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On the perturbation functions and similarity orbits

by

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Abstract. We show that the essential spectral radius $\varrho_e(T)$ of $T \in B(H)$ can be calculated by the formula $\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}) : X$ an invertible operator$\}$, where $\mathcal{F}_{\sharp\cdot\sharp}(T)$ is a Φ_1-perturbation function introduced by Mbekhta [J. Operator Theory 51 (2004)]. Also, we show that if $\mathcal{G}_{\sharp\cdot\sharp}(T)$ is a Φ_2-perturbation function [loc. cit.] and if T is a Fredholm operator, then $\operatorname{dist}(0, \sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X$ an invertible operator$\}$.

1. Terminology and introduction. Let $(H, \|\cdot\|)$ be a complex, infinite-dimensional Hilbert space and let $\mathcal{N}$ denote the set of all norms $\sharp\cdot\sharp$ on H that are equivalent to $\|\cdot\|$, and derived from an inner product $\prec\cdot,\cdot\succ$ on H, that is, $\sharp x\sharp = \sqrt{\prec x, x\succ}$ for all $x \in H$ (1).

Let $B(H)$ be the Banach algebra of all bounded linear operators on H and let $K(H)$ be its ideal of compact operators. If $T \in B(H)$ and $\sharp\cdot\sharp \in \mathcal{N}$, we will denote by $\sharp T\sharp$ the operator-norm of T relative to $\sharp\cdot\sharp$.

We denote by $N(T)$ the kernel and by $R(T)$ the range of $T \in B(H)$. The spectrum of T is denoted by $\sigma(T)$, and the adjoint by T^*. An operator $T \in B(H)$ is called *Fredholm* (resp. *semi-Fredholm*) if $R(T)$ is closed and $\max\{\dim N(T), \operatorname{codim} R(T)\} < \infty$ (resp. $\min\{\dim N(T), \operatorname{codim} R(T)\} < \infty$). We denote by $\Phi(H)$ (resp. $\Phi_\pm(H)$) the set of all Fredholm (resp. semi-Fredholm) operators. Set $C(H) = B(H)/K(H)$, the *Calkin algebra* (see [3, 4]); it is well known that $C(H)$ is a C^*-algebra.

The *essential spectrum* of T is $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(H)\}$, and the *semi-Fredholm spectrum* of T is $\sigma_\pm(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_\pm(H)\}$. Recall that the *essential spectral radius* of T is $\varrho_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$.

If T a semi-Fredholm operator, then the *index* of T is defined as

$$\operatorname{ind}(T) = \dim N(T) - \operatorname{codim} R(T).$$

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(1) From the polar identity, it follows that the inner product is unique:

$$4\prec x, y\succ = \sharp x + y\sharp^2 - \sharp x - y\sharp^2 + i\sharp x + iy\sharp^2 - i\sharp x - iy\sharp^2.$$

Let $\Phi_{\pm}^n$ denote the set of semi-Fredholm operators with $\mathrm{ind}(T) = n \in \mathbb{Z} \cup \{+\infty, -\infty\}$. Finally, let $G(H)$ be the group of all invertible elements in $B(H)$.

The rest of this paper is organized as follows. In the next section we shall show that for a Φ_1-perturbation function $\mathcal{F}_{\sharp\cdot\sharp}$, the infimum of $\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\}$ is equal to $\varrho_e(T)$. In Section 3 we prove that if T is a Fredholm operator and if $\mathcal{G}_{\sharp\cdot\sharp}(T)$ is a Φ_2-perturbation function, then the supremum of $\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\}$ is equal to $\mathrm{dist}(0, \sigma_e(T))$.

2. Similarity orbits and Φ_1-perturbation functions. Recently, Mbekhta [8] has introduced the following definition.

Definition 2.1 ([8, Definition 2.1]). Let $\sharp \cdot \sharp \in \mathcal{N}$. A *$\Phi_1$-perturbation function* on $B(H)$ is a function $\mathcal{F}_{\sharp\cdot\sharp}$ which associates to each $T \in B(H)$ a real number $\mathcal{F}_{\sharp\cdot\sharp}(T) \geq 0$ such that:

(a) $\mathcal{F}_{\sharp\cdot\sharp}(T + K) = \mathcal{F}_{\sharp\cdot\sharp}(T)$ for all $K \in K(H)$;
(b) $\mathcal{F}_{\sharp\cdot\sharp}(I) = 1$;
(c) $\min\{\mathcal{F}_{\sharp\cdot\sharp}(ST), \mathcal{F}_{\sharp\cdot\sharp}(TS)\} \leq \sharp S \sharp\, \mathcal{F}_{\sharp\cdot\sharp}(T)$ for all $T, S \in B(H)$;
(d) if $|\lambda| > \mathcal{F}_{\sharp\cdot\sharp}(T)$ then $T - \lambda I$ is Fredholm.

Remark. The definition given by Galaz-Fontes [5] for a perturbation function is a particular case of the above definition.

From now on, we shall denote by $\mathcal{F}_{\sharp\cdot\sharp}$ a Φ_1-perturbation function with $\sharp \cdot \sharp \in \mathcal{N}$.

In the proof of the following lemma, we use a method introduced by Mbekhta [7].

Lemma 2.2. *Let $T \in B(H)$ and $\varepsilon > 0$. Then there exists $W_\varepsilon \in B(H)$ such that*

$$\mathcal{F}_{\sharp\cdot\sharp}(e^{W_\varepsilon} T e^{-W_\varepsilon}) \leq \varrho_e(T) + \varepsilon.$$

Proof. By [10, Lemma 6], there exists a finite rank operator K_ε such that

$$\varrho(T + K_\varepsilon) \leq \varrho_e(T) + \varepsilon/2.$$

Since $\varrho\big(\frac{T+K_\varepsilon}{\varrho_e(T)+\varepsilon}\big) < 1$, it follows from the Rota theorem [12, Theorem 2] that there exists $X_\varepsilon \in B(H)$ invertible such that

$$(*) \qquad \sharp X_\varepsilon (T + K_\varepsilon) X_\varepsilon^{-1} \sharp \leq \varrho_e(T) + \varepsilon.$$

Let $X_\varepsilon = UP_\varepsilon$ be the polar decomposition of X_ε with $P_\varepsilon = (X_\varepsilon^* X_\varepsilon)^{1/2}$. Recall that U is unitary, and P_ε is positive and invertible. Since $\sigma(P_\varepsilon) \subseteq]0, +\infty[$, log is a continuous real function on $\sigma(P_\varepsilon)$. It follows from the symbolic calculus that there is a self-adjoint $W_\varepsilon \in B(H)$ such that $P_\varepsilon = e^{W_\varepsilon}$. Thus $P_\varepsilon^{-1} = e^{-W_\varepsilon}$. Since U is unitary, we see that $\sharp X_\varepsilon (T + K_\varepsilon) X_\varepsilon^{-1} \sharp =$

$\sharp e^{W_\varepsilon}(T+K_\varepsilon)e^{-W_\varepsilon}\sharp$. By property (a) of Definition 2.1, it follows that

$$\mathcal{F}_{\sharp\cdot\sharp}(e^{W_\varepsilon}Te^{-W_\varepsilon}) = \mathcal{F}_{\sharp\cdot\sharp}(e^{W_\varepsilon}(T+K_\varepsilon)e^{-W_\varepsilon}).$$

Using properties (b) and (c) of Definition 2.1, we deduce that

$$\begin{aligned}\mathcal{F}_{\sharp\cdot\sharp}(e^{W_\varepsilon}(T+K_\varepsilon)e^{-W_\varepsilon}) &\le \sharp e^{W_\varepsilon}(T+K_\varepsilon)e^{-W_\varepsilon}\sharp \\ &\le \sharp X_\varepsilon(T+K_\varepsilon)X_\varepsilon^{-1}\sharp \le \varrho_e(T)+\varepsilon.\end{aligned}$$

Therefore, $\mathcal{F}_{\sharp\cdot\sharp}(e^{W_\varepsilon}Te^{-W_\varepsilon}) \le \varrho_e(T)+\varepsilon$. ∎

Remark. In the above proof, we used the notion of adjoint operator, which depends on the scalar product associated to the norm $\sharp\cdot\sharp$.

Theorem 2.3. *Let $T \in B(H)$. Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\}.$$

Proof. First, by the property (d) of $\mathcal{F}_{\sharp\cdot\sharp}(T)$ (see Definition 2.1), for all invertible operators X we have

$$\varrho_e(XTX^{-1}) \le \mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}).$$

Since $\varrho_e(XTX^{-1}) = \varrho_e(T)$, we obtain

$$\varrho_e(T) \le \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\}.$$

Conversely, given $\varepsilon > 0$, by Lemma 2.2 there exists $W_\varepsilon \in B(H)$ such that

$$\mathcal{F}_{\sharp\cdot\sharp}(e^{W_\varepsilon}Te^{-W_\varepsilon}) \le \varrho_e(T)+\varepsilon.$$

Since e^{W_ε} is invertible, we deduce that

$$\inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\} \le \inf\{\varrho_e(T)+\varepsilon : \varepsilon > 0\} = \varrho_e(T). \ ∎$$

Remark. If $\mathcal{F}_{\sharp\cdot\sharp}(\cdot) = \sharp\cdot\sharp_e$, the result we obtain is the same as in [11], when the C^*-algebra is $B(H)$ and $I = K(H)$.

From the first part of the above proof and Lemma 2.2, we obtain the following theorem.

Theorem 2.4. *Let $T \in B(H)$. Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(e^{X}Te^{-X}) : X \in B(H)\}.$$

Remark. If $\mathcal{F}_{\sharp\cdot\sharp}(\cdot) = \sharp\cdot\sharp_e$, we obtain the result of [9] in the particular case when the C^*-algebra is $C(H) = B(H)/K(H)$.

Theorems 2.3 and 2.4 have the following consequence.

Corollary 2.5. *Let $T \in B(H)$. Then*

$$\begin{aligned}\varrho_e(T) &= \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H),\ \sharp\cdot\sharp \in \mathcal{N}\} \\ &= \inf\{\mathcal{F}_{\sharp\cdot\sharp}(e^{X}Te^{-X}) : X \in B(H),\ \sharp\cdot\sharp \in \mathcal{N}\}.\end{aligned}$$

Consider the natural map $\pi : B(H) \to C(H) = B(H)/K(H)$. Let $X \in \Phi(H)$. We say that $X_\pi \in B(H)$ is a π-*inverse* of X if $\pi(X_\pi)$ is the inverse of $\pi(T)$ in $C(H)$, i.e.

$$\pi(X)\pi(X_\pi) = \pi(X_\pi)\pi(X) = \pi(I). \tag{2.1}$$

From (2.1), it is easily seen that

$$\sigma_e(T) = \sigma_e(XTX_\pi) = \sigma_e(X_\pi TX), \tag{2.2}$$

$$\varrho_e(T) = \varrho_e(XTX_\pi) = \varrho_e(X_\pi TX). \tag{2.3}$$

Corollary 2.6. *Let $T \in B(H)$. Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX_\pi) : X \in \Phi(H)\}.$$

Proof. Since $G(H) \subseteq \Phi(H)$, it follows from Theorem 2.3 that

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\} \ge \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX_\pi) : X \in \Phi(H)\}.$$

Conversely, by the property (d) of $\mathcal{F}_{\sharp\cdot\sharp}$ (see Definition 2.1), for all $X \in \Phi(H)$ we have

$$\varrho_e(XTX_\pi) \le \mathcal{F}_{\sharp\cdot\sharp}(XTX_\pi). \tag{2.4}$$

The result follows from (2.4) and (2.3). ∎

Corollary 2.7. *Let $T \in B(H)$. Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX_\pi) : X \in \Phi(H),\ \sharp\cdot\sharp \in \mathcal{N}\}.$$

We will show similar results for left and right invertible operators. First we need some notation. Let $G_l(H)$ denote the set of all left invertible operators:

$$G_l(H) = \{X \in B(H) : \exists L \in B(H) \text{ such that } LX = I\},$$

and $G_r(H)$ the set of all right invertible operators:

$$G_r(H) = \{X \in B(H) : \exists R \in B(H) \text{ such that } XR = I\}.$$

We shall denote by X^l (resp. X^r) a left (resp. right) inverse of $X \in G_l(H)$ (resp. $X \in G_r(H)$).

Corollary 2.8. *Let $T \in B(H)$. Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^l) : X \in G_l(H),\ \mathrm{ind}(X) \in \mathbb{Z}_-\}.$$

Proof. Since $G(H) \subseteq \{X \in G_l(H) : \mathrm{ind}(X) \in \mathbb{Z}_-\} \subseteq \Phi(H)$, it follows from Theorem 2.3 and Corollary 2.6 that

$$\begin{aligned}\varrho_e(T) &= \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\}\\ &\ge \{\mathcal{F}_{\sharp\cdot\sharp}(XTX^l) : X \in G_l(H),\ \mathrm{ind}(X) \in \mathbb{Z}_-\}\\ &\ge \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX_\pi) : X \in \Phi(H)\} = \varrho_e(T). \ ∎\end{aligned}$$

Corollary 2.9. *Let* $T \in B(H)$. *Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^l) : X \in G_l(H),\, \mathrm{ind}(X) \in \mathbb{Z}_-,\, \sharp\cdot\sharp \in \mathcal{N}\}.$$

For right invertible operators we have the following corollaries.

Corollary 2.10. *Let* $T \in B(H)$. *Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^r) : X \in G_r(H),\, \mathrm{ind}(X) \in \mathbb{N}\}.$$

Corollary 2.11. *Let* $T \in B(H)$. *Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^r) : X \in G_r(H),\, \mathrm{ind}(X) \in \mathbb{N},\, \sharp\cdot\sharp \in \mathcal{N}\}.$$

We denote by $G_\pm(H) = G_l(H) \cup G_r(H)$ the set of all *semi-invertible* operators. When $X \in G_\pm(H)$, we simply write $X^\pm$ for a left inverse or a right inverse of X.

The proof of the following is exactly the same as the proof of Corollary 2.8.

Corollary 2.12. *Let* $T \in B(H)$. *Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^\pm) : X \in G_\pm(H),\, \mathrm{ind}(X) \in \mathbb{Z}\}.$$

Corollary 2.13. *Let* $T \in B(H)$. *Then*

$$\varrho_e(T) = \inf\{\mathcal{F}_{\sharp\cdot\sharp}(XTX^\pm) : X \in G_\pm(H),\, \mathrm{ind}(X) \in \mathbb{Z},\, \sharp\cdot\sharp \in \mathcal{N}\}.$$

3. Similarity orbits and $\varPhi_2$-perturbation functions. We denote by $\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G_l(H)\}$ the *left spectrum* and by $\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G_r(H)\}$ the *right spectrum*. Moreover, $\varPhi_\pm^n(T) = \{\lambda \in \mathbb{C} : T - \lambda I \in \varPhi_\pm^n\}$, with $n \in \mathbb{Z} \cup \{+\infty, -\infty\}$.

The following definition was introduced by Mbekhta [8].

Definition 3.1 ([8, Definition 3.4]). Let $\sharp\cdot\sharp \in \mathcal{N}$. A *$\varPhi_2$-perturbation function* on $B(H)$ is a function $\mathcal{G}_{\sharp\cdot\sharp}$ which associates to each $T \in B(H)$ a real number $\mathcal{G}_{\sharp\cdot\sharp}(T) \geq 0$ such that:

(a) $\mathcal{G}_{\sharp\cdot\sharp}(T + K) = \mathcal{G}_{\sharp\cdot\sharp}(T)$ for all $K \in K(H)$;
(b) $\mathcal{G}_{\sharp\cdot\sharp}(I) = 1$;
(c) $\min\{\mathcal{G}_{\sharp\cdot\sharp}(ST), \mathcal{G}_{\sharp\cdot\sharp}(TS)\} \leq \sharp S\sharp\, \mathcal{G}_{\sharp\cdot\sharp}(T)$ for all $T, S \in B(H)$;
(d) if $T \in \varPhi(H)$ and $|\lambda| < \mathcal{G}_{\sharp\cdot\sharp}(T)$, then $T - \lambda I \in \varPhi(H)$.

We shall denote by $\mathcal{G}_{\sharp\cdot\sharp}$ a $\varPhi_2$-perturbation function with $\sharp\cdot\sharp \in \mathcal{N}$.

The following theorem is the main result of this section.

Theorem 3.2. *Let* $T \in \varPhi(H)$. *Then*

$$\mathrm{dist}(0, \sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\}.$$

For the proof we need some lemmas.

LEMMA 3.3. *Let $S \in B(H)$. If $\lambda_0 \in \sigma_e(S)^c \cap \partial[\sigma_l(S)]$, then λ_0 is an isolated point of $\sigma_l(S)$.*

Proof. The result follows from [3, Theorem 3.2.10] (see also [6, Theorem V.1.6 and Corollary V.1.7]). ∎

LEMMA 3.4. *Let $T \in \Phi(H)$ and let K be a compact operator such that*

$$\sigma_e(T) = [\sigma_l(T+K) \cap \sigma_r(T+K)] \cup \Phi_{\pm}^{+\infty}(T) \cup \Phi_{\pm}^{-\infty}(T).$$

Then $\partial(\sigma_l(T+K)) \cap [\sigma_e(T)]^c = \emptyset$.

Proof. Suppose there exists $\lambda_0 \in \partial(\sigma_l(T+K)) \cap [\sigma_e(T)]^c$. Lemma 3.3 asserts that λ_0 is an isolated point of $\sigma_l(T+K)$. This proves that $T+K-\lambda_0$ is a right invertible operator, because otherwise $\lambda_0 \in \sigma_l(T+K) \cap \sigma_r(T+K) \subseteq \sigma_e(T)$, which is a contradiction. Now, since $T+K-\lambda_0$ is right invertible, we see that $\operatorname{ind}(T+K-\lambda_0 I) \geq 0$. But $\lambda_0 \in \partial(\sigma_l(T+K))$, which implies that $\operatorname{ind}(T+K-\lambda_0 I) < 0$, a contradiction.

LEMMA 3.5. *Let $T \in \Phi(H)$ and let K be a compact operator as in Lemma 3.4. If $0 \notin \sigma_l(T+K)$, then $\operatorname{dist}(0, \sigma_e(T)) = \operatorname{dist}(0, \sigma_l(T+K))$.*

Proof. First, it is easy to see that $\partial[\sigma_e(T)] \subseteq \sigma_l(T+K) \cap \sigma_r(T+K)$. Therefore,

$$\operatorname{dist}(0, \sigma_e(T)) = \operatorname{dist}(0, \sigma_l(T+K) \cap \sigma_r(T+K)).$$

We consider the case where $0 \notin \sigma_r(T+K)$. Since $\partial(\sigma_r(T+K)) \subseteq \sigma_l(T+K)$ and $\partial(\sigma_l(T+K)) \subseteq \sigma_r(T+K)$, we obtain

$$\begin{aligned}\operatorname{dist}(0, \sigma_e(T)) &= \operatorname{dist}(0, \sigma_l(T+K) \cap \sigma_r(T+K)) \\ &= \operatorname{dist}(0, \sigma_l(T+K)) = \operatorname{dist}(0, \sigma_r(T+K)).\end{aligned}$$

On the other hand, if $0 \in \sigma_r(T+K)$, it was shown in Lemma 3.4 that $\partial(\sigma_l(T+K)) \cap \sigma_e(T)^c = \emptyset$. Thus, $\partial(\sigma_l(T+K)) \subseteq \sigma_e(T)$. Therefore,

$$\begin{aligned}\operatorname{dist}(0, \sigma_e(T)) &\leq \operatorname{dist}(0, \partial(\sigma_l(T+K)) \leq \operatorname{dist}(0, \sigma_l(T+K)) \\ &\leq \operatorname{dist}(0, \sigma_l(T+K) \cap \sigma_r(T+K)) \leq \operatorname{dist}(0, \sigma_e(T)).\end{aligned}$$

This proves the lemma. ∎

Proof of Theorem 3.2. First, we show that

$$\operatorname{dist}(0, \sigma_e(T)) \geq \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\}.$$

Let $X \in B(H)$ be an invertible operator, and let $\lambda \in \mathbb{C}$ be such that $|\lambda| < \mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1})$. Since $X(T-\lambda)X^{-1} = XTX^{-1} - \lambda \in \Phi(H)$, we see that $T - \lambda$ is Fredholm. Therefore,

$$\operatorname{dist}(0, \sigma_e(T)) \geq \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\}.$$

Conversely, Theorem 4.5 of [1] asserts that there is $K \in K(H)$ such that $\sigma_{\pm}(T) = \sigma_l(T+K) \cap \sigma_r(T+K)$. But

$$\sigma_e(T) = \sigma_{\pm}(T) \cup \Phi_{\pm}^{+\infty}(T) \cup \Phi_{\pm}^{-\infty}(T),$$

so

$$\sigma_e(T) = [\sigma_l(T+K) \cap \sigma_r(T+K)] \cup \Phi_{\pm}^{+\infty}(T) \cup \Phi_{\pm}^{-\infty}(T).$$

Since $0 \notin \sigma_e(T)$, we obtain $0 \notin \sigma_l(T+K)$ or $0 \notin \sigma_r(T+K)$. We will suppose that $0 \notin \sigma_l(T+K)$; the other case is similar. It was shown in Lemma 3.5 that $\mathrm{dist}(0, \sigma_e(T)) = \mathrm{dist}(0, \sigma_l(T+K))$. Corollary 2.6 of [2] implies that

$$(*) \quad \mathrm{dist}(0, \sigma_e(T)) = \mathrm{dist}(0, \sigma_l(T+K)) = \sup\{1/\varrho(S) : S(T+K) = I\}.$$

On the other hand, let $S \in B(H)$ be a left inverse of $T+K$ and let $\varepsilon > 0$. Since $\varrho\big(\frac{S}{\varrho(S)+\varepsilon}\big) < 1$, it follows from the Rota theorem [12, Theorem 2] that there exists an invertible operator Z_ε such that

$$(**) \qquad \sharp Z_\varepsilon S {Z_\varepsilon}^{-1} \sharp \le \varrho(S) + \varepsilon.$$

Consider the polar decomposition $Z_\varepsilon = UP_\varepsilon$, where $P_\varepsilon = (Z_\varepsilon^* Z_\varepsilon)^{1/2}$ and U is the partial isometry with $N(U) = N(Z_\varepsilon)$ and $R(U) = R(Z_\varepsilon)$. This implies that U is unitary. Recall that P_ε is positive and invertible. Since $\sigma(P_\varepsilon) \subseteq \,]0, +\infty[$, log is a continuous real function on $\sigma(P_\varepsilon)$. It follows from the symbolic calculus that there is a self-adjoint $W_\varepsilon \in B(H)$ such that $P_\varepsilon = e^{W_\varepsilon}$. Thus $P_\varepsilon^{-1} = e^{-W_\varepsilon}$. It is obvious that

$$[e^{W_\varepsilon}(T+K)e^{-W_\varepsilon}][e^{W_\varepsilon} S e^{-W_\varepsilon}][e^{W_\varepsilon}(T+K)e^{-W_\varepsilon}] = e^{W_\varepsilon}(T+K)e^{-W_\varepsilon}.$$

It follows from [8, Lemme 3.18] that

$$\mathcal{G}_{\sharp\cdot\sharp}(e^{W_\varepsilon} T e^{-W_\varepsilon}) = \mathcal{G}_{\sharp\cdot\sharp}(e^{W_\varepsilon}(T+K)e^{-W_\varepsilon}) \ge \frac{1}{\sharp e^{W_\varepsilon} S e^{-W_\varepsilon} \sharp_e}.$$

But $\sharp e^{W_\varepsilon} S e^{-W_\varepsilon} \sharp \ge \sharp e^{W_\varepsilon} S e^{-W_\varepsilon} \sharp_e$, so

$$\mathcal{G}_{\sharp\cdot\sharp}(e^{W_\varepsilon} T e^{-W_\varepsilon}) \ge \frac{1}{\sharp e^{W_\varepsilon} S e^{-W_\varepsilon} \sharp}.$$

Since $Z_\varepsilon = UP_\varepsilon = Ue^{W_\varepsilon}$ and U is a unitary operator, we deduce that

$$\mathcal{G}_{\sharp\cdot\sharp}(e^{W_\varepsilon} T e^{-W_\varepsilon}) \ge \frac{1}{\sharp Z_\varepsilon S Z_\varepsilon^{-1} \sharp}.$$

It follows from $(**)$ that

$$\sup_{\varepsilon>0}\{\mathcal{G}_{\sharp\cdot\sharp}(e^{W_\varepsilon} T e^{-W_\varepsilon})\} \ge \sup_{\varepsilon>0}\left\{\frac{1}{\sharp Z_\varepsilon S Z_\varepsilon^{-1} \sharp}\right\} \ge \frac{1}{\varrho(S)}.$$

But

$$\sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\} \ge \sup_{\varepsilon>0}\{\mathcal{G}_{\sharp\cdot\sharp}(e^{W_\varepsilon} T e^{-W_\varepsilon})\}.$$

We deduce that

$$(***)\qquad \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\} \geq 1/\varrho(S).$$

Since $(***)$ holds for all left inverses of $T+K$, we obtain

$$\sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\} \geq \sup\{1/\varrho(S) : S(T+K) = I\}.$$

It follows from $(*)$ that

$$\sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\} \geq \operatorname{dist}(0, \sigma_e(T)). \ \blacksquare$$

It is easy to see that the above proof yields the following result.

THEOREM 3.6. *Let $T \in \Phi(H)$. Then*

$$\operatorname{dist}(0, \sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(e^X T e^{-X}) : X \in B(H)\}.$$

COROLLARY 3.7. *Let $T \in \Phi(H)$. Then*

$$\begin{aligned}\operatorname{dist}(0, \sigma_e(T)) &= \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H),\ \sharp\cdot\sharp \in \mathcal{N}\} \\ &= \sup\{\mathcal{G}_{\sharp\cdot\sharp}(e^X T e^{-X}) : X \in B(H),\ \sharp\cdot\sharp \in \mathcal{N}\}.\end{aligned}$$

COROLLARY 3.8. *Let $T \in \Phi(H)$. Then*

$$\operatorname{dist}(0, \sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX_\pi) : X \in \Phi(H)\}.$$

Proof. Let $X \in \Phi(H)$ and let $\lambda \in \mathbb{C}$ be such that

$$|\lambda| < \mathcal{G}_{\sharp\cdot\sharp}(XTX_\pi).$$

It follows from the fact that $X(T-\lambda)X_\pi = XTX_\pi - \lambda XX_\pi \in \Phi(H)$ and the relation (2.2) that $T - \lambda \in \Phi(H)$. Then by Theorem 3.2,

$$\begin{aligned}\operatorname{dist}(0, \sigma_e(T)) &\geq \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX_\pi) : X \in \Phi(H)\} \\ &\geq \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\} \geq \operatorname{dist}(0, \sigma_e(T)). \ \blacksquare\end{aligned}$$

COROLLARY 3.9. *Let $T \in \Phi(H)$. Then*

$$\operatorname{dist}(0, \sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX_\pi) : X \in \Phi(H),\ \sharp\cdot\sharp \in \mathcal{N}\}.$$

COROLLARY 3.10. *Let $T \in \Phi(H)$. Then*

$$\operatorname{dist}(0, \sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^l) : X \in G_l(H),\ \operatorname{ind}(X) \in \mathbb{Z}_-\}.$$

Proof. We deduce from Corollary 3.8 that

$$\begin{aligned}\operatorname{dist}(0, \sigma_e(T)) &= \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX_\pi) : X \in \Phi(H)\} \\ &\geq \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^l) : X \in G_l(H), \operatorname{ind}(X) \in \mathbb{Z}_-\}.\end{aligned}$$

By Theorem 3.2, we conclude that

$$\begin{aligned}\operatorname{dist}(0, \sigma_e(T)) &= \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{-1}) : X \in G(H)\} \\ &\leq \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^l) : X \in G_l(H),\ \operatorname{ind}(X) \in \mathbb{Z}_-\}. \ \blacksquare\end{aligned}$$

We also have the following corollary.

Corollary 3.11. *Let $T \in \Phi(H)$. Then*

$$\mathrm{dist}(0,\sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^l) : X \in G_l(H),\ \mathrm{ind}(X) \in \mathbb{Z}_-,\ \sharp\cdot\sharp \in \mathcal{N}\}.$$

For right invertible operators we have the following corollaries.

Corollary 3.12. *Let $T \in \Phi(H)$. Then*

$$\mathrm{dist}(0,\sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^r) : X \in G_r(H),\ \mathrm{ind}(X) \in \mathbb{N}\}.$$

Corollary 3.13. *Let $T \in \Phi(H)$. Then*

$$\mathrm{dist}(0,\sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^r) : X \in G_r(H),\ \mathrm{ind}(X) \in \mathbb{N},\ \sharp\cdot\sharp \in \mathcal{N}\}.$$

The proof of the following corollary is exactly the same as the proof of Corollary 3.10.

Corollary 3.14. *Let $T \in \Phi(H)$. Then*

$$\mathrm{dist}(0,\sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{\pm}) : X \in G_{\pm}(H),\ \mathrm{ind}(X) \in \mathbb{Z}\}.$$

We easily obtain the following.

Corollary 3.15. *Let $T \in \Phi(H)$. Then*

$$\mathrm{dist}(0,\sigma_e(T)) = \sup\{\mathcal{G}_{\sharp\cdot\sharp}(XTX^{\pm}) : X \in G_{\pm}(H),\ \mathrm{ind}(X) \in \mathbb{Z},\ \sharp\cdot\sharp \in \mathcal{N}\}.$$

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