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Linear maps on $M_n(\mathbb{C})$ preserving the local spectral radius

by

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Abstract. Let x_0 be a nonzero vector in $\mathbb{C}^n$. We show that a linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves the local spectral radius at x_0 if and only if there is $\alpha \in \mathbb{C}$ of modulus one and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\Phi(T) = \alpha ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

1. Introduction. Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on a complex Banach space X, and let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. The *local resolvent set*, $\varrho_T(x)$, of an operator $T \in \mathcal{B}(X)$ at a point x is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\Phi : U \to X$ such that $(T-\lambda)\Phi(\lambda) = x$ $(\lambda \in U)$. The local spectrum of T at x is defined by $\sigma_T(x) := \mathbb{C} \setminus \varrho_T(x)$, and is obviously a closed subset of $\sigma(T)$, the spectrum of T. The *local spectral radius* of T at x is defined by

$$r_T(x) := \limsup_{n\to+\infty} \|T^n x\|^{1/n},$$

and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. Recall that $T \in \mathcal{B}(X)$ is said to have the *single-valued extension property* provided that for every open subset U of $\mathbb{C}$, the equation

$$(T-\lambda)\Phi(\lambda) = 0 \quad (\lambda \in U)$$

has no nontrivial analytic solution Φ. Every operator $T \in \mathcal{B}(X)$ for which the interior of its point spectrum, $\sigma_{\mathrm{p}}(T)$, is empty enjoys this property.

In [4], A. Bourhim and T. Ransford proved that the only additive map Φ from $\mathcal{B}(X)$ to itself for which

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$$\sigma_{\Phi(T)}(x) = \sigma_T(x) \quad (T \in \mathcal{B}(X),\, x \in X)$$

is the identity of $\mathcal{B}(X)$, and investigated several extensions of this result. Recently M. González and M. Mbekhta described the linear maps from $\mathcal{B}(X)$ to itself which preserve the local spectrum at a fixed nonzero vector $x_0 \in X$ but only when $X = \mathbb{C}^n$ is a finite-dimensional space; see [7]. They proved that a linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves the local spectrum at a nonzero vector $x_0 \in \mathbb{C}^n$ if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\Phi(T) = ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

In this paper, we treat a more general situation where one considers linear maps $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ that preserve the local spectral radius at a fixed nonzero vector. When X is a finite-dimensional space, we prove the following result.

Theorem 1.1. *Let x_0 be a nonzero fixed vector of $\mathbb{C}^n$. A linear map Φ from $M_n(\mathbb{C})$ into itself preserves the local spectral radius at $x_0 \in \mathbb{C}^n$, i.e.,*

$$r_T(x_0) = r_{\Phi(T)}(x_0) \quad (T \in M_n(\mathbb{C})),$$

if and only if there exist a scalar α of modulus 1 and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\Phi(T) = \alpha ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

As a corollary, we recapture the main result of M. González and M. Mbekhta [7].

Corollary 1.2. *Let x_0 be a nonzero fixed vector of $\mathbb{C}^n$. A linear map Φ from $M_n(\mathbb{C})$ into itself preserves the local spectrum at x_0, i.e.,*

$$\sigma_T(x_0) = \sigma_{\Phi(T)}(x_0) \qquad (T \in M_n(\mathbb{C})),$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\Phi(T) = ATA^{-1}$ for all $T \in M_n(\mathbb{C})$.

2. The proof of the main result. We first fix some more notation. The duality between the Banach space X and its dual, X^*, will be denoted by $\langle \cdot, \cdot \rangle$. For $x \in X$ and $f \in X^*$, we denote as usual by $x \otimes f$ the rank one operator on X given by $u \mapsto \langle u, f \rangle x$.

The proof of the main result, Theorem 1.1, relies on several lemmas. The first two of these hold even when X is an infinite-dimensional complex Banach space.

Lemma 2.1. *Let $A \in \mathcal{B}(X)$ be an invertible operator. The automorphism $\Phi : T \mapsto ATA^{-1}$ preserves the local spectral radius at $x \in X$ if and only if*

$$x \in \bigcup_{\lambda \in \sigma_{\mathrm{p}}(A)} \ker(A - \lambda).$$

Proof. Let $x \in X$, and let $T \in \mathcal{B}(X)$. We have

$$\begin{aligned} r_{ATA^{-1}}(Ax) &= \limsup_{n\to+\infty} \|(ATA^{-1})^n Ax\|^{1/n} = \limsup_{n\to+\infty} \|AT^n x\|^{1/n} \\ &\le \limsup_{n\to+\infty} \|T^n x\|^{1/n} = r_T(x). \end{aligned}$$

Similarly, we have

$$r_T(x) \le r_{ATA^{-1}}(Ax),$$

and so

$$r_T(x) = r_{ATA^{-1}}(Ax) \quad (T \in \mathcal{B}(X)).$$

Thus if $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, then

$$r_T(x) = r_{ATA^{-1}}(x) \quad (T \in \mathcal{B}(X)). \tag{2.1}$$

Conversely, assume that (2.1) holds, and suppose by way of contradiction that x and Ax are linearly independent. So, there is a linear functional $f \in X^*$ such that $\langle x, f\rangle = 1$ and $\langle A^{-1}x, f\rangle = 0$. Take $T := x \otimes f$, and note that $Tx = x$ and $ATA^{-1}x = 0$. This shows $1 = r_T(x) \ne r_{ATA^{-1}}(x) = 0$, and gives a contradiction. Hence, $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$. ∎

LEMMA 2.2. *Assume that the dimension of X is at least two. Let x_0 be a nonzero vector of X, and let $A : X^* \to X$ be a bijective operator. The antiautomorphism $\Phi : T \mapsto AT^*A^{-1}$ does not preserve the local spectral radius at x_0.*

Proof. Suppose by way of contradiction that $r_{\Phi(T)}(x_0) = r_T(x_0)$ for all $T \in \mathcal{B}(X)$. Just as at the beginning of the proof of Lemma 2.1, we get

$$r_{T^*}(A^{-1}x_0) = r_{\Phi(T)}(x_0) = r_T(x_0)$$

for all $T \in \mathcal{B}(X)$. Let $y \in X$ be a nonzero element such that $y \notin \mathbb{C}x_0$ and $\langle y, A^{-1}x_0\rangle = 1$, and let f be a linear functional on X such that

$$\langle x_0, f\rangle = 0 \quad \text{and} \quad \langle y, f\rangle = 1.$$

Set $T := y \otimes f$, and note that $Tx_0 = 0$ and that

$$T^{*n}(A^{-1}x_0) = \langle y, A^{-1}x_0\rangle \cdot \langle y, f\rangle^{n-1} f = f$$

for all $n \ge 1$. Thus

$$r_T(x_0) = 0 \quad \text{and} \quad r_{T^*}(A^{-1}x_0) = 1.$$

This gives a contradiction and finishes the proof. ∎

REMARK 2.3. Just as in the proof of the above lemma, one can see that when $X = H$ is a complex Hilbert space and $A : H \to H$ is a bijective operator, the antiautomorphism $\Phi : T \mapsto AT^{\mathrm{tr}}A^{-1}$ does not preserve the local spectral radius at a nonzero fixed vector $x_0 \in H$. Here, T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal basis.

Let $\mathcal{A}$ be a complex Banach algebra. The *spectral radius* of an element $a \in \mathcal{A}$, denoted by $r(a)$, is the maximum modulus of its spectrum $\sigma(a)$, i.e.,

$$r(a) = \max\{|\lambda| : \lambda \in \sigma(a)\},$$

and coincides with the limit of the convergent sequence $(\|a^n\|^{1/n})_{n\geq 1}$.

LEMMA 2.4. *The spectral radius function*

$$r : M_n(\mathbb{C}) \to [0, +\infty), \qquad A \mapsto r(A),$$

is continuous.

Proof. See [1, Corollary 3.4.5]. ∎

An operator $T \in \mathcal{B}(X)$ is said to be *cyclic* with *cyclic vector* $x \in X$ provided that the linear span of $\{T^k x : k \geq 0\}$ is dense in X. For a nonzero vector $x_0 \in \mathbb{C}^n$ and a matrix $T \in M_n(\mathbb{C})$, the following are equivalent:

(i) T is cyclic with cyclic vector x_0.
(ii) The vectors $x_0, Tx_0, T^2x_0, \ldots, T^{n-1}x_0$ form a basis of $\mathbb{C}^n$.
(iii) For every $k = 1, \ldots, n-1$, we have

$$\operatorname{dist}\Big(T^k x_0, \bigvee\{x_0, Tx_0, \ldots, T^{k-1}x_0\}\Big) > 0.$$

Here, $\bigvee$ denotes the linear span.

Now, assume that $T \in M_n(\mathbb{C})$ is a matrix having exactly n distinct eigenvalues, and let us show that such a matrix is cyclic. Indeed, since T is a diagonalizable matrix and cyclicity is preserved under similarity, we may and will suppose that T is a diagonal matrix with distinct diagonal entries $\lambda_1, \ldots, \lambda_n$. Let x_0 be the vector of $\mathbb{C}^n$ with all coordinates 1, and assume that there are scalars $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that

$$\alpha_0 x_0 + \alpha_1 T x_0 + \cdots + \alpha_n T^{n-1} x_0 = 0.$$

We shall prove that each α_k equals zero. From the above identity, it follows that

$$\begin{cases} \alpha_0 + \alpha_1\lambda_1 + \alpha_2\lambda_1^2 + \cdots + \alpha_{n-1}\lambda_1^{n-1} = 0, \\ \alpha_0 + \alpha_1\lambda_2 + \alpha_2\lambda_2^2 + \cdots + \alpha_{n-1}\lambda_2^{n-1} = 0, \\ \vdots \\ \alpha_0 + \alpha_1\lambda_n + \alpha_2\lambda_n^2 + \cdots + \alpha_{n-1}\lambda_n^{n-1} = 0. \end{cases}$$

This shows that the polynomial

$$p(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_{n-1} z^{n-1}$$

has n distinct zeros and implies that $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$. Thus the matrix T is cyclic and x_0 is a cyclic vector for T.

Since the set of all matrices which have n distinct eigenvalues is dense in $M_n(\mathbb{C})$, it should have become clear from what has been done above that the set of all cyclic matrices is dense in $M_n(\mathbb{C})$. The next lemma says, in fact, a little more and shows that the set of all cyclic matrices with a fixed common cyclic vector is dense in $M_n(\mathbb{C})$. Our arguments are influenced by the ones given in [7].

LEMMA 2.5. *Let x_0 be a fixed nonzero vector of $\mathbb{C}^n$. The set of all cyclic matrices with cyclic vector x_0 is dense in $M_n(\mathbb{C})$.*

Proof. We may and will suppose that $n > 1$. Let $T \in M_n(\mathbb{C})$ be an $n \times n$ matrix, and let $\varepsilon > 0$. It suffices to show that there is a cyclic $n \times n$ matrix S with cyclic vector x_0 such that $\|T - S\| < \varepsilon$. To do that we shall employ the above statement (iii) together with an induction process. First observe that there is $S_1 \in M_n(\mathbb{C})$ such that $S_1 x_0 \notin \bigvee\{x_0\}$ and $\|T - S_1\| < \varepsilon$ and suppose that for a positive integer k for which $1 < k < n$ there is $S_k \in M_n(\mathbb{C})$ such that $\|S_k - T\| < \varepsilon$ and

$$\min_{1 \le j \le k} \operatorname{dist}\Big(S_k^j x_0, \bigvee\{x_0, S_k^1 x_0, \ldots, S_k^{j-1} x_0\}\Big) > 0.$$

Set $X_{k-1} = \bigvee\{x_0, S_k^1 x_0, \ldots, S_k^{k-1} x_0\}$ and let Y be the vector subspace of $\mathbb{C}^n$ such that $\mathbb{C}^n = X_{k-1} \oplus Y$. If $S_k^k x_0 \notin X_{k-1}$, we set $S_{k+1} = S_k$. Otherwise, choose a matrix $Q \in M_n(\mathbb{C})$ vanishing on X_{k-1} such that $\|Q\| < \varepsilon - \|S_k - T\|$ and $Q S_k^k x_0 \in Y \setminus \{0\}$. We therefore set $S_{k+1} := S_k + Q$ and note that in both cases, we have $\|T - S_{k+1}\| < \varepsilon$ and

$$\min_{1 \le j \le k+1} \operatorname{dist}\Big(S_{k+1}^j x_0, \bigvee\{x_0, S_{k+1}^1 x_0, \ldots, S_{k+1}^{j-1} x_0\}\Big) > 0,$$

as desired. An induction process finishes the proof. ∎

Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras, and denote by $\operatorname{rad}(\mathcal{A})$ the radical of $\mathcal{A}$. When $\operatorname{rad}(\mathcal{A}) = \{0\}$, the algebra $\mathcal{A}$ is said to be *semisimple*. Examples of semisimple Banach algebras include $M_n(\mathbb{C})$ and $\mathcal{B}(X)$; see [1]. A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be *spectrally bounded* if there exists a positive constant M such that

$$r(\Phi(a)) \le M r(a) \tag{2.2}$$

for all $a \in \mathcal{A}$. This concept was introduced in [13], and the question when such a map has to be a Jordan homomorphism has been studied in great detail by a number of authors; see for instance [5], [6], [10]–[12], and [14]–[16].

We also need the following lemma which we believe to be of interest in its own right. It shows that a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is injective provided that $\mathcal{A}$ is semisimple and the reverse inequality of (2.2) holds.

LEMMA 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear map. If there exists a positive constant M such that*

$$r(a) \le M r(\Phi(a)) \tag{2.3}$$

for all $a \in \mathcal{A}$, *then* $\ker(\Phi) \subset \mathrm{rad}(\mathcal{A})$. *If*, *in particular*, $\mathcal{A}$ *is semisimple*, *then* Φ *is injective.*

Proof. Let $a_0 \in \mathcal{A}$ be such that $\Phi(a_0) = 0$. To prove that $a_0 \in \mathrm{rad}(\mathcal{A})$, pick an arbitrary $a \in \mathcal{A}$, and note that for every $\lambda \in \mathbb{C}$, we have

$$r(\lambda a_0 + a) \leq M r(\Phi(\lambda a_0 + a)) = M r(\Phi(a)).$$

Since $\lambda \mapsto r(\lambda a_0 + a)$ is a subharmonic function on $\mathbb{C}$, Liouville's theorem implies that $r(\lambda a_0 + a) = r(a)$ for all $\lambda \in \mathbb{C}$. In particular,

$$r(a_0 + a) = r(a).$$

Since a is an arbitrary element of $\mathcal{A}$, it follows from Zemánek's spectral characterization of the radical [1, Theorem 5.3.1] that $a_0 \in \mathrm{rad}(\mathcal{A})$. ∎

Now, we are able to prove the main result.

Proof of Theorem 1.1. We may and will assume that $n \geq 2$, as otherwise there is nothing to prove.

The "if" part is of course trivial. To prove the "only if" part, assume that Φ preserves the local spectral radius at x_0, and let $T \in M_n(\mathbb{C})$ be a cyclic matrix with a cyclic vector x_0. Let λ be an eigenvalue of T such that $|\lambda| = r(T)$, and let u be a corresponding eigenvector of norm 1. Since there is a nonzero polynomial p such that $u = p(T)x_0$, we have

$$r(T)^k = |\lambda|^k = \|T^k u\| = \|p(T)T^k x_0\| \leq \|p(T)\|\,\|T^k x_0\|$$

for all positive integers k. This gives

$$r(T) \leq \|p(T)\|^{1/k} \|T^k x_0\|^{1/k}$$

for all positive integers k and implies that $r(T) \leq r_T(x_0)$. Since the reverse inequality always holds for any operator, we have $r(T) = r_T(x_0)$ and

$$r(T) = r_T(x_0) = r_{\Phi(T)}(x_0) \leq r(\Phi(T)).$$

By Lemmas 2.4 and 2.5, this inequality holds true, in fact, for all $S \in M_n(\mathbb{C})$, i.e.,

$$r(S) \leq r(\Phi(S)) \tag{2.4}$$

for all $S \in M_n(\mathbb{C})$. Now, Lemma 2.6 tells us that Φ is a bijective linear map and its inverse Φ^{-1} preserves as well the local spectral radius at x_0. By what has been shown above, we also have

$$r(S) \leq r(\Phi^{-1}(S)) \tag{2.5}$$

for all $S \in M_n(\mathbb{C})$. By combining (2.4) and (2.5), we, in fact, have

$$r(\Phi(S)) = r(S) \tag{2.6}$$

for all $S \in M_n(\mathbb{C})$. By [3, Theorem 1], there exist an $\alpha \in \mathbb{C}$ of modulus 1 and an invertible matrix $A \in M_n(\mathbb{C})$ such that either

$$\Phi(T) = \alpha ATA^{-1} \quad (T \in M_n(\mathbb{C}))$$

or

$$\Phi(T) = \alpha AT^{\mathrm{tr}}A^{-1} \quad (T \in M_n(\mathbb{C})).$$

Lemma 2.2 excludes the second form of Φ. Finally, Lemma 2.1 implies that $Ax_0 = \lambda x_0$ for some nonzero $\lambda \in \mathbb{C}$. Dividing A by λ if necessary, we may assume that $Ax_0 = x_0$. ∎

3. Concluding remarks. In [2], M. Brešar, A. Fošner, and P. Šemrl extended Sourour's result which describes linear bijective maps from $\mathcal{B}(X)$ to $\mathcal{B}(Y)$ that preserve invertibility [17]. They characterized linear bijective invertibility preserving maps from an arbitrary semisimple Banach algebra with large socle to another one. B. Kuzma [8], motivated by Sourour's question [17] whether a linear unital map from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ which preserves invertibility is necessarily injective, showed that the answer is affirmative when "invertibility" is replaced by "noninvertibility". In fact, he proved the following result.

THEOREM 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple unital complex Banach algebras, and let Φ be a linear map from $\mathcal{A}$ onto $\mathcal{B}$ preserving noninvertibility; i.e., $\Phi(a)$ is not invertible in $\mathcal{B}$ whenever a is not invertible in $\mathcal{A}$. Then Φ is a bounded bijective linear map and is a Jordan isomorphism followed by left multiplication by a fixed invertible element in $\mathcal{B}$ provided that* $\operatorname{soc}(\mathcal{B})$ *is an essential ideal of $\mathcal{B}$.*

Proof. The second part of this theorem is a consequence of the bijectivity of Φ and the main result of [2]. To establish the injectivity of Φ, B. Kuzma made use of the scarcity lemma; see [1, Theorem 3.4.25, and Corollary 3.4.18]. This holds, in fact, at once by applying Lemma 2.6. Indeed, by surjectivity, $\Phi(a) = 1$ for some $a \in \mathcal{A}$ which is necessarily invertible, and the map Ψ defined by

$$\Psi(x) = \Phi(ax) \quad (x \in \mathcal{A})$$

is a linear unital surjective mapping preserving noninvertibility. Thus,

$$\sigma(x) \subset \sigma(\Psi(x)) \quad (x \in \mathcal{A}),$$

and

$$r(x) \le r(\Psi(x)) \quad (x \in \mathcal{A}).$$

Now, Lemma 2.6 implies that Ψ is injective, and so too is Φ. ∎

We close this paper with a complete description of the local spectrum and local spectral radius of an $n \times n$ matrix; see for instance [7] or [18].

Remark 3.2. Let $T \in M_n(\mathbb{C})$ be an $n \times n$ matrix. Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of T and denote by $E_1, \ldots, E_r$ the corresponding root spaces. We have

$$\mathbb{C}^n = E_1 \oplus \cdots \oplus E_r \quad \text{and} \quad T = T_1 \oplus \cdots \oplus T_r,$$

where T_i is the restriction of T to E_i. It follows that for every $x \in \mathbb{C}^n$,

$$\sigma_T(x) = \bigcup_{1 \le i \le r} \sigma_{T_i}(P_i x) = \{\lambda_i : 1 \le i \le r \text{ and } P_i(x) \ne 0\},$$

where $P_i : \mathbb{C}^n \to E_i$ is the canonical projection. Therefore,

$$\begin{aligned} r_T(x) &= \max\{r_{T_i}(P_i x) : 1 \le i \le r\} \\ &= \max\{|\lambda_i| : 1 \le i \le r \text{ with } P_i(x) \ne 0\}. \end{aligned}$$

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