ell_p^n(C_p^n))}.$$

Denote the first component by $T^{(1)}$ and the second by $T^{(2)}$. We note that $\prod_{\mathcal{U}} F$ is a quotient space of $\prod_{\mathcal{U}} R_p(L_p(0,1))$. Denote the quotient map by q. Then we can find a separable subspace $Y \subset \prod_{\mathcal{U}} R_p(L_p(0,1))$ such that the image of $T^{(2)}$ is cb-isomorphic to $q(Y)$. According to Proposition 2.13, we can assume that Y is contained in $R_p(L_p(\mathcal{N}))$ for some commutative von Neumann algebra $\mathcal{N}$. Moreover, $\prod_{\mathcal{U}} C_p$ is a homogeneous Hilbert space which carries the C_p structure. Thus every separable subspace is completely isometric to C_p. Therefore, we can find an embedding of $\ell_p(C_p)$ in $C_p \oplus_p Y/\ker(q)$. Following the argument in Proposition 2.11, we see that for the first component $T^{(1)}$ and every $\varepsilon > 0$ there exists an n such that $\|T^{(1)}|_{\{(x_k)|x_1=\cdots=x_n=0\}}\|_{\mathrm{cb}} \le \varepsilon$. Thus C_p will be cb-isomorphic to a subspace of a quotient of $R_p(L_p(0,1))$. This contradicts Lemma 2.7. ∎

THEOREM 2.15. *Let σ tend to 0 and be such that for all $n \in \mathbb{N}$ there is a subset A_n of cardinality n such that $\sigma_i = \alpha_n$ for $i \in A_n$ and*

$$\lim_n n^{1/r}\alpha_n = \infty.$$

Then $X_{p,c_p}(\sigma)$ does not admit a cb-factorization through $C_p \oplus_p F$, where F is a quotient of a subspace of $\prod_{\mathcal{U}} R_p(L_p(0,1))$.

Proof. Supposing the contrary we can write $\mathrm{id} = T + S$, where T factors through a quotient F of $\prod_{\mathcal{U}} R_p(L_p(0,1))$ and S factors through C_p. We denote by Q the projection onto the C_p coordinate in $X_{p,c_p}(\sigma) \subset \ell_p \oplus_p C_p$. Using Lemma 2.10, we can decompose $S = S_1 + S_2$ so that $S_1 : \ell_p \to X_{p,c_p}$ is a completely bounded operator and $S_2 : C_p \to X_{p,c_p}$ is completely bounded. For a fixed index $i \in I$ we consider

$$(e_i, \sigma_i e_i) = S(e_i, \sigma_i e_i) + T(e_i, \sigma_i e_i) = \sigma_i S_2(0, e_i) + S_1(e_i, 0) + T(e_i, \sigma_i e_i).$$

Thus

$$1 \le \|S_1(e_i, 0) + T(e_i, \sigma_i e_i)\| + \sigma_i \|S_2\|.$$

Hence for $i \ge i_0$ we get $\sigma_i \|S_2\| \le 1/2$ and therefore

$$1/2 \le \|S_1(e_i, 0) + T(e_i, \sigma_i e_i)\|.$$

Write

$$S_1(e_i, 0) + T(e_i, \sigma_i e_i) = (y, \sigma y).$$

We have the following alternative: If $\|y\|_p \le \|y\sigma\|_2$, then

$$1/2 \le (\|y\|_p^p + \|y\sigma\|_2^p)^{1/p} \le 2\|y\sigma\|_2.$$

Hence

$$1/4 \le \|y\sigma\|_2.$$

If $\|y\sigma\|_2 \le \|y\|_p$, we get $1/4 \le \|y\|_p$ and thus

$$\sigma_i \|y\|_p / 4 \le \|y\sigma\|_p \le \|y\sigma\|_2.$$

In both cases we deduce

$$\sigma_i/4 \le \|QS_1(e_i, 0) + QT(e_i, \sigma_i e_i)\|_2.$$

Now we decompose $QT = T_1 + T_2$, with T_1 acting on ℓ_p and T_2 acting on C_p according to Lemma 2.10. Let $n \in \mathbb{N}$ to be determined later and assume that $\sigma_i = \alpha_n$ is constant on a set A_n of cardinality n. Recall that

$$\Big(\sum_i \|QS_1(e_i)\|_2^r\Big)^{1/r} \le \|QS_1\| \le C_1$$

and

$$\Big(\sum_i \|T_1(e_i)\|_2^r\Big)^{1/r} \le \|T_1\| \le C_2.$$

Thus we get, for $C_3 = \|T_2\|$,

$$\begin{aligned}
\frac{\alpha_n n^{1/r}}{4} &\le \Big(\sum_{i\in A_n} \|QS_1(e_i,0) + QT(e_i,\sigma_i e_i)\|_2^r\Big)^{1/r} \\
&\le C_1 + C_2 + \Big(\sum_{i\in A_n} \|T_2(0,\sigma_i e_i)\|_2^r\Big)^{1/r} \\
&\le C_1 + C_2 + \Big(\sum_{i\in A_n,\, \|T_2(0,e_i)\|\le 1/16} \|T_2(0,\sigma_i e_i)\|_2^r\Big)^{1/r} \\
&\quad + \Big(\sum_{i\in A_n,\, \|T_2(0,e_i)\|> 1/16} \|T_2(0,\sigma_i e_i)\|_2^r\Big)^{1/r} \\
&\le C_1 + C_2 + \alpha_n \frac{1}{16} n^{1/r} + \alpha_n C_3 \operatorname{card}\{i \in A_n \mid \|T_2(0,e_i)\| > 1/16\}.
\end{aligned}$$

Hence for n so large that $8(C_1 + C_2) \le \alpha_n n^{1/r}$ we get

$$\frac{1}{16C_3} n^{1/r} \le \operatorname{card}\{i \in A_n \mid \|T_2(0,e_i)\| > 1/16\}.$$

Hence we can find a subset B_n of cardinality $n/C_3^r 16^r$ such that for all $i \in B_n$ we have

$$\|T_2(0,e_i)\|_2 > 1/16.$$

Now we consider the map $w : \ell_2(B_n) \to \ell_2$ defined by $w(e_i) = T_2(0,e_i)$. Defining $\delta = C_3^{-1} 32^{-2}$ and $n' = \operatorname{card} B_n$ we deduce, for the approximation numbers of w,

$$\begin{aligned}
\frac{1}{16}\sqrt{n'} \le \pi_2(w) \le \Big(\sum_{k=1}^{n'} a_k(w)^2\Big)^{1/2} &\le \sqrt{\delta}\sqrt{n'}\|T_2\| + a_{\delta n'}(w)\sqrt{n'} \\
&\le \frac{1}{32}\sqrt{n'} + a_{\delta n'}(w)\sqrt{n'}.
\end{aligned}$$

Therefore with $\delta' = C_3^{-r} 16^{-r}$ we obtain

$$1/32 \le a_{\delta n'}(w) = a_{\delta\delta' n}(w).$$

Let $u : \ell_2(B_n) \to C_p \cong \ell_2$ be defined by $u(e_i) = QT(e_i, \sigma_i e_i)$. In order to obtain a lower estimate for a proportional approximation number of u we observe

$$\alpha_n w(e_i) = T_2(0, \sigma_i e_i) = QT(e_i, \sigma_i e_i) - T_1(e_i, 0) = u(e_i) - T_1(e_i, 0).$$

Since T_1 is bounded on ℓ_p, the map $T_1' : \ell_2 \to \ell_2$ defined by $e_i \mapsto T_1(e_i, 0)$ factors through the inclusions map $\mathrm{id}_{2,p} : \ell_2 \to \ell_p$:

$$\alpha_n w - u = T_1 \mathrm{id}_{p,2},$$

Let us recall a result of Carl on the Weyl numbers of $\mathrm{id}_{p',2}$:

$$k^{1/r} x_k(\mathrm{id} : \ell_{p'} \to \ell_2) \le c_0.$$

Therefore we have

$$\begin{aligned}\alpha_n/32 &\le a_{\delta\delta' n}(\alpha_n w) = a_{\delta\delta' n}(u + \alpha_n w - u)\\ &\le a_{(\delta\delta'/2)n}(u) + a_{(\delta\delta'/2)n}(T_1 \mathrm{id}_{p,2}) = a_{(\delta\delta'/2)n}(u) + \left(\frac{2n}{\delta\delta'}\right)^{-1/r} c_0 \|T_1\|.\end{aligned}$$

Hence for n large enough that $n^{1/r}\alpha_n \ge 128 c_0 \|T_1\|/\delta\delta'$ we obtain

$$\alpha_n/64 \le a_{(\delta\delta'/2)n}(u).$$

It follows that we can find a linear map $W : \ell_2 \to \ell_2$ and a $k = (\delta\delta'/2)n$-dimensional subspace $H \subset \ell_2(B_n)$ such that $\|W\| \le 64\alpha_n^{-1}$ and $WQTP_H = \mathrm{id}_H$.

Note that the cb-norm of the identity mapping $\mathrm{id} : C_p \to X_{p,c_p}$ is completely contractive and thus we obtain

$$\mathrm{id}_H = WQT \,\mathrm{id}\, P_H.$$

According to our assumption $T = w_1 v_1$, where $v_1 : X_{p,c_p}(\sigma) \to F$, $w_1 : F \to X_{p,c_p}(\sigma)$, and F is a quotient of a subspace of $\prod_{\mathcal{U}} R_p(L_p(0,1))$. We deduce from Lemma 2.8 that

$$\begin{aligned}(\delta\delta'/2)n^{1/r} = k^{1/r} &\le c_p \inf_{E \in QS(\prod_{\mathcal{U}} R_p(L_p(0,1)))} d_{\mathrm{cb}}(C_p^k, E)\\ &\le c_p \|W\|_{\mathrm{cb}} \|v_1\|_{\mathrm{cb}} \|w_1\|_{\mathrm{cb}} \le \alpha_n^{-1} c_p \|v_1\|_{\mathrm{cb}} \|w_1\|_{\mathrm{cb}}.\end{aligned}$$

Using once more $\lim_n n^{1/r}\alpha_n = \infty$, we get a contradiction and the assertion is proved. ■

THEOREM 2.16. *If $V \subseteq \ell_p \oplus_p C_p \oplus_p R_p$ is a rectangular $\mathcal{OL}_p$-space, then there exists an increasing sequence (X_j) of finite-dimensional subspaces of V with dense union and non-negative integers k_j, m_j, n_j and a constant K such that*

$$d_{\mathrm{cb}}(X_j, \ell_p^{k_j} \oplus_p C_p^{n_j} \oplus_p R_p^{m_j}) \le K \quad \textit{for all } j \in \mathbb{N}. \tag{2.2}$$

In particular, V is cb-isomorphic to a cb-complemented subspace of $L_p(0,1) \oplus_p C_p \oplus_p R_p$.

If $V \subseteq \ell_p \oplus_p C_p$, the R_p-terms in (2.2) *disappear and V is cb-isomorphic to a cb-complemented subspace of $L_p(0,1) \oplus_p C_p$. Similarly if $V \subseteq \ell_p \oplus_p R_p$.*

Proof. Since V is a rectangular $\mathcal{OL}_p$-space there is an increasing sequence (X_j) of finite-dimensional subspace with dense union and numbers $k(j)$, $n_j(i)$ and $m_j(i)$ and a constant K_1 so that

$$d_{\mathrm{cb}}\Big(X_j, \Big(\bigoplus_{i=1}^{k(j)} S_p^{n_j(i),m_j(i)}\Big)_p\Big) \le K_1 \quad \text{for all } j \in \mathbb{N}.$$

For every $n \in \mathbb{N}$ we define

$$h(n) = \sup\{m_j(i) \mid n_j(i) \ge n\}.$$

If $h(n) \ge n$ for all $n \in \mathbb{N}$, clearly (S_p^n) embeds cb-uniformly into V and hence S_p is isomorphic to a subspace of an ultrapower of $\ell_p \oplus_p C_p \oplus_p R_p$ which is a Banach lattice of cotype p. This contradicts [22, Theorem 2.1]. Hence there is an $n_0 \in \mathbb{N}$ such that $h(n_0) < n_0$. If $n_j(i) \le n_0$, then

$$d_{\mathrm{cb}}(S_p^{n_j(i),m_j(i)}, \ell_p^{n_j(i)}(R_p^{m_j(i)})) \le n_0^{1/r},$$

and if $n_j(i) \ge n_0$, then $m_j(i) < n_0$ and hence

$$d_{\mathrm{cb}}(S_p^{n_j(i),m_j(i)}, \ell_p^{m_j(i)}(C_P^{n_j(i)})) \le n_0^{1/r}.$$

We can therefore find a constant K_2 and numbers k_j', $n_j'(i)$, and $m_j'(i)$ so that

$$d_{\mathrm{cb}}\Big(X_j, \Big(\bigoplus_{i=1}^{k_j'} C_p^{n_j'(i)}\Big)_p \oplus_p \Big(\bigoplus_{i=1}^{k_j'} R_p^{m_j'(i)}\Big)_p\Big) \le K_2 \quad \text{for all } j \in \mathbb{N}.$$

For every n and j we put $A_j(n) = \{i \le k_j' \mid n_j'(i) \ge n\}$ and $f(n) = \sup_j |A_j(n)|$. If $f(n) \ge n$ for all $n \in \mathbb{N}$, then clearly $(\ell_p^n(C_p^n))$ embeds cb-uniformly into $V \subseteq \ell_p \oplus_p C_p \oplus_p R_p$, which contradicts Corollary 2.14. Hence there is an n_0 so that $|A_j(n_0)| < n_0$ for all $j \in \mathbb{N}$. For every j we then get

$$d_{\mathrm{cb}}\Big(\Big(\bigoplus_{i \in A_j(n_0)} C_p^{n_j'}\Big)_p, C_p^{\sum_{i \in A_j(n_0)} n_j'(i)}\Big) \le n_0^{1/r},$$

$$d_{\mathrm{cb}}\Big(\bigoplus_{i \notin A_j(n_0)} C_P^{n_j'(i)}, \ell_p^{\sum_{i \notin A_j(n_0)} n_j'(i)}\Big) \le n_0^{1/r}.$$

Treating the R_p-terms in the same way we find that there is a constant K and numbers k_j, n_j, and m_j so that

$$d_{\mathrm{cb}}(X_j, \ell_p^{k_j} \oplus_p C_p^{n_j} \oplus_p R_p^{m_j}) \le K \quad \text{for all } j \in \mathbb{N},$$

which proves (2.2). Using an ultraproduct construction as in [5, Section 10.3] we deduce that there is an ultrafilter $\mathcal{U}$ so that V is cb-complemented in $\prod_{\mathcal{U}} \ell_p \oplus_p \prod_{\mathcal{U}} C_p \oplus_p \prod_{\mathcal{U}} R_p$. Since $\prod_{\mathcal{U}} \ell_p$ is cb-isometrically isomorphic to $L_p(\mathcal{N})$ for some commutative $\mathcal{N}$, and C_p and R_p are homogeneous, the separability of V implies that it is cb-complemented in $L_p(\mathcal{N}_1) \oplus_p C_p \oplus_p R_p$

with $(\mathcal{N}_1)_*$ separable. Decomposing $\mathcal{N}_1$ into discrete and continuous parts we find that $L_p(\mathcal{N}_1)$ is cb-contractively complemented in $L_p(0,1)$ and hence V is cb-isomorphic to a cb-complemented subspace of $L_p(0,1) \oplus_p C_p \oplus_p R_p$.

Since (R_p^n) does not embed cb-uniformly into $\ell_p \oplus_p C_p$ by Lemma 2.7, it is readily seen that if $V \subseteq \ell_p \oplus_p C_p$, then the R_p-components disappear in the argument above, and the ultraproduct construction shows that V is cb-isomorphic to a cb-complemented subspace of $L_p(0,1) \oplus_p C_p$. ■

As a corollary we obtain

THEOREM 2.17. *If σ satisfies* (1.5) *and* (1.6), *then the spaces $X_{p,c_p}(\sigma)$ and $X_{p,r_p}(\sigma)$ are not rectangular $\mathcal{OL}_p$-spaces.*

Proof. Suppose that $X_{p,c_p}(\sigma)$ is a rectangular $\mathcal{OL}_p$-space. Theorem 2.16 then shows that it is cb-complemented in $L_p(0,1) \oplus_p C_p$. By Theorem 1.3 we can assume that σ satisfies the additional assumptions in Theorem 2.15, and hence that theorem yields a contradiction. ■

THEOREM 2.18. *If σ satisfies* (1.5) *and* (1.6) *and*

$$V \in \{R_p \oplus_p X_{p,c_p}(\sigma), \ell_p(R_p) \oplus_p X_{p,c_p}(\sigma), X_{p,r_p}(\sigma) \oplus_p X_{p,c_p}(\sigma)\},$$

then V is not a rectangular $\mathcal{OL}_p$-space.

Proof. Suppose $V = \ell_p(R_p) \oplus_p X_{p,c_p}(\sigma)$. The proof of Theorem 2.16 shows that V is cb-complemented in $C_p \oplus_p \prod_{\mathcal{U}} R_p(\ell_p)$, which contradicts Theorem 2.15 since $X_{p,c_p}(\sigma)$ is cb-complemented in V. The other cases follow directly from Theorem 2.16. ■

PROPOSITION 2.19. *Assume that σ satisfies* (1.5) *and* (1.6), *and let $\mathcal{U}$ be a free ultrafilter on the integers.*

(i) *If $V \in \{X_{p,c_p}(\sigma), R_p \oplus_p X_{p,c_p}(\sigma), X_{p,r_p}(\sigma) \oplus_p X_{p,c_p}(\sigma)\}$, then $\ell_p(R_p) \oplus_p X_{p,c_p}(\sigma)$ does not embed into $\prod_{\mathcal{U}} V$.*
(ii) *$X_{p,r_p}(\sigma) \oplus_p X_{p,c_p}(\sigma)$ is not cb-isomorphic to a cb-complemented subspace of $\prod_{\mathcal{U}}(R_p \oplus_p X_{p,c_p}(\sigma))$.*

In particular, *the spaces $X_{p,c_p}(\sigma)$, $R_p \oplus_p X_{p,c_p}(\sigma)$, $X_{p,r_p}(\sigma) \oplus_p X_{p,c_p}(\sigma)$, $\ell_p(R_p) \oplus_p X_{p,c_p}(\sigma)$ are mutually not cb-isomorphic.*

Proof. To prove (i), we observe that $V \subset \ell_p \oplus_p C_p \oplus_p R_p$. Thus the assertion follows from the row version of Corollary 2.14. To get (ii) we note that $R_p \oplus_p X_{p,c_p}(\sigma)$ is complemented in $R_p \oplus_p L_p([0,1]; C_p)$. According to Proposition 2.13 a separable complemented subspace of $\prod_{\mathcal{U}} R_p \oplus_p L_p([0,1]; C_p)$ is cb-complemented in $R_p \oplus_p C_p(L_p(\mathcal{N}))$ for a commutative $\mathcal{N}$. But the row version of Theorem 2.15 excludes this for $X_{p,r_p}(\sigma)$. ■

REMARK 2.20. If $W \in \{\ell_p(R_p), \ell_p(R_p) \oplus_p X_{p,c_p}(\sigma), R_p \oplus_p X_{p,c_p}(\sigma), X_{p,r_p}(\sigma) \oplus_p X_{p,c_p}(\sigma)\}$, then W contains R_p cb-isomorphically which does not cb-embed into an ultrapower of $L_p([0,1];C_p)$. However, $X_{p,c_p}(\sigma) \subseteq L_p([0,1];C_p)$ and hence W does not cb-embed into an ultrapower of $X_{p,c_p}(\sigma)$.

Consequently, none of the spaces above nor those from Proposition 2.19 can be paved with local pieces of any of the others except for trivial reasons. It is easily seen that we can also add $\ell_p(C_p) \oplus_p X_{p,r_p}(\sigma)$ and the rectangular $\mathcal{OL}_p$-space $\ell_p(C_p) \oplus_p X_{p,c_p}(\sigma)$ to this list.

At the end of this section we want to compare the space $X_{p,c_p}(\sigma)$, $X_{p,r_p}(\varrho)$ with their intersection in interpolation sense. Let $2 < p < \infty$ and let $\sigma = (\sigma_n)$ and $\varrho = (\varrho_n)$ be two positive sequences. In analogy with the spaces defined in Section 1 we let $X(\sigma, \varrho)$ be the subspace of $S_p \oplus_p C_p \oplus_p R_p$ defined as the closed linear span of the sequence $\{e_{nn} \oplus_p \sigma_n e_{n1} \oplus_p \varrho_n e_{1n}\}$. Note that $X(\sigma,\varrho)$ is the interpolation space $X_{p,c_p}(\sigma) \cap X_{p,r_p}(\varrho)$. We shall show that if σ and ϱ satisfy (1.5) and (1.6), then $X_p(\sigma,\varrho)$ is a rectangular $\mathcal{OL}_p$-space if and only if it is cb-isomorphic to $X_p(\sigma)$, $X_p(\sigma) \oplus_p C_p$, $X_p(\sigma) \oplus_p R_p$ or $X_p(\sigma) \oplus_p C_p \oplus_p R_p$. We first investigate the space $X_p(\alpha,\beta)$ where $\alpha > 0$ and $\beta > 0$ are constants. We have:

Proposition 2.21. *There is a constant $K = K(p)$ such that if T is a cb-isomorphism of $X_p(\alpha,\beta)$ into $L_p(0,1) \oplus_p C_p \oplus_p R_p$ and P is a cb-projection of $L_p(0,1) \oplus_p C_p \oplus_p R_p$ onto $T(X_p(\alpha,\beta))$, then either*

$$\max(\alpha,\beta) \le K\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}} \min(\alpha,\beta) \tag{2.3}$$

or

$$\frac{1}{2\min(\alpha,\beta)} \le K\|P\|_{\mathrm{cb}}\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}}. \tag{2.4}$$

If T is a cb-isomorphism of $X_{p,c_p}(\alpha)$ into $L_p(0,1) \oplus_p C_p$ and P is a cb-projection of $L_p(0,1) \oplus_p C_p$ onto $T(X_{p,c_p}(\alpha))$, then

$$\frac{1}{2\alpha} \le K\|P\|_{\mathrm{cb}}\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}}. \tag{2.5}$$

Similarly for $X_{p,r_p}(\sigma)$.

Proof. Assume that $\beta \le \alpha$ (the other case can be proved similarly), let Q_1 be the natural projection of $L_p(0,1)\oplus_p C_p \oplus_p R_p$ onto $L_p(0,1)$, and Q_2 the natural projection of $L_p(0,1)\oplus_p C_p \oplus_p R_p$ onto $C_p \oplus_p R_p$. If (f_n) denotes the canonical basis of $X_p(\alpha,\beta)$, we put $h_n = Q_1 T f_n$ for all $n \in \mathbb{N}$. Since $f_n \to 0$ weakly, so does (h_n) and we can therefore extract a martingale subsequence of (h_n) and then use the argument in [14] to extract a further subsequence, still called (h_n), so that there exist constants $K_1 = K_1(p) \ge 1$, $b_1 \ge 0$ and $b_2 \ge 0$ such that

$$\Big\|\sum_k a_k h_k\Big\|_{S_p[L_p(0,1)]} \sim_{K_1} \max\Big\{b_1 \sum_k \|a_k\|_p^p, b_2\Big\|\Big(\sum_k a_k^* a_k\Big)^{1/2}\Big\|_{S_p}, b_2\Big\|\Big(\sum_k a_k a_k^*\Big)^{1/2}\Big\|_{S_p}\Big\}$$

for all finite sequences $(a_k) \subseteq S_p$. Plugging in the vectors $a_k = e_{1k}$, we get, for every $n \in \mathbb{N}$,

$$\max(b_1 n^{1/p}, b_2 n^{1/p}, b_2 n^{1/2}) \le K_1 \|T\|_{\mathrm{cb}} \max(n^{1/p}, \alpha n^{1/p}, \beta n^{1/2}),$$

which implies that $b_2 \le K_1 \beta$.

As in Corollary 2.10, there is a constant K_2 only depending on p such that the operator $Q_2 T$ has a cb-extension $S : S_p \oplus_p C_p \oplus_p R_p \to C_p \oplus_p R_p$ with $\|S\|_{\mathrm{cb}} \le K_2 \|T\|_{\mathrm{cb}}$. Hence for all $n \in \mathbb{N}$,

$$Tf_n = h_n + Se_{nn} + \alpha Se_{n1} + \beta Se_{1n}.$$

By [26], $\sum_{n=1}^{\infty} \|Se_{nn}\|^r < \infty$, and if Q denotes the canonical projection of $S_p \oplus_p C_p \oplus_p R_p$ onto R_p we find that the operator $QT^{-1}PS|C_p$ is $(r,2)$-summing and therefore also $\sum_{n=1}^{\infty} \|QT^{-1}PSe_{n1}\|^r < \infty$. In particular, we can find an $n_0 \in \mathbb{N}$ such that

$$\|T^{-1}PSe_{nn}\| + \frac{\alpha}{\beta} \|QT^{-1}PSe_{n1}\| \le \frac{1}{4} \quad \text{for all } n \ge n_0. \tag{2.6}$$

If (F_n) denotes the biorthogonal system to (f_n), then clearly $|F_n(T^{-1}PSe_{n1})| \le (1/\beta)\|QT^{-1}PSe_{n1}\|$ and hence (2.6) gives

$$\begin{aligned} 1 &\le |F_n(T^{-1}Ph_n)| + \beta |F_n(T^{-1}PSe_{1n})| + 1/4 \\ &\le |F_n(T^{-1}Ph_n)| + K_2 \beta \|P\|_{\mathrm{cb}} \| \, \|T\|_{\mathrm{cb}} \|T^{-1}\|_{\mathrm{cb}} + 1/4 \end{aligned}$$

for all $n \ge n_0$. If we now assume that $K_2 \beta \|T\|_{\mathrm{cb}} \|T^{-1}\|_{\mathrm{cb}} \|P\|_{\mathrm{cb}} < 1/2$, then by the above $1/4 \le |F_n(T^{-1}Ph_n)|$ for all $n \ge n_0$.

By interpolation there exists a constant $K_3 = K_3(p)$ such that if U denotes the diagonal of $T^{-1}P|[h_n]$ with respect to the bases (f_n) and (h_n), then U is cb-bounded with $\|U\|_{\mathrm{cb}} \le K_3 \|T^{-1}\|_{\mathrm{cb}} \|P\|_{\mathrm{cb}}$, and hence for all $(a_k) \subseteq S_p$ and all $n \ge n_0$ we get

$$\begin{aligned} \frac{1}{4} \Big\| \sum_{k=n_0}^{n} a_k \otimes f_k \Big\| &\le \Big\| U\Big(\sum_{k=n_0}^{n} a_k \otimes h_k \Big) \Big\|_{S_p[L_p(0,1)]} \\ &\le K_3 \|T^{-1}\|_{\mathrm{cb}} P\|_{\mathrm{cb}} \Big\| \sum_{k=n_0}^{n} a_k \otimes h_k \Big\|_{S_p[L_p(0,1)]}. \end{aligned}$$

If we plug in the vectors $a_k = e_{k1}$ in this inequality we get

$$\frac{1}{4} \max\{(n-n_0)^{1/p}, \alpha(n-n_0)^{1/2}, \beta(n-n_0)^{1/p}\}$$
$$\le K_1 K_3 \|T\|_{\mathrm{cb}} \|T^{-1}\|_{\mathrm{cb}} \|P\|_{\mathrm{cb}} \max\{b_1(n-n_0)^{1/p}, b_2(n-n_0)^{1/2}, b_2(n-n_0)^{1/p}\}$$

and therefore $\alpha \le K_1 K_3 \|T\|_{\mathrm{cb}} \|T^{-1}\|_{\mathrm{cb}} \|P\|_{\mathrm{cb}} b_2 \le K_1^2 K_3 \|T\| \|T^{-1}\|_{\mathrm{cb}} \|P\|_{\mathrm{cb}} \beta$. Hence we have proved the proposition with $K = \max(K_1^2 K_3, K_2)$.

To prove the statement for $X_{p,c_p}(\alpha)$ we go through the argument above, but we omit the R_p-coordinate, and adjust the sequence (h_n) to the new situation. Then we drop the argument with the projection Q. The first part will then show that $b_2 \le K_1\alpha$. If $K_2\alpha\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}}\|P\|_{\mathrm{cb}} < 1/2$, then the second part will show that $\alpha \le K_1K_3\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}}\|P\|_{\mathrm{cb}}b_2$. Hence (f_n) is cb-equivalent to (h_n), which is a contradiction because $X_{p,c_p}(\alpha)$ is cb-isomorphic to C_p, which does not embed into $L_p(0,1)$ by Lemma 2.4. ∎

We need the following two lemmas:

LEMMA 2.22. *Let $2 \le p < \infty$ and let σ and ϱ be two sequences such that there exist $\delta > 0$ and $\varepsilon > 0$ with $\sigma_n \le \delta\varrho_n$ for all $n \in \mathbb{N}$ and $\sum_{\sigma_n\le\varepsilon}\sigma_n^r < \infty$. If $X_p(\sigma,\varrho)$ is cb-isomorphic to a cb-complemented subspace of $L_p(0,1) \oplus_p C_p \oplus_p R_p$, then there exist $0 \le K, M, N \le \infty$ such that $X_p(\sigma,\varrho)$ is cb-isomorphic to $l_p^N \oplus_p (C_p \cap R_p)^M \oplus_p R_p^K$. If $\varrho_n \to 0$, the last two summands do not occur.*

Proof. Assume that $X_p(\sigma,\varrho)$ is an $\mathcal{OL}_p$-space, put

$$A = \{n \in \mathbb{N} \mid \sigma_n \le \varepsilon\}, \qquad B = \{n \in \mathbb{N} \mid \sigma_n > \varepsilon\},$$

and let $\sigma_A = \{\sigma_n \mid n \in A\}$ and $\sigma_B = \{\sigma_n \mid n \in B\}$. In a similar manner we define ϱ_A and ϱ_B. Clearly we can write

$$X_p(\sigma,\varrho) = X_p(\sigma_A,\varrho_A) \oplus X_p(\sigma_B,\varrho_B).$$

If $\liminf \varrho_A(n) > 0$, then $X_p(\sigma_A,\varrho_A)$ is cb-isomorphic to $R_p^{|A|}$ (which is cb-isomorphic to $\ell_p^{|A|}$ in case A is finite). Assume next that $\liminf \varrho_A(n) = 0$. If ϱ_A satisfies (1.6), then $X_p(\sigma_A,\varrho_A)$ is cb-isomorphic to $X_{p,r_p}(\varrho_A)$, which contradicts Theorem 2.17, and hence there is an $\varepsilon_1 > 0$ such that $\sum_{\varrho_A(n)\le\varepsilon_1}\varrho_A(n)^r < \infty$. We may assume that $\varepsilon_1 = \varepsilon$ and conclude that $X_p(\sigma,\varrho)$ is cb-isomorphic to $\ell_p^{|A|}$. If $n \in B$, then $\varepsilon < \sigma_n \le \delta\varrho_n$ so that $X_p(\sigma_B,\varrho_B)$ is cb-isomorphic to $(C_p \cap R_p)^{|B|}$.

Summing up we have found that there exist $0 \le K, M, N \le \infty$ such that $X_p(\sigma,\varrho)$ is cb-isomorphic to $\ell_p^N \oplus_p (C_p \cap R_p)^M \oplus_p R_p^K$. ∎

LEMMA 2.23. *Let $2 < p < \infty$ and let σ and ϱ be two sequences such that $X_p(\sigma,\varrho)$ is cb-complemented in $L_p(0,1)\oplus_p C_p \oplus_p R_p$. Then $\{\varrho_n \mid \sigma_n \ge \varepsilon\}$ does not satisfy (1.6) for any $\varepsilon > 0$. The same holds with σ and ϱ interchanged.*

Proof. Suppose that there is an $\varepsilon > 0$ such that $\{\varrho_n \mid \sigma \ge \varepsilon\}$ satisfies (1.6). Then it also satisfies (1.5), and if $\beta > 0$ is arbitrary, we can find a sequence (B_k) of mutually disjoint finite subsets of $\mathbb{N}$ so that

$$\beta \le \Big(\sum_{n\in B_k,\,\sigma_n\ge\varepsilon}\varrho_n^r\Big)^{1/r} \le 2\beta.$$

For every $k \in \mathbb{N}$ we put $\alpha_k = (\sum_{n\in B_k,\,\sigma_n\ge\varepsilon}\sigma_n^r)^{1/r}$ and arguing as in Proposition 1.2 we find that $X_p((\alpha_k),\beta)$ is cb-complemented in $X_p(\sigma,\varrho)$. Clearly

$\alpha = \liminf \alpha_k \geq \varepsilon$, and if we choose a subsequence (α_{k_m}) tending sufficiently fast to α we conclude that $X_p(\alpha,\beta)$ is cb-complemented in $X_p(\sigma,\varrho)$, and hence also in $L_p(0,1) \oplus_p C_p \oplus_p R_p$. This contradicts (2.3) and (2.4) for β small enough. ∎

We are now able to prove:

THEOREM 2.24. *Let σ and ϱ be two sequences satisfying* (1.5) *and* (1.6). *If $X_p(\sigma,\varrho)$ is a rectangular $\mathcal{OL}_p$-space, then it is cb-isomorphic to $X_p(\sigma)$, $X_p(\sigma) \oplus_p R_p$, $X_p(\sigma) \oplus_p C_p$ or $X_p(\sigma) \oplus_p C_p \oplus_p R_p$. If in addition $\sigma_n \to 0$ and $\varrho_n \to 0$, then $X_p(\sigma,\varrho)$ is cb-isomorphic to $X_p(\sigma)$.*

Proof. If $X_p(\sigma,\varrho)$ is a rectangular $\mathcal{OL}_p$-space, Theorem 2.17 shows that it is cb-isomorphic to a cb-complemented subspace of $L_p(0,1) \oplus_p C_p \oplus_p R_p$. Suppose that for all $\varepsilon > 0$ and all $\delta > 0$ we have $\sum_{\{\varrho_n \leq \delta\sigma_n, \varrho_n \leq \varepsilon\}} \varrho_n^r = \infty$. We shall show that this leads to a contradiction. Let $\delta > 0$ and $A = \{n \in \mathbb{N} \mid \varrho_n \leq \delta\sigma_n\}$, and define σ_A and ϱ_A as in Lemma 2.22. Clearly ϱ_A satisfies (1.5) and (1.6). If $\sum_{\{\sigma_A(n) \leq \varepsilon\}} \sigma_A(n)^r < \infty$ for some $\varepsilon > 0$, then also $\sum_{\{\sigma_A(n)\leq\varepsilon\}} \varrho_A(n)^r < \infty$ and therefore $\{\varrho_A(n) \mid \sigma_A(n) > \varepsilon\}$ satisfies (1.6), which contradicts Lemma 2.23. Hence also σ_A satisfies (1.5) and (1.6). Let now $\alpha > 0$, choose mutually disjoint finite sets $A_k \subseteq \mathbb{N}$ such that for all $k \in \mathbb{N}$,

$$\alpha \leq \Big(\sum_{n\in A_k} \sigma_A(n)^r\Big)^{1/r} \leq 2\alpha,$$

and put $\beta_k = (\sum_{n\in A_k} \varrho_A(n)^r)^{1/r}$ for all $k \in \mathbb{N}$. Again Proposition 1.2 shows that $X_p(\alpha,(\beta_k))$ is cb-isomorphic to a cb-complemented subspace of $X_p(\sigma,\varrho)$, and by choosing a subsequence of (β_k) tending sufficiently fast to $\beta = \liminf \beta_k > 0$ we find that $X_p(\alpha,\beta)$ is cb-isomorphic to a cb-complemented subspace of $X_p(\sigma,\varrho)$. If $\beta = 0$ we have $X_p(\alpha,\beta) = X_{p,c_p}(\alpha)$, contrary to (2.5) of Proposition 2.21 for α small enough. If $\beta > 0$, then $\beta \leq 2\delta\alpha$, contrary to (2.3) of Proposition 2.21 for δ small enough. By choosing α small enough, (2.4) is violated and we have reached a contradiction.

Interchanging the roles of σ and ϱ in the argument above we can conclude that there are $\varepsilon > 0$ and $\delta > 0$ such that

$$\sum_{\{\sigma_n\leq\delta\varrho_n,\,\sigma_n\leq\varepsilon\}} \sigma_n^r < \infty, \tag{2.7}$$

$$\sum_{\{\varrho_n\leq\delta\sigma_n,\,\varrho_n\leq\varepsilon\}} \varrho_n^r < \infty. \tag{2.8}$$

Let A be as above and put

$$B = \{n \in \mathbb{N} \mid \delta\varrho_n < \sigma_n < (1/\delta)\varrho_n\}, \qquad D = \{n \in \mathbb{N} \mid \sigma_n \leq \delta\varrho_n\},$$

and define the sequences (σ_A), (σ_B), (σ_D), (ϱ_A), (ϱ_B), and (ϱ_D) as before. We can then write

$$X_p(\sigma,\varrho) = X_p(\sigma_A,\varrho_A) \oplus X_p(\sigma_B,\varrho_B) \oplus X_p(\sigma_D,\varrho_D).$$

By Lemma 2.22, $X_p(\sigma_A,\varrho_A) \oplus X_p(\sigma_D,\varrho_D)$ is cb-isomorphic to $l_p^N \oplus_p (C_p \cap R_p)^M \oplus_p C_p^K \oplus_p R_p^L$ for some $0 \le k, L, M, N \le \infty$. $X_p(\sigma_B,\varrho_B)$ is cb-isomorphic to $X_p(\sigma_B)$ and since σ_B satisfies (1.5) and (1.6) it contains cb-complemented copies of $l_p \oplus_p (C_p \cap R_p)$, which shows that $X_p(\sigma,\varrho)$ is cb-isomorphic to $X_p(\sigma_B) \oplus_p C_p^K \oplus_p R_p^L$. This finishes the proof since clearly $X_p(\sigma_B)$ is cb-isomorphic to $X_p(\sigma)$. Obviously the C_p- and R_p-terms do not appear in case $\sigma_n \to 0$ and $\varrho_n \to 0$. ∎

3. Operator space properties of the matricial Rosenthal spaces. In this section we will discuss the operator space structure of the matricial Rosenthal spaces. As before we let $p > 2$, $1/2 = 1/p + 1/r$, and let σ be a sequence with $\sigma_n \ge 0$. (ξ_n) denotes the unit vector basis of ℓ_2. Throughout the rest of the paper we let $\mathcal{R}$ denote the hyperfinite II_1 factor defined as the σ-weak closure of the infinite tensor product $\bigotimes_{n\in\mathbb{N}} M_2$ in the GNS-construction with respect to the trace $\tau_{\mathcal{R}} = \bigotimes_{n\in\mathbb{N}} \mathrm{tr}/2$.

We start with the following result on $Y_p(\sigma)$:

PROPOSITION 3.1. *$Y_p(\sigma)$ is complemented in $L_p(\mathcal{R})$.*

Proof. Let μ denote the Lebesgue measure on $(0,\infty)$ and let $A_n \subset (0,\infty)$ be disjoint sets with $\mu(A_n) = \sigma_n^r$ for all $n \in \mathbb{N}$. We consider the subspace $V \subset L_p((0,\infty);S_p) \cap L_2^{r_p\cap c_p}((0,\infty);S_2)$ defined as the closure of $\{\sum_n \mu(A_n)^{-1/p} 1_{A_n} x_n \mid x_n \in S_p^n\}$.

Given $X_n \in S_p \otimes S_p^n$, we have

$$\Big\|\sum_n \mu(A_n)^{-1/p} 1_{A_n} X_n\Big\|_{L_p(S_p)} = \Big(\sum_n \|X_n\|_p^p\Big)^{1/p}.$$

Further,

$$\begin{aligned}\Big\|\sum_n \mu(A_n)^{-1/p} 1_{A_n} X_n\Big\|_{S_p[L_2^{c_p}]} &= \Big\|\Big(\sum_n \mu(A_n)^{1-2/p} (\mathrm{id}\otimes \mathrm{tr})(X_n^* X_n)\Big)^{1/2}\Big\|_{S_p}\\ &= \Big\|\Big(\sum_n \sigma_n^2 (\mathrm{id}\otimes \mathrm{tr})(X_n^* X_n)\Big)^{1/2}\Big\|_{S_p}.\end{aligned}$$

The calculation for the row term is similar. Comparing this with (1.22) we conclude that V is cb-isomorphic to $Y_p(\sigma)$.

For every $n \in \mathbb{N}$ we let p_n denote the orthogonal projection of ℓ_2 onto $\mathrm{span}\{\xi_n \mid n(n-1)/2 + 1 \le k \le n(n+1)/2\}$. Since $B = \{\sum_n 1_{A_n} \otimes x_n \mid x_n = p_n x_n p_n\}$ is a von Neumann subalgebra of $L_\infty((0,\infty);B(\ell_2))$ and the restriction of the trace is normal on B, we deduce from [28] that there is the conditional expectation

$$E(x) = \sum_n 1_{A_n} \otimes \int_{A_n} p_n x(t) p_n \frac{dt}{\mu(A_n)}$$

which is completely contractive on $L_p((0,\infty); S_p)$ for all $1 \le p \le \infty$. Clearly E is a projection onto V and hence V is cb-complemented in $L_p((0,\infty); S_p) \cap L_2^{c_p \cap r_p}((0,\infty); S_2)$. According to [9] the latter space is cb-isomorphic to $L_p(\mathcal{R})$, and the assertion is proved. ∎

Remark 3.2. According to [9], the spaces

$$L_p((0,\infty); S_p) \cap L_2^{c_p}((0,\infty); S_2) \quad \text{and} \quad L_p((0,\infty); S_p) \cap L_2^{r_p}((0,\infty); S_2)$$

are cb-isomorphic to completely complemented subspaces in $L_p(\mathcal{R} \otimes B(\ell_2))$ and hence the same argument as above shows that Y_{p,c_p} and Y_{p,r_p} are cb-isomorphic to cb-complemented subspaces of $L_p(\mathcal{R} \otimes B(\ell_2))$. However, in general we cannot expect a cb-embedding into $L_p(\mathcal{R})$. Indeed, from Theorem 1.5 it follows that if σ satisfies (1.5) and (1.6), then S_p cb-embeds into $Z_p(\sigma)$ but it does not embed into $L_p(\mathcal{R})$ according to a result of Sukochev [27]. Hence $Z_p(\sigma)$ does not cb-embed into $L_p(\mathcal{R})$.

Corollary 3.3. *The spaces $Y_p(\sigma)$, $Y_{p,c_p}(\sigma)$, and $Y_{p,r_p}(\sigma)$ have the γ_p-AP.*

Proof. Since $L_p(\mathcal{R} \otimes B(\ell_2))$ is the L_p-space of an injective von Neumann algebra, this space has the γ_p-AP. The γ_p-AP passes to complemented subspaces. ∎

We now turn our attention to the space $Z_p(\sigma)$ but for this we need some preliminary results.

Let $m, n \in \mathbb{N}$ and let D be a positive $m \times m$ diagonal matrix with $\mathrm{tr}(D) = 1$. We define $Z_p^m(n, D)$ to be the subspace of $S_p^m \oplus_p C_p^{m^2} \oplus_p R_p^{m^2}$ defined by

$$Z_p^m(n, D) = \{(x, n^{1/r} x D^{1/r}, n^{1/r} D^{1/r} x) \mid x \in S_P^m\}.$$

Here we consider $xD^{1/r}$ as an element of $C_p^m(C_p^m) = C_p^{m^2}$, and $D^{1/r}x$ as an element of $R_p^m(R_p^m) = R_p^{m^2}$. The spaces $Z_{p,c_p}^m(n, D)$ and $Z_{p,r_p}^m(n, D)$ are defined similarly as subspaces of $S_p^m \oplus_p C_p^{m^2}$, respectively $S_p^m \oplus_p R_p^{m^2}$.

For every $1 \le i \le n$ we define $\Psi_i : S_p^m \to S_p^{m^n} = S_p^{\otimes n}$ by

$$\Psi_i(x) = D^{1/p} \otimes \cdots \otimes D^{1/p} \otimes x \otimes D^{1/p} \otimes \cdots \otimes D^{1/p}$$

for all $x \in S_p^m$, where x is the ith factor. Further, we put

$$U_p(x) = n^{-1/p} \sum_{i=1}^n \varepsilon_i \Psi_i(x) \quad \text{for all } x \in S_p^m,$$

$$U_{p,c}(x) = n^{-1/p} \sum_{i=1}^{n} \varepsilon_i \Psi_i(x) \otimes e_{i1} \quad \text{for all } x \in S_p^m,$$
$$U_{p,r}(x) = n^{-1/p} \sum_{i=1}^{n} \varepsilon_i \Psi_i(x) \otimes e_{1i} \quad \text{for all } x \in S_p^m,$$

where (ε_i) is the sequence of Rademacher functions on $[0,1]$.

Theorem 3.4. *U_p acts as a cb-isomorphism of $Z_p^m(n, D)$ onto its image which is cb-complemented in $L_p([0,1]; S_p^{mn})$ with cb-norms only depending on p. Similarly $Z_{p,c_p}^m(n, D)$ (resp., $Z_{p,r_p}^m(n, D)$) is cb-complemented in $L_p([0,1]; S_p^{mn} \otimes C_p^n)$ (resp., $L_p([0,1]; S_p^{mn} \otimes R_p^n)$) via the map $U_{p,c}$ (resp., $U_{p,r}$).*

Proof. Let $\{x_{jk} \mid 1 \le j, k \le m\} \subseteq S_p$. For every $1 \le i \le n$ put

$$Y_i = \varepsilon_i \sum_{j,k} x_j k \otimes \Psi_i(e_{jk}).$$

Then the Y_i's are independent in the sense of [14] and have mean zero.

Therefore, if we put $E(x \otimes y) = \operatorname{tr}(D^{1-2/p}x)y$ for all $x \in S_p^{mn}$ and all $y \in S_p$ and let "$\sim$" denote a two-sided inequality with constants only depending on p, then [14, Theorem 1.2] gives

$$\begin{aligned}(3.1)\quad \Big\| \sum_{j,k} x_{jk} \otimes U_p(e_{jk}) \Big\|_{S_p[L_p(S_p^{mn})]} &= n^{-1/p} \Big\| \sum_{i=1}^{n} Y_i \Big\|_{S_p[L_p(S_p^{mn})]} \\ \sim n^{-1/p} \max \Big\{ \Big(\sum_{i=1}^{n} \|Y_i\|^p_{S_p[L_p(S_p^{mn})]} \Big)^{1/p}&, \Big\| \Big(\sum_{i=1}^{n} E(Y_i^* Y_i) \Big)^{1/p} \Big\|_{S_p}, \\ &\Big\| \Big(\sum_{i=1}^{n} E(Y_i Y_i^*) \Big)^{1/p} \Big\|_{S_p} \Big\}.\end{aligned}$$

For all $i \le n$ we easily get

$$\|Y_i\|_{S_p[L_p(S_p^{mn})]} = \Big\| \sum_{j,k} x_{jk} \otimes e_{jk} \Big\|_{S_p[S_p^m]}.$$

Further,

$$\begin{aligned}\Big\| \Big(\sum_{i=1}^{n} E(Y_i^* Y_i) \Big)^{1/2} \Big\|_{S_p} &= n^{1/2} \Big\| \Big(\sum_{j,k} \sigma_k^{1-2/p} x_{jk}^* x_{jk} \Big)^{1/2} \Big\|_{S_p} \\ &= n^{1/2} \Big\| \sum_{k=1}^{m} \sigma_k^{1/r} \sum_{j=1}^{m} x_{jk} \otimes e_{jk} \Big\|_{S_p[C_p^{m^2}]}\end{aligned}$$

and similarly

$$\Big\|\Big(\sum_{i=1}^{n} E(Y_iY_i^*)\Big)^{1/2}\Big\|_{S_p} = n^{1/2}\Big\|\sum_{j=1}^{n}\sigma_j^{1/r}\sum_{k=1}^{m} x_{jk}\otimes e_{jk}\Big\|_{S_p[R_p^{m^2}]}.$$

Combining these calculations with (3.1) we deduce that U is a cb-isomorphism of $Z_p^m(n,D)$ onto its image.

For every $1\le i\le n$ we define $\Psi_i' : S_{p'}^m \to S_{p'}^{m^n}$ by

$$\Psi_i'(x) = D^{1/p}\otimes\cdots\otimes D^{1/p}\otimes x\otimes D^{1/p}\otimes\cdots\otimes D^{1/p}$$

for every $x\in S_{p'}$, where x is the ith factor, and $U_{p'} = \sum_{i=1}^n \varepsilon_i\Psi_i'(x)$ for all $x\in S_{p'}$. Using [14, Theorem 4.3] we can show in a similar manner that $U_{p'}$ acts as a cb-bounded operator from $Z_p^m(n,D)^*$ to $L_{p'}([0,1];S_p^{m^n})$ It is readily verified that $U_pU_{p'}^*$ is a cb-bounded projection of $L_p([0,1];S_p^{m^n})$ onto the range of U_p.

The argument for $U_{p,c}$ and $U_{p,r}$ is similar. ∎

We are now able to prove:

THEOREM 3.5. *Let $2\le p,r<\infty$ be such that $1/2 = 1/p+1/r$. If σ is a sequence of positive numbers such that $\sigma\notin\ell_r$ and* $\liminf_n \sigma_n = 0$, *then $Y_p(\sigma)$, $Y_{p,r_p}(\sigma)$, $Y_{p,c_p}(\sigma)$, $Z_p(\sigma)$, $Z_{p,r}(\sigma)$, and $Z_{p,c}(\sigma)$ are $\mathcal{COS}_p$-spaces.*

Proof. Consider

$$s_j = \sum_{k=1}^{j}\sigma_k^r.$$

By assumption s_j tends to ∞ and hence we can find a subsequence (j_k) and integers n_k such that

$$n_k \le s_{j_k} \le n_k+1.$$

By definition Z_p, $Z_{p,c}$, $Z_{p,r}$ are the closures of $\bigcup_k Z_p^{j_k}$, $\bigcup_k Z_{p,c}^{j_k}$, $\bigcup_k Z_{p,r}^{j_k}$, respectively. Fix $k\in\mathbb{N}$ and define $\varrho_k = s_{j_k}^{-1}(\sigma_j^r)_{j\le j_k}$. The map

$$w(x) = (x, n_k^{1/r}xD_{\varrho_k}^{1/r}, n_k^{1/r}D_{\varrho_k}^{1/r}x)$$

yields an isomorphism between $Z_p^{j_k}(\sigma)$ and $Z_p(n_k, D_{\varrho_k})$. Indeed, for $\sigma_k = (\sigma_j)_{j\le j_k}$ we have

$$n_k^{1/r}D_{\varrho_k}^{1/r} = \left(\frac{n_k}{s_{j_k}}\right)^{1/r} D_{\sigma_k}$$

and

$$1\le \left(\frac{n_k}{s_{j_k}}\right)^{1/r} \le \left(1+\frac{1}{n_k}\right)^{1/r} \le 2.$$

Hence by Theorem 3.4, $Z_p^{j_k}(\sigma)$ has the γ_p-AP with a constant only depending σ and p, and therefore $Z_p(\sigma)$ has the γ_p-AP. Similarly for $Z_{p,c_p}(\sigma)$ and

$Z_{p,r_p}(\sigma)$. The spaces $Y_p(\sigma)$, $Y_{p,c_p}(\sigma)$, and $Y_{p,r_p}(\sigma)$ have the γ_p-AP by Corollary 3.3. Since $\liminf_n \sigma_n = 0$, we can find a subsequence $\sigma' = \sigma_{n_k}$ such that $(\sigma_{n_k}) \in \ell_r$. Then the map $M_r : S_p \to C_p(\mathbb{N}^2)$ defined by $M_r(x) = xD_{\sigma'}$ is completely bounded, and similarly $M_l : S_p \to R_p(\mathbb{N}^2)$ defined by $L_l(x) = D_{\sigma'}x$ is completely bounded. If $A = \{n_k : k \in \mathbb{N}\}$, then the subspace $Z_A = \{(x_{ij}) \mid i,j \in A\}$ is cb-isomorphic to S_p and complemented in $Z_p(\sigma)$, $Z_{p,c}(\sigma)$, and $Z_{p,r}(\sigma)$, respectively. By the definition of $Y_p(\sigma)$ we deduce that $Y_A = \{(x_k)_k \mid k \in A,\ x_k \in M_{m_k}\}$ is cb-isomorphic to $(\sum_{k\in A} \oplus_p S_p^{m_k})_p$ and cb-complemented. Thus all these spaces contain S_p^n's uniformly complemented. According to [11, Theorem 2.2], these spaces are $\mathcal{COS}_p$-spaces. ∎

4. Uncomplemented copies of some $\mathcal{OL}_p$-spaces. Throughout this section, $2 < p < \infty$, unless specified otherwise.

THEOREM 4.1. *Let X and Y be subspaces of rectangular $\mathcal{OL}_p$-spaces such that X is completely isomorphic to a subspace of Y. Then $\ell_p(Y)$ (respectively, $S_p[Y]$) contains an uncomplemented completely isomorphic copy of $\ell_p(X)$ (respectively, $S_p[X]$).*

Before proving the theorem, we formulate a corollary of it.

COROLLARY 4.2.

(a) *Suppose X is one of the following operator spaces: ℓ_p, S_p, $\mathcal{K}_p$, or $L_p(\mathcal{R})$. Then X contains an uncomplemented copy of itself.*
(b) *Suppose $\mathcal{N}$ is a group von Neumann algebra with QWEP, and X is either $\ell_p(L_p(\mathcal{N}))$ or $S_p[L_p(\mathcal{N})]$. Then X contains an uncomplemented copy of itself.*

Proof. All the spaces listed in (a) and (b) are $\mathcal{OL}_p$-spaces (see [12] for the spaces from (b)). Moreover, any of the spaces X listed in (a) is completely isomorphic to $\ell_p(X)$, by Pełczyński's decomposition method. The same argument shows that for $\mathcal{N}$ as in (b), $S_p[L_p(\mathcal{N})]$ is completely isomorphic to $\ell_p(S_p[L_p(\mathcal{N})])$. ∎

To establish Theorem 4.1, consider a finite-dimensional version of the Rosenthal space. More precisely, if $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers, then we let $X_p^m(\sigma)$ be the linear span of the first m vectors of the canonical basis of $X_p(\sigma)$. By Corollary 2.2 there exist $\lambda > 0$ and a sequence $(k_m)_{m\in\mathbb{N}}$ such that $\ell_p^{k_m}$ contains a λ-completely complemented λ-completely isomorphic copy of $X_p^m(\sigma)$.

Now suppose the sequence (σ_n) satisfies (1.5) and (1.6). By [26], if P_m is a projection from $\ell_p^m \oplus_p R_p^m \oplus_p C_p^m$ onto the "natural" copy of $X_p^m(\sigma)$, then $\lim_m \|P_m\| = \infty$. By [20] (see also [23]), $\ell_p^m \oplus_p R_p^m \oplus_p C_p^m$ embeds into ℓ_p^{3m} c_p-completely isomorphically. Thus, there exists a sequence (T_m)

of complete contractions $T_m : X_p^m(\sigma) \to \ell_p^{3^m}$ such that $\|T_m^{-1}\|_{\mathrm{cb}} \le c_p$, and $\lim_m \|Q_m\| = \infty$ whenever Q_m is a projection from $\ell_p^{3^m}$ onto range(T_m).

The properties of the spaces $X_p^m(\sigma)$ yield:

LEMMA 4.3. *ℓ_p contains an uncomplemented completely isomorphic copy of itself.*

Proof. Suppose the sequence (σ_m) satisfies (1.5) and (1.6). Consider the spaces $Y = (\sum_m \ell_p^{3^m})_p$ and $Z = (\sum_m T_m(X_p^m(\sigma)))_p$. By the discussion preceding the statement of this lemma, Z is an uncomplemented subspace of Y. Moreover, Y is completely isometric to ℓ_p. It remains to show that Z is completely isomorphic to ℓ_p. To this end, note that Z is completely isomorphic to a completely complemented subspace of $(\sum_m \ell_p^{k_m})_p \sim \ell_p$. Moreover, Y contains a completely complemented copy of ℓ_p. As $\ell_p = \ell_p(\ell_p)$, we complete the proof by applying the Pełczyński decomposition method. ∎

We need yet another lemma.

LEMMA 4.4. *Suppose X is a rectangular $\mathcal{OL}_p$-space, and T is a complete isomorphism from ℓ_p onto a subspace. Then $T \otimes \mathrm{id}_X$ is a complete isomorphism from $\ell_p(X)$ onto its range, viewed as a subspace of $\ell_p(X)$.*

Proof. We can assume that T is a complete contraction and let $c = \|T^{-1}\|_{\mathrm{cb}}$. It suffices to show that $T \otimes \mathrm{id}_{S_p^N} : \ell_p(S_p^N) \to \ell_p(S_p^N)$ is a complete contraction, and $\|(T \otimes \mathrm{id}_{S_p^N})^{-1}\|_{cb} \le c$. To complete the proof identify $\ell_p(S_p^N)$ with $S_p^N[\ell_p]$ and apply Proposition 0.1. ∎

REMARK 4.5. The same result also holds for complete isomorphisms from S_p onto its subspaces.

Proof of Theorem 4.1 Suppose X and Y are subspaces of rectangular $\mathcal{OL}_p$-spaces and $S : X \to Y$ is a complete isomorphism. Let $T : \ell_p \to \ell_p$ be a complete isomorphism with an uncomplemented range (such a T exists, by Lemma 4.3). By Lemma 4.4, $T \otimes S$ determines a complete isomorphism from $\ell_p(X)$ onto a subspace of $\ell_p(Y)$. It remains to show that range$(T \otimes S)$ is uncomplemented. Indeed, suppose for the sake of contradiction that there exists a projection P from $\ell_p(Y)$ onto range$(T \otimes S)$. Pick $x \in X \backslash \{0\}$ and denote by Q a bounded projection onto span(Sx). As T is a complete isomorphism, $\widetilde{Q} = I_{\mathrm{range}(T)} \otimes Q$ is a completely bounded projection from range$(T \otimes S)$ onto range$(T) \otimes$ span(Sx). Hence $\widetilde{Q} \circ P|_{\ell_p \otimes \mathrm{span}(Sx)}$ is a bounded projection from $\ell_p \otimes$ span(Sx) onto range$(T) \otimes$ span(Sx), which contradicts the fact that range(T) is uncomplemented. ∎

COROLLARY 4.6. *Suppose $\mathcal{N}$ is a von Neumann algebra equipped with a normal semifinite faithful trace which is not of type I. Then there exists an uncomplemented subspace X of $L_p(\mathcal{N})$ completely isomorphic to $L_p(\mathcal{R})$.*

Proof. By [11] (see also [21]), $L_p(\mathcal{N})$ contains a (completely contractively complemented) subspace Y, completely isometric to $L_p(\mathcal{R})$. By Theorem 4.1, Y contains an uncomplemented copy of $L_p(\mathcal{R})$. ■

COROLLARY 4.7.

(1) *Every infinite-dimensional rectangular $\mathcal{OL}_p$-space contains an uncomplemented copy of ℓ_p.*
(2) *Every infinite-dimensional $\mathcal{OS}_p$-space contains an uncomplemented copy of $(\sum_n S_p^n)_p$.*

Proof. By [11] any $\mathcal{OL}_p$-space X (with $1 < p < \infty$) embeds completely isometrically (and even completely contractively complementedly) into $\prod_{\mathcal{U}} S_p$, where $\mathcal{U}$ is an ultrafilter. By [24] and [25], X contains a completely isomorphic (and even completely complemented) subspace Y, completely isomorphic to ℓ_p. Moreover, if X is an $\mathcal{OS}_p$-space, then it contains a subspace Y, completely isomorphic to $(\sum_n S_p^n)_p$. In either case an application of Theorem 4.1 completes the proof. ■

References

[1] J. Arazy, *On large subspaces of the Schatten p-classes*, Compos. Math. 41 (1980), 297–336.

[2] J. Arazy and J. Lindenstrauss, *Some linear topological properties of the spaces C_p of operators on Hilbert space*, ibid. 30 (1975), 81–111.

[3] J. Bourgain, H. P. Rosenthal, and G. Schechtman, *An ordinal L_p-index for Banach spaces, with application to complemented subspaces of L_p*, Ann. of Math. 114 (1981), 193–228.

[4] E. G. Effros and Z.-J. Ruan, *$\mathcal{OL}_p$ spaces*, in: Contemp. Math. 228, Amer. Math. Soc., 1998, 51–77.

[5] —, —, *Operator Spaces*, London Math. Soc. Monogr. 23, Oxford Univ. Press, 2000.

[6] U. Haagerup, H. P. Rosenthal and F. A. Sukochev, *Banach embedding properties of non-commutative L_p-spaces*, Mem. Amer. Math. Soc. 163 (2003), no. 766.

[7] F. Hansen and G. K. Pedersen, *Perturbation formulas for traces on C^*-algebras*, Publ. RIMS Kyoto Univ. 31 (1995), 169–178.

[8] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces*, ibid. 217 (1979).

[9] M. Junge, *Doob's inequality for non-commutative martingales*, J. Reine Angew. Math. 549 (2002), 149–190.

[10] —, *Fubini's theorem for ultraproducts of non-commutative L_p-spaces*, Canad. J. Math. 56 (2004), 983–1021.

[11] M. Junge, N. J. Nielsen, Z.-J. Ruan, and Q. Xu, *$\mathcal{OL}_p$ spaces—The local structure of non-commutative L_p-spaces I*, Adv. Math. 187 (2004), 257–319.

[12] M. Junge and Z.-J. Ruan, *Approximation properties for non-commutative L_p-spaces associated with discrete groups*, Duke Math. J. 117 (2003), 313–341.

[13] M. Junge and Q. Xu, *Non-commutative Burkholder–Rosenthal inequalities*, Ann. Probab. 31 (2003), 948–995.

[14] M. Junge and Q. Xu, *Non-commutative Burkholder–Rosenthal inequalities II: Applications*, preprint.
[15] R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol. II. Advanced Theory*, Amer. Math. Soc., Providence, RI, 1997.
[16] J.-P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge Univ. Press, Cambridge, 1985.
[17] J. Lindenstrauss and H. P. Rosenthal, *The $\mathcal{L}_p$-spaces*, Israel J. Math. 7 (1969), 325–349.
[18] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Ergeb. Math. Grenzgeb. 92, Springer, Berlin, 1977.
[19] —, —, *Classical Banach Spaces II. Function Spaces*, Ergeb. Math. Grenzgeb. 97, Springer, Berlin, 1979.
[20] F. Lust-Piquard, *Inégalités de Khintchine dans C_p $(1 < p < \infty)$*, C. R. Acad. Sci. Paris 303 (1986), 289–292.
[21] J. Marcolino, *La stabilité des espaces L_p non-commutatifs*, Math. Scand. 81 (1997), 212–219.
[22] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compos. Math. 37 (1978), 3–19.
[23] —, *Non-Commutative Vector Valued L_p-Spaces and Completely p-Summing Maps*, Astérisque 247 (1998).
[24] Y. Raynaud, *On ultrapowers of non-commutative L_p spaces*, J. Operator Theory 48 (2002), 41–68.
[25] Y. Raynaud and Q. Xu, *On subspaces of non-commutative L_p-spaces*, J. Funct. Anal. 203 (2003), 149–196.
[26] H. P. Rosenthal, *On the subspaces of L_p $(p > 2)$ spanned by sequences of independent random variables*, Israel J. Math. 8 (1970), 273–303.
[27] F. A. Sukochev, *Non-isomorphisms of L_p-spaces associated with finite or infinite von Neumann algebras*, Proc. Amer. Math. Soc. 124 (1996), 1517–1527.
[28] M. Takesaki, *Theory of Operator Algebras I*, Springer, New York, 2001.

Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green Street
Urbana, IL 61801, U.S.A.
E-mail: junge@math.uiuc.edu

Department of Mathematics
and Computer Science
University of Southern Denmark
Campusvej 55
DK-5230 Odense M, Denmark
E-mail: njn@imada.sdu.dk

Department of Mathematics
University of California, Irvine
103 MSTB
Irvine, CA 92697-3875, U.S.A.
E-mail: toikhber@math.uci.edu

Received January 17, 2007
Revised version February 27, 2008 (6089)