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1-amenability of $\mathcal{A}(X)$ for Banach spaces with 1-unconditional bases

by

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Abstract. The main result of the note is a characterization of 1-amenability of Banach algebras of approximable operators for a class of Banach spaces with 1-unconditional bases in terms of a new basis property. It is also shown that amenability and symmetric amenability are equivalent concepts for Banach algebras of approximable operators, and that a type of Banach space that was long suspected to lack property $\mathbb{A}$ has in fact the property. Some further ideas on the problem of whether or not amenability (in this setting) implies property $\mathbb{A}$ are discussed.

1. Introduction. The homological notion of an amenable Banach algebra was introduced by Johnson in [J1] and it has been the object of much study ever since. A Banach algebra $\mathcal{A}$ is said to be *amenable* if every continuous derivation from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule is inner, or equivalently, if the first (continuous) cohomology group of $\mathcal{A}$ with coefficients in any dual $\mathcal{A}$-bimodule is trivial.

In the case of Banach algebras that sit on some particular structure there has been much interest in characterizing amenability in terms of properties of the underlying structure. The best example of this kind is the celebrated result of Johnson that the group algebra, $L^1(G)$, of a locally compact group G is amenable if and only if the group G is amenable [J1, Theorem 2.5]. One would expect an analogous characterization to be possible for the Banach algebras $\mathcal{A}(X)$ and $\mathcal{B}(X)$ of approximable and bounded operators, respectively, acting on a Banach space X, but to date, we do not know of any such result.

In this note, our main concern will be the Banach algebra $\mathcal{A}(X)$. The study of amenability of this algebra was initiated in [J1] and then continued in [GJW], where a certain Banach space property, which the authors called property $\mathbb{A}$, was identified and shown to be a sufficient condition for

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amenability of $\mathcal{A}(X)$. Later on, in [BG2], we showed that a weaker version of property $\mathbb{A}$ was, in fact, equivalent to amenability of $\mathcal{A}(X)$ whenever X is a π-space. This new condition was compared with the Følner condition for group amenability. However, still this characterization was not entirely satisfactory. It reduced the problem of amenability of $\mathcal{A}(X)$ to the existence of a bounded net of so-called generalized diagonals. These objects are elements of projective tensor products of the form $\mathcal{F}(Y,Z)\,\widehat{\otimes}\,\mathcal{F}(Z,Y)$, where Y and Z are finite-dimensional subspaces of X. Although they are well described from the algebraic point of view, the intractability of the projective norm (at least for practical purposes) led us to abandon any hope of being able to further translate our condition into a geometric property of the underlying Banach space.

In this paper, we shall be able to translate 1-amenability of $\mathcal{A}(X)$ for a wide class of Banach spaces with 1-unconditional bases into a property of the basis which is much in the spirit of property $\mathbb{A}$. It has long been known that local unconditional structure of the underlying space is not required for amenability of $\mathcal{A}(X)$ ([GJW, Corollary 2.7]). However, in its full generality, the problem mentioned at the end of the previous paragraph seems to be rather difficult. For this reason, we have chosen to start by looking at Banach spaces with 1-unconditional bases. It is known that there are Banach spaces with 1-unconditional bases whose algebras of approximable operators are not amenable ([BG2, Theorem 5.6]) while, on the other hand, if X has a subsymmetric basis (or even less, if X has a 1-unconditional basis such that the right and left shift operators with respect to it are power bounded) then $\mathcal{A}(X)$ is amenable. So even in this restrictive setting the answer to our problem is far from trivial.

We should point out that also the 1-amenability assumption will play a key role in our results. It is essentially this condition that will allow us to translate the existence of a sequence of generalized diagonals into a property of the given 1-unconditional basis. In doing so, we shall rely heavily on approximate versions of known results about norm-one projections on Banach spaces with 1-unconditional bases as well as on known results about the extreme properties of the identity element of a Banach algebra. We do not know if whenever $\mathcal{A}(X)$ is amenable, an equivalent renorming, $\widetilde{X}$, of X exists such that $\mathcal{A}(\widetilde{X})$ is 1-amenable and, in case of X having a 1-unconditional basis, such that, in addition, the unconditionality constant be preserved. If the answer to this last question were negative (as we suspect) then it seems very unlikely that our results will still hold if one removes the 1-unconditionality and 1-amenability assumptions.

Lastly, the note has been organized as follows. In the next section, we will cover some of the background material and terminology needed throughout these pages. Section 3 might seem to the reader like a small digression, but

we feel it belongs in here. In it, we will look at the relationship between amenability and symmetric amenability in the context of Banach algebras of approximable operators. The notion of symmetric amenability was introduced in [J3] and was expected to be (in this context) stronger than amenability and closely related to property $\mathbb{A}$. We will show in Section 3 that this is not the case. We will then turn our attention, in Section 4, to the finite-dimensional situation of our main problem. It will be shown in this section that if X is finite-dimensional and has a 1-unconditional basis then 1-amenability of $\mathcal{A}(X)$ is equivalent to X having property $\mathbb{A}$ with constant 1.

In Section 5, we will prove the main result of the note, i.e., our characterization of 1-amenability of $\mathcal{A}(X)$ for a Banach space X with a 1-unconditional basis in terms of a basis property. Unfortunately, apart from 1-unconditionality, other assumptions on the basis will be needed. We will introduce and discuss them also in Section 5. The last section of the paper, Section 6, will be devoted to the question of whether property $\mathbb{A}$ is a necessary condition for amenability of $\mathcal{A}(X)$. This question goes back to [GJW] and is closely related to the main problem of this note. Although we have been unable to answer it, we expect the results of this paper, and in particular those from the last section, will help addressing this seemingly simpler question.

2. Background material and notation. The notation and terminology that we will use in what follows are standard and mostly consistent with those used in previous papers. Let us recall here some of them. Others will be introduced or recalled later on as they are needed.

To simplify the statement of the results, we shall denote by ℓ_∞ the linear space, usually denoted by c_0, of all bounded scalar sequences tending to zero. Given a normed space X we denote by X' its topological dual. If X and Y are isomorphic (resp., isometric) normed spaces, we write this as $X \simeq Y$ (resp., $X \cong Y$), and denote by $d(X,Y)$ the *Banach–Mazur distance* between them, that is, the infimum of the numbers $\|T\|\,\|T^{-1}\|$, where T is an isomorphism between X and Y.

The adjoint of an operator $T : X \to Y$ is denoted by T^* and we write $\operatorname{rg} T$ (resp., $\operatorname{rk} T$) for the range (resp., rank) of T. The identity operator on a normed space X will be denoted by id_X or just id if the space X is clear from context. Given a linear map $T : X \to Y$ and linear subspaces $E \subseteq X$ and $F \subseteq Y$ we will denote by $T|_E$ and $T|^F$, respectively, the restriction to E and corestriction to F of T.

We write $\|\cdot\|_\wedge$ (resp., $\|\cdot\|$) for the projective (resp., operator) norm. Given a subset S of a Banach space X, we write $\operatorname{sp}(S)$ for its linear span and $\overline{\operatorname{co}}(S)$ for the norm-closure of its convex hull. In the case of an indexed

subset $\{e_i : i \in I\}$ of X, we write $[e_i]_{i\in I}$ (or $[e_i : i \in I]$) for the closure of its linear span. We denote by $X_{[1]}$ the (closed) unit ball of a normed space X. We write $(x_i)_{i=1}^n$ for the sequence consisting of the elements $x_1, \dots, x_n$, or just (x_i) if n is clear from context.

Given Banach spaces X and Y we write $\mathcal{F}(X, Y)$ for the normed space of all finite-rank operators from X to Y endowed with the operator norm. When appropriate we will identify $\mathcal{F}(X, Y)$ with $X' \otimes Y$, so that, for $f \in X'$ and $y \in Y$, the rank-1 operator $x \mapsto f(x)y$ will be denoted by $f \otimes y$. When $X = Y$ we simply write $\mathcal{F}(X)$. We will denote by $\mathrm{tr} : X' \mathbin{\widehat{\otimes}} X \to \mathbb{C}$ the trace functional, i.e., the linear functional defined on elementary tensors by $\mathrm{tr}(f \otimes x) := f(x)$.

There is an intrinsic characterization of amenability that is particularly useful in this setting and will be crucial to the results of this paper. Precisely, a Banach algebra $\mathcal{A}$ is amenable if and only if it has a *bounded approximate diagonal*, i.e., a bounded net (d_α) in $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$ such that

$$\pi(d_\alpha)a \to a \quad \text{and} \quad ad_\alpha - d_\alpha a \to 0 \qquad (a \in \mathcal{A}),$$

where $\pi : \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \to \mathcal{A}$, $a \otimes b \mapsto ab$ [J2, Lemma 1.2 and Theorem 1.3]. The Banach algebra $\mathcal{A}$ is said to be K*-amenable* if it has an approximate diagonal of bound K. The smallest such K is called the *amenability constant* of $\mathcal{A}$. Recall that a *bounded approximate identity* (b.a.i. for short) for a Banach algebra $\mathcal{A}$ is a bounded net (e_α) in $\mathcal{A}$ such that $\lim_\alpha e_\alpha a = \lim_\alpha a e_\alpha = a$ $(a \in \mathcal{A})$. So, in particular, if (d_α) is a bounded approximate diagonal for $\mathcal{A}$ then $(\pi(d_\alpha))$ is a b.a.i. for $\mathcal{A}$.

Let $^\circ : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ be defined by $(a\otimes b)^\circ := b\otimes a$ $(a\otimes b \in \mathcal{A}\widehat{\otimes}\mathcal{A})$. The algebra $\mathcal{A}$ is said to be *symmetrically amenable* if it has a bounded approximate diagonal (d_α) all of whose elements satisfy $d_\alpha^\circ = d_\alpha$. An approximate diagonal with this last property is said to be *symmetric*. In general, symmetric amenability is stronger than amenability. For commutative Banach algebras and group algebras they agree ([J3, Corollary 2.3 and Theorem 4.1]) and we will see below that they also agree for Banach algebras of approximable operators.

A Banach space X is said to have *property* $\mathbb{A}$ if there is in $\mathcal{A}(X)$ a bounded net (p_α) of projections such that:

(i) (p_α) converges strongly to id_X;
(ii) (p_α^*) converges strongly to $\mathrm{id}_{X'}$; and
(iii) for every α there is a finite group $G_\alpha \subset \mathcal{A}(\mathrm{rg}\, p_\alpha)_{[K]}$ such that $\mathrm{sp}(G_\alpha) = \mathcal{A}(\mathrm{rg}\, p_\alpha)$, where K is a constant independent of α.

It is easy to show that, in this case, the elements $|G_\alpha|^{-1} \sum_{T\in G_\alpha} T \otimes T^{-1}$ form a symmetric approximate diagonal for $\mathcal{A}(X)$ of bound K. We shall refer to the smallest constant K for which (i)–(iii) hold as the *property* $\mathbb{A}$ *constant*

of X. Recall also that a Banach space X such that $\mathcal{A}(X)$ contains a bounded net of projections converging strongly to id_X is said to be a *π-space*.

Given a pair of finite-dimensional Banach spaces X and Y, an element d of $\mathcal{F}(Y,X)\widehat{\otimes}\mathcal{F}(X,Y)$ is called a *generalized diagonal* for $\mathcal{F}(X)$ if $\pi(d)=\mathrm{id}_X$ and $Wd=dW$ $(W\in\mathcal{F}(X))$. If $(x_k)_{k=1}^m$ and $(y_i)_{i=1}^n$ are bases for X and Y, respectively, and d is a generalized diagonal, then d can be represented in the form $\sum_{i,j}a_{i,j}\sum_k(y_j^*\otimes x_k)\otimes(x_k^*\otimes y_i)$, where the $a_{i,j}$'s are scalars such that $\sum_i a_{i,i}=1$. Conversely, any element of this form is a generalized diagonal for $\mathcal{F}(X)$. There is in $\mathcal{F}(X)\widehat{\otimes}\mathcal{F}(X)$ a unique generalized diagonal for $\mathcal{F}(X)$, d say, with the additional property that d° is also a generalized diagonal for $\mathcal{F}(X)$. We shall refer to this unique element as the *symmetric diagonal* for $\mathcal{F}(X)$.

It was shown in [BG2, Corollary 3.3] that *if X is a Banach space with a bounded net $(p_\alpha)_{\alpha\in I}$ of projections converging strongly to id_X and such that $(p_\alpha^*)_{\alpha\in I}$ converges strongly to $\mathrm{id}_{X'}$ then $\mathcal{A}(X)$ is amenable if and only if there is a constant $K\geq 1$ such that, for every $\alpha\in I$, there is $\beta\in I$ such that $\mathcal{F}(\operatorname{rg}p_\beta,\operatorname{rg}p_\alpha)\,\widehat{\otimes}\,\mathcal{F}(\operatorname{rg}p_\alpha,\operatorname{rg}p_\beta)$ contains a generalized diagonal d_α for $\mathcal{F}(\operatorname{rg}p_\alpha)$ of norm $\leq K$. Moreover, if $\mathcal{A}(X)$ is 1-amenable and $\lim_\alpha\|p_\alpha\|=1$ then, given an arbitrary net (ε_α) of positive real numbers converging to zero, the d_α's can be chosen to satisfy $\|d_\alpha\|_\wedge\leq 1+\varepsilon_\alpha$*. The latter was not explicitly stated in [BG2], but it is implicit in the proof of [BG2, Proposition 3.1].

Recall that a Schauder basis (x_i) for a Banach space X is said to be 1-*unconditional* if

$$\Big\|\sum_i\alpha_i x_i\Big\|\leq\Big\|\sum_i\beta_i x_i\Big\|$$

whenever (α_i) and (β_i) are (eventually null) scalar sequences with $|\alpha_i|\leq|\beta_i|$ $(i\in\mathbb{N})$. If $\|x_i\|=1$ $(i\in\mathbb{N})$ then the basis is said to be *normalized*. Given a basis (x_i) we write x_k^* for the kth biorthogonal functional associated with it. If the associated sequence (x_i^*) of biorthogonal functionals forms a basis for X' then we call the basis *shrinking*.

If X is a Banach space with a Schauder basis (x_i) then we denote by $\operatorname{supp}x$ the support of a vector $x\in X$, i.e., the set $\{i\in\mathbb{N}:x_i^*(x)\neq 0\}$. Vectors x and y will be said to be *disjointly supported* if $\operatorname{supp}x\cap\operatorname{supp}y=\emptyset$. By a *support functional* for a given vector $x\in X$ we always mean a norm-one linear functional $f\in X'$ such that $f(x)=\|x\|$. As is customary while working with sequence spaces, if E and F are non-empty subsets of $\mathbb{N}$, we write $E<F$ to mean $\max E<\min F$ and denote also by E the projection $\sum_i\alpha_i x_i\mapsto\sum_{i\in E}\alpha_i x_i$.

Given a Banach space X with a 1-unconditional Schauder basis (x_i), there is, for each pair of positive integers $m<n$, a natural isometric Banach algebra embedding $\mathcal{A}([x_i]_{i=1}^m)\hookrightarrow\mathcal{A}([x_i]_{i=1}^n)$ induced by the natural

embedding $[x_i]_{i=1}^m \hookrightarrow [x_i]_{i=1}^n$. Letting m and n vary we obtain a direct system of Banach algebras and isometric Banach algebra homomorphisms. Its inductive limit is also a Banach algebra that we shall denote by $\mathcal{A}_0(X)$. In general, $\mathcal{A}_0(X)$ is properly contained in $\mathcal{A}(X)$, but if (x_i) is shrinking then $\mathcal{A}_0(X) = \mathcal{A}(X)$. When $\mathcal{A}(X)$ is selfinduced (see [BG1, Section 1.2] for the definition) the amenability of $\mathcal{A}(X)$ can be deduced from that of $\mathcal{A}_0(X)$ ([BG1, Lemma 2.2]).

Since we will rely on [BG2, Corollary 3.3] we will need the natural projections with respect to a given basis for X to form a b.a.i. for $\mathcal{A}(X)$. However, rather than assuming this (or equivalently, that the basis is shrinking), it seems more natural to us to state our results for $\mathcal{A}_0(X)$, and this is precisely what we will do.

Lastly, in what follows, operator always means continuous linear operator and we assume all our Banach spaces to be over the complex field. We write $\mathbb{R}_+$ for the set $\{t \in \mathbb{R} : t > 0\}$.

3. Symmetric amenability of $\mathcal{A}(X)$. There is no obvious reason at all to expect that whenever a bounded approximate diagonal for $\mathcal{A}(X)$ exists there is also a symmetric one. However, as we will see next, this happens to be the case.

THEOREM 3.1. *Let X be a Banach space. Then $\mathcal{A}(X)$ is amenable if and only if it is symmetrically amenable.*

Proof. In what follows, mainly to simplify notation, we shall identify $\mathcal{F}(X)$ with $X' \otimes X$ (see fifth paragraph of the previous section). It is with this identification in mind that we shall apply $\|\cdot\|_\wedge$ and tr to operators in $\mathcal{F}(X)$.

Of course, we only need to show that if $\mathcal{A}(X)$ is amenable then it is symmetrically amenable. To this end, let $(d_\alpha) \subset \mathcal{F}(X) \otimes \mathcal{F}(X)$ be a bounded approximate diagonal for $\mathcal{A} := \mathcal{A}(X)$, so $(\pi(d_\alpha))$ is a b.a.i. for $\mathcal{A}$. For each α, set $e_\alpha = \pi(d_\alpha)$ and choose a projection $p_\alpha \in \mathcal{A}$ so that $p_\alpha e_\alpha = e_\alpha$ and

$$\frac{\|e_\alpha(e_\alpha - 2\,\mathrm{id})\|_\wedge \|p_\alpha\|}{\operatorname{tr} p_\alpha} \le \varepsilon_\alpha := 1/\operatorname{rk} e_\alpha.$$

This is possible because for every finite-dimensional subspace E of X there exists a projection onto it of norm no greater than $\sqrt{\dim E}$ (see [Pi1, Theorem 1.14]). Also note that $\lim_\alpha \varepsilon_\alpha = 0$. Choose γ $(= \gamma(\alpha))$ 'greater' than α such that

$$\left| 1 - \frac{\operatorname{tr}(p_\alpha e_\gamma)}{\operatorname{tr} p_\alpha} \right| \le \varepsilon_\alpha,$$

and such that, for some $\lambda_\alpha \in \mathbb{C}$,

$$\|\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)p_\alpha - \lambda_\alpha p_\alpha\|_\wedge \le \varepsilon_\alpha,$$

where $f_\alpha = \mathrm{id} - e_\alpha$. That the latter is possible follows on noting that we have $wd_\gamma - d_\gamma w \to 0 \Rightarrow w\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ) - \pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)w \to 0$, for

$$\begin{aligned}\|w\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ) - \pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)w\| &= \|\pi(w(f_\alpha d_\gamma^\circ f_\alpha)^\circ - (f_\alpha d_\gamma^\circ f_\alpha)^\circ w)\| \\ &\le \|w(f_\alpha d_\gamma^\circ f_\alpha)^\circ - (f_\alpha d_\gamma^\circ f_\alpha)^\circ w\|_\wedge \\ &\le \|f_\alpha\|^2 \|wd_\gamma - d_\gamma w\|_\wedge.\end{aligned}$$

Therefore, we can get $p_\alpha\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)p_\alpha$ as close to the center of $\mathcal{A}_\alpha := p_\alpha\mathcal{A}p_\alpha$ (i.e., as close to a multiple of p_α) as we wish. Since

$$p_\alpha\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)p_\alpha - \pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)p_\alpha = (p_\alpha\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ) - \pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)p_\alpha)p_\alpha$$

and $\mathcal{A}_\alpha$ is finite-dimensional, the same holds for $\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)p_\alpha$.

For each α, define

$$\delta_\alpha := (f_\alpha d_\gamma^\circ f_\alpha)^\circ + f_\alpha d_\gamma^\circ f_\alpha.$$

We show next that the net (δ_α) is a (bounded) symmetric approximate diagonal for $\mathcal{A}$. That $\delta_\alpha^\circ = \delta_\alpha$ for every α is easy to check. To see that $w\delta_\alpha - \delta_\alpha w \underset{\alpha}{\to} 0$ $(w \in \mathcal{A})$, simply note that

$$\begin{aligned}\|w\delta_\alpha - \delta_\alpha w\|_\wedge &\le \|w(f_\alpha d_\gamma^\circ f_\alpha)^\circ - (f_\alpha d_\gamma^\circ f_\alpha)^\circ w\|_\wedge + \|wf_\alpha d_\gamma^\circ f_\alpha - f_\alpha d_\gamma^\circ f_\alpha w\|_\wedge \\ &\le \|f_\alpha\|^2\|wd_\gamma - d_\gamma w\|_\wedge + \|f_\alpha\|\,\|d_\gamma\|_\wedge(\|wf_\alpha\| + \|f_\alpha w\|),\end{aligned}$$

and the last expression tends to zero with α for every $w \in \mathcal{A}$.

It remains to show that $(\pi(\delta_\alpha))$ is a b.a.i. for $\mathcal{A}$. For this, first note that

$$\begin{aligned}\operatorname{tr}\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ p_\alpha) &= \operatorname{tr}\pi(f_\alpha(p_\alpha d_\gamma)^\circ f_\alpha) = \operatorname{tr}(f_\alpha^2\pi((p_\alpha d_\gamma)^\circ)) \\ &= \operatorname{tr}(p_\alpha e_\gamma) + \operatorname{tr}(e_\alpha(e_\alpha - 2\,\mathrm{id})\pi((p_\alpha d_\gamma)^\circ)),\end{aligned}$$

so we have

$$\begin{aligned}(1-\lambda_\alpha)\operatorname{tr}p_\alpha = [\operatorname{tr}p_\alpha - \operatorname{tr}(p_\alpha e_\gamma)] + [\operatorname{tr}\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ p_\alpha) - \lambda_\alpha \operatorname{tr}p_\alpha] \\ - \operatorname{tr}(e_\alpha(e_\alpha - 2\,\mathrm{id})\pi((p_\alpha d_\gamma)^\circ)),\end{aligned}$$

and in turn,

$$|1 - \lambda_\alpha| \le \varepsilon_\alpha(2 + \|d_\gamma\|_\wedge).$$

For every $w \in \mathcal{A}$, let $w_\alpha := e_\alpha w$. Then

$$\begin{aligned}\|\pi(\delta_\alpha)w_\alpha - w_\alpha\| &= \|\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)w_\alpha + f_\alpha\pi(d_\gamma^\circ)f_\alpha w_\alpha - w_\alpha\| \\ &\le \|\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)w_\alpha - w_\alpha\| + \|f_\alpha\|\,\|d_\gamma\|_\wedge\|f_\alpha w_\alpha\|,\end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
(3.2)\qquad \|\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)w_\alpha - w_\alpha\| &\\
&\le \|\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)p_\alpha w_\alpha - \lambda_\alpha p_\alpha w_\alpha\| + \|\lambda_\alpha w_\alpha - w_\alpha\| \\
&\le \|\pi((f_\alpha d_\gamma^\circ f_\alpha)^\circ)p_\alpha - \lambda_\alpha p_\alpha\|\,\|w_\alpha\| + |1-\lambda_\alpha|\,\|w_\alpha\| \\
&\le \varepsilon_\alpha(3 + \|d_\gamma\|_\wedge)\|w_\alpha\|.
\end{aligned}
$$

Combining (3.1) and (3.2) we readily see that $\|\pi(\delta_\alpha)w_\alpha - w_\alpha\| \underset{\alpha}{\to} 0$.

To finish the proof simply note that

$$
\begin{aligned}
\|\pi(\delta_\alpha)w - w\| &\le \|\pi(\delta_\alpha)w - \pi(\delta_\alpha)w_\alpha\| + \|\pi(\delta_\alpha)w_\alpha - w_\alpha\| + \|w_\alpha - w\| \\
&\le (1 + 2\|f_\alpha\|^2\|d_\gamma\|_\wedge)\|w_\alpha - w\| + \|\pi(\delta_\alpha)w_\alpha - w_\alpha\|,
\end{aligned}
$$

and the last expression converges to 0 with α. ∎

Thus, it will make no difference whatsoever, while working with Banach algebras of approximable operators, to assume that the algebra is symmetrically amenable instead of just amenable. However, this does not mean that, in the case of π-spaces, we shall be able to find an approximate diagonal whose elements are simultaneously generalized diagonals and symmetric tensors. Indeed, our proof of Theorem 3.1 suggests that, in general, one might need to give up the property of each $\pi(d_\alpha)$ being a projection in order to symmetrize the terms of the approximate diagonal. In what follows, the existence of an approximate diagonal whose terms are generalized diagonals will be a far more useful property to us, and so we shall not be concerned again with the symmetric aspect.

4. 1-Amenability (the finite case). The results of this section are particular cases of those to be discussed in the next section. However, the proofs in the finite-dimensional case are rather simple and therefore more transparent. We expect, by doing things in this way, to ease the reading of the next section, where fairly often, due to technicalities proper to the infinite-dimensional setting, the arguments will be long and involved. In addition, it seems worth pointing out that the answer to the question of whether 1-amenability implies property $\mathbb{A}$ turns out to be 'almost' affirmative in the finite-dimensional case, as we shall see in Theorem 4.4 and Corollary 4.5 below.

Our first lemma is well known. Recall first that, given a convex subset C of a vector space, a point $x \in C$ is said to be an *extreme point* of C if whenever $x \in [x_1, x_2] \subset C$ either $x = x_1$ or $x = x_2$.

LEMMA 4.1. *Let x be an extreme point of a convex subset C of a Banach space. If (x_i) and (λ_i) are sequences (possibly finite) in C and $\mathbb{R}_+$, respectively, such that $x = \sum_i \lambda_i x_i$ and $\sum_i \lambda_i = 1$ then $x_j = x$ $(j \in \mathbb{N})$.*

Clearly, in the finite-dimensional situation, the net (p_α) figuring in the definition of property $\mathbb{A}$ can be replaced by the identity map, and (i) and (ii) hold trivially. So only the third condition is of interest in this case and our next two lemmas deal essentially with it.

LEMMA 4.2. *Let $\mathcal{A}$ be a Banach algebra and let $e \in \mathcal{A}$ be a non-zero idempotent. If $\sum_i u_i \otimes v_i \in e\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}e$ is such that $\sum_i \|u_i\| \, \|v_i\| = 1$ and $\sum_i u_i v_i = e$ then $u_i v_i = \|u_i\| \, \|v_i\| e$ for every i.*

Proof. First note that

$$1 = \sum_i \|u_i\| \, \|v_i\| \geq \sum_i \|u_i v_i\| \geq \|e\| \geq 1,$$

so $\sum_i \|u_i v_i\| = 1$ and $\|u_i\| \, \|v_i\| = \|u_i v_i\|$ for every i. Next note that e is the identity of $e\mathcal{A}e$, so it must be an extreme point of its unit ball (see Section 3 of [Lu]). Since $\sum_i \|u_i v_i\| (\|u_i v_i\|^{-1} u_i v_i) = e$, we must have, by the previous lemma, $u_i v_i = \|u_i v_i\| e = \|u_i\| \, \|v_i\| e$ for every i. ∎

LEMMA 4.3. *Let $\mathcal{A}$ be a finite-dimensional Banach algebra, let $e \in \mathcal{A}$ be an idempotent and let $\sum_i u_i \otimes v_i \in e\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}e$ be such that $\sum_i v_i u_i$ is invertible and $\sum_i a u_i \otimes v_i = \sum_i u_i \otimes v_i a$ $(a \in e\mathcal{A}e)$. Then $\operatorname{sp}\{u_i\} = e\mathcal{A}$ and $\operatorname{sp}\{v_i\} = \mathcal{A}e$.*

Proof. Let $\sum_i u_i \otimes v_i$ be as in the hypotheses. Without loss of generality, suppose $v_1, \ldots, v_n$ form a basis for $\operatorname{sp}\{v_i\}$, so $v_i = \alpha_{i,1} v_1 + \cdots + \alpha_{i,n} v_n$ for every $i > n$. Then $\sum_i u_i \otimes v_i = \sum_{i=1}^n (u_i + \sum_{k>n} \alpha_{k,i} u_k) \otimes v_i$. Let $\widetilde{u}_i := u_i + \sum_{k>n} \alpha_{k,i} u_k$ $(1 \leq i \leq n)$, and let $\mathcal{J} = \operatorname{sp}\{\widetilde{u}_i\}$. Clearly, $\mathcal{J} \subseteq \operatorname{sp}\{u_i\} \subseteq e\mathcal{A}$. Next note that, for every $a \in e\mathcal{A}e$ and every $1 \leq i \leq n$, $a\widetilde{u}_i \in \mathcal{J}$. Indeed, suppose otherwise, that there are $1 \leq i_0 \leq n$ and $a \in \mathcal{A}$ such that $a\widetilde{u}_{i_0} \notin \mathcal{J}$. Choose $\phi \in (e\mathcal{A})'$ and $\psi \in (\mathcal{A}e)'$ such that $\phi(\mathcal{J}) = \{0\}$, $\phi(a\widetilde{u}_{i_0}) = 1$ and $\psi(v_i) = \delta_{ii_0}$ $(1 \leq i \leq n)$. Then, applying the linear functional $\phi \otimes \psi$ to both sides of the identity $\sum_i a\widetilde{u}_i \otimes v_i = \sum_i \widetilde{u}_i \otimes v_i a$, we reach the absurd $1 = 0$.

Let u be the invertible element $\sum_i v_i u_i$. Then, for every $a \in \mathcal{A}$, we find, using the result from the previous paragraph, that

$$ea = (eau^{-1})u = \sum_i (eau^{-1}) v_i u_i = \sum_{i=1}^n (eau^{-1}) v_i \widetilde{u}_i \in \mathcal{J},$$

i.e., $e\mathcal{A} \subseteq \mathcal{J}$. It follows that $\operatorname{sp}\{u_i\} = e\mathcal{A}$, as claimed.

That $\operatorname{sp}\{v_i\} = \mathcal{A}e$ is proved analogously. ∎

Notice that the assumption that $\sum_i v_i u_i$ is invertible cannot just be dropped from the hypotheses of the last lemma. For instance, if X is a finite-dimensional Banach space with basis $(x_i)_{i=1}^n$ then $\Delta := \sum_{k=1}^n (x_1^* \otimes x_k) \otimes (x_k^* \otimes x_1)$ is a diagonal for $\mathcal{A}(X)$, but neither $\operatorname{sp}\{x_k^* \otimes x_1 : 1 \leq k \leq n\} = \mathcal{A}(X)$ nor $\operatorname{sp}\{x_1^* \otimes x_k : 1 \leq k \leq n\} = \mathcal{A}(X)$.

Now the main result of the section reads as follows.

THEOREM 4.4. *A finite-dimensional Banach algebra $\mathcal{A}$ is 1-amenable if and only if there is a group G of invertible elements in $\mathcal{A}_{[1]}$ such that* $\mathrm{sp}(G) = \mathcal{A}$.

Proof. Of course, as $\mathcal{A}$ is finite-dimensional, amenability and contractibility of $\mathcal{A}$ are the same thing, so we can talk of a diagonal instead of an approximate one.

It is well known that if there is a group G of invertible elements in $\mathcal{A}_{[1]}$ whose linear span is the whole of $\mathcal{A}$ then $\mathcal{A}$ is 1-amenable. Simply replace G, if necessary, by its norm-closure $\overline{G}$ and take as a norm-1 diagonal the element $\mu(\overline{G})^{-1} \int_{\overline{G}} g \otimes g^{-1}\, d\mu(g)$, where μ denotes the Haar measure on the group $\overline{G}$ endowed with the topology induced by the norm.

In the opposite direction, if $\mathcal{A}$ is 1-amenable then there is a diagonal Δ for $\mathcal{A}$ of norm 1. Furthermore, as $\mathcal{A}$ is finite-dimensional, Δ admits a finite representation $\sum_{i=1}^{n} u_i \otimes v_i$ such that $\| \sum_i u_i \otimes v_i \|_\wedge = \sum_i \|u_i\| \, \|v_i\|$ (see [TJ, Proposition 8.6]). Let e be the identity of $\mathcal{A}$. Then, by Lemma 4.2, $u_i v_i = \|u_i\| \, \|v_i\| e$ $(1 \le i \le n)$. It follows from this, combined with the finite-dimensionality of $\mathcal{A}$, that $\|v_i\|^{-1} v_i = (\|u_i\|^{-1} u_i)^{-1}$ $(1 \le i \le n)$, so also $\sum_i v_i u_i = e$. The group $G = \langle \|u_i\|^{-1} u_i : 1 \le i \le n \rangle$ is clearly contained in $\mathcal{A}_{[1]}$ and, by Lemma 4.3, its linear span must be the whole of $\mathcal{A}$. ∎

We do not know whether Theorem 4.4 remains true if one replaces in it 1 by a larger constant. We think this is very unlikely to happen but we do not have any examples. Clearly, the answer to this question would be positive whenever there is a renorming $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ with amenability constant 1.

Also, we are unable to conclude anything, in general, about the finiteness of the group G appearing in Theorem 4.4. In certain cases, though, this is possible, as our next result shows. Recall first that a *Hilbert* (or *Hilbertian*) *component* of a Banach space X with a 1-unconditional basis (x_i) is simply any maximal (in the sense of inclusion) linear subspace of the form $[x_i : i \in J]$, $J \subseteq \mathbb{N}$, which is isometrically isomorphic to a Hilbert space. (This definition of Hilbert component is not the usual one, but it is equivalent and it will suffice for our subsequent work. For the standard definition, which involves the concept of a Hermitian projection, we refer the reader to [KW, Sections 3–5].) Hilbert components of dimension greater than 1 are said to be *non-trivial*. It is known (see [KW, Theorem 5.3]) that the Hilbert components of a Banach space with a 1-unconditional basis form a (topological) direct sum decomposition of the space.

COROLLARY 4.5. *Let X be a finite-dimensional Banach space with a normalized 1-unconditional basis (x_i) and without non-trivial Hilbert com-*

ponents. Then $\mathcal{A}(X)$ has property $\mathbb{A}$ with constant 1 if and only if $\mathcal{A}(X)$ is 1-amenable.

Proof. In view of Theorem 4.4, all that is needed is to show that if $\mathcal{A}$ is 1-amenable then G can be chosen to be finite. Let $G = \langle \|u_i\|^{-1}u_i : 1 \le i \le n\rangle$ be the group constructed in the proof of Theorem 4.4, so each $\|u_i\|^{-1}u_i$ is a linear isometry. In the special case under consideration, all linear isometries $T : X \to X$ are of the form $T(\sum_i \alpha_i x_i) = \sum_i \lambda_i \alpha_i x_{\pi(i)}$, where π is a permutation of the index set and the λ_i's are scalars of modulus one (see [KW, Theorem 6.1]). Thus, each element $\|u_i\|^{-1}u_i$ can be written as the product of a permutation operator, which we shall denote by Π_i, and a diagonal linear isometry. Clearly, we must have $\|\Pi_i\| = 1$ $(1 \le i \le n)$. For every $\eta \in \{-1,1\}^{\dim X}$, let $U_\eta : X \to X$ be defined by $U_\eta(\sum_i \alpha_i x_i) := \sum_i \eta_i \alpha_i x_i$. It is easy to see that $G_0 := \langle U_\eta \Pi_i : 1 \le i \le n,\ \eta \in \{-1,1\}^{\dim X}\rangle$ is finite, that $G_0 \subset \mathcal{A}(X)_{[1]}$ and that $\operatorname{sp} G_0 = \mathcal{A}(X)$ (the latter follows on noting that, as $\operatorname{sp} G = \mathcal{A}(X)$, there is, for every pair $1 \le i, j \le \dim X$, an element $\|u_k\|^{-1}u_k \in G$ such that $x_i^*(u_k x_j) \neq 0$, and therefore the same must hold with respect to G_0). The rest is clear. ∎

The absence of non-trivial Hilbertian components is not a major requirement in Corollary 4.5, and can be dropped without affecting the rest of it. Indeed, by [ST, 2.3], if $X_1, \dots, X_m$ are the Hilbert components of X then there is a normed space Y with a 1-unconditional basis $(y_i)_{i=1}^m$ such that $\|(x_i)_{i=1}^m\|_X = \big\|\sum_{i=1}^m \|x_i\| y_i\big\|_Y$ $((x_i) \in \bigoplus_{i=1}^m X_i)$ and every isometry $T : X \to X$ is of the form $T((x_i)_{i=1}^m) = (U_i x_{\pi(i)})_{i=1}^m$, where π is a permutation and $U_1, \dots, U_m$ are unitary operators. If $\mathcal{A}(X)$ is 1-amenable then, combining these facts with the fact that the group G of the proof of Corollary 4.5 must span the whole of $\mathcal{A}(X)$, one finds that all Hilbert components must be of the same dimension, and also that, if necessary, the group G can be replaced by a finite one that still spans $\mathcal{A}(X)$. (For this last, one simply needs to note that every operator $T : X \to X$ of the form $T((x_i)_{i=1}^m) := (U_i x_i)_{i=1}^m$, where $U_1, \dots, U_m$ are unitaries, is an isometry.) Unfortunately, it is not the same in the infinite-dimensional case, where instead one needs to deal with non-surjective (almost) isometric maps. For this reason, in this note, we shall restrict our attention to Banach spaces all of whose Hilbert components are trivial.

5. 1-Amenability of $\mathcal{A}_0(X)$ when X has a 1-unconditional basis. We now extend the results from the previous section to the infinite-dimensional setting. From the characterization of amenability of $\mathcal{A}(X)$ for π-spaces, highlighted in Section 2, it follows easily that if X is a Banach space with a 1-unconditional basis (x_i) and $X_n := [x_i]_{i=1}^n$ $(n \in \mathbb{N})$ then $\mathcal{A}_0(X)$ is 1-amenable if and only if, for every $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and a generalized

diagonal $d_n \in \mathcal{F}(X_m, X_n) \mathbin{\widehat{\otimes}} \mathcal{F}(X_n, X_m)$ such that $\lim_n \|d_n\|_\wedge = 1$. Clearly, Lemmas 4.3 and 4.2 concern the more general situation of a generalized diagonal, but Lemma 4.3, which is central to the results from the previous section, assumes, in addition, that the norm of the generalized diagonal is one. We do not know whether the d_n's can always be chosen to have norm one, and consequently we need to modify the results from the previous section to take into account this new difficulty. It is apparent, though, from the proof of [BG2, Proposition 3.1] that, although finding d_n's of norm one might not always be possible, it is always possible (by choosing m large enough) to find d_n's with norms as close to 1 as we wish, and we shall take advantage of this fact.

Our first two lemmas are analogs of Lemmas 4.1 and 4.2, respectively. Recall that, given a closed convex subset C of a Banach space X, a point $x \in C$ is said to be an *exposed point* of C if there is a real-linear, real-valued, norm-one functional f on X such that $f(x) > f(c)$ $(c \in C \setminus \{x\})$.

LEMMA 5.1. *Let C be a compact convex subset of a Banach space X and suppose x is an exposed point of C. For every $\tau > 0$ there exists a constant $\varkappa_\tau$ such that whenever $(x_i) \subset C$ and $(\lambda_i) \subset \mathbb{R}_+$ are sequences (possibly finite) satisfying $x = \sum_i \lambda_i x_i$ and $\sum_i \lambda_i \geq 1$ one has*

$$\sum_{i:\, \|x - x_i\| > \tau} \lambda_i \leq \varkappa_\tau\Big(\sum_i \lambda_i - 1\Big).$$

Proof. Let $\tau > 0$ arbitrary and let x, (x_i) and (λ_i) be as in the hypotheses. Let $u_i = x - x_i$ $(i \in \mathbb{N})$ and let $I = \{i : \|u_i\| > \tau\}$. As x is an exposed point of C, there is a real-linear, real-valued, norm-one functional f on X such that $f(x) > f(c)$ $(c \in C \setminus \{x\})$. Compactness of C guarantees that $f(x) > \sup_{c \in C_\tau} f(c)$, where $C_\tau := \overline{C \setminus (x + \tau X_{[1]})}$. So,

$$\begin{aligned}\Big(\sum_i \lambda_i - 1\Big)\|x\| &= \Big\|\sum_i \lambda_i x - \sum_i \lambda_i x_i\Big\| = \Big\|\sum_i \lambda_i u_i\Big\| \\ &\geq f\Big(\sum_i \lambda_i u_i\Big) \geq \sum_{i \in I} \lambda_i f(u_i) \\ &\geq \Big(f(x) - \sup_{c \in C_\tau} f(c)\Big) \sum_{i \in I} \lambda_i.\end{aligned}$$

We can take $\varkappa_\tau = \|x\|/(f(x) - \sup_{c \in C_\tau} f(c))$. ∎

It is not hard to see that an exposed point is always an extreme point but, unfortunately, the converse is not true, and the conclusion of Lemma 5.1 does not seem to hold under the weaker assumption of x being an extreme point. Fortunately, for any finite-dimensional unital Banach algebra the identity element is an exposed point of its unit ball. (This seems to be known but we

have not found any reference. For completeness, a proof is given as part of the proof of our next lemma.)

LEMMA 5.2. *Let $\mathcal{A}$ be a Banach algebra and let $e \in \mathcal{A}$ be a non-zero idempotent such that $e\mathcal{A}e$ is finite-dimensional. Then for every $\tau > 0$ there exists a constant $\varkappa_\tau$ such that whenever $\sum_i u_i \otimes v_i \in e\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}e$ satisfies $\sum_i u_i v_i = e$ one has*

$$\sum_{i \in I_\tau} \|u_i\| \, \|v_i\| \le \varkappa_\tau\Big(\sum_i \|u_i\| \, \|v_i\| - 1\Big),$$

where $I_\tau = \big\{i : \|(\|u_i\|^{-1}u_i)(\|v_i\|^{-1}v_i) - e\| > \tau\big\}$.

Proof. Let C be the unit ball of $e\mathcal{A}e =: \mathfrak{A}$. Let us start by showing that e is an exposed point of C. It is well known ([BK, Theorem 2]) that the set of all states of $\mathfrak{A}$ is total. Choose a basis of states $f_1, \dots, f_n$ for $\mathfrak{A}'$, choose $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ such that $\sum_i \lambda_i = 1$, and define $f := \sum_i \lambda_i f_i$. Then

$$\operatorname{Re} f(e) = 1 > \operatorname{Re} f(a) \quad (a \in C \setminus \{e\}).$$

To see this, suppose $\operatorname{Re} f(a) = 1$ for some $a \in C$. Then $f(a) = 1$, so $0 = f(e-a) = \sum_i \lambda_i f_i(e-a)$. Since $\operatorname{Re} f_i(e-a) \ge 0$ $(1 \le i \le n)$, it follows from the last equality that $\operatorname{Re} f_i(e-a) = 0$ $(1 \le i \le n)$. In turn, using the fact that $f_i(e) = 1$ $(1 \le i \le n)$, we deduce that $f_i(e-a) = 0$ $(1 \le i \le n)$, and as the f_i's form a basis for $\mathfrak{A}'$, we must have $e = a$. (The same argument can be easily adapted to any unital Banach algebra whose set of states contains a countable, weak*-dense subset. One simply replaces the basis of states in the above argument by this countable subset. The rest of the argument, apart from the obvious modifications, remains the same.)

Now, let $x = e$ and, for every $\tau > 0$, define $\varkappa_\tau$ as in the previous lemma. If $\sum_i u_i \otimes v_i$ is such that $\sum_i u_i v_i = e$ then

$$\sum_i \|u_i\| \, \|v_i\| \ge \sum_i \|u_i v_i\| \ge \|e\| \ge 1$$

and

$$e = \sum_i \|u_i\| \|v_i\| (\|u_i\|^{-1}u_i)(\|v_i\|^{-1}v_i),$$

so $\sum_{i \in I_\tau} \|u_i\| \, \|v_i\| \le \varkappa_\tau(\sum_i \|u_i\| \, \|v_i\| - 1)$. ∎

Notice that the finite-dimensionality assumption of Lemma 5.2 is important. For instance, in $\mathcal{B}(\ell_2)$, the identity is an exposed point of the unit ball [Gr, Theorem 1], but it is not denting [GS, Theorem 1] (a point x of a subset C of a Banach space X is *denting* if, for every $\delta > 0$, $x \notin \overline{\operatorname{co}}(C \setminus \{x + \delta X_{[1]}\})$), and therefore a functional as in the proof of Lemma 5.1 cannot exist in this case.

As mentioned in the introduction, apart from 1-unconditionality, other assumptions on the basis will be needed for our approach to work. We dis-

cuss next the most significant of these restrictions. Recall first that a 1-unconditional basis (x_i) for a Banach space X is said to be *strictly monotone* if $\|x+y\| > \|x\|$ for every pair $x, y \in X_+$, where $X_+ := \{\sum_i \alpha_i x_i \in X : \alpha_i \geq 0\ (i \in \mathbb{N})\}$.

Now, let X and Y be finite-dimensional Banach spaces with 1-unconditional bases (x_i) and (y_j), respectively, and let $u : Y \to X$ and $v : X \to Y$ be linear maps such that $uv = \mathrm{id}_X$ and $\|u\|\,\|v\| = 1$. Then $v|^{\mathrm{rg}\, v}$ is a surjective isometry and vu is a norm-one projection onto $\mathrm{rg}\, v$. Suppose X has no non-trivial Hilbert components and the basis (y_j) is strictly monotone. First note that since X has no non-trivial Hilbert components, the same must hold for $\mathrm{rg}\, v$, and hence, as the sequence (vx_i) is a 1-unconditional basis for $\mathrm{rg}\, v$, each subspace $\langle vx_i \rangle$ must be a Hilbert component of $\mathrm{rg}\, v$. It is easy to see that $\mathrm{rg}\, v = \mathrm{rg}\, vu$, and so, by [Ra1, Theorem 3.1], the vx_i's must be disjointly supported with respect to (y_j). It was precisely a particular case of this result that allowed us to translate the existence of a norm-1 diagonal into property $\mathbb{A}$, in the proof of Corollary 4.5. Now, it is apparent from Lemma 5.2 that, in the present situation, we will not be dealing necessarily with isometries, and so we will also need an approximate version of this result. To this end, we first introduce the following 'uniform' version of the concept of a strictly monotone basis.

Definition 5.3. Let X be a Banach space with a normalized 1-unconditional basis (x_i), and let $\mu_X : [0,1] \to [0,1]$ be defined by

$$\mu_X(t) := \inf\{\|x+ty\| - 1 : x, y \in X_+ \text{ and } \|x\| = \|y\| = 1\}.$$

(It is easy to see that μ_X is non-decreasing and that $\lim_{t\to 0} \mu_X(t) = 0$.) We shall say that the basis (x_i) is *uniformly strictly monotone* (u.s.m. for short) if $\mu_X(t) > 0$ $(1 \geq t > 0)$.

Any normalized 1-unconditional basis in a *uniformly convex* Banach space X, i.e., such that $\delta_X(t) := \inf\{1 - \|(x+y)/2\| : \|x\| = \|y\| = 1 \text{ and } \|x-y\| \geq t\} > 0$ $(2 \geq t > 0)$, is uniformly strictly monotone. To see this, let X be uniformly convex and suppose towards a contradiction that $\mu_X(t) = 0$ for some $1 \geq t > 0$. For every $n \in \mathbb{N}$, choose $x_n, y_n \in X_+$ such that $\|x_n\| = \|y_n\| = 1$ and $\|x_n + ty_n\| - 1 \leq 1/n$, and set $\xi_n = (x_n + ty_n)/\|x_n + ty_n\|$. Then, for n large enough, we should have $\|\xi_n - x_n\| \geq t/2$, and hence $\|(x_n + \xi_n)/2\| \leq 1 - \delta_X(t/2)$, while on the other hand, $\liminf \|(x_n+\xi_n)/2\| = \liminf \|x_n + (t/2)y_n\| \geq 1$. Notice, though, that uniformly convex Banach spaces are not the only ones which can possibly have uniformly strictly monotone Schauder bases. For instance, $\mu_{\ell_1}(t) = t$ $(1 \geq t > 0)$ and ℓ_1 is not uniformly convex.

Before stating and proving our approximate version of the result discussed prior to Definition 5.3, we should recall a few things. First, the *nu-*

merical range of an operator T acting on a Banach space X is defined to be the set

$$\mathcal{V}(T) := \{f(Tx) : x \in X,\ f \in X',\ \|x\| = \|f\| = f(x) = 1\}$$

and its *numerical radius* is defined to be the number

$$|\mathcal{V}(T)| := \sup\{|z| : z \in \mathcal{V}(T)\}.$$

When X is a complex Banach space one has $\|T\| \le 4|\mathcal{V}(T)|$ for every $T \in \mathcal{B}(X)$ ([Lu, Theorem 5]). Recall also that a linear operator T such that $\mathcal{V}(T) \subset \mathbb{R}$ is said to be *Hermitian*. In a Banach space with a 1-unconditional basis and without non-trivial Hilbertian components, the Hermitian operators are precisely those whose representation with respect to the given 1-unconditional basis is diagonal (see [KW, Theorem 6.1]).

LEMMA 5.4. *Let X and Y be finite-dimensional Banach spaces with normalized 1-unconditional bases $(x_i)_{i=1}^m$ and $(y_i)_{i=1}^n$, respectively. Suppose X has no non-trivial Hilbertian components and (y_i) is u.s.m. Then for every $\varepsilon > 0$ there exists $\delta > 0$, independent of n, such that if $V : Y \to X$ and $W : X \to Y$ are norm-one operators satisfying $\|VW - \mathrm{id}_X\| \le \delta$ then there are pairwise disjoint subsets $E_1, \dots, E_m$ of $\{1, \dots, n\}$ such that*

$$\|Wx_i - E_i W x_i\| \le \varepsilon \quad (1 \le i \le m).$$

Proof. First note that, since X has no non-trivial Hilbert components, $x_i^*(Tx_j) = 0$ $(1 \le i \ne j \le m)$ for every Hermitian operator $T \in \mathcal{B}(X)$. Combining this fact with the finite-dimensionality of $\mathcal{B}(X)$, one easily deduces that for every $\eta > 0$ there is a $\bar{\delta}$ such that if $T \in \mathcal{B}(X)$ satisfies

$$\mathcal{V}(T) \subset [-1, 1] + \bar{\delta}\mathbb{C}_{[1]} \quad (=: G(\bar{\delta}))$$

then $|x_i^*(Tx_j)| < \eta$ $(1 \le i \ne j \le m)$. Indeed, otherwise, for some $\eta > 0$, there would be a sequence (T_n) in $\mathcal{B}(X)$ such that $\mathcal{V}(T_n) \subset G(1/n)$ yet $\max\{|x_i^*(T_n x_j)| : 1 \le i \ne j \le m\} \ge \eta$ $(n \in \mathbb{N})$. Since $\mathcal{B}(X)$ is finite-dimensional and (T_n) is bounded, one could assume, by passing to a subsequence if necessary, that (T_n) converges to some $T \in \mathcal{B}(X)$. Clearly, T would be a Hermitian operator, and therefore $\lim_n x_i^*(T_n x_j) = x_i^*(Tx_j) = 0$ $(1 \le i \ne j \le m)$, which is a contradiction.

Now, let $\varepsilon > 0$ be arbitrary. Choose $\bar{\varepsilon} > 0$ small enough, so that $\mu_Y(t) \le 8\bar{\varepsilon} \Rightarrow t \le \varepsilon/(2m)$; choose $0 < \bar{\delta} < 9\sqrt{\bar{\varepsilon}}$ so that for every $T \in \mathcal{B}(X)$ satisfying $\mathcal{V}(T) \subset G(\bar{\delta})$ one has $|x_i^*(Tx_j)| < \bar{\varepsilon}/(m+1)$ $(1 \le i \ne j \le m)$; and choose $0 < \delta < \min\{1/2, (\bar{\delta}/9)^2\}$. Let $V : Y \to X$ and $W : X \to Y$ be norm-one operators such that $\|VW - \mathrm{id}_X\| \le \delta$, let $\xi_i = Wx_i$ and $\xi_i^* = x_i^* V$ $(1 \le i \le m)$, and let $P = WV = \sum_{i=1}^m \xi_i^* \otimes \xi_i$.

Let $(x,f)\in X\times X'$ be such that $\|x\|=\|f\|=f(x)=1$. Then

$$\big|1-\|fV\|^{-1}fV(Wx)\big|\le\frac{\big|\|fV\|-1\big|}{\|fV\|}+\frac{|f(x)-fV(Wx)|}{\|fV\|}\le\frac{2\delta}{1-\delta}\le 4\delta,$$

and, by the Bishop–Phelps–Bollobás theorem ([BD, Section 16, Theorem 1]) applied to the pair $(Wx,\|fV\|^{-1}fV)\in Y\times Y'$, there is $(y,g)\in Y\times Y'$ such that $\|y\|=\|g\|=g(y)=1$, $\|Wx-y\|\le 4\sqrt{\delta}$ and $\big\|\,\|fV\|^{-1}fV-g\big\|\le 4\sqrt{\delta}$. Note the last implies that

$$\|fV-g\|\le\big\|fV-\|fV\|^{-1}fV\big\|+\big\|\,\|fV\|^{-1}fV-g\big\|\le\delta+4\sqrt{\delta}\le 5\sqrt{\delta}.$$

If D is a norm-one operator on Y then

$$\begin{aligned}|f(VDWx)-g(Dy)|&\le|f(VDWx)-g(DWx)|+|g(DWx)-g(Dy)|\\&\le\|fV-g\|+\|Wx-y\|\le 9\sqrt{\delta},\end{aligned}$$

and so if D is in addition Hermitian, then $\mathcal{V}(VDW)\subset G(9\sqrt{\delta})$. Thus, for every norm-one Hermitian $D\in\mathcal{B}(Y)$, $\mathcal{V}(VDW)\subset G(\bar{\delta})$, and in turn, by our choice of $\bar{\delta}$,

$$|\xi_i^*(D\xi_j)|=|x_i^*(VDWx_j)|<\frac{\bar{\varepsilon}}{m+1}\qquad(1\le i\ne j\le m).$$

Now, let $1\le k\ne l\le m$, $\xi_k=\sum_i\alpha_i y_i$ and $\xi_l=\sum_i\beta_i y_i$. Define $\xi:=\sum_i\gamma_i y_i$, where $\gamma_i:=\alpha_i|\beta_i|/(2|\alpha_i|)$ if $|\alpha_i|>|\beta_i|$ and $\gamma_i:=\alpha_i/2$ otherwise. Note that for $E=\{1\le i\le n:|\alpha_i|>|\beta_i|\}$ one has

$$\|\xi_k-E\xi_k\|\le 2\|\xi\|\quad\text{and}\quad\|E\xi_l\|\le 2\|\xi\|. \tag{5.1}$$

Let D_k, D_{l1} and D_{l2} be Hermitian maps in $\mathcal{B}(Y)$ of norm ≤ 1 and such that $D_k\xi_k=\xi=(D_{l1}+iD_{l2})\xi_l$. Then

$$\begin{aligned}\|P\xi\|&=\Big\|\sum_{i=1}^m\xi_i^*(\xi)\xi_i\Big\|\\&\le\Big\|\sum_{i\ne k}\xi_i^*(D_k\xi_k)\xi_i+(\xi_k^*(D_{l1}\xi_l)+i\xi_k^*(D_{l2}\xi_l))\xi_k\Big\|\\&\le\sum_{i\ne k}|\xi_i^*(D_k\xi_k)|\,\|\xi_i\|+(|\xi_k^*(D_{l1}\xi_l)|+|\xi_k^*(D_{l2}\xi_l)|)\|\xi_k\|\le\bar{\varepsilon}.\end{aligned}$$

It follows from this and the definition of μ_Y that

$$\begin{aligned}\left(1+\mu_Y\left(\frac{\|\xi\|}{\|\xi_k-\xi\|}\right)\right)\|\xi_k-\xi\|&\le\|\xi_k\|\le\|P\xi_k\|+\delta\\&<\|P(\xi_k-\xi)+P\xi\|+\bar{\varepsilon}\le\|\xi_k-\xi\|+2\bar{\varepsilon}.\end{aligned}$$

Thus,

$$\mu_Y\left(\frac{\|\xi\|}{\|\xi_k-\xi\|}\right)\le\frac{2\bar{\varepsilon}}{\|\xi_k-\xi\|}\le 8\bar{\varepsilon},$$

where we have used that $2\|\xi_k - \xi\| \geq \|\xi_k\| \geq 1 - \delta \geq 1/2$. By our choice of $\bar{\varepsilon}$,

$$\|\xi\| \leq \frac{\varepsilon}{2m}\|\xi_k - \xi\| \leq \frac{\varepsilon}{2m}\|\xi_k\| \leq \frac{\varepsilon}{2m},$$

and combining this last with (5.1), we arrive at

$$\|\xi_k - E\xi_k\| \leq \frac{\varepsilon}{m} \quad \text{and} \quad \|E\xi_l\| \leq \frac{\varepsilon}{m}.$$

Now, for each pair $1 \leq i < j \leq m$, choose a subset $E_{i,j}$ of $\{1, \ldots, n\}$ with

$$\|\xi_i - E_{i,j}\xi_i\| \leq \frac{\varepsilon}{m} \quad \text{and} \quad \|E_{i,j}\xi_j\| \leq \frac{\varepsilon}{m}$$

(which is possible, by our previous result). Define $E_{i,j} := E_{j,i}^c$ for $i > j$, and for $1 \leq i \leq m$, define $E_i := \bigcap_{j \neq i} E_{i,j}$. Then

$$\|\xi_i - E_i\xi_i\| = \|E_i^c\xi_i\| = \Big\|\Big(\bigcup_j E_{i,j}^c\Big)\xi_i\Big\| \leq \sum_j \|E_{i,j}^c\xi_i\| < \varepsilon \quad (1 \leq i \leq m),$$

where the superscript 'c' stands for complement. Moreover,

$$E_k \cap E_l = \Big(\bigcap_{j \neq k} E_{k,j}\Big) \cap \Big(\bigcap_{j \neq l} E_{l,j}\Big) \subset E_{k,l} \cap E_{l,k} = \emptyset \quad (1 \leq k \neq l \leq m),$$

so, in addition, the sets $E_1, \ldots, E_m$ are pairwise disjoint, as needed. This concludes the proof of the lemma. ∎

Recall that a *biorthogonal system* $\{x_i, f_i\}_{i=1}^n$ for a Banach space X is a pair of sequences $(x_i)_{i=1}^n \subset X$ and $(f_i)_{i=1}^n \subset X'$ such that $f_i(x_j) = \delta_{i,j}$ $(1 \leq i, j \leq n)$. We shall say that the biorthogonal system $\{y_i, g_i\}_{i=1}^n$ is *normalized* if $(y_i)_{i=1}^n$ and $(g_i)_{i=1}^n$ are normalized, that it is *C-complemented* if $\|\sum_i g_i \otimes y_i\| \leq C$, and that it is *$C$-equivalent* to a biorthogonal system $\{x_i, f_i\}_{i=1}^n$ if, for every scalar sequence $(\alpha_i)_{i=1}^n$,

$$\Big\|\sum_i \alpha_i y_i\Big\| \leq C\Big\|\sum_i \alpha_i x_i\Big\| \quad \text{and} \quad \Big\|\sum_i \alpha_i g_i|_{\mathcal{Y}}\Big\| \leq C\Big\|\sum_i \alpha_i f_i|_{\mathcal{X}}\Big\|,$$

where $\mathcal{X} = [x_i]_{i=1}^n$ and $\mathcal{Y} = [y_i]_{i=1}^n$. Note that two biorthogonal systems $\{x_i, f_i\}_{i=1}^n$ and $\{y_i, g_i\}_{i=1}^n$ are C-equivalent if and only if, for every scalar sequence $(\alpha_i)_{i=1}^n$,

$$\frac{1}{C}\Big\|\sum_i \alpha_i x_i\Big\| \leq \Big\|\sum_i \alpha_i y_i\Big\| \leq C\Big\|\sum_i \alpha_i x_i\Big\|.$$

To see this, simply note that if $T : \mathcal{X}' \to \mathcal{Y}'$ and $S : \mathcal{Y} \to \mathcal{X}$ are the linear maps defined by $Tf_i|_{\mathcal{X}} := g_i|_{\mathcal{Y}}$ $(1 \leq i \leq n)$ and $Sy_i := x_i$ $(1 \leq i \leq n)$, respectively, then $T^* = S$, so $\|S\| = \|T\|$. Thus, C-equivalence of biorthogonal systems is a symmetric relation.

We shall call a biorthogonal system $\{y_i, g_i\}_{i=1}^n$ in a Banach space X with Schauder basis (x_i) a *block biorthogonal system* (b.b.s. for short) if the y_i's are disjointly supported with respect to (x_i), each g_i is a support functional

for y_i and $\operatorname{supp} g_i = \operatorname{supp} y_i$ $(1 \le i \le n)$. (Note that we do not require $\operatorname{supp} y_1 < \cdots < \operatorname{supp} y_n$.)

With this, we are at last ready to state and prove our main result.

THEOREM 5.5. *Let X be a Banach space with a u.s.m. basis (x_i) and without non-trivial Hilbert components. Then $\mathcal{A}_0(X)$ is 1-amenable if and only if, given an arbitrary null sequence $(\varepsilon_n) \subset \mathbb{R}_+$, there are, for every $n \in \mathbb{N}$, normalized block biorthogonal systems $\{x_{j,1}, f_{j,1}\}_{j=1}^n$, $\{x_{j,2}, f_{j,2}\}_{j=1}^n$, $\ldots, \{x_{j,N_n}, f_{j,N_n}\}_{j=1}^n$, $(1+\varepsilon_n)$-complemented and $(1+\varepsilon_n)$-equivalent to $\{x_j, x_j^*\}_{j=1}^n$, and positive scalars $\lambda_1, \ldots, \lambda_{N_n}$ satisfying $\sum_i \lambda_i = 1$, such that*

$$\sum_k \Big| \sum_{i=1}^{N_n} \lambda_i f_{1,i}(x_k)\, x_k^*(x_{1,i}) - \sum_{i=1}^{N_n} \lambda_i f_{j,i}(x_k)\, x_k^*(x_{j,i}) \Big| \le \varepsilon_n \tag{5.2}$$

for every $1 < j \le n$.

Proof. Suppose first that (x_i) is a basis satisfying the conditions of the theorem, and let (ε_n) be a non-negative sequence such that $\lim_n n\varepsilon_n = 0$. For every $n \in \mathbb{N}$, define

$$d_n := \frac{1}{2^{n+K_n}} \sum_{\xi \in \{-1,1\}^n} \sum_{\eta \in \{-1,1\}^{K_n}} \sum_{i=1}^{N_n} \lambda_i\, U_\xi R_i U_\eta \otimes U_\eta S_i U_\xi,$$

where

$$K_n := \max \bigcup_{\substack{1 \le j \le n \\ 1 \le i \le N_n}} \operatorname{supp} x_{j,i}, \qquad U_\xi := \sum_{j=1}^n \xi_j x_j^* \otimes x_j, \qquad U_\eta := \sum_{k=1}^{K_n} \eta_k x_k^* \otimes x_k,$$

$$R_i := \sum_{j=1}^n f_{j,i} \otimes x_j \quad \text{and} \quad S_i := \sum_{j=1}^n x_j^* \otimes x_{j,i} \qquad (1 \le i \le N_n),$$

and the $x_{j,i}$'s and $f_{j,i}$'s have been chosen to satisfy the hypotheses of the theorem with respect to the sequence (ε_n). We show next that the sequence (d_n) is an approximate diagonal for $\mathcal{A}_0(X)$ such that $\|d_n\|_\wedge \le (1+\varepsilon_n)^3$ $(n \in \mathbb{N})$.

It follows easily from our definition of d_n that

$$\pi(d_n) = \sum_{j=1}^n x_j^* \otimes x_j =: P_n.$$

Let

$$\widetilde{d}_n := \sum_{\substack{1 \le j \le n \\ 1 \le k \le K_n}} \Big(\sum_{i=1}^{N_n} \lambda_i f_{1,i}(x_k)\, x_k^*(x_{1,i}) \Big) (x_k^* \otimes x_j) \otimes (x_j^* \otimes x_k) \quad (n \in \mathbb{N}),$$

so $\widetilde{d}_n$ is a generalized diagonal for $\mathcal{F}(X_n)$. Then note that

$$\begin{aligned}
d_n &= \frac{1}{2^{n+K_n}} \sum_{\xi} \sum_{\eta} \sum_{i=1}^{N_n} \lambda_i \Big(\sum_{h=1}^{n} f_{h,i} U_\eta \otimes \xi_h x_h \Big) \otimes \Big(\sum_{j=1}^{n} \xi_j x_j^* \otimes U_\eta x_{j,i} \Big) \\
&= \frac{1}{2^{K_n}} \sum_{\eta} \sum_{i=1}^{N_n} \lambda_i \sum_{1 \le h,j \le n} \Big(\frac{1}{2^n} \sum_{\xi} \xi_j \xi_h \Big) (f_{h,i} U_\eta \otimes x_h) \otimes (x_j^* \otimes U_\eta x_{j,i}) \\
&= \sum_{i=1}^{N_n} \lambda_i \sum_{1 \le j \le n} \sum_{1 \le k,l \le K_n} \left(\frac{1}{2^{K_n}} \sum_{\eta} \eta_k \eta_l \right) (f_{j,i}(x_k) x_k^* \otimes x_j) \otimes (x_j^* \otimes x_l^*(x_{j,i}) x_l) \\
&= \sum_{\substack{1 \le j \le n \\ 1 \le k \le K_n}} \Big(\sum_{i=1}^{N_n} \lambda_i f_{j,i}(x_k) x_k^*(x_{j,i}) \Big) (x_k^* \otimes x_j) \otimes (x_j^* \otimes x_k).
\end{aligned}$$

By (5.2), $\|\widetilde{d}_n - d_n\|_\wedge \le n\varepsilon_n$, and so, for every $W \in \mathcal{F}(X_n)$,

$$\|W d_n - d_n W\|_\wedge \le \|W\| \, \|d_n - \widetilde{d}_n\|_\wedge + \|\widetilde{d}_n - d_n\|_\wedge \|W\| \le 2n\varepsilon_n \|W\|.$$

Since $\mathcal{A}_0(X)$ is the inductive limit of the algebras $\mathcal{F}(X_n)$ we must have $P_n W P_n \to W$ $(W \in \mathcal{A}_0(X))$. Combining this fact with the previous inequality one easily concludes that $W d_n - d_n W \to 0$ $(W \in \mathcal{A}_0(X))$.

Lastly, as $\{x_{j,i}, f_{j,i}\}_{j=1}^n$ is $(1+\varepsilon_n)$-complemented and $(1+\varepsilon_n)$-equivalent to $\{x_j, x_j^*\}_{j=1}^n$, one has

$$\begin{aligned}
\|R_i(x)\| &= \Big\| \sum_j f_{j,i}(x) x_j \Big\| \\
&\le (1+\varepsilon_n) \Big\| \sum_j f_{j,i}(x) x_{j,i} \Big\| \le (1+\varepsilon_n)^2 \|x\| \quad (x \in X),
\end{aligned}$$

and also

$$\begin{aligned}
\|S_i(x)\| &= \Big\| \sum_j x_j^*(x) x_{j,i} \Big\| \\
&\le (1+\varepsilon_n) \Big\| \sum_j x_j^*(x) x_j \Big\| \le (1+\varepsilon_n) \|x\| \quad (x \in X),
\end{aligned}$$

so

$$\|d_n\|_\wedge = \sum_i \lambda_i \|R_i\| \, \|S_i\| \le \max_i \|R_i\| \max_i \|S_i\| \le (1+\varepsilon_n)^3.$$

Thus, if (x_i) satisfies the conditions of the theorem then $\mathcal{A}_0(X)$ is 1-amenable.

Conversely, suppose $\mathcal{A}_0(X)$ is 1-amenable and let (ε_n) be a positive sequence bounded by 1 and converging to zero. (Note that the general case, in which (ε_n) is an arbitrary positive sequence converging to zero, follows im-

mediately from this one.) For each ε_n, choose $0 < \tau_n < \varepsilon_n^2/(30n)^2$ such that for every pair of norm-one linear maps $V : X_n \to X_m$ and $W : X_m \to X_n$ satisfying $\|VW - P_n\| \le \tau_n$ there are disjoint subsets $E_1, \dots, E_n$ of $\{1, \dots, m\}$ such that $\|Wx_j - E_jWx_j\| \le \varepsilon_n^2/(30n)^2$ $(1 \le j \le n)$, which is possible by Lemma 5.4 and the uniform strict monotonicity of the basis (x_i). Next, let $\varkappa(\tau_n)$ be as in Lemma 5.2, and choose $\delta_n > 0$ small enough so that $8(1 + 2\varkappa(\tau_n))\delta_n \le \varepsilon_n$. Lastly, choose a bounded approximate diagonal $(d_n)_{n\in\mathbb{N}}$ for $\mathcal{A}_0(X)$ such that

(i) $d_n \in \mathcal{F}(X_{K_n}, X_n) \mathbin{\widehat{\otimes}} \mathcal{F}(X_n, X_{K_n})$ for some $K_n \in \mathbb{N}$;
(ii) $\pi(d_n) = P_n$;
(iii) $Wd_n = d_nW$ $(W \in P_n\mathcal{A}(X)P_n)$; and
(iv) $\|d_n\|_\wedge \le 1 + \delta_n$.

As indicated in Section 2, the proof that such an approximate diagonal exists is exactly the same as that of [BG2, Proposition 3.1].

Now, fix n. As in the proof of Theorem 4.4, the finite-dimensionality of X_n and X_{K_n} implies that d_n admits a finite representation $\sum_{i=1}^{M_n} R_i \otimes S_i$ such that $\sum_{i=1}^{M_n} R_iS_i = P_n$ and $\sum_{i=1}^{M_n} \|R_i\|\,\|S_i\| \le 1 + \delta_n$. We shall assume that $\|R_i\|\,\|S_i\| \ne 0$ $(1 \le i \le M_n)$. By Lemma 5.2,

$$\sum_{i\in I_{\tau_n}} \|R_i\|\,\|S_i\| \le \varkappa(\tau_n)\delta_n, \tag{5.3}$$

where

$$I_{\tau_n} = \{i : \big\|(\|R_i\|^{-1}R_i)(\|S_i\|^{-1}S_i) - P_n\big\| > \tau_n\}.$$

For each $1 \le i \le M_n$, let $V_i = \|R_i\|^{-1}R_i$ and $W_i = \|S_i\|^{-1}S_i$. Note, in particular, that $\|V_iW_i - P_n\| \le \tau_n$ $(i \in I_{\tau_n}^c)$. Hence, by our choice of τ_n, for every $i \in I_{\tau_n}^c$, there exist disjoint subsets $E_{1,i}, \dots, E_{n,i}$ of $\{1, \dots, K_n\}$ such that $\|W_ix_j - E_{j,i}W_ix_j\| \le \varepsilon_n^2/(30n)^2$ $(1 \le j \le n)$. Let $y_{j,i} := E_{j,i}W_ix_j$ and $g_{j,i} := x_j^*V_iE_{j,i}$ $(i \in I_{\tau_n}^c,\ 1 \le j \le n)$. Then

$$\begin{aligned}|1 - g_{j,i}(y_{j,i})| &\le |x_j^*(x_j) - x_j^*V_i(W_ix_j)| + |x_j^*V_i(W_ix_j - y_{j,i})| \\ &\le \tau_n + \left(\frac{\varepsilon_n}{30n}\right)^2 \le 2\left(\frac{\varepsilon_n}{30n}\right)^2,\end{aligned}$$

and

$$\begin{aligned}&\big|1 - \|g_{j,i}\|^{-1}g_{j,i}(y_{j,i})\big| \\ &\quad\le |1 - g_{j,i}(y_{j,i})| + \frac{\big|1 - \|g_{j,i}\|\big|}{\|g_{j,i}\|} \le |1 - g_{j,i}(y_{j,i})| + \frac{|1 - g_{j,i}(y_{j,i})|}{1 - |1 - g_{j,i}(y_{j,i})|} \\ &\quad\le 2\left(\frac{\varepsilon_n}{30n}\right)^2 + \frac{2\left(\frac{\varepsilon_n}{30n}\right)^2}{1 - 2\left(\frac{\varepsilon_n}{30n}\right)^2} \le \left(\frac{\varepsilon_n}{15n}\right)^2 \frac{1 - \frac{1}{900}}{1 - \frac{2}{900}} \le \left(\frac{\varepsilon_n}{14n}\right)^2.\end{aligned}$$

By the Bishop–Phelps–Bollobás theorem ([BD, Section 16, Theorem 1]), there are pairs $(x_{j,i}, f_{j,i}) \in [x_j : j \in E_{j,i}] \times [x_j^* : j \in E_{j,i}]$ $(i \in I_{\tau_n}^c,\ 1 \le j \le n)$ such that $\|x_{j,i}\| = \|f_{j,i}\| = f_{j,i}(x_{j,i}) = 1$, $\|y_{j,i} - x_{j,i}\| \le \varepsilon_n/(7n)$ and $\big\| \|g_{j,i}\|^{-1} g_{j,i} - f_{j,i}\big\| \le \varepsilon_n/(7n)$. Note that

$$\begin{aligned}\|f_{j,i} - g_{j,i}\| &\le \big\|g_{j,i} - \|g_{j,i}\|^{-1} g_{j,i}\big\| + \frac{\varepsilon_n}{7n} \\ &\le |1 - g_{j,i}(y_{j,i})| + \frac{\varepsilon_n}{7n} \le 2\Big(\frac{\varepsilon_n}{30n}\Big)^2 + \frac{\varepsilon_n}{7n}.\end{aligned}$$

Then, for every scalar sequence $\alpha_1, \dots, \alpha_n$, we have

$$\begin{aligned}\Big\| \sum_j \alpha_j x_{j,i}\Big\| &\le \sum_j |\alpha_j| \, \|x_{j,i} - y_{j,i}\| + \sum_j |\alpha_j| \, \|y_{j,i} - W_i x_j\| + \Big\| \sum_j \alpha_j W_i x_j\Big\| \\ &\le \left(\frac{\varepsilon_n}{7n} + \left(\frac{\varepsilon_n}{30n}\right)^2\right) \sum_j |\alpha_j| + \Big\| \sum_j \alpha_j x_j\Big\| \\ &\le \left(1 + \frac{\varepsilon_n}{7} + \frac{\varepsilon_n^2}{900n}\right) \Big\| \sum_j \alpha_j x_j\Big\|,\end{aligned}$$

and

$$\begin{aligned}(1 - \tau_n)&\Big\| \sum_j \alpha_j x_j\Big\| \\ &\le \Big\| V_i W_i \Big(\sum_j \alpha_j x_j\Big)\Big\| \le \Big\| \sum_j \alpha_j W_i x_j\Big\| \\ &\le \sum_j |\alpha_j| \, \|W_i x_j - y_{j,i}\| + \sum_j |\alpha_j| \, \|y_{j,i} - x_{j,i}\| + \Big\| \sum_j \alpha_j x_{j,i}\Big\| \\ &\le \left(1 + \frac{\varepsilon_n}{7} + \frac{\varepsilon_n^2}{900n}\right) \Big\| \sum_j \alpha_j x_{j,i}\Big\|,\end{aligned}$$

which in turn, since $\varepsilon_n \to 0$, yields

$$\frac{1}{1 + \varepsilon_n} \Big\| \sum_j \alpha_j x_j\Big\| \le \Big\| \sum_j \alpha_j x_{j,i}\Big\| \le (1 + \varepsilon_n) \Big\| \sum_j \alpha_j x_j\Big\|.$$

Thus, each b.b.s. $\{x_{j,i}, f_{j,i}\}_{j=1}^n$ $(i \in I_{\tau_n}^c)$ is $(1+\varepsilon_n)$-equivalent to $\{x_j, x_j^*\}_{j=1}^n$.

To show that each b.b.s. $\{x_{j,i}, f_{j,i}\}_{j=1}^n$ $(i \in I_{\tau_n}^c)$ is $(1+\varepsilon_n)$-complemented, we argue as follows. First, for every $\xi \in \{-1, 1\}^n$, define $E_{i,\xi} := \sum_j \xi_j E_{j,i}$. Then define $\Phi_{i,\xi} : X' \otimes X \to X' \otimes X$ by

$$\Phi_{i,\xi}(f \otimes x) := f V_i E_{i,\xi} \otimes E_{i,\xi} W_i x,$$

and define $\Phi_i := 2^{-n} \sum_\xi \Phi_{i,\xi}$. Note that each $\Phi_{i,\xi}$ has norm $\|V_i\| \, \|W_i\| \le 1$,

so $\|\Phi_i\| \le 1$. We then have

$$\begin{aligned}
1 &\ge \Big\|\Phi_i\Big(\sum_j x_j^* \otimes x_j\Big)\Big\| \\
&= \Big\|\sum_j \sum_{k,l} 2^{-n}\Big(\sum_\xi \xi_k\xi_l\Big) x_j^* V_i E_{k,i} \otimes E_{l,i} W_i x_j\Big\| \\
&= \Big\|\sum_j \sum_k x_j^* V_i E_{k,i} \otimes E_{k,i} W_i x_j\Big\| \\
&\ge \Big\|\sum_j g_{j,i} \otimes y_{j,i}\Big\| - \sum_{k\ne j} \|E_{k,i} W_i x_j\| \\
&\ge \Big\|\sum_j f_{j,i} \otimes x_{j,i}\Big\| - \sum_j \|f_{j,i} - g_{j,i}\| \\
&\quad - \sum_j \|x_{j,i} - y_{j,i}\| - (n-1)\sum_j \|(\mathrm{id} - E_{j,i}) W_i x_j\|,
\end{aligned}$$

which combined with our previous estimates gives

$$\begin{aligned}
&\Big\|\sum_j f_{j,i} \otimes x_{j,i}\Big\| \\
&\qquad \le 1 + n\Big(\frac{\varepsilon_n}{7n}\Big) + n\Big(\frac{\varepsilon_n}{7n} + 2\Big(\frac{\varepsilon_n}{30n}\Big)^2\Big) + n^2\Big(\frac{\varepsilon_n}{30n}\Big)^2 \le 1 + \varepsilon_n,
\end{aligned}$$

as required.

To finish our proof we show that

$$\sum_k \Big|\sum_{i\in I_{\tau_n}^c} \lambda_i f_{1,i}(x_k) x_k^*(x_{1,i}) - \sum_{i\in I_{\tau_n}^c} \lambda_i f_{j,i}(x_k) x_k^*(x_{j,i})\Big| \le \varepsilon_n \quad (1 \le j \le n).$$

So $N_n = |I_{\tau_n}^c|$.

First, the fact that d_n is a generalized diagonal for $\mathcal{F}(X_n)$ easily implies that

$$(x_1^* \otimes x_k) d_n (x_k^* \otimes x_1) = (x_j^* \otimes x_k) d_n (x_k^* \otimes x_j) \quad (1 < j \le n,\ 1 \le k \le K_n).$$

Applying the linear functional $\mathcal{F}(X) \otimes \mathcal{F}(X) \to \mathbb{C}$, $R \otimes S \mapsto \mathrm{tr}(R)\,\mathrm{tr}(S)$, on both sides of the last identity one obtains

$$\sum_{i=1}^{M_n} x_1^*(R_i x_k)\, x_k^*(S_i x_1) = \sum_{i=1}^{M_n} x_j^*(R_i x_k)\, x_k^*(S_i x_j),$$

or equivalently,

$$\sum_{i=1}^{M_n} \widetilde{\lambda}_i x_1^*(V_i x_k)\, x_k^*(W_i x_1) = \sum_{i=1}^{M_n} \widetilde{\lambda}_i x_j^*(V_i x_k)\, x_k^*(W_i x_j) \quad (1 < j \le n), \tag{5.4}$$

where $\widetilde{\lambda}_i = \|R_i\|\,\|S_i\|$ $(1 \le i \le M_n)$. Define $\lambda_i := \widetilde{\lambda}_i / \sum_{l \in I^c_{\tau_n}} \widetilde{\lambda}_l$ $(i \in I^c_{\tau_n})$. Using (5.4), one obtains

$$
\begin{aligned}
(5.5)\qquad &\Big| \sum_{i \in I^c_{\tau_n}} \lambda_i f_{1,i}(x_k) x_k^*(x_{1,i}) - \sum_{i \in I^c_{\tau_n}} \lambda_i f_{j,i}(x_k) x_k^*(x_{j,i}) \Big| \\
&\qquad \le \sum_{i \in I^c_{\tau_n}} |\lambda_i f_{1,i}(x_k) x_k^*(x_{1,i}) - \widetilde{\lambda}_i x_1^*(V_i x_k) x_k^*(W_i x_1)| \\
&\qquad\quad + \sum_{i \in I^c_{\tau_n}} |\widetilde{\lambda}_i x_j^*(V_i x_k) x_k^*(W_i x_j) - \lambda_i f_{j,i}(x_k) x_k^*(x_{j,i})| \\
&\qquad\quad + \sum_{i \in I_{\tau_n}} \widetilde{\lambda}_i |x_1^*(V_i x_k) x_k^*(W_i x_1)| + \sum_{i \in I_{\tau_n}} \widetilde{\lambda}_i |x_j^*(V_i x_k)\, x_k^*(W_i x_j)|.
\end{aligned}
$$

For each $1 \le j \le n$, choose $\eta_j \in \mathbb{T}^{K_n}$ (where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$) such that

$$
\begin{aligned}
&\sum_k |\lambda_i f_{j,i}(x_k) x_k^*(x_{j,i}) - \widetilde{\lambda}_i x_j^*(V_i x_k)\, x_k^*(W_i x_j)| \\
&\qquad\qquad = |\lambda_i f_{j,i}(U_{\eta_j} x_{j,i}) - \widetilde{\lambda}_i x_j^*(V_i U_{\eta_j} W_i x_j)|.
\end{aligned}
$$

Then note that

$$
\begin{aligned}
&|f_{j,i}(U_{\eta_j} x_{j,i}) - x_j^*(V_i U_{\eta_j} W_i x_j)| \\
&\qquad \le |f_{j,i}(U_{\eta_j} x_{j,i}) - f_{j,i}(U_{\eta_j} y_{j,i})| + |f_{j,i}(U_{\eta_j} y_{j,i}) - x_j^*(V_i U_{\eta_j} y_{j,i})| \\
&\qquad\quad + |x_j^*(V_i U_{\eta_j} y_{j,i}) - x_j^*(V_i U_{\eta_j} W_i x_j)| \\
&\qquad \le \|x_{j,i} - y_{j,i}\| + \|f_{j,i} - g_{j,i}\| + \|y_{j,i} - W_i x_j\| \le \frac{2\varepsilon_n}{7n} + 3\Big(\frac{\varepsilon_n}{30n}\Big)^2.
\end{aligned}
$$

Combining this last estimate with the previous identity and taking (5.3) into account we obtain, for every $1 \le j \le n$,

$$
\begin{aligned}
(5.6)\qquad &\sum_k |\lambda_i f_{j,i}(x_k) x_k^*(x_{j,i}) - \widetilde{\lambda}_i x_j^*(V_i x_k) x_k^*(W_i x_j)| \\
&\qquad \le \lambda_i |f_{j,i}(U_{\eta_j} x_{j,i}) - x_j^*(V_i U_{\eta_j} W_i x_j)| + |\lambda_i - \widetilde{\lambda}_i| |x_j^*(V_i U_{\eta_j} W_i x_j)| \\
&\qquad \le \lambda_i \Big(\frac{2\varepsilon_n}{7n} + 3\Big(\frac{\varepsilon_n}{30n}\Big)^2 \Big) + \lambda_i \Big| 1 - \sum_{l \in I^c_{\tau_n}} \widetilde{\lambda}_l \Big| \\
&\qquad \le \lambda_i \Big(\frac{2\varepsilon_n}{7n} + 3\Big(\frac{\varepsilon_n}{30n}\Big)^2 + (1 + \varkappa(\tau_n))\delta_n \Big).
\end{aligned}
$$

Similarly, for every $1 \le j \le n$, there is a $\zeta_j \in \mathbb{T}^{K_n}$ such that

$$
(5.7)\qquad \sum_k |x_j^*(V_i x_k)\, x_k^*(W_i x_j)| = |x_j^*(V_i U_{\zeta_j} W_i x_j)| \le 1.
$$

Finally, combining (5.6) and (5.7) with (5.5), and once again taking into account (5.3), we obtain

$$\begin{aligned}\sum_k \Big| \sum_{i\in I^c_{\tau_n}} \lambda_i f_{1,i}(x_k)x^*_k(x_{1,i}) - \sum_{i\in I^c_{\tau_n}} \lambda_i f_{j,i}(x_k)x^*_k(x_{j,i})\Big| \\ \le 2\sum_{i\in I^c_{\tau_n}} \lambda_i\bigg(\frac{2\varepsilon_n}{7n} + 3\bigg(\frac{\varepsilon_n}{30n}\bigg)^2 + (1+\varkappa(\tau_n))\delta_n\bigg) + 2\sum_{i\in I_{\tau_n}} \widetilde{\lambda}_i \\ \le \frac{4\varepsilon_n}{7n} + 6\bigg(\frac{\varepsilon_n}{30n}\bigg)^2 + 2(1+2\varkappa(\tau_n))\delta_n \\ \le \frac{4\varepsilon_n}{7n} + 6\bigg(\frac{\varepsilon_n}{30n}\bigg)^2 + \frac{\varepsilon_n}{4} < \varepsilon_n.\end{aligned}$$

This concludes the proof of the theorem. ∎

Remark 5.6. It might be worth pointing out that condition (5.2) is equivalent to the following:

$$\max_{\eta\in\{-1,1\}^{\mathbb{N}}} \Big\{ \sum_{i=1}^{N_n} \lambda_i f_{1,i}(U_\eta x_{1,i}) - \sum_{i=1}^{N_n} \lambda_i f_{j,i}(U_\eta x_{j,i})\Big\} \le \varepsilon_n \quad (1<j\le n).$$

Indeed, since each $f_{j,i}$ is a support functional for $x_{j,i}$ and the basis (x_i) is 1-unconditional, there is $\eta\in\{-1,1\}^{\mathbb{N}}$ such that

$$\begin{aligned}\sum_k \Big| \sum_{i=1}^{N_n} \lambda_i f_{1,i}(x_k)x^*_k(x_{1,i}) - \sum_{i=1}^{N_n} \lambda_i f_{j,i}(x_k)x^*_k(x_{j,i})\Big| \\ = \sum_k \eta_k\Big(\sum_{i=1}^{N_n} \lambda_i f_{1,i}(x_k)x^*_k(x_{1,i}) - \sum_{i=1}^{N_n} \lambda_i f_{j,i}(x_k)x^*_k(x_{j,i})\Big) \\ = \sum_{i=1}^{N_n} \lambda_i f_{1,i}(U_\eta x_{1,i}) - \sum_{i=1}^{N_n} \lambda_i f_{j,i}(U_\eta x_{j,i}) \\ \le \max_{\xi\in\{-1,1\}^{\mathbb{N}}} \Big\{ \sum_{i=1}^{N_n} \lambda_i f_{1,i}(U_\xi x_{1,i}) - \sum_{i=1}^{N_n} \lambda_i f_{j,i}(U_\xi x_{j,i})\Big\},\end{aligned}$$

and the opposite inequality is proved similarly. Notice also that since the basis (x_i) is 1-unconditional the block biorthogonal systems of Theorem 5.5 can be chosen to be positive, i.e., with each vector in them having all its coordinates with respect to either (x_i) or (x^*_i) real positive.

In the following two examples, the corresponding algebras of approximable operators are well known to be amenable. Their only purpose is to illustrate the conditions of Theorem 5.5 in some concrete situations.

EXAMPLE 5.7. Let X be a Banach space with a normalized 1-unconditional basis (x_i) such that the left and right shift operators with respect to it are both norm-one. Then for every null sequence (ε_n) there is a sequence (N_n) of positive integers such that, for every $n \in \mathbb{N}$, the block biorthogonal systems $\{x_{i+j}, x^*_{i+j}\}_{j=1}^n$, $0 \le i \le N_n - 1$, satisfy the conditions of Theorem 5.5 with respect to the given sequence (ε_n). Indeed, each b.b.s. $\{x_{i+j}, x^*_{i+j}\}_{j=1}^n$ is clearly 1-complemented and 1-equivalent to $\{x_j, x^*_j\}_{j=1}^n$ (the latter because both shifts are norm-one). As for (5.2), note that

$$\begin{aligned}\sum_k \Big| \frac{1}{N} \sum_{i=0}^{N-1} x^*_{i+1}(x_k)\, x^*_k(x_{i+1}) - \frac{1}{N} \sum_{i=0}^{N-1} x^*_{i+j}(x_k)\, x^*_k(x_{i+j}) \Big| \\ = \frac{1}{N} \sum_k \Big| \sum_{i=1}^{N} \delta_{i,k} - \sum_{i=j}^{N+j-1} \delta_{i,k} \Big| \\ = \frac{1}{N} \sum_k \Big| \sum_{i=1}^{j-1} \delta_{i,k} - \sum_{i=N+1}^{N+j-1} \delta_{i,k} \Big| \\ = 2\,\frac{j-1}{N} \quad (1 \le j \le n,\ n \in \mathbb{N},\ N \in \mathbb{N}).\end{aligned}$$

So, to have (5.2) satisfied with respect to (ε_n), it is enough to choose $N_n \ge 2n/\varepsilon_n$ $(n \in \mathbb{N})$. Clearly, this will not affect the other two conditions.

EXAMPLE 5.8. Let $X := (\bigoplus_i \ell_p^i)_{\ell_q}$, $1 \le p \ne q \le \infty$, and let $(\varepsilon_n) \subset \mathbb{R}_+$ be an arbitrary null sequence. It is convenient to write down the unit vector basis of X in the form $x_1^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}, \dots$, where $x_j^{(i)}$ $(1 \le j \le i,\ i \in \mathbb{N})$ stands for the jth vector of the unit vector basis of the ith summand of X. To simplify notations let us denote by $f_j^{(i)}$ the associated biorthogonal functional. Then, for every $n \in \mathbb{N}$, there is a positive integer N_n such that the block biorthogonal systems

$$\begin{aligned}\{&x_j^{(i)}, x_j^{(i+1)}, x_{j+1}^{(i+1)}, \dots, x_j^{(i+n)}, x_{j+1}^{(i+n)}, \dots, x_{j+n}^{(i+n)}, \\ &f_j^{(i)}, f_j^{(i+1)}, f_{j+1}^{(i+1)}, \dots, f_j^{(i+n)}, f_{j+1}^{(i+n)}, \dots, f_{j+n}^{(i+n)}\} \\ &\qquad\qquad (1 \le j \le i,\ 1 \le i \le N_n)\end{aligned}$$

are 1-complemented, 1-equivalent to

$$\begin{aligned}\{&x_1^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_1^{(n+1)}, x_2^{(n+1)}, \dots, x_{n+1}^{(n+1)}, \\ &f_1^{(1)}, f_1^{(2)}, f_2^{(2)}, \dots, f_1^{(n+1)}, f_2^{(n+1)}, \dots, f_{n+1}^{(n+1)}\},\end{aligned}$$

and satisfy (5.2). Again, only this last needs a proof. For this, note that, in the present situation, the left hand side of (5.2) takes the form

$$\frac{2}{N_n(N_n+1)}\sum_{l\in\mathbb{N}}\sum_{k=1}^{l}\Big|\sum_{i=1}^{N_n}\sum_{j=1}^{i} f_j^{(i)}(x_k^{(l)})\,f_k^{(l)}(x_j^{(i)}) - \sum_{i=\varkappa+1}^{N_n+\varkappa}\sum_{j=m+1}^{i-\varkappa+m} f_j^{(i)}(x_k^{(l)})\,f_k^{(l)}(x_j^{(i)})\Big|,$$

where $0\le m\le\varkappa$ and $0\le\varkappa\le n$. After some straightforward simplifications this last expression becomes

$$\begin{aligned}\frac{2}{N_n(N_n+1)}\sum_{l\in\mathbb{N}}\sum_{k=1}^{l}\Big|&\sum_{i=1}^{\varkappa}\sum_{j=1}^{i}\delta_{i,l}\delta_{j,k}+\sum_{i=\varkappa+1}^{N_n}\sum_{j=1}^{m}\delta_{i,l}\delta_{j,k}\\ &+\sum_{i=\varkappa+1}^{N_n}\sum_{j=i-\varkappa+m+1}^{i}\delta_{i,l}\delta_{j,k}-\sum_{i=N_n+1}^{N_n+\varkappa}\sum_{j=m+1}^{i-\varkappa+m}\delta_{i,l}\delta_{j,k}\Big|\\ &\le\frac{2}{N_n(N_n+1)}[\varkappa^2+(\varkappa-1)(N_n-\varkappa)+\varkappa N_n]\le\frac{4n}{N_n+1},\end{aligned}$$

which can clearly be made smaller than any $\varepsilon>0$ by choosing N_n large enough.

With this, we have verified the conditions of the theorem for every n of the form $k(k+1)/2$ ($k\in\mathbb{N}$). But this is enough, for, in general, if the conditions of Theorem 5.5 are satisfied for every positive integer in an increasing sequence (n_k) with respect to some null sequence $(\varepsilon_k)\subset\mathbb{R}_+$ and $n_{k-1}<m<n_k$ for some $k>1$, then the block biorthogonal systems $\{x_{j,1},f_{j,1}\}_{j=1}^m,\dots,\{x_{j,N_{n_k}},f_{j,N_{n_k}}\}_{j=1}^m$, obtained from those corresponding to n_k by removing the last n_k-m vectors, will be $(1+\varepsilon_k)$-equivalent to $\{x_j,x_j^*\}_{j=1}^m$, $(1+\varepsilon_k)$-complemented and still satisfy (5.2) for ε_k. From this observation one easily obtains the desired conclusion. We leave the details to the reader.

We should notice that the same kind of argument can be used for any Banach space of the form $X=(\bigoplus_i \ell_p^{n_i})_{\ell_q}$, where $1\le p\ne q\le\infty$ and $\lim_i n_i=\infty$. But, in the general case, the details are a bit more involved.

We finish the section with a few observations. First of all, note that uniform strict monotonicity of the basis is not a necessary condition for 1-amenability. For instance, $\mathcal{A}(\ell_\infty)$ is 1-amenable, but the unit vector basis of ℓ_∞ is not u.s.m. (recall our convention about the meaning of ℓ_∞). We do not know whether the result of Theorem 5.5 (or a similar one) will still hold if one removes this assumption.

Second, it is apparent that our proof of Theorem 5.5 relies on a 'good' description of the norm-one projections on X in terms of the unconditional basis (x_i). Unfortunately, the only results of this kind that the author is aware of depend heavily on the assumption that the unconditionality constant of

the basis is 1 (see, for instance, the survey [Ra2]). So Theorem 5.5 may well be the best one can hope for regarding translation of amenability into a basis property.

Lastly, it seems very unlikely to the author that, even for Banach spaces with u.s.m. bases, the conditions of Theorem 5.5 can be equivalent to property $\mathbb{A}$, but we do not have any relevant example. In the next section, we shall say more about this problem. Here, just to give an idea of the sort of difficulty involved, let us look at a simple case in which 1-amenability implies property $\mathbb{A}$. First, suppose $d \in \mathcal{F}(Y,X)\,\widehat{\otimes}\,\mathcal{F}(X,Y)$ is a norm-one generalized diagonal such that $wd^\circ = d^\circ w$ $(w \in \mathcal{F}(Y))$. This implies, in particular, that $\pi(d^\circ)$ is in the center of $\mathcal{F}(Y)$, i.e., it is a multiple of id_Y. Let $\sum_{i=1}^m \lambda_i R_i \otimes S_i$ be a finite representation for d such that $\sum_i \lambda_i = 1$ and $\|R_i\| = \|S_i\| = 1$ $(1 \le i \le m)$. From Lemma 4.2 we know that $R_i S_i = \mathrm{id}_X$ for every i. Define $\Phi : \mathcal{F}(Y,X)\,\widehat{\otimes}\,\mathcal{F}(X,Y) \to \mathcal{F}(X)\,\widehat{\otimes}\,\mathcal{F}(X)$ by

$$\Phi(U \otimes V) := US_1 \otimes R_1 V \qquad (U \otimes V \in \mathcal{F}(Y,X)\,\widehat{\otimes}\,\mathcal{F}(X,Y)).$$

Then note that

$$\begin{aligned}
wd = dw \;\Rightarrow\; w\Phi(d) &= \Phi(d)w \qquad (w \in \mathcal{F}(X)),\\
w\Phi(d)^\circ = wR_1 d^\circ S_1 &= R_1(S_1 w R_1) d^\circ S_1\\
&= R_1 d^\circ (S_1 w R_1) S_1 = R_1 d^\circ S_1 w = \Phi(d)^\circ w \quad (w \in \mathcal{F}(X)),\\
\pi(\Phi(d)^\circ) = R_1 \pi(d^\circ) S_1 &= R_1 \left(\frac{\mathrm{tr}(\pi(d))}{\dim Y} \mathrm{id}_Y \right) S_1 = \frac{\dim X}{\dim Y} \mathrm{id}_X,
\end{aligned}$$

and

$$\pi(\Phi(d)) = \frac{\mathrm{tr}(\pi(\Phi(d)))}{\dim X}\mathrm{id}_X = \frac{\mathrm{tr}(\pi(\Phi(d)^\circ))}{\dim X}\mathrm{id}_X = \frac{\dim X}{\dim Y}\mathrm{id}_X,$$

so $\Delta := (\dim Y/\dim X)\Phi(d)$ is the corresponding symmetric diagonal for $\mathcal{F}(X)$ and $\|\Delta\|_\wedge \le \dim Y/\dim X$. It follows easily that if X is an infinite-dimensional Banach space and $\mathcal{A}(X)$ has an approximate diagonal $(d_\alpha)_{\alpha\in I}$ such that, for every $\alpha \in I$,

(i) $\pi(d_\alpha)$ is a projection,
(ii) $wd_\alpha = d_\alpha w$ $(w \in \mathcal{F}(\mathrm{rg}(\pi(d_\alpha))))$,
(iii) $wd_\alpha^\circ = d_\alpha^\circ w$ $(w \in \mathcal{F}(\mathrm{rg}(\pi(d_\alpha^\circ))))$,
(iv) $\|d_\alpha\|_\wedge = 1$, and
(v) $\mathrm{rk}(\pi(d_\alpha^\circ)) \le C\,\mathrm{rk}(\pi(d_\alpha))$ for some constant C independent of α,

then $\mathcal{A}(X)$ has property $\mathbb{A}$ with constant $\le C$. Note that the first and second assumptions on (d_α) are not significant restrictions. Indeed, as indicated in Section 2, if X is a π-space then (d_α) can always be chosen to satisfy both of them.

6. On the existence of a Banach space X without property $\mathbb{A}$ and with $\mathcal{A}(X)$ amenable. In [GJW, Theorem 6.5], it was shown that every Banach space of the form $(\bigoplus_{k=1}^{\infty}\ell_p^{n_k})_{\ell_q}$, where $1\le p\ne q\le\infty$ and $\limsup_k n_k=\infty$, carries an amenable algebra of approximable operators. However, the question of whether or not these spaces have property $\mathbb{A}$ was left open in that paper. For this reason, it was expected that Banach spaces of this form could provide an affirmative answer to the problem in the title of this section. Let us start by showing that this is not the case. In fact, quite the opposite, every Banach space of this form has property $\mathbb{A}$.

The idea of the proof is simple. For instance, if $n_k=k$ ($k\in\mathbb{N}$) then, for every $n\in\mathbb{N}$,

$$\Big(\bigoplus_{k=1}^{2n}\ell_p^k\Big)_{\ell_q}\cong\Big(\bigoplus_{k=1}^{n}(\ell_p^k\oplus_q\ell_p^{2n-k+1})\Big)_{\ell_q}$$

and

$$d\Big(\Big(\bigoplus_{k=1}^{n}(\ell_p^k\oplus_q\ell_p^{2n-k+1})\Big)_{\ell_q},\Big(\bigoplus_{k=1}^{n}\ell_p^{2n+1}\Big)_{\ell_q}\Big)\le 2^{(q-1)/q}.$$

Since the symmetric diagonal for $\mathcal{A}((\bigoplus_{k=1}^{n}\ell_p^{2n+1})_{\ell_q})$ has norm 1 (see [BG2, Proposition 3.7]), the symmetric diagonal for $\mathcal{A}((\bigoplus_{k=1}^{n}(\ell_p^k\oplus_q\ell_p^{2n-k+1}))_{\ell_q})$ should have norm $\le 2^{2(q-1)/q}$, and so $(\bigoplus_{k=1}^{\infty}\ell_p^k)_{\ell_q}$ has property $\mathbb{A}$ with constant $\le 2^{2(q-1)/q}$.

We shall prove here a result slightly stronger than the one mentioned above. For this, we shall rely on the following particular case of [JRZ, Lemma 2.4].

Lemma 6.1. *Let X be a Banach space and let $P:X\to E$ be a finite-rank projection onto the subspace E of X. If F is a linear subspace of X such that $\dim F=\dim E$ and $\|(P-\mathrm{id})|_F\|\le\varepsilon$, where $\varepsilon\in(0,1)$ satisfies $(1-\varepsilon)^{-1}\varepsilon\dim E<1$, then there is a projection $U:X\to F$ such that*

$$\|U-P\|\le(1-\varepsilon)^{-1}\varepsilon\dim E\|P\|.$$

Recall that, given an infinite-dimensional Banach space E with a 1-unconditional Schauder basis $\mathbf{e}=(e_i)$ and a sequence of Banach spaces $(X_i,\|\cdot\|_i)$, the Banach space $(\bigoplus_i X_i)_{\mathbf{e}}$ is defined to be the linear space $\{(x_i)\in\Pi_i X_i:\text{ the series }\sum_i\|x_i\|_i e_i\text{ converges in }E\}$, endowed with the norm $\|(x_i)\|=\big\|\sum_i\|x_i\|_i e_i\big\|_E$. If $X_i=\{0\}$ for every $i>m$ then we also write $(\bigoplus_{i=1}^{m}X_i)_{\mathbf{e}}$. Furthermore, if for some Banach space X and $m\in\mathbb{N}$, $X_i=X$ for every $1\le i\le m$ and $X_i=\{0\}$ for every $i>m$ (resp. $X_i=X$ for every $i\in\mathbb{N}$) then we write $E^m(X)$ (resp., $E(X)$) instead of $(\bigoplus_{i=1}^{m}X_i)_{\mathbf{e}}$ (resp. $(\bigoplus_i X_i)_{\mathbf{e}}$).

PROPOSITION 6.2. *Let X be a Banach space and let (p_i) be a bounded sequence of projections in $\mathcal{A}(X)$ with a subsequence converging strongly to id_X. Let $X_i := \operatorname{rg} p_i$ $(i \in \mathbb{N})$ and let E be a Banach space with a 1-unconditional shrinking Schauder basis $\mathbf{e} := (e_i)$. If $\mathcal{A}(E(X))$ has property $\mathbb{A}$ with constant M then $\mathcal{A}((\bigoplus_i X_i)_{\mathbf{e}})$ has property $\mathbb{A}$ with constant*

$$\le 4M\Big(2\sup_i \|p_i\| + M\Big)^2\Big(\sup_i \|p_i\| + M\Big)^2.$$

Proof. Let $(d_\alpha)_{\alpha\in J}$ be an approximate diagonal for $\mathcal{A}(E(X))$ of norm $\le M$ such that, for every $\alpha \in J$, d_α is the symmetric diagonal for $\mathcal{A}(\operatorname{rg} Q_\alpha)$, where $Q_\alpha := \pi(d_\alpha)$. Note that each Q_α must be a projection. Also, let $\pi_i : E(X) \to X$ (resp., $\imath_i : X \to E(X)$) stand for the ith coordinate projection (resp., embedding), and let $P_k := \sum_{i=1}^k \imath_i p_i \pi_i$ $(k \in \mathbb{N})$. Clearly, to prove the desired result, it will suffice to exhibit a bounded sequence $(q_k)_{k\in\mathbb{N}}$ of projections in $\mathcal{A}((\bigoplus_i X_i)_{\mathbf{e}})$ converging strongly to the identity map on $(\bigoplus_i X_i)_{\mathbf{e}}$ and such that, for every $k \in \mathbb{N}$, there is an α $(= \alpha(k)) \in J$ such that $d(\operatorname{rg} q_k, \operatorname{rg} Q_\alpha) \le C$ for some constant C independent of k.

For each $k \in \mathbb{N}$ we construct q_k as follows. First choose $\alpha = \alpha(k) \in J$ such that $\operatorname{rg} P_k Q_\alpha P_k = \operatorname{rg} P_k$ and

$$\big\|(P_k - \mathrm{id})|_{\operatorname{rg} Q_\alpha P_k}\big\| < 1/(\operatorname{rk} P_k)^2.$$

Then choose positive integers $n_1 < \cdots < n_N$ greater than k such that if $R_k = \sum_{i=1}^N \imath_i p_{n_i} \pi_i$ then $\operatorname{rg} Q_\alpha R_k Q_\alpha = \operatorname{rg} Q_\alpha$, $\operatorname{rg} P_k R_k Q_\alpha P_k = \operatorname{rg} P_k$,

$$\big\|(P_k - \mathrm{id})|_{\operatorname{rg} R_k Q_\alpha P_k}\big\| \le 1/(\operatorname{rk} P_k)^2 \quad \text{and} \quad \big\|(Q_\alpha - \mathrm{id})|_{\operatorname{rg} R_k Q_\alpha}\big\| \le 1/(\operatorname{rk} Q_\alpha)^2.$$

(The existence of such integers is guaranteed by the fact that (p_i) contains a subsequence that converges strongly to id_X.) By Lemma 6.1, for every k large enough, there are projections $U_k : E(X) \to \operatorname{rg} R_k Q_\alpha P_k$ and $V_k : E(X) \to \operatorname{rg} R_k Q_\alpha$ such that

$$\|U_k - P_k\| \le (\operatorname{rk} P_k - 1)^{-1}\|P_k\| \quad \text{and} \quad \|V_k - Q_\alpha\| \le (\operatorname{rk} Q_\alpha - 1)^{-1}\|Q_\alpha\|.$$

In addition, it follows from our choice of α and R_k above that

$$d(\operatorname{rg} U_k, \operatorname{rg} P_k) = d(\operatorname{rg} R_k Q_\alpha P_k, \operatorname{rg} P_k R_k Q_\alpha P_k) \le \frac{\operatorname{rk} P_k + 1}{\operatorname{rk} P_k - 1},$$

and

$$d(\operatorname{rg} Q_\alpha, \operatorname{rg} V_k) = d(\operatorname{rg} Q_\alpha R_k Q_\alpha, \operatorname{rg} R_k Q_\alpha) \le \frac{\operatorname{rk} Q_\alpha + 1}{\operatorname{rk} Q_\alpha - 1},$$

where we have used the fact that if $\|Tx - x\| \le \varepsilon\|x\|$ $(x \in F)$ for some $0 < \varepsilon < 1$, then $\|T|_F\| \le 1+\varepsilon$, $\|(T|_F)^{-1}\| \le 1-\varepsilon$ and $d(F, TF) \le (1+\varepsilon)/(1-\varepsilon)$.

Define $Y_k := \operatorname{rg} R_k$ and $\widetilde{Y}_k := \{(x_i) \in (\bigoplus_i X_i)_{\mathbf{e}} : x_i = 0 \text{ if } i \ne n_j \ (1 \le j \le N)\}$. Clearly, $Y_k \cong \widetilde{Y}_k$, so $d(Y_k, \widetilde{Y}_k) = 1$. Let us write $\widetilde{U}_k$ and $\widetilde{V}_k$ for the projections on $\widetilde{Y}_k$ corresponding to $U_k|_{Y_k}$ and $V_k|_{Y_k}$, respectively. Also,

let us write Z_k for $(\bigoplus_{i=1}^k X_i)_{\mathbf{e}}$. Note that $\operatorname{rg} P_k \cong Z_k$, so $d(\operatorname{rg} P_k, Z_k) = 1$. Moreover, if $R : \operatorname{rg} U_k \to Z_k$ and $S : \operatorname{rg}(V_k - U_k) \to \operatorname{rg}(\widetilde{V}_k - \widetilde{U}_k)$ are linear isomorphisms such that $\|R\| = 1$, $\|R^{-1}\| = d(\operatorname{rg} U_k, Z_k)$ and $\|S\| = \|S^{-1}\| = 1$ then, for every $x \in \operatorname{rg} U_k$ and every $y \in \operatorname{rg}(V_k - U_k)$,

$$\begin{aligned}\|x+y\| &\le (\|R^{-1}\|+1)\|Rx+Sy\| \\ &\le (\|R^{-1}\|+1)(\|RU_k\|+\|V_k-U_k\|)\|x+y\| \\ &\le (1+d(\operatorname{rg} U_k, Z_k))(\|U_k\|+\|V_k-U_k\|)\|x+y\|,\end{aligned}$$

so

$$\begin{aligned}d(\operatorname{rg} U_k + \operatorname{rg}(V_k-U_k), Z_k + \operatorname{rg}(\widetilde{V}_k - \widetilde{U}_k)) \\ \le (1+d(\operatorname{rg} U_k, Z_k))(\|U_k\|+\|V_k-U_k\|).\end{aligned}$$

Thus,

$$\begin{aligned}d(\operatorname{rg} Q_\alpha, Z_k + \operatorname{rg}(\widetilde{V}_k - \widetilde{U}_k)) \\ &\le d(\operatorname{rg} Q_\alpha, \operatorname{rg} V_k)\, d(\operatorname{rg} U_k + \operatorname{rg}(V_k-U_k), Z_k + \operatorname{rg}(\widetilde{V}_k - \widetilde{U}_k)) \\ &\le d(\operatorname{rg} Q_\alpha, \operatorname{rg} V_k)\,(1+d(\operatorname{rg} U_k, Z_k))(\|U_k\|+\|V_k-U_k\|),\end{aligned}$$

and so, given an arbitrary $\varepsilon > 0$, we should have, for k large enough,

$$d(\operatorname{rg} Q_\alpha, Z_k + \operatorname{rg}(\widetilde{V}_k - \widetilde{U}_k)) \le (2+\varepsilon)(\|P_k\| + \|Q_\alpha - P_k\|).$$

We then define q_k to be the composite of the natural projection from $(\bigoplus_i X_i)_{\mathbf{e}}$ onto $Z_k + \widetilde{Y}_k$ with the projection $\mathrm{id}_{Z_k} \oplus (\widetilde{V}_k - \widetilde{U}_k) : Z_k + \widetilde{Y}_k \to Z_k + \operatorname{rg}(\widetilde{V}_k - \widetilde{U}_k)$. One easily sees that the sequence (q_k) satisfies all required conditions.

As for the property $\mathbb{A}$ constant of $\mathcal{A}((\bigoplus_i X_i)_{\mathbf{e}})$, simply note that if $\sum_i R_{i,\alpha} \otimes S_{i,\alpha} \in \mathcal{F}(\operatorname{rg} Q_\alpha) \otimes \mathcal{F}(\operatorname{rg} Q_\alpha)$ is a representation of d_α ($\alpha \in J$) and $T_k : \operatorname{rg} Q_{\alpha(k)} \to \operatorname{rg} q_k$ is a linear isomorphism ($k \in \mathbb{N}$) then $\sum_i T_k R_{i,\alpha(k)} T_k^{-1} q_k \otimes T_k S_{i,\alpha(k)} T_k^{-1} q_k$ is the symmetric diagonal for $\mathcal{A}(\operatorname{rg} q_k)$. Thus, the property $\mathbb{A}$ constant of $\mathcal{A}((\bigoplus_i X_i)_{\mathbf{e}})$ cannot exceed

$$M \liminf_k d(\operatorname{rg} q_k, \operatorname{rg} Q_\alpha)^2 \|q_k\|^2 \le 4M \Big(2 \sup_i \|p_i\| + M\Big)^2 \Big(\sup_i \|p_i\| + M\Big)^2. \ \blacksquare$$

Although the estimates provided by Proposition 6.2 are far from sharp, we suspect that, in general, if E and X are incomparable Banach spaces then the property $\mathbb{A}$ constant of $(\bigoplus_k [x_i]_{i=1}^{n_k})_{\mathbf{e}}$ will be greater than the property $\mathbb{A}$ constants of both E and X. We do not have a proof for this, not even in the case where both E and X are ℓ_p-spaces. However, assuming this were true for ℓ_p-spaces, and bearing in mind that $\mathcal{A}((\bigoplus_k \ell_p^{n_k})_{\ell_q})$ is 1-amenable whenever $\limsup_k n_k = \infty$ ([BG2, Example 3.6]), one might wonder whether iterating this construction and taking some kind of direct limit one could produce a Banach space X carrying an amenable algebra of approximable operators

and yet lacking property $\mathbb{A}$. Our next result shows why this idea is unlikely to work.

COROLLARY 6.3. *Let $1 \le p_1, \dots, p_{m+1} \le \infty$, and let $(n_{i,1})$, $(n_{i,2}), \dots,$ $(n_{i,m})$ be sequences of non-negative integers. Define $E_1 := \ell_{p_1}$ and $E_{k+1} := (\bigoplus_i \ell_{p_{k+1}}^{n_{i,k}})_{\mathbf{e}(k)}$ $(1 \le k \le m)$, where $\mathbf{e}(k)$ is the natural unit vector basis of E_k. If $\limsup_i n_{i,m} = \sup_i n_{i,m}$ then E_{m+1} has property $\mathbb{A}$ with constant ≤ 144.*

Proof. Define $E := \ell_{p_1}(\ell_{p_2}(\cdots(\ell_{p_m})\cdots))$. It is known that $\mathcal{A}(E)$ has property $\mathbb{A}$ with constant 1. (This follows easily by induction combined with an argument very similar to that used in the proof of [BG2, Proposition 3.7].) It is clear that the unit vector basis of E can be reordered so as to contain the unit vector basis of E_k as a subsequence. Denote by $\mathbf{e}$ the new (reordered) basis for E. Then one can realize E_{m+1} as a space of the form $(\bigoplus_i \ell_{p_{m+1}}^{n_i})_{\mathbf{e}}$ for some sequence (n_i) of non-negative integers. Of course, one may have $n_i = 0$ for infinitely many values of i. Now, let $\gamma := \sup_i n_{i,m}$ (if $(n_{i,m})$ is unbounded then let $\gamma = \infty$). By hypothesis, $(n_{i,m})$ contains a subsequence converging to γ, and since (n_i) contains $(n_{i,m})$ as a subsequence, also (n_i) has a subsequence converging to γ. In turn, this means that $(q_i) \subset \mathcal{A}(\ell_{p_{m+1}}^{\gamma})$, where q_i stands for the natural projection onto the linear span of the first n_i vectors of the unit vector basis of $\ell_{p_{m+1}}^{\gamma}$, has a subsequence converging strongly to the identity map on $\ell_{p_{m+1}}^{\gamma}$ (if $\gamma = \infty$ then $\ell_{p_{m+1}}^{\gamma} = \ell_{p_{m+1}}$). Thus, by Proposition 6.2, E_{m+1} has property $\mathbb{A}$ with constant $\le 4(2+1)^2(1+1)^2$. (Actually, in this case, it is not hard to see, from the proof of Proposition 6.2, that the property $\mathbb{A}$ constant of E_{m+1} should be ≤ 4.) ∎

REMARK 6.4. Note that starting with numbers $1 \le p_1, \dots, p_{m+1} \le \infty$ and sequences $(n_{k,1})$, $(n_{k,2}), \dots,$ $(n_{k,m})$ as in the statement of Corollary 6.3 one could, alternatively, define $F_1 := \ell_{p_1}$ and then $F_{k+1} := (\bigoplus_i [e_{j,k}]_{j=1}^{n_{i,k}})_{\ell_{p_{k+1}}}$ $(1 \le k \le m)$, where $(e_{j,k})_{j\in\mathbb{N}}$ is the natural unit vector basis for F_k. However, it is not hard to see that if one drops the assumption on $(n_{i,m})$ then this construction will produce exactly the same family of Banach spaces as the construction considered in Corollary 6.3.

In view of the above results, one is naturally led to ask whether there are other families of Banach spaces that could yield the desired example. Our next result will provide a class of Banach spaces, essentially different from the one considered thus far, all of whose members lack property $\mathbb{A}$. We think it could contain examples of Banach spaces X for which $\mathcal{A}(X)$ is amenable, although, so far, we have failed to find any. Recall first that an *operator ideal norm* on the operator ideal $\mathcal{F}$ of all finite-rank operators between Banach spaces is a function $\gamma : \mathcal{F} \to [0, \infty[$ that satisfies the following:

(a) $\gamma|_{\mathcal{F}(X,Y)}$ is a norm for every pair of Banach spaces X and Y;
(b) $\gamma(\mathrm{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}) = 1$; and

(c) if $A : Y \to Y_0$ and $B : X_0 \to X$ are bounded operators and $T \in \mathcal{F}(X,Y)$ then $\gamma(ATB) \le \|A\|\gamma(T)\|B\|$.

PROPOSITION 6.5. *Let X be a Banach space. Suppose that for some $\delta > 0$ and some operator ideal norm γ on $\mathcal{F}$ there is a map $P \mapsto Q_P$ from the set of all projections in $\mathcal{A}(X)$ into itself such that, for every projection $P \in \mathcal{A}(X)$, $PQ_P = Q_P = Q_P P$ and $\operatorname{rk} Q_P \ge \delta \operatorname{rk} P$; and, whenever $(P_\alpha) \subset \mathcal{A}(X)$ is a bounded net of projections converging strongly to id_X, $\limsup_\alpha \gamma(P_\alpha)/\gamma(Q_{P_\alpha}) = \infty$. Then $\mathcal{A}(X)$ cannot have a bounded approximate diagonal $(d_\alpha)_{\alpha\in I}$ such that, for every $\alpha \in I$,*

(i) $P_\alpha := \pi(d_\alpha)$ *is a projection;*
(ii) $\pi(d_\alpha^\circ) = \pi(d_\alpha)$*; and*
(iii) $P_\alpha d_\alpha = d_\alpha$ *and* $W d_\alpha = d_\alpha W$ $(W \in P_\alpha \mathcal{A}(X) P_\alpha)$.

Proof. Suppose towards a contradiction that $\mathcal{A}(X)$ has a bounded approximate diagonal $(d_\alpha)_{\alpha\in I}$ satisfying conditions (i)–(iii) above.

For each $\alpha \in I$, let $\sum_i R_{i,\alpha} \otimes S_{i,\alpha}$ be a representation of d_α such that $\sum_i \|R_{i,\alpha}\| \, \|S_{i,\alpha}\| \le 2\|d_\alpha\|_\wedge$, and let $Q_\alpha = Q_{P_\alpha}$. Condition (iii) implies that $\pi(\sum_i R_{i,\alpha} \otimes Q_\alpha S_{i,\alpha}) = cP_\alpha$ for some scalar c. Moreover, $c \ge \delta$, for

$$\begin{aligned} c(\operatorname{rk} P_\alpha) = \operatorname{tr}(cP_\alpha) &= \operatorname{tr}\Big(\pi\Big(\sum_i R_{i,\alpha} \otimes Q_\alpha S_{i,\alpha}\Big)\Big) \\ &= \operatorname{tr}\Big(\sum_i R_{i,\alpha} Q_\alpha S_{i,\alpha}\Big) = \operatorname{tr}\Big(\Big(\sum_i S_{i,\alpha} R_{i,\alpha}\Big) Q_\alpha\Big) = \operatorname{rk} Q_\alpha \end{aligned}$$

(where we have used (i) and (ii)). Then note that

$$\gamma(cP_\alpha) \le \sum_i \gamma(R_{i,\alpha} Q_\alpha S_{i,\alpha}) \le \sum_i \|R_{i,\alpha}\| \gamma(Q_\alpha) \|S_{i,\alpha}\| \le 2\|d_\alpha\|_\wedge \gamma(Q_\alpha),$$

and hence

$$\gamma(P_\alpha)/\gamma(Q_\alpha) \le 2\|d_\alpha\|_\wedge/\delta \quad (\alpha \in I),$$

which contradicts our hypothesis. ∎

The conditions of Proposition 6.5 have been modeled on the defining properties of weak Hilbert spaces. Recall that *weak Hilbert spaces* are Banach spaces which are both weak type and weak cotype 2. A well known characterization of them states that a weak Hilbert space is a Banach space X such that for every $0 < \delta < 1$, there is a constant $C(\delta)$ with the following property: every finite-dimensional subspace $E \subset X$ contains a subspace $F \subset E$ with $\dim F \ge \delta \dim E$ such that $d(F, \ell_2^{\dim F}) \le C(\delta)$ and there is a projection $P : X \to F$ with $\|P\| \le C(\delta)$ ([Pi2, Theorem 12.2]). So, in particular, every weak Hilbert space non-isomorphic to a Hilbert space satisfies the conditions of Proposition 6.5 with $\gamma = \gamma_2$ (recall $\gamma_2(T : X \to Y) = \inf\{\|R\| \, \|S\| : R : \ell_2 \to Y, \, S : X \to \ell_2 \text{ and } RS = T\}$). The class of weak Hilbert spaces ranges from the classical Hilbert spaces,

all of which have property $\mathbb{A}$ with constant 1, to the 2-convexified Tsirelson spaces, which fail to be amenable. These spaces have been extensively studied. However, if one restricts attention to this class then the problem gets slightly more difficult, as indicated by our next result, which may be seen as an extension of [CS, Theorem Ae15].

PROPOSITION 6.6. *Let H be a weak Hilbert space. If $\mathcal{A}(H)$ has a bounded approximate diagonal $(d_\alpha)_{\alpha\in I}$ such that, for every $\alpha \in I$,*

(i) *$P_\alpha := \pi(d_\alpha)$ is a projection;*
(ii) *$\pi(d_\alpha^\circ)$ is a scalar multiple of a projection; and*
(iii) *$P_\alpha d_\alpha = d_\alpha$ and $W d_\alpha = d_\alpha W$ $(W \in P_\alpha \mathcal{A}(X) P_\alpha)$,*

then H is isomorphic to a Hilbert space.

Proof. Suppose towards a contradiction that $\mathcal{A}(H)$ has a bounded approximate diagonal $(d_\alpha)_{\alpha\in I}$ with the indicated properties. Set $H_\alpha = \operatorname{rg} P_\alpha$ $(\alpha \in I)$. Recall that, since H is a weak Hilbert space, there are constants $0 < \delta \le 1$ and $C \ge 1$ such that for every finite-dimensional subspace X of H there exist a linear subspace $Y \subseteq X$ with $\dim Y \ge \delta \dim X$ and $d(Y, \ell_2^{\dim Y}) \le C$, and a projection $Q : H \to Y$ with $\|Q\| \le C$.

Let $d_\alpha = \sum_{i=1}^{m(\alpha)} R_i^\alpha \otimes S_i^\alpha$, where the R_i^α's and the S_i^α's have been chosen in such a way that $\sum_i \|R_i^\alpha\|^2 = 1$ and $(\sum_i \|S_i^\alpha\|^2)^{1/2} = \|d_\alpha\|_\wedge$, and let $H_\alpha^\circ = \operatorname{rg} \pi(d_\alpha^\circ)$. By the previous paragraph, there exist a subspace K_α of H_α° and a projection $Q_\alpha : H \to K_\alpha$ such that $\dim K_\alpha \ge \delta \dim H_\alpha^\circ$, $d(K_\alpha, \ell_2^{\dim K_\alpha}) \le C$ and $\|Q_\alpha\| \le C$. Define $\widetilde{d}_\alpha := \sum_i R_i^\alpha \jmath_\alpha \otimes Q_\alpha S_i^\alpha$, where we have written $\jmath_\alpha$ for the natural inclusion of K_α into H. For each i, let $X_i^\alpha \cong \ell_2^{\dim K_\alpha}$ and let $T_i : K_\alpha \to X_i^\alpha$ be a norm-1 linear isomorphism such that $\|T_i^{-1}\| = d(K_\alpha, X_i^\alpha)$. Define $X_\alpha := (\bigoplus_i X_i^\alpha)_{\ell_2}$, and let $\pi_i : X_\alpha \to X_i^\alpha$ and $\imath_i : X_i^\alpha \to X_\alpha$ be the ith coordinate projection and embedding, respectively. Lastly, define $R := \sum_i R_i^\alpha \jmath_\alpha T_i^{-1} \pi_i$ and $S := \sum_i \imath_i T_i Q_\alpha S_i^\alpha$. Clearly, $RS = \pi(\widetilde{d}_\alpha) = \varkappa P_\alpha$ for some scalar $\varkappa$. Let $\operatorname{tr}(\pi(d_\alpha^\circ)) = c \dim H_\alpha^\circ$ and note that $\operatorname{tr}(\pi(d_\alpha^\circ)) = \operatorname{tr}(\pi(d_\alpha)) = \dim H_\alpha$, so

$$\begin{aligned}\varkappa \dim H_\alpha &= \operatorname{tr}(\pi(\widetilde{d}_\alpha)) = \operatorname{tr}(\pi(\widetilde{d}_\alpha^\circ)) \\ &= \operatorname{tr}(Q_\alpha \pi(d_\alpha^\circ)\jmath) = \frac{\dim H_\alpha}{\dim H_\alpha^\circ} \operatorname{tr}(\mathrm{id}_{K_\alpha}) \ge \delta \dim H_\alpha,\end{aligned}$$

i.e., $\varkappa \ge \delta$. Moreover, for every $(x_1, \dots, x_m) \in X_\alpha$,

$$\begin{aligned}\|R(x_1, \dots, x_m)\| &= \Big\| \sum_i R_i^\alpha \jmath_\alpha T_i^{-1}(x_i) \Big\| \\ &\le \Big(\sum_i \|R_i^\alpha \jmath_\alpha T_i^{-1}\|^2 \Big)^{1/2} \Big(\sum_i \|x_i\|^2 \Big)^{1/2} \\ &\le C \|(x_1, \dots, x_m)\|,\end{aligned}$$

and, for every $h \in H_\alpha$,

$$\|S(h)\| = \Big(\sum_i \|T_i Q_\alpha S_i^\alpha(h)\|^2 \Big)^{1/2} \le C \|d_\alpha\|_\wedge \|h\|.$$

Thus, $d(H_\alpha, \ell_2^{\dim H_\alpha}) = d(H_\alpha, \operatorname{rg} R) \le \varkappa^{-1} \|R\| \, \|S\| \le \delta^{-1} C^2 \|d_\alpha\|_\wedge$.

Since the net (P_α) converges strongly to id_H, it follows easily from our last estimate that $d(E, \ell_2^{\dim E}) \le \delta^{-1} C^2 \sup_\alpha \|d_\alpha\|_\wedge$ for every finite-dimensional subspace E of H. In turn, by [LP, Proposition 5.2], we conclude that H is isomorphic to a Hilbert space. ∎

We finish the note with the following question:

Is it possible for a weak Hilbert space which is not isomorphic to a Hilbert space to carry an amenable Banach algebra of approximable operators?

In view of the above, an affirmative answer to this question would imply a positive answer to the problem in the title of the section.

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