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Types of tightness in spaces with unconditional basis

by

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Abstract. We present a reflexive Banach space with an unconditional basis which is quasi-minimal and tight by range, i.e. of type (4) in Ferenczi–Rosendal's list related to Gowers' classification program of Banach spaces, but in contrast to the recently constructed space of type (4), our space is also tight with constants, thus essentially extending the list of known examples in Gowers' program. The space is defined on the basis of a boundedly modified mixed Tsirelson space with the use of a special coding function.

1. Introduction. The "loose" classification program for Banach spaces was started by W. T. Gowers in the celebrated paper [G4]. The goal is to identify classes of Banach spaces which are

- hereditary, i.e. if a space belongs to a given class, then any of its closed infinite-dimensional subspaces belongs to the same class,
- inevitable, i.e. any Banach space contains an infinite-dimensional subspace in one of those classes,
- defined in terms of the richness of the family of bounded operators on/in the space.

The program was inspired by Gowers' famous dichotomy [G3] exhibiting the first two classes: spaces with an unconditional basis and hereditary indecomposable spaces. Recall that a space is called *hereditarily indecomposable* (HI) if none of its closed infinite-dimensional subspaces is a direct sum of further two closed infinite-dimensional subspaces.

Research now concentrates on identifying classes in terms of the family of isomorphisms defined in a space. The richness of this family can be described using various "minimality" conditions, whereas the lack of certain types of isomorphic embeddings of subspaces of a given space is described by different types of "tightness" of the space under consideration.

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Recall that a Banach space is *minimal* if it embeds isomorphically into any of its closed infinite-dimensional subspaces. Relaxing this notion one obtains *quasi-minimality*: any two infinite-dimensional subspaces of a given space contain further two isomorphic infinite-dimensional subspaces. Relaxing the notions of minimality or adding additional requirements on the choice of isomorphic subspaces in the case of quasi-minimality leads to different types of minimality, contrasted in [G4, FR1] with different types of tightness, categorized in [FR1].

Recall that a subspace Y of a Banach space X with a basis (e_n) is *tight in* X if there is a sequence of successive subsets $I_1 < I_2 < \cdots$ of $\mathbb{N}$ such that the support of any isomorphic copy of Y in X intersects all but finitely many I_n's. The space X is called *tight* if any of its subspaces is tight in X. Adding requirements on the subsets I_n with respect to a given Y one obtains more specific notions, in particular *tightness by support* where the subsets I_n witnessing tightness of a subspace Y spanned by a block sequence (x_n) are chosen to be the supports of x_n. W. T. Gowers in [G4] shows that every Banach space contains either a quasi-minimal subspace or a subspace which is tight by support. The counterpart for minimality is tightness—by [FR1] every Banach space contains a subspace which is either tight or minimal.

A natural relaxing of the notion of tightness by support, called *tightness by range*, ensuring that one can choose the subsets I_n to be the ranges of x_n (recall that the *range* of a vector is the smallest interval containing the support of this vector), has also its dichotomy counterpart in a stronger form of quasi-minimality, called sequential minimality [FR1]. A Banach space X is *sequentially minimal* if it is block saturated with block sequences (x_n) with the following property: any subspace of X contains a sequence equivalent to a subsequence of (x_n).

Finally, on the side of minimality type properties, one can relax the notion of minimality requiring that the space X considered is only finitely represented in any of its infinite-dimensional subspaces. Such *local minimality* is contrasted in a dichotomy in [FR1] with *tightness with constants* giving strict control on the embedding constants—for any $K \in \mathbb{N}$ the subspace Y does not embed with constant K into $[e_i : i \notin I_K]$.

Obvious observations relate some of the properties listed above to the HI/unconditional dichotomy—in particular clearly any HI space is quasi-minimal and any basis which is tight by support is unconditional. V. Ferenczi and C. Rosendal [FR1] presented a list of classes within Gowers' classification program, and according to this list they examined in [FR2] the spaces already known. The list of examples of the main classes was completed by the recent works of V. Ferenczi and Th. Schlumprecht [FS] and of S. A. Argyros and the present authors [AMP]. We now recall the list of classes developed in [FR1] as stated in [FS], indicating also some already known examples.

THEOREM 1.1 (Ferenczi–Rosendal classification). *Any infinite-dimensional Banach space contains a subspace with a basis from one of the following classes:*

(1) *HI, tight by range* (*Gowers space of* [G2] *with asymptotically unconditional basis, by* [FR2]),
(2) *HI, tight, sequentially minimal* (*a version of Gowers–Maurey space, by* [FS]),
(3) *tight by support* (*Gowers space of* [G1] *with an unconditional basis, by* [FR2]),
(4) *with an unconditional basis, tight by range, quasi-minimal* (*an unconditional version of Gowers HI space of* [G2] *with an asymptotically unconditional basis, by* [AMP]),
(5) *with an unconditional basis, tight, sequentially minimal* (*Tsirelson space, by* [FR2]),
(6) *with an unconditional basis, minimal* (ℓ_p, c_0, *the dual to Tsirelson space, by* [CJT]*; Schlumprecht space, by* [AS]).

The full Ferenczi–Rosendal list [FR1, Theorem 8.4] splits each of the above classes with respect to the dichotomy: local minimality versus tightness with constants. A further splitting of some of the above classes involves the asymptotic structure of the space, i.e. the dichotomy: strong ℓ_p-asymptoticity versus uniform inhomogeneity of [T].

The aim of the present paper is to examine the class (4) with respect to its local structure. We briefly sketch the proof of local minimality of the space $\mathcal{X}_{(4)}$ constructed in [AMP] and concentrate on the construction of an example at the other extreme of the class (4). Namely we prove the following.

THEOREM 1.2. *There exists a reflexive space* $\mathcal{X}_{\mathrm{cr}}$ *with an unconditional basis which is quasi-minimal, tight by range and tight with constants.*

As mentioned above, the example of [AMP] is an unconditional version of the Gowers HI space with an asymptotically unconditional basis [G1], using the standard framework of a space constructed on the basis of a mixed Tsirelson space, defined by a norming set closed under certain operations. The standard operations include taking averages of certain block sequences (like $(\mathcal{A}_n, \theta_n)$-operations or $(\mathcal{S}_n, \theta_n)$-operations), projections on subsets of $\mathbb{N}$ or intervals in $\mathbb{N}$, change of signs etc. Taking averages could be restricted to a special family of block sequences, picked usually by means of a so-called "coding function"; this method, introduced by B. Maurey and H. Rosenthal, and exploited by W. T. Gowers and B. Maurey, led to the construction of the first HI space [GM].

In the case of the space $\mathcal{X}_{(4)}$, in [AMP] its norming set is closed under change of signs, certain $(\mathcal{A}_{n_j}, 1/m_j)$-operations, projections on intervals, and—in order to ensure tightness by range—a special "Gowers operation" used in [G1], a scaled projection on $\mathcal{S}_1$ sets. This structure also allows the usual reasoning that yields the presence of finite-dimensional copies of ℓ_∞ in every subspace of $\mathcal{X}_{(4)}$ (cf. [M]), thus also ensures local minimality.

The typical way to provide tightness with constants of a space is to base its construction on the Schreier families instead of $(\mathcal{A}_n)$. The strong asymptotic structure of the space, by [FR1], provides the desired type of tightness. However, using the Schreier families in the definition of the norming set neutralizes the effect of the "Gowers operation" used in the example of [AMP]. Therefore in order to construct the space $\mathcal{X}_{\mathrm{cr}}$ with the desired properties one needs other tools which would "spoil" the sequential minimality of regular modified mixed Tsirelson spaces defined by the Schreier families, proved in [KMP].

We present here a variant of a standard construction on the basis of boundedly modified mixed Tsirelson spaces [ADKM]. The norming set of the Banach space $\mathcal{X}_{\mathrm{cr}}$ to be constructed is closed under change of signs, projection on intervals, and $(\mathcal{S}_{n_j}, 1/m_j)$-operations on certain sequences, partly defined by a carefully chosen coding function. As mentioned earlier, tightness with constants is ensured by a strong asymptotic structure of the space, quasi-minimality follows from the regularity of the operations applied, whereas tightness by range follows from the use of the coding function. The key point in the choice of the coding function is its "complexity level"—high enough to spoil sequential minimality and ensure tightness by range, but still low enough to preserve the quasi-minimality of the space. Our construction exhibits many possibilities—within the framework of spaces built on the basis of mixed Tsirelson spaces—of designing the properties of a space by means only of the coding function involved in the definition of the norming set.

We now describe the contents of the paper. We recall the standard notation in the second section. The third section is devoted to the definition and basic properties of our space $\mathcal{X}_{\mathrm{cr}}$, while in the fourth section we prove the quasi-minimality and tightness properties of $\mathcal{X}_{\mathrm{cr}}$. In the last section we sketch the proof of the local minimality of the space $\mathcal{X}_{(4)}$ of [AMP].

2. Preliminaries. We recall the basic definitions and standard notation.

By a *tree* we shall mean a non-empty partially ordered set $(\mathcal{T}, \leq)$ for which the set $\{y \in \mathcal{T} : y \leq x\}$ is linearly ordered and finite for each $x \in \mathcal{T}$. If $\mathcal{T}' \subseteq \mathcal{T}$ then we say that $(\mathcal{T}', \leq)$ is a *subtree* of $(\mathcal{T}, \leq)$. The tree $\mathcal{T}$ is called *finite* if the set $\mathcal{T}$ is finite. The *root* is the smallest element of the tree (if it exists). A *branch* in $\mathcal{T}$ is a maximal linearly ordered set in $\mathcal{T}$. The *immediate successors* of $x \in \mathcal{T}$, denoted by $\mathrm{succ}(x)$, are all the nodes $y \in \mathcal{T}$ such that

$x < y$ but there is no $z \in \mathcal{T}$ with $x < z < y$. If X is a linear space, then a *tree in* X is a tree whose nodes are vectors in X.

Let X be a Banach space with a basis (e_i). The *support* of a vector $x = \sum_i x_i e_i$ is the set $\operatorname{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$; the *range* of x, denoted by $\operatorname{range} x$, is the minimal interval containing $\operatorname{supp} x$. Given any $x = \sum_i a_i e_i$ and finite $E \subset \mathbb{N}$ put $Ex = \sum_{i \in E} a_i e_i$. We write $x < y$ for vectors $x, y \in X$ if $\max \operatorname{supp} x < \min \operatorname{supp} y$. A *block sequence* is any sequence $(x_i) \subset X$ satisfying $x_1 < x_2 < \cdots$. A closed subspace spanned by an infinite block sequence $(x_n)_{n \in \mathbb{N}}$ is called a *block subspace* and is denoted by $[x_n : n \in \mathbb{N}]$.

We shall consider two hierarchies of families of finite subsets of $\mathbb{N}$: the families $(\mathcal{A}_n)_{n \in \mathbb{N}}$, defined as $\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \leq n\}$ for each $n \in \mathbb{N}$, and the *Schreier families* $(\mathcal{S}_n)_{n \in \mathbb{N}}$, introduced in [AA], defined by induction:

$$\mathcal{S}_0 = \{\{k\} : k \in \mathbb{N}\} \cup \{\emptyset\},$$
$$\mathcal{S}_{n+1} = \{F_1 \cup \cdots \cup F_k : k \leq F_1 < \cdots < F_k,\ F_1, \ldots, F_k \in \mathcal{S}_n\}, \qquad n \in \mathbb{N}.$$

We can also define modified Schreier families $(\mathcal{S}_n^M)_{n \in \mathbb{N}}$ by replacing in the definition above the condition "$F_1 < \cdots < F_k$" by "$F_1, \ldots, F_k$ are pairwise disjoint". The following observation proves that these families coincide.

Lemma 2.1 ([ADKM, Lemma 1.2]). *For any $n \in \mathbb{N}$ we have $\mathcal{S}_n = \mathcal{S}_n^M$.*

Fix a family $\mathcal{M}$ of finite subsets of $\mathbb{N}$. We say that a sequence $(E_1, \ldots, E_k)$ of subsets of $\mathbb{N}$ is

(1) $\mathcal{M}$-*admissible* if $E_1 < \cdots < E_k$ and $(\min E_i)_{i=1}^k \in \mathcal{M}$,
(2) $\mathcal{M}$-*allowable* if $(E_i)_{i=1}^k$ are pairwise disjoint and $(\min E_i)_{i=1}^k \in \mathcal{M}$.

Let X be a Banach space with a basis. We say that a sequence $x_1, \ldots, x_n$ is $\mathcal{M}$-*admissible* (resp. *allowable*) if $(\operatorname{supp} x_i)_{i=1}^n$ is $\mathcal{M}$-admissible (resp. allowable).

The $(\mathcal{M}, \theta)$-*operation*, with $0 < \theta \leq 1$, is the operation associating with a sequence $(x_1, \ldots, x_k)$ with $(\min \operatorname{supp} x_i)_{i=1}^k \in \mathcal{M}$ the vector $\theta(x_1 + \cdots + x_k)$.

Fix sequences $(\theta_n)_n \subset (0,1)$, $(k_n) \nearrow +\infty$ and $(\mathcal{M}_n)$ with either $\mathcal{M}_n = \mathcal{A}_{k_n}$ for all n or $\mathcal{M}_n = \mathcal{S}_{k_n}$ for all n. The *mixed Tsirelson space* $T[(\mathcal{M}_n, \theta_n)_n]$ is defined to be the completion of $c_{00}(\mathbb{N})$ endowed with the norm whose norming set K is the smallest subset of $c_{00}(\mathbb{N})$ containing $(\pm e_n)_n$, where $(e_n)_n$ is the canonical basis of $c_{00}(\mathbb{N})$, and closed under all $(\mathcal{M}_n, \theta_n,)$-operations on block sequences. If one allows also $(\mathcal{M}_n, \theta_n)$-operations on sequences of vectors with pairwise disjoint supports for some $n \in \mathbb{N}$, one gets (boundedly) *modified mixed Tsirelson spaces* (cf. [ADM]).

The first famous member of the family of spaces $T[(\mathcal{A}_{k_n}, \theta_n)_n]$ is the Schlumprecht space [S], the first space known to be arbitrarily distortable (see also [M] for a study of this class of spaces). The spaces $T[(\mathcal{S}_{k_n}, \theta_n)_n]$ were introduced in [AD]. Allowing some $(\mathcal{M}_n, \theta_n)$-operations on special block sequences, defined by means of a suitably chosen coding function, led to the Gowers–Maurey construction of the first known HI space [GM]. Adding in the definition of the norming set other operations of a different kind allowed building spaces enjoying extreme properties, like the HI asymptotically unconditional space of Gowers [G2] or a quasi-minimal and tight-by-range space with an unconditional basis [AMP].

3. Definition of the space $\mathcal{X}_{\mathrm{cr}}$. The space we shall define is constructed on the basis of the boundedly modified mixed Tsirelson space $T_M[(\mathcal{S}_{n_j}, 1/m_j)_j]$ with the use of an additional coding function. First we describe the basic ingredients of the construction.

We fix two sequences $(m_j)_j$ and $(n_j)_j$ of natural numbers defined recursively as follows. We set $m_1 = 2$ and $m_{j+1} = m_j^5$ and $n_1 = 4$ and $n_{j+1} = 15 s_j n_j$ where $s_j = \log_2(m_{j+1}^3)$, $j \geq 1$.

Let X be a Banach space with a basis (e_i) satisfying

$$\frac{1}{m_{2j}} \sum_i \|E_i x\| \leq \|x\| \text{ for any } x \in X,\ j \in \mathbb{N},\ (E_i)\ \mathcal{S}_{n_{2j}}\text{-admissible.} \tag{3.1}$$

We now recall standard facts on vectors of a special type in such a space.

Definition 3.1 (Special convex combination). Fix $\varepsilon > 0$ and $n \in \mathbb{N}$.

We call a vector $y = \sum_{i \in F} a_i e_i$ an (n, ε)-*basic special convex combination* (basic scc) if $F \in \mathcal{S}_n$ and the scalars $(a_i) \subset [0,1]$ satisfy $\sum_{i \in F} a_i = 1$ and $\sum_{i \in G} a_i < \varepsilon$ for any $G \in \mathcal{S}_{n-1}$.

We call a vector $x = \sum_{i \in F} a_i x_i$ an (n, ε)-*special convex combination* (scc) of (x_i) if the vector $y = \sum_{i \in F} a_i e_{\min \operatorname{supp} x_i}$ is an (n, ε)-basic scc.

We call a scc $x = \sum_{i \in F} a_i x_i$ of a normalized block sequence (x_i) a *seminormalized scc* if $\|x\| \geq 1/2$.

It is well known (see [ATo, Prop. 2.3]) that for every $n \in \mathbb{N}$, $\varepsilon > 0$ and every $L \subset \mathbb{N}$ there exists an (n, ε)-basic scc $x = \sum_{n \in F} a_n e_n$ such that F is a maximal $\mathcal{S}_n$-subset of L. The next lemma provides seminormalized scc's in every block subspace.

Lemma 3.2 ([ADM, Lemma 4.5]). *For every $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $l(n, \varepsilon) \in \mathbb{N}$ such that for any block sequence (x_i) there is $F \in \mathcal{S}_{l(n,\varepsilon)}$ such that there is an (n, ε)-scc x supported on $(x_i)_{i \in F}$ with $\|x\| \geq 1/2$.*

Recall that for any $n \in \mathbb{N}$ and $\varepsilon > 0$ the constant $l(n, \varepsilon) \geq n$ depends only on the sequences (n_j) and (m_j).

Given any $n \in \mathbb{N}$ let $\rho(n)$ be the constant $l(n_{2s}, m_{2s}^{-2})$ obtained from the above lemma, where $s \in \mathbb{N}$ is minimal with

$$n^2 \leq m_{2s}. \tag{3.2}$$

We fix a partition of $\mathbb{N}$ into two infinite sets L_1, L_2. Let

$$\mathcal{G} = \{(E_1, \ldots, E_n) : E_1 < \cdots < E_n \text{ intervals of } \mathbb{N},\ n \in \mathbb{N}\}$$

and take a 1-1 *coding function* $\sigma : \mathcal{G} \to 2L_2$ such that for any sequence $(E_1, \ldots, E_n) \in \mathcal{G}$, $n \geq 2$, we have

$$n_{\sigma(E_1,\ldots,E_n)} \geq \rho(\max E_n) + \max E_n. \tag{3.3}$$

Let W be the smallest subset of $c_{00}(\mathbb{N})$ such that

(α) $(\pm e_n)_n \in W$, where $(e_n)_n$ is the canonical basis of $c_{00}(\mathbb{N})$,
(β) for any $f \in W$ and $g \in c_{00}(\mathbb{N})$ with $|f| = |g|$ also $g \in W$,
(γ) W is closed under the $(\mathcal{S}_{n_{2j}}, m_{2j}^{-1})$-operations on any allowable sequences,
(δ) W is closed under the $(\mathcal{S}_{n_{2j+1}}, m_{2j+1}^{-1})$-operations on $(2j+1)$-dependent sequences.

In order to complete the definition we need to define dependent sequences.

Definition 3.3 (Dependent sequence). A block sequence $(f_i)_{i\in F} \subset c_{00}(\mathbb{N})$ is said to be $(2j+1)$-*dependent* if it is $\mathcal{S}_{n_{2j+1}}$-admissible, each f_i is of the form $f_i = m_{2j_i}^{-1} \sum_{k\in K_i} f_{i,k}$, and for some sequences $(E_r)_{r\in A} \in \mathcal{G}$, with the index set $A \subset \mathbb{N}$ represented as a union $\bigcup\{A_k : k \in K_i,\ i \in F\}$ of intervals, the following hold:

(1) $w(f_1) = m_{2j_1}^{-1}$, $j_1 \in L_1$ and $m_{2j_1} > n_{2j+1}$,
(2) $2j_{i+1} = \sigma(E_j : j \in A_k,\ k \in K_l,\ l \leq i)$ for any $i < \max F$,
(3) $\operatorname{supp} f_{i,k} \subset \bigcup_{j\in A_k} E_j$ for any $k \in K_i$, $i \in F$,
(4) $A_{\min K_i}$ is a singleton for each $i \in F$, and $(E_r)_{r\in A_k}$ is $\mathcal{S}_{\rho(\max E_{\max A_{k-1}})}$-admissible for any $k \in K_i$, $k > \min K_i$, $i \in F$.

Any functional of the form $m_{2j+1}^{-1} \sum_{i\in F} f_i$, where $(f_i)_{i\in F}$ is a dependent sequence, is called a *special functional.*

Notice that as $(E_r)_r \in \mathcal{G}$, by (3) each family $(f_{i,k})_{k\in K_i}$ is $\mathcal{S}_{n_{2j_i}}$-admissible.

Let $\mathcal{X}_{\mathrm{cr}}$ be the completion of $c_{00}(\mathbb{N})$ with the norm $\|\cdot\|$ for which W is a norming set, i.e. $\|\cdot\| = \sup\{|f(\cdot)| : f \in W\}$.

Remark 3.4. (a) The canonical basis $(e_n)_n$ of $\mathcal{X}_{\mathrm{cr}}$ is 1-sign unconditional by (β).

(b) Note that in the definition of dependent sequences the admissibility of the functional chosen in the $(i+1)$th step depends not on the supports or ranges of the previously chosen functionals, but on the choice of some intervals containing the ranges of the previously chosen functionals.

(c) By (3) of the definition of special functionals we can choose $f_{i,k} = 0$. Also by (3) any restriction of a special functional to a subset of $\mathbb{N}$ is also a special functional. This property also easily implies that the set W is closed under projections on subsets of $\mathbb{N}$, and $(e_n)_n$ is a 1-unconditional basis.

(d) The space $\mathcal{X}_{\rm cr}$ satisfies (3.1) by (γ).

(e) Reflexivity of $\mathcal{X}_{\rm cr}$ can be proved by repeating the argument of [AD].

(f) The norming set W of $\mathcal{X}_{\rm cr}$ is contained in the norming set of the modified mixed Tsirelson space $T_M[(\mathcal{S}_{n_j}, 1/m_j)_j]$ (cf. [ADM]).

As in the previous cases of norming sets defined to be closed under certain operations, every functional $f \in W$ admits a tree-analysis, which in the present case is described as follows.

Definition 3.5 (Tree-analysis of a functional). Let $f \in W$. A family $(f_\alpha)_{\alpha\in\mathcal{T}}$, where $\mathcal{T}$ is a rooted finite tree, is a *tree-analysis* of f if:

(1) $f = f_0$ where 0 denotes the root of $\mathcal{T}$.

(2) If α is a maximal element of $\mathcal{T}$ then $f_\alpha = \pm e_n^*$ for some $n \in \mathbb{N}$. If $\alpha \in \mathcal{T}$ is not maximal, then one of the next two conditions holds.

(3) $f_\alpha = \frac{1}{m_j}\sum_{\beta\in\mathrm{succ}(\alpha)} f_\beta$ with $(f_\beta)_{\beta\in\mathrm{succ}(\alpha)}$ $\mathcal{S}_{n_j}$-allowable, for $j \in 2\mathbb{N}$.

(4) $f_\alpha = \frac{1}{m_j}\sum_{\beta\in\mathrm{succ}(\alpha)} f_\beta$ with $(f_\beta)_{\beta\in\mathrm{succ}(\alpha)}$ $\mathcal{S}_{n_j}$-admissible, for $j \in 2\mathbb{N}+1$.

In cases (3) and (4) we define the weight $w(f_\alpha)$ of f_α to be m_j^{-1}.

For any $0 \neq \alpha \in \mathcal{T}$ we set $\mathrm{tag}(\alpha) = \prod_{\beta\prec\alpha} w(f_\beta)$ and define $\mathrm{ord}(\alpha)$ to be the length of the branch linking α and the root 0.

Lemma 3.6 ([ADM, Lemma 4.6]). *Let $j \in \mathbb{N}$ and let $f \in W$ be a norming functional with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}}$. Let*

$$\mathcal{F} = \Big\{\alpha \in \mathcal{T} : \prod_{\beta\prec\alpha} w(f_\beta) > 1/m_j^2 \text{ and } w(f_\beta) \geq 1/m_{j-1} \text{ for all } \beta \prec \alpha\Big\}.$$

Then for any subset $\mathcal{G}$ of $\mathcal{F}$ of incomparable nodes the set $\{f_\alpha : \alpha \in \mathcal{G}\}$ is $\mathcal{S}_{\frac{1}{5}n_j}$-allowable and for any $\alpha \in \mathcal{F}$ we have $\mathrm{ord}(\alpha) \leq m_j$.

Proof. For every $\alpha \in \mathcal{G}$ by the assumptions we get

$$\frac{1}{m_j^2} < \prod_{\beta\prec\alpha} w(f_\beta) \leq \Big(\frac{1}{m_1}\Big)^{\mathrm{ord}(\alpha)} \Rightarrow \mathrm{ord}(\alpha) \leq 2\log_{m_1}(m_j).$$

Since $n_\beta \leq n_{j-1}$ for all $\beta \prec \alpha$, it follows that for every $s \leq 2\log_{m_1}(m_j)$ the nodes of $\mathcal{G}$ at the sth level of the tree are at most $\mathcal{S}_{sn_{j-1}}$-allowable. It follows that all nodes of $\mathcal{G}$ are at most $\mathcal{S}_{2\log_{m_1}(m_j)n_{j-1}}$-allowable, thus also $\mathcal{S}_{\frac{1}{5}n_j}$-allowable. ∎

Lemma 3.7 ([ADM]). *For any (n_j, m_j^{-2})-scc $x = \sum_{i\in F} a_i x_i \in \mathcal{X}_{\rm cr}$ with $\|x_i\| \leq C$ for every $i \in F$ and any $\mathcal{S}_{n_j-1}$-allowable family $(f_p)_{p\in A} \subset W$ of norming functionals we have*

$$\sum_{p\in A} f_p(x) \leq 3C.$$

DEFINITION 3.8. Fix $C > 0$. A block sequence (x_k) is called a *C-rapidly increasing sequence* (C-RIS) if $\|x_k\| \le C$ for each k and there exists a strictly increasing sequence $(j_k) \subset \mathbb{N}$ such that

(1) $\max \operatorname{supp} x_k \le m_{j_{k+1}}/m_{j_k}$ for any k,
(2) $|f(x_k)| \le Cw(f)$ for every $f \in W$ with $w(f) > m_{j_k}^{-1}$ and any k.

By repeating the proof of [ADM, Prop. 4.12] we obtain the following

LEMMA 3.9. *For any (n_j, m_j^{-2})-scc $x = \sum_k a_k x_k$ of a C-RIS (x_k) defined by a sequence (j_k) with $j+2 < j_1$ and any norming functional $f \in W$ with weight $w(f) = m_s^{-1}$ we have*

$$|f(x)| \le \begin{cases} 14C/(m_s m_j) & \text{if } s < j, \\ 8C/m_j & \text{if } s = j, \\ 8C/m_j^2 & \text{if } s > j. \end{cases}$$

In particular $\|x\| \le 8C/m_j$ and for any $\mathcal{S}_{n_{2s}}$-allowable family $(f_\alpha)_{\alpha \in A} \subset W$ with $2s < j$ we have

$$\sum_{\alpha \in A} f_\alpha(m_j x) \le 14C. \tag{3.4}$$

Notice that by the above lemma a sequence of scaled vectors $(m_{j_k} x_k)$, where each x_k is an $(n_{j_k}, m_{j_k}^{-2})$-scc of some C-RIS, satisfying (1) of Def. 3.8, is also a $14C$-RIS. As by Lemmas 3.2 and 3.7 any block subspace of $\mathcal{X}_{\mathrm{cr}}$ contains a 2-RIS of seminormalized scc's, by Lemma 3.9 any block subspace contains also for any $j \in \mathbb{N}$ a scaled $(n_{2j+1}, m_{2j+1}^{-2})$-scc of a 28-RIS.

By repeating the proof of [ADM, Lemma 4.10] with the use of the above estimate we obtain the following.

LEMMA 3.10. *Let $j > 5$, $u = m_{2j+1} \sum_k a_k x_k$ be a scaled $(n_{2j+1}, m_{2j+1}^{-2})$-scc of a 28-RIS (x_k) defined by a sequence (j_k) with $j + 2 < j_1$. Then any norming functional f with a tree-analysis $(f_\alpha)_{\alpha \in \mathcal{T}}$ such that $w(f_\alpha) > 1/m_{2j+1}$ for any $\alpha \in \mathcal{T}$ satisfies*

$$f(u) \le 1/m_{2j}.$$

4. Properties of the space $\mathcal{X}_{\mathrm{cr}}$. In this section we study the minimality properties of $\mathcal{X}_{\mathrm{cr}}$. We deal first with tightness with constants, as it follows immediately from [FR1].

DEFINITION 4.1 ([FR1]). A Banach space X with a basis (e_n) is called *tight with constants* if for any infinite-dimensional subspace Y of X there is a sequence of successive intervals $I_1 < I_2 < \cdots$ such for any $K \in \mathbb{N}$ the subspace Y does not embed with constant K into $[e_i : i \notin I_K]$.

Recall that a Banach space with a basis is *ℓ_1-strongly asymptotic* if any $\mathcal{S}_1$-allowable sequence $(x_1, \ldots, x_n)$ of normalized vectors is C-equivalent to the u.v.b. of ℓ_1^n, for any $n \in \mathbb{N}$ and some universal $C \geq 1$. By (γ) in the definition of its norming set and Remark 3.4, the space $\mathcal{X}_{\mathrm{cr}}$ is ℓ_1-strongly asymptotic. Since $\mathcal{X}_{\mathrm{cr}}$ is also reflexive, by [FR1, Prop. 4.2] we obtain the following.

THEOREM 4.2. *The space $\mathcal{X}_{\mathrm{cr}}$ is tight with constants.*

We now pass to the proof of quasi-minimality; adapting it suitably we shall also deduce tightness by range. Recall that a Banach space is *quasi-minimal* if any two of its infinite-dimensional subspaces have further two infinite-dimensional subspaces which are isomorphic.

THEOREM 4.3. *The space $\mathcal{X}_{\mathrm{cr}}$ is quasi-minimal.*

Proof. Given two block subspaces Y, Z of X we pick a block RIS (u_n) and (v_n) satisfying:

(A) $u_n = m_{2j_n+1} \sum_{i\in I_n} b_i y_i$ is a scaled $(n_{2j_n+1}, m_{2j_n+1}^{-2})$-scc of a 28-RIS (y_i),
$v_n = m_{2j_n+1} \sum_{i\in I_n} b_i z_i$ is a scaled $(n_{2j_n+1}, m_{2j_n+1}^{-2})$-scc of a 28-RIS (z_i),
$\min\operatorname{supp} u_n, \min\operatorname{supp} v_n > m_{2j_n+1}$ for all $n \in \mathbb{N}$,
$\sum_n m_{2j_n}^{-1} < 100^{-1}$,

(B) $y_i = m_{2j_i} \sum_{k\in K_i} b_{i,k} y_{i,k}$ is a scaled $(n_{2j_i}, m_{2j_i}^{-2})$-scc of a 2-RIS $(y_{i,k})_{k\in K_i}$,
$z_i = m_{2j_i} \sum_{k\in K_i} b_{i,k} z_{i,k}$ is a scaled $(n_{2j_i}, m_{2j_i}^{-2})$-scc of a 2-RIS $(z_{i,k})_{k\in K_i}$,
$2j_i > 2j_n + 3$, $y_i^* = \frac{1}{m_{2j_i}} \sum_{k\in K_i} y_{i,k}^*$, $z_i^* = \frac{1}{m_{2j_i}} \sum_{k\in K_i} z_{i,k}^*$, for each $i \in I_n$, $n \in \mathbb{N}$,

(C) $y_{i,k}$ is an $(n_{2j_{i,k}}, m_{2j_{i,k}}^{-2})$-scc with $\|y_{i,k}\| \geq 1/2$ supported on some block sequence equivalent to a block sequence in Y,
$z_{i,k}$ is an $(n_{2j_{i,k}}, m_{2j_{i,k}}^{-2})$-scc with $\|z_{i,k}\| \geq 1/2$ supported on some block sequence equivalent to a block sequence in Z,
$y_{i,k}^*(y_{i,k}) = 1 = z_{i,k}^*(z_{i,k})$, $\operatorname{range} y_{i,k}^* = \operatorname{range} y_{i,k} = \operatorname{range} z_{i,k}^* = \operatorname{range} z_{i,k}$ for each $k \in K_i$, $i \in I_n$, $n \in \mathbb{N}$,

(D) $(y_i^*)_{i\in I_n}$, $(z_i^*)_{i\in I_n}$ are $(2j_n+1)$-dependent sequences, defined by the collection of intervals $(\operatorname{range} y_{i,k})_{k\in K_i, i\in I_n}$, $n \in \mathbb{N}$ (in the notation of Def. 3.3 for each $k \in K_i$, $i \in I_n$ we take as the collection $(E_j)_{j\in A_k}$ just one interval, $\operatorname{range} y_{i,k}$).

The construction is straightforward: using Lemma 3.2 we first construct two infinite 2-RIS $(\hat{y}_k)_{k\in\mathbb{N}} \subset Y$ and $(\hat{z}_k)_{k\in\mathbb{N}} \subset Z$ of scc's with norm at least $1/2$ with a common sequence $(j_k)_k$. Allowing a small perturbation we can also assume that $\operatorname{range} \hat{y}_k = \operatorname{range} \hat{z}_k$. Then we define (y_i) and (z_i) as 28-RIS's of scaled scc's of $(\hat{y}_k)_k$ and $(\hat{z}_k)_k$ respectively, with the same coefficients with respect to $(y_{i,k})_{k\in K_i} \subset (\hat{y}_k)_k$ and $(z_{i,k})_{k\in K_i} \subset (\hat{z}_k)_k$. We repeat the procedure, building sequences of scaled scc's $(u_n)_n$ and $(v_n)_n$ on $(y_i)_i$ and $(z_i)_i$ respectively.

Note that by the definition of the coding function (3.3), using that $\max E_{i,\max K_i} = \max \operatorname{supp} y_i$ we get, for each $i \in I_n, n \in \mathbb{N}$,

(E) $\quad n_{2j_{i+1}} > \rho(\max \operatorname{supp} y_i) + \max \operatorname{supp} y_i > \rho(\max \operatorname{supp} y_i) + 3n_{2j_n+1}.$

We claim that the sequences (u_n) and (v_n) are equivalent. Take any non-negative scalars (a_n) with $\|\sum_n a_n u_n\| = 1$, let $u = \sum_n a_n u_n$ and take a norming functional f with a tree-analysis $(f_\alpha)_{\alpha\in\mathcal{T}}$ such that $f(u) = 1$. Since the norming set W is invariant under changing signs of coefficients by condition (β) in the definition of the norming set W, we can assume that all coefficients of the vectors (u_n), (v_n) and the functional f are non-negative.

By modifying the tree-analysis of f we shall construct a tree-analysis of some norming functional g such that $g(\sum_n a_n z_n) \geq 1/6$. We shall first make some reductions, erasing nodes of $\mathcal{T}$ with some controllable error. After the reductions we define suitable replacements of certain nodes f_α, $\alpha \in \mathcal{T}$, in order to define g.

First we introduce some notation. For any collection $\mathcal{E}$ of nodes of $\mathcal{T}$ we shall write $\operatorname{supp}\mathcal{E} = \bigcup_{\alpha\in\mathcal{E}} \operatorname{supp} f_\alpha$. We shall prove several reductions, enabling us to restrict the tree-analysis of f to the nodes convenient for the replacement procedure.

First Reduction. *For any $n \in \mathbb{N}$ let*

$$\mathcal{P}_n = \{\alpha \in \mathcal{T} : \operatorname{supp} f_\alpha \cap \operatorname{range} u_n \neq \emptyset \textit{ and } \alpha \in \mathcal{T} \textit{ is minimal with } w(f_\alpha) \leq m_{2j_n+1}^{-1}\}.$$

With error $2m_{2j_n}^{-1}$ we can assume that $(f_\alpha|_{\operatorname{range} u_n})_{\alpha\in\mathcal{P}_n}$ is $\mathcal{S}_{\frac{1}{5}n_{2j_n+1}}$-allowable and $\operatorname{supp} f|_{\operatorname{range} u_n} \subset \operatorname{supp}\mathcal{P}_n$ *and for any $\alpha \in \mathcal{P}_n$ we have* $\operatorname{ord}(\alpha) \leq m_{2j_n+1}$.

Proof. Notice first that by Lemma 3.10 we have

$$(f - f|_{\operatorname{supp}\mathcal{P}_n})(u_n) \leq \frac{1}{m_{2j_n}},$$

hence with error $m_{2j_n}^{-1}$ we can assume that $\operatorname{supp} f|_{\operatorname{range} u_n} \subset \operatorname{supp}\mathcal{P}_n$. Now let

$$\mathcal{P}_{n,1} = \Big\{\alpha \in \mathcal{P}_n : \prod_{\beta\prec\alpha} w(f_\beta) \leq m_{2j_n+1}^{-2}\Big\}.$$

For every $\alpha \in \mathcal{P}_{n,1}$ choose $\beta_\alpha \prec \alpha$ such that

$$\prod_{\gamma\prec\beta_\alpha} w(f_\gamma) > m_{2j_n+1}^{-2} \quad \text{and} \quad \prod_{\gamma\preceq\beta_\alpha} w(f_\gamma) \leq m_{2j_n+1}^{-2}.$$

It follows that

$$\frac{1}{m_{2j_n+1}^2} < \prod_{\gamma\prec\beta_\alpha} w(f_\gamma) \leq w(f_{\beta_\alpha})^{-1} \prod_{\gamma\preceq\beta_\alpha} w(f_\gamma) \leq \frac{m_{2j_n}}{m_{2j_n+1}^2}. \tag{4.1}$$

Note that if $\alpha, \alpha_1 \in \mathcal{P}_{n,1}$ the nodes $\beta_\alpha, \beta_{\alpha_1}$ are either incomparable or equal. Set

$$\mathcal{R}_n = \{\beta_\alpha : \alpha \in \mathcal{P}_{n,1}\}.$$

By Lemma 3.6 we see that

$$\{f_\beta : \beta \in \mathcal{R}_n\} \text{ is } \mathcal{S}_{n_{2j_n+1}-1}\text{-allowable.} \tag{4.2}$$

Consequently, using (4.1), (4.2) and Lemma 3.7 for the scc $\sum_i b_i y_i$, we obtain

$$\begin{aligned} f|_{\operatorname{supp}\mathcal{P}_{n,1}}\Big(m_{2j_n+1}\sum_{i\in I_n} b_i y_i\Big) &\le m_{2j_n+1}\sum_{\beta\in\mathcal{R}_n}\Big(\prod_{\gamma\prec\beta_\alpha} w(f_\gamma)\Big) f_\beta\Big(\sum_{i\in I_n} b_i y_i\Big) \\ &\le \frac{m_{2j_n}}{m_{2j_n+1}}\sum_{\beta\in\mathcal{R}_n} f_\beta\Big(\sum_{i\in I_n} b_i y_i\Big) \le 3\cdot 28\frac{1}{m_{2j_n}^2} \le \frac{1}{m_{2j_n}}. \end{aligned}$$

Now Lemma 3.6 applied to the family $\{f_\alpha : \alpha \in \mathcal{P}_n \setminus \mathcal{P}_{n,1}\}$ finishes the proof of the first reduction.

Second Reduction. *For any $n \in \mathbb{N}$, with error $m_{2j_n}^{-1}$ we can assume that $w(f_\alpha) = m_{2j_n+1}^{-1}$ for any $\alpha \in \mathcal{P}_n$.*

Proof. Set $\mathcal{P}_{n,2} = \{\alpha \in \mathcal{P}_n : w(f_\alpha) < m_{2j_n+1}^{-1}\}$. For any $\alpha \in \mathcal{P}_{n,2}$ pick $i_\alpha \in I_n$ with $m_{2j_{i_\alpha}}^{-1} \ge w(f_\alpha) > m_{2j_{i_\alpha+1}}^{-1}$. If $w(f_\alpha) \le m_{2j_i}^{-1}$ for any $i \in I_n$ let $i_\alpha = \max I_n$; if $w(f_\alpha) > m_{2j_i}^{-1}$ for all $i \in I_n$, put $i_\alpha = 0$.

Split $f|_{\operatorname{supp}\mathcal{P}_{n,2}}(u_n)$ in the following way:

$$\begin{aligned} f|_{\operatorname{supp}\mathcal{P}_{n,2}}(u_n) \le m_{2j_n+1}&\sum_{i\in I_n} b_i \sum_{\alpha\in\mathcal{P}_{n,2}:\, i_\alpha>i} f_\alpha(y_i) \\ &+ m_{2j_n+1}\sum_{i\in I_n} b_i \sum_{\alpha\in\mathcal{P}_{n,2}:\, i-2\le i_\alpha\le i} \operatorname{tag}(f_\alpha) f_\alpha(y_i) \\ &+ m_{2j_n+1}\sum_{i\in I_n} b_i \sum_{\alpha\in\mathcal{P}_{n,2}:\, i_\alpha<i-2} f_\alpha(y_i) \end{aligned}$$

Fix $i \in I_n$ and compute, using condition (1) of Def. 3.8 for the last estimate:

$$\begin{aligned} \sum_{\alpha\in\mathcal{P}_n:\, i_\alpha>i} f_\alpha(y_i) &\le \sum_{\alpha\in\mathcal{P}_n:\, i_\alpha>i} w(f_\alpha) \sum_{\gamma\in\operatorname{succ}(\alpha)} f_\gamma(y_i) \\ &\le \sum_{\alpha\in\mathcal{P}_n:\, i_\alpha>i} \frac{1}{m_{2j_{i_\alpha}}} \sum_{\gamma\in\operatorname{succ}(\alpha)} f_\gamma(y_i) \\ &\le \sum_{\alpha\in\mathcal{P}_n:\, i_\alpha>i} \frac{1}{m_{2j_{i+1}}} \sum_{\gamma\in\operatorname{succ}(\alpha)} f_\gamma(y_i) \\ &\le \frac{1}{m_{2j_{i+1}}} 28 \max\operatorname{supp} y_i \le \frac{28}{m_{2j_i}}. \end{aligned}$$

For the second part notice that for each $\alpha \in \mathcal{P}_n$ there are at most three i's in I_n with $i_\alpha \in \{i-2, i-1, i\}$. Denote the set of all such i's by J_n. As by the first reduction $(f_\alpha|_{\mathrm{range}\, u_n})_{\alpha\in\mathcal{P}_n}$ is $\mathcal{S}_{\frac{1}{5}n_{2j_n+1}}$-admissible, using that u_n is a scaled $(n_{2j_n+1}, m_{2j_n+1}^{-2})$-scc we obtain

$$m_{2j_n+1}\sum_{i\in I_n} b_i \sum_{\alpha\in\mathcal{P}_{n,2}:\, i-2\le i_\alpha\le i} \mathrm{tag}(f_\alpha) f_\alpha(y_i) \le \Big\| m_{2j_n+1}\sum_{i\in\mathcal{J}_n} b_i y_i\Big\| \le \frac{6\cdot 28}{m_{2j_n+1}}.$$

For the third part notice that if $i_\alpha \le i-3$, then $w(f_\alpha) \ge m_{2j_{i-2}}^{-1}$. Fix again $i \in I_n$, and estimate, by the definition of $\mathcal{P}_{n,2}$,

$$\sum_{\alpha\in\mathcal{P}_{n,2}:\, i_\alpha<i-2} f_\alpha(y_i) \le \frac{1}{m_{2j_n+2}} \sum_{E\in\mathcal{H}_i} \|Ey_i\|,$$

where $\mathcal{H}_i = \{\mathrm{supp}\, f_\gamma \cap \mathrm{supp}\, y_i : \gamma \in \mathrm{succ}(\alpha),\ i-2 > i_\alpha\}$. Notice that each $\mathcal{H}_i$ is $\mathcal{S}_{n_{2j_n+1}+n_{2j_{i-2}}}$-allowable, thus also $\mathcal{S}_{n_{2j_i-2}}$-allowable, hence by (3.4), $\sum_{E\in\mathcal{H}_i}\|Ey_i\| \le 14\cdot 28$ for each $i \in I_n$. Putting together the above estimates we obtain

$$\begin{aligned} f|_{\mathrm{supp}\,\mathcal{P}_{n,2}}(u_n) &\le m_{2j_n+1}\sum_{i\in I_n} b_i \frac{28}{m_{2j_i}} + \frac{6\cdot 28}{m_{2j_n+1}} + m_{2j_n+1}\sum_{i\in I_n} b_i \frac{14\cdot 28}{m_{2j_n+2}} \\ &\le \frac{1}{m_{2j_n}} \end{aligned}$$

as $\sum_i b_i = 1$. Thus erasing the set $\mathcal{P}_{n,2}$ we finish the proof of the second reduction.

THIRD REDUCTION. *For any $n \in \mathbb{N}$ with error $m_{2j_n}^{-1}$ we can assume that for any $\alpha \in \mathcal{P}_n$ the special functional $f_\alpha|_{\mathrm{range}\, u_n}$ is defined by the sequence $(\mathrm{range}\, y_{i,k})_{k\in K_i, i\in I_n}$, in particular*

$$f_\alpha|_{\mathrm{range}\, u_n} = \frac{1}{m_{2j_n+1}} \sum_{\mathrm{supp}\, f_\alpha\cap\mathrm{supp}\, y_i\neq\emptyset} f_i^\alpha$$

with $f_i^\alpha = \frac{1}{m_{2j_i}}\sum_{k\in K_i} f_{i,k}^\alpha$ and $\mathrm{supp}\, f_{i,k}^\alpha \subset \mathrm{range}\, y_{i,k}$ for each $k \in K_i$, $i \in I_n$.

Proof. Recall that by the definition of a dependent sequence, for any $\beta \in \mathrm{succ}(\mathcal{P}_n)$ we have $w(f_\beta) = m_{2s}^{-1}$ for some $s \in \mathbb{N}$. Fix $i \in I_n$ and let

$$\begin{aligned} \mathcal{P}_{n,i} &= \{\beta \in \mathrm{succ}(\mathcal{P}_n) : w(f_\beta) < m_{2j_i}^{-1}\}, \\ \mathcal{Q}_{n,i} &= \{\beta \in \mathrm{succ}(\mathcal{P}_n) : w(f_\beta) > m_{2j_i-2}^{-1}\}. \end{aligned}$$

Notice that by the first and second reductions the family $(f_\beta)_{\beta\in\mathcal{P}_{n,i}}$ is $\mathcal{S}_{\frac{6}{5}n_{2j_n+1}}$-allowable, thus also $\mathcal{S}_{n_{2j_i-1}}$-allowable. Moreover y_i is a scaled $(n_{2j_i}, m_{2j_i}^{-2})$-scc. Therefore we can repeat the reasoning from the second reduction to obtain $f|_{\mathrm{supp}\,\mathcal{P}_{n,i}}(y_i) \le 2m_{2j_i-1}^{-1}$.

By the first and second reductions the family $\{f_\gamma : \gamma \in \mathrm{succ}(\beta),\ \beta \in \mathcal{Q}_{n,i}\}$ is $\mathcal{S}_{\frac{6}{5}n_{2j_n+1}+n_{2j_i-4}}$-allowable, thus $\mathcal{S}_{n_{2j_i-2}}$-allowable. Thus by (3.4),

$$f|_{\mathrm{supp}\,\mathcal{Q}_{n,i}}(y_i) \le \frac{1}{m_{2j_n+1}} \sum_{\beta\in\mathcal{Q}_{n,i}} w(f_\beta) \sum_{\gamma\in\mathrm{succ}(\beta)} f_\gamma(y_i) \le \frac{14\cdot 28}{m^3_{2j_n+1}},$$

as any f_β is an immediate descendant of some special functional f_α, $\alpha \in \mathcal{P}_n$, with $w(f_\alpha) = m^{-1}_{2j_n+1}$ by the second reduction.

Let $\tilde{\mathcal{P}}_{n,3} = \bigcup_i (\mathrm{supp}\,\mathcal{P}_{n,i} \cap \mathrm{supp}\, y_i)$ and $\tilde{\mathcal{Q}}_{n,3} = \bigcup_i (\mathrm{supp}\,\mathcal{Q}_{n,i} \cap \mathrm{supp}\, y_i)$. Then by the above,

$$\begin{aligned} f|_{\tilde{\mathcal{P}}_{n,3}\cup\tilde{\mathcal{Q}}_{n,3}}(u_n) &\le m_{2j_n+1} \sum_{i\in I_n} \frac{2b_i}{m_{2j_i-1}} + 14\cdot 28 m_{2j_n+1} \sum_{i\in I_n} \frac{b_i}{m^3_{2j_n+1}} \\ &\le \frac{1}{m_{2j_n+1}}. \end{aligned}$$

Now notice that for any $\alpha \in \mathcal{P}_n$ there is at most one $i \in I_n$ with $w(f_\beta) = m^{-1}_{2j_i-2}$ for some $\beta \in \mathrm{succ}(\alpha)$. Denote the set of such i's by K_n. Therefore, by the first reduction, as u_n is a scaled $(n_{2j_n+1}, m^{-1}_{2j_n+1})$-scc,

$$m_{2j_n+1} \sum_{i\in I_n} b_i \sum_{\beta\in\mathrm{succ}(\mathcal{P}_n):\, w(f_\beta)=m^{-1}_{2j_i-2}} f_\beta(y_i) \le m_{2j_n+1} \Big\| \sum_{i\in K_n} b_i y_i \Big\| \le \frac{2\cdot 28}{m_{2j_n+1}}.$$

Summing up, with error $57m^{-1}_{2j_n+1}$ we can assume that for any $n \in \mathbb{N}$ and $\alpha \in \mathcal{P}_n$ we have $w(f_\beta) = m^{-1}_{2j_i}$ for any $\beta \in \mathrm{succ}(\alpha)$ with $\mathrm{supp}\, f_\beta \cap \mathrm{supp}\, y_i \neq \emptyset$. In particular, $\mathrm{supp}\, f_\beta$, $\beta \in \mathrm{succ}(\mathcal{P}_n)$, intersects at most one of $\mathrm{supp}\, y_i$'s.

By (2) in Def. 3.3 and (D) it follows that $f_\alpha|_{[1,\dots,\max\mathrm{supp}\, u_n]}$, $\alpha \in \mathcal{P}_n$, is a special functional defined by the intervals $(\mathrm{range}\, y_{i,k})_{k\in K_i, i\in I_n}$. In order to achieve that suitable $f^\alpha_{i,k}$ satisfy $\mathrm{supp}\, f^\alpha_{i,k} \subset \mathrm{range}\, y_{i,k}$ we shall make one more correction.

For any $\alpha \in \mathcal{P}_n$ let $i_\alpha = \max\{i \in I_n : \mathrm{supp}\, f_\alpha \cap \mathrm{supp}\, y_i \neq \emptyset\}$. Put $F_n = \{i_\alpha : \alpha \in \mathcal{P}_n\}$. Notice that, as u_n is a scaled $(n_{2j_n+1}, m^{-2}_{2j_n+1})$-scc and $(f_\alpha|_{\mathrm{range}\, u_n})_{\alpha\in\mathcal{P}_n}$ is $\mathcal{S}_{n_{2j_n+1}-1}$-allowable by the first reduction, we obtain

$$f\Big(m_{2j_n+1} \sum_{i\in F_n} b_i y_i\Big) \le \Big\| m_{2j_n+1} \sum_{i\in F_n} b_i y_i \Big\| \le \frac{2}{m_{2j_n+1}}.$$

Therefore, after erasing $\mathrm{supp}\, f|_{\bigcup_{i\in F_n} \mathrm{supp}\, y_i}$ with error $2m^{-1}_{2j_n+1}$ by (3) of Def. 3.3 we finish the proof of the third reduction.

Fourth Reduction. *Given $n \in \mathbb{N}$ let*

$$\mathcal{D}_n = \{\xi \in \mathcal{T} : \xi \prec \alpha \text{ for some } \alpha \in \mathcal{P}_n \text{ and } f_\xi \text{ is a special functional}\}.$$

For $\xi \in \mathcal{D}_n$ let $(E^\xi_j)_{j\in A_l, l\in G_s, s\in F}$ be the sequence of intervals that determines f_ξ (see Def. 3.3). With error $m^{-1}_{2j_n}$ we can assume that for any $i \in I_n$, $k \in K_i$

and any $\xi \in \mathcal{D}_n$ *with* $f_\xi(y_{i,k}) \neq 0$ *there are* $j \in A_l$, $l \in G_s$, $s \in F$ *such that* $\operatorname{range} y_{i,k} \subset E_j^\xi$.

Proof. For any $\xi \in \mathcal{D}_n$ let $i_\xi \in I_n$ be minimal with $\operatorname{supp} f_\xi \cap \operatorname{range} y_{i_\xi} \neq \emptyset$.

First notice that the family $(y_{i_\xi})_{\xi \in \mathcal{D}_n}$ is $\mathcal{S}_{n_{2j_n+1}-1}$-admissible. Indeed, for any $\xi \in \mathcal{D}_n$ we have $\operatorname{supp} f_\xi \cap \operatorname{range} u_n = \bigcup\{\operatorname{supp} f_\alpha : \xi \prec \alpha \in \mathcal{P}_n\} \cap \operatorname{range} u_n$ by the first reduction, thus

$$(\min \operatorname{supp} f_\xi|_{\operatorname{range} u_n})_{\xi \in \mathcal{D}_n} \subset (\min \operatorname{supp} f_\alpha|_{\operatorname{range} u_n})_{\alpha \in \mathcal{P}_n}. \tag{4.3}$$

Hence by the first reduction the family $(y_{i_\xi})_{\xi \in \mathcal{D}_n \setminus \{\xi_0\}}$, where $f_{\xi_0}|_{\operatorname{range} u_n}$ has the smallest min supp among $f_\xi|_{\operatorname{range} u_n}$, $\xi \in \mathcal{D}_n$, is $\mathcal{S}_{\frac{1}{5}n_{2j_n+1}}$-allowable, which yields the desired observation.

Therefore, as u_n is a scaled $(n_{2j_n+1}, m_{2j_n+1}^{-1})$-scc, we obtain

$$\Big\| m_{2j_n+1} \sum_{\xi \in \mathcal{D}_n} b_{i_\xi} y_{i_\xi} \Big\| \leq \frac{2 \cdot 28}{m_{2j_n+1}}.$$

Thus with the above error we may assume that for all $\xi \in \mathcal{D}_n$ there is some $i > i_\xi$ with $f_\xi(y_i) \neq 0$. Let now $\overline{E}_j^\xi = E_j^\xi \cap (\max \operatorname{supp} y_{i_\xi}, \max \operatorname{supp} u_n]$ for any $\xi \in \mathcal{D}_n$ and element E_j^ξ of a sequence defining f_ξ. It follows that for any $\xi \in \mathcal{D}_n$ we have

$$\min \operatorname{supp} f_\xi|_{\operatorname{range} u_n} \leq \min \bigcup_{j \in A_l,\, l \in G_s,\, s \in F} \overline{E}_j^\xi. \tag{4.4}$$

Notice that for any $\alpha \in \mathcal{P}_n$ there can be at most m_{2j_n+1} many $\xi \in \mathcal{D}_n$ with $\min \operatorname{supp} f_\alpha|_{\operatorname{range} u_n} = \min \operatorname{supp} f_\xi|_{\operatorname{range} u_n}$, as this relation implies that $\xi \prec \alpha$ and $\operatorname{ord}(\alpha) \leq m_{2j_n+1}$ by the first reduction. As $\min \operatorname{supp} u_n > m_{2j_n+1}$, by (4.4) for any $\alpha \in \mathcal{P}_n$ we have

$$\Big\{ \min \bigcup_{j \in A_l,\, l \in G_s,\, s \in F} \overline{E}_j^\xi : \min \operatorname{supp} f_\xi|_{\operatorname{range} u_n} = \min \operatorname{supp} f_\alpha|_{\operatorname{range} u_n} \Big\} \in \mathcal{S}_1. \tag{4.5}$$

Therefore by (4.3), (4.5), the first reduction and Lemma 2.1 we obtain

$$\Big\{ \min \bigcup_{j \in A_l,\, l \in G_s,\, s \in F} \overline{E}_j^\xi : \xi \in \mathcal{D}_n \Big\} \in \mathcal{S}^M_{\frac{1}{5}n_{2j_n+1}+1} = \mathcal{S}_{\frac{1}{5}n_{2j_n+1}+1}. \tag{4.6}$$

On the other hand by Def. 3.3 for any $(E_j^\xi)_{j \in A_l, l \in G_s, s \in F}$ with $\xi \in \mathcal{D}_n$, any union $\bigcup_{j \in A_l} E_j^\xi$ contains $\operatorname{supp} f_\gamma$ for some $\gamma \in \operatorname{succ}(\operatorname{succ} \xi)$ provided $f_\xi|_{\bigcup_{j \in A_l} E_j^\xi} \neq 0$. Denote the set of all such l's by $\overline{G}_s$. We shall prove that with the declared error for any interval E_j^ξ, for some $j \in A_l$, $l \in \overline{G}_s$, $s \in F$ defining some $\xi \in \mathcal{D}_n$ and any $i \in I_n$, $k \in K_i$ we have either $\operatorname{range} y_{i,k} \cap E_j^\xi = \emptyset$ or $\operatorname{range} y_{i,k} \subset E_j^\xi$, which ends the proof of the fourth reduction.

Take any $l \in \overline{G}_s$, $s \in F$ attached to some $\xi \in \mathcal{D}_n$ and consider $\gamma \in \operatorname{succ}(\operatorname{succ}\xi)$ with $\bigcup_{j\in A_l} E_j^\xi \supset \operatorname{supp} f_\gamma$. Since $\operatorname{supp} f|_{\operatorname{range} u_n} \subset \operatorname{range}\mathcal{P}_n$ by the first reduction, and by the definition of $\mathcal{P}_n$, for such γ we have either $\gamma \preceq \alpha$ for some $\alpha \in \mathcal{P}_n$, or $\operatorname{range} f_\gamma \cap \operatorname{range} u_n = \emptyset$. Therefore by the first reduction and as the functionals $(f_\gamma)_{\gamma\in\operatorname{succ}(\operatorname{succ}(\xi))}$ have successive supports, for any $\xi \in \mathcal{D}_n$ we have

$$\Big\{\min\Big(\bigcup_{j\in A_l} E_j^\xi \cap \operatorname{range} u_n\Big) : l \in \overline{G}_s,\ s \in F\Big\} \in \mathcal{S}_{\frac{1}{5}n_{2j_n+1}+1}. \tag{4.7}$$

Putting together (4.6) and (4.7), by Lemma 2.1 we obtain

$$\Big\{\min\Big(\bigcup_{j\in A_l} \overline{E}_j^\xi \cap \operatorname{range} u_n\Big) : l \in \overline{G}_s,\ s \in F,\ \xi \in \mathcal{D}_n\Big\} \in \mathcal{S}_{\frac{2}{5}n_{2j_n+1}+2}. \tag{4.8}$$

Set

$$J_n = \{\min I_n\} \cup \Big\{i \in I_n : \min \bigcup_{j\in A_l} \overline{E}_j^\xi \in (\max\operatorname{supp} y_{i-1}, \max\operatorname{supp} y_i] \text{ for some } l \in \overline{G}_s,\ s \in F,\ \xi \in \mathcal{D}_n\Big\}.$$

Using (4.8), as u_n is a scaled $(n_{2j_n+1}, m_{2j_n+1}^{-1})$-scc, we obtain

$$\Big\|m_{2j_n+1}\sum_{i\in J_n} b_i y_i\Big\| \le \frac{2\cdot 28}{m_{2j_n+1}}.$$

Thus for any $i \in I_n \setminus J_n$ and $(E_j^\xi)_{j\in A_l}$ with $l \in \overline{G}_s$, $s \in F$, $\xi \in \mathcal{D}_n$, if $\bigcup_{j\in A_l} \overline{E}_j^\xi \cap \operatorname{range} y_i \ne \emptyset$, then $\min\bigcup_{j\in A_l} \overline{E}_j^\xi \le \max\operatorname{supp} y_{i-1}$. By (4) in Def. 3.3 it follows that the family $(\overline{E}_j^\xi \cap \operatorname{range} y_i)_{j\in A_l}$ is $\mathcal{S}_{\rho(\max\operatorname{supp} y_{i-1})}$-admissible and consequently by (4.8) and Lemma 2.1 for any $i \in I_n$ we have

$$\{\min(\overline{E}_j^\xi \cap \operatorname{range} y_i) : j\in A_l,\ l\in\overline{G}_s,\ s\in F,\ \xi\in\mathcal{D}_n\} \in \mathcal{S}_{\rho(\max\operatorname{supp} y_{i-1})+n_{2j_n+1}}.$$

As y_i is a scaled $(n_{2j_i}, m_{2j_i}^{-1})$-scc, by condition (E) we obtain

$$\Big\|m_{2j_i}\sum_{k\in L_i} b_{i,k} y_{i,k}\Big\| \le \frac{2\cdot 2}{m_{2j_i}},$$

where L_i denotes the set of all $k \in K_i$ such that $\min \overline{E}_j^\xi$ or $\max \overline{E}_j^\xi$ belongs to the interval $(\max\operatorname{supp} y_{i,k-1}, \max\operatorname{supp} y_{i,k}]$ (in case $k > \min K_i$) or to $(\max\operatorname{supp} y_{i-1,\max K_{i-1}}, \max\operatorname{supp} y_{i,\min K_i}]$ (in case $k = \min K_i$) for some element E_j^ξ of a sequence $(E_j^\xi)_{j\in A_l, l\in\overline{G}_s, s\in F}$ defining f_ξ for some $\xi \in \mathcal{D}_n$. It follows that

$$\Big\|m_{2j_n+1}\sum_{i\in I_n\setminus J_n} b_i m_{2j_i}\sum_{k\in L_i} b_{i,k} y_{i,k}\Big\| \le m_{2j_n+1}\sum_{i\in I_n}\frac{4b_i}{m_{2j_i}} \le \frac{1}{m_{2j_n+1}}.$$

As $\min E_j^\xi, \max E_j^\xi \in \{\min \overline{E}_j^\xi, \max \overline{E}_j^\xi\} \cup [1, \max\operatorname{supp} y_{i_\xi}] \cup (\max\operatorname{supp} u_n, \infty)$, after erasing from the support of f the union of the supports of the vectors $(y_{i_\xi})_{\xi\in\mathcal{D}_n}$, $(y_i)_{i\in J_n}$ and $(y_{i,k})_{k\in L_i, i\in I_n}$ with error $113 m_{2j_n+1}^{-1} < m_{2j_n}^{-1}$ we conclude that for any interval E_j^ξ with $j \in A_l$, $l \in \overline{G}_s$, $s \in F$ defining some $\xi \in \mathcal{D}_n$ and any $i \in I_n$, $k \in K_i$ we have either $f_\xi(y_{i,k}) = 0$ or $\operatorname{range} y_{i,k} \subset E_j^\xi$, which proves the fourth reduction.

The total error we paid for the reductions is $\sum_n 5 m_{2j_n}^{-1} \le 1/2$ by (A).

Replacement. Let $\tilde f$ denote the restriction of f obtained by the above reduction. By the above $\tilde f(\sum_n a_n u_n) \ge 1/2$.

Fix now $i \in I_n$, $k \in J_i$ and denote by $\Gamma_{i,k}$ the collection of all $\gamma \in \mathcal{T}$ with $\gamma \in \operatorname{succ}(\operatorname{succ}(\alpha))$, for some $\alpha \in \mathcal{P}_n$, with $\operatorname{supp} f_\gamma \subset \operatorname{range} y_{i,k}$. By the third reduction $\operatorname{supp} \Gamma_{i,k} \supset \operatorname{supp} y_{i,k} \cap \operatorname{supp} f$. We pick $\gamma_{i,k} \in \Gamma_{i,k}$ with the largest $\operatorname{tag}(\gamma_{i,k})$, erase all other f_γ with $\gamma \in \Gamma_{i,k}$ and replace $f_{\gamma_{i,k}}$ by $z_{i,k}^*$. Denote the new functional defined by the modified tree by g.

Notice first that the replacement is correct, i.e. $g \in W$, since the change does not affect the sequences $(E_j)_j = (\operatorname{range} y_{i,k})_{i,k}$ defining the special functionals f_α, $\alpha \in \mathcal{P}_n$, nor any other sequence in $\mathcal{D}_n$ by the fourth reduction. Indeed, assume $\gamma_{i,k} \succ \xi$ for some $\xi \in \mathcal{D}_n$. Then $\operatorname{supp} f_\xi \cap \operatorname{range} y_{i,k} \neq \emptyset$, thus by the fourth reduction $\operatorname{range} z_{i,k}^* = \operatorname{range} y_{i,k} \subset E_j^\xi$ for any element E_j^ξ of a sequence defining f_ξ.

Notice also that for $\gamma_1 \neq \gamma_2$ with $\gamma_1, \gamma_2 \in \Gamma_{i,k}$, $\gamma_1 \in \operatorname{succ}(\beta_1)$ and $\gamma_2 \in \operatorname{succ}(\beta_2)$ we deduce, by the definition of a special functional and the third reduction, that β_1, β_2 are incomparable. Therefore $(f_\gamma)_{\gamma\in\Gamma_{i,k}}$ is $\mathcal{S}_{n_{2j_n+1}-1+n_{2j_i}}$-allowable, and using Lemma 3.7 we obtain

$$\tilde f(y_{i,k}) = \sum_{\gamma\in\Gamma_{i,k}} \operatorname{tag}(\gamma) f_\gamma(y_{i,k}) \le 3\operatorname{tag}(\gamma_{i,k}) = 3\operatorname{tag}(\gamma_{i,k}) z_{i,k}^*(z_{i,k}).$$

Thus we have $1/2 \le \tilde f(\sum_n a_n u_n) \le 3g(\sum_n a_n v_n)$, which ends the proof. ∎

Definition 4.4 ([FR1]). A Banach space with a basis (e_n) is called *tight by range* if for any block subspace Y of X spanned by a block sequence (y_n), Y does not embed into $[e_i : i \notin \bigcup_n \operatorname{range} y_n]$.

It was shown in [FR1] that a Banach space is tight by range iff any two of its block subspaces with disjoint ranges are incomparable.

Theorem 4.5. *The space $\mathcal{X}_{\mathrm{cr}}$ is tight by range.*

Proof. Let $(z_r)_r$ be a block sequence. We show that there exists no bounded operator T such that $\operatorname{supp} Tz_r \cap \operatorname{range} z_r = \emptyset$ and T can be extended to an isomorphism from $[z_r : r \in \mathbb{N}]$ to X; this will prove that $\mathcal{X}_{\mathrm{cr}}$ is tight by range.

By standard arguments we may assume that $\|T\| \le 1$, $(Tz_r)_j$ is a block sequence and $\operatorname{range}(z_r+Tz_r) < \operatorname{range}(z_{r+1}+Tz_{r+1})$ for every $r \in \mathbb{N}$. Passing to a further subsequence we may assume that either $\min\operatorname{supp} z_r < \min\operatorname{supp} Tz_r$ for all r, or $\min\operatorname{supp} z_r > \min\operatorname{supp} Tz_r$ for all r. Notice that if $\sum_r a_r z_r$ is an (n,ε)-scc, then in the first case $\sum_r a_r Tz_r$ is also an (n,ε)-scc, while in the second $\sum_r a_r Tz_r$ is an (n,ε)-scc up to the first element. With this observation we can adapt here the argument of the proof of quasi-minimality of $\mathcal{X}_{\mathrm{cr}}$.

For any fixed $j \in \mathbb{N}$ we construct a special sequence $(x_i^*)_{i\in F}$ and a block sequence $(x_i)_{i\in F} \subset [z_r : r \in \mathbb{N}]$ such that

(A′) $(x_i)_{i\in F}$ is a 28-RIS, $x = m_{2j+1}\sum_{i\in F} b_i x_i$ is a scaled $(n_{2j+1}, m_{2j+1}^{-2})$-scc of $(x_i)_{i\in F}$, $x^* = m_{2j+1}^{-1}\sum_{i\in F} x_i^*$ is a special functional in W,

(B′) $x_i = m_{2j_i}\sum_{k\in K_i} b_{i,k}x_{i,k}$ is a scaled $(n_{2j_i}, m_{2j_i}^{-3})$-scc of a 2-RIS $(x_{i,k})$, $x_i^* = m_{2j_i}^{-1}\sum_{k\in K_i} x_{i,k}^*$ for any $i \in F$,

(C′) $x_{i,k}$ is a normalized $(n_{2j_{i,k}}, m_{2j_{i,k}}^{-2})$-scc supported on $(z_r)_{r\in A_k}$, $x_{i,k}^*(x_{i,k}) = 1$, $\operatorname{range} x_{i,k} = \operatorname{range} x_{i,k}^*$ for any $k \in K_i$, $i \in F$,

(D′) the sequence $(x_i^*)_{i\in F}$ is a $(2j+1)$-dependent sequence defined by $(E_r)_{r\in\tilde{A}_k,\, k\in K_i,\, i\in F}$, where $\tilde{A}_k = A_k$ and $E_r = \operatorname{range} z_r$ for any $r \in \tilde{A}_k$, $k \in K_i \setminus \{\min K_i\}$, $i \in F$, and $\tilde{A}_{\min K_i}$ is a singleton indexing the interval $\operatorname{range} x_{i,\min K_i}$ for each $i \in F$,

(E′) $A_k \in \mathcal{S}_{\rho(\max\operatorname{supp} x_{i,k-1})}$ for any $k \in K_i \setminus \{\min K_i\}$, $i \in F$.

Notice that condition (E′) ensures that in (D′) we have a correctly defined dependent sequence. We can ensure conditions (B′), (C′) and (E′) by Lemma 3.2 and the definition of the function ρ. Indeed, having chosen $x_{i,k-1}$ for $i \in F$ and $k \in K_i \setminus \{\min K_i\}$ we can choose the next element $x_{i,k}$ of a RIS with weight $m_{j_{i,k}}$ satisfying $\max\operatorname{supp} x_{i,k-1} \le m_{j_{i,k}}/m_{j_{i,k-1}}$ and supported on $[z_r : r \in A_k]$ for some $A_k \in \mathcal{S}_{\rho(\max\operatorname{supp} x_{i,k-1})}$ by the definition of ρ and the condition (3.2), as $\rho(\max\operatorname{supp} x_{i,k-1})$ enables one to choose an (n_{2s}, m_{2s}^{-2})-scc with weight $m_{2s} \ge \max\operatorname{supp} x_{i,k-1}^2 \ge \max\operatorname{supp} x_{i,k-1} m_{j_{i,k-1}}$.

Notice that the construction of x differs from the choice of the vectors u_n in one respect—in the speed of growth of (m_{2j_i}). In the previous case we demanded high speed of growth in condition (E), here we tame the speed of growth (m_{2j_i}) as much as possible, in order to obtain condition (E′) and in consequence to be able to use $(\operatorname{range} z_j)_j$ as the intervals defining special functionals. Recall that in the previous case we took as intervals defining special functionals the sets $(\operatorname{range} y_{i,k})_{i,k}$, so we used vectors at a "higher" level. Again by (β) we can assume that all coefficients of f and x are nonnegative.

Notice that by (A′) we have $\|x\| \geq x^*(x) \geq 1$. In order to estimate $\|Tx\|$ we take $f \in W$ with a tree-analysis and repeat the first, second and third reductions from the proof of the previous theorem for one vector x instead of a linear combination of (u_n). Recall that condition (E) was required only in the last reduction, which we shall not repeat here. Within the first, second and third reductions we define as before

$$\mathcal{P} = \{\alpha \in \mathcal{T} : \operatorname{supp} f_\alpha \cap \operatorname{supp} Tx \neq \emptyset \text{ and } \alpha \text{ is minimal in } \mathcal{T} \text{ with } w(f_\alpha) \leq m_{2j+1}^{-1}\},$$

inferring after reductions that there is a restriction $\tilde{f}$ of the functional f such that

(1) $f(Tx) \leq \tilde{f}(Tx) + 4m_{2j}^{-1}$,
(2) $\operatorname{supp} \tilde{f} \subset \operatorname{supp} \tilde{\mathcal{P}}$, where

$$\tilde{\mathcal{P}} = \{\alpha \in \mathcal{T} : \operatorname{supp} f_\alpha \cap \operatorname{supp} Tx \neq \emptyset \text{ and } \alpha \text{ is minimal in } \mathcal{T} \text{ with } w(f_\alpha) = m_{2j+1}^{-1}\},$$

(3) for any $\alpha \in \tilde{\mathcal{P}}$ the special functional $f_\alpha|_{\operatorname{range} Tx}$ is defined by the sequence $(E_r)_{r \in \tilde{A}_k, k \in K_i, i \in F}$, in particular

$$f_\alpha|_{\operatorname{range} Tx} = \frac{1}{m_{2j+1}} \sum_{\operatorname{supp} f_\alpha \cap \operatorname{supp} Tx_i \neq \emptyset} f_i^\alpha$$

with $f_i^\alpha = \frac{1}{m_{2j_i}} \sum_{k \in K_i} f_{i,k}^\alpha$ and $\operatorname{supp} f_{i,k}^\alpha \subset \bigcup_{j \in A_k} \operatorname{range} z_j$ for each $k \in K_i \setminus \{\min K_i\}$, $i \in F$.

Therefore, as $\operatorname{supp} Tz_j \cap \operatorname{range} z_j = \emptyset$, we obtain

$$\tilde{f}(Tx_i) \leq \sum_{\alpha \in \tilde{\mathcal{P}}} \frac{w(f_i^\alpha)}{m_{2j_i}} f_{i,\min K_i}^\alpha (Tx_i) \leq \frac{16}{m_{2j_i}}$$

as $\sum_{\alpha \in \tilde{\mathcal{P}}} w(f_i^\alpha) f_{i,\min K_i}^\alpha$ is a norming functional obtained from $\tilde{f}$ by replacing in its tree-analysis each f_i^α by $f_{i,\min K_i}^\alpha$. Finally we have

$$\|Tx\| \leq \frac{3}{m_{2j}} + \sum_{i \in F} \frac{16}{m_{2j_i}} \leq \frac{4}{m_{2j}}.$$

Since $j \in \mathbb{N}$ is arbitrarily large, the above shows that T is not an isomorphism onto its image. ∎

Remark 4.6. Consider a Banach space $\mathcal{Y}$ satisfying conditions (α)–(δ) with respect to $(\mathcal{A}_n)_{n \in \mathbb{N}}$-admissible sets instead of Schreier admissible or allowable sets. Then by repeating the reasoning above we obtain another example of a Banach space with an unconditional basis, which is quasi-minimal and tight by range, as in [AMP]. As in the example of [AMP] the space $\mathcal{Y}$ is also locally minimal, being saturated with ℓ_∞^n's (see the next section).

5. Local minimality of the space $\mathcal{X}_{(4)}$ of [AMP]. We first briefly recall the construction of the norming set W_4 of the space $\mathcal{X}_{(4)}$ constructed in [AMP]. We fix two sequences $(m_j)_j$ and $(n_j)_j$ of natural numbers and a partition of $\mathbb{N}$ into two infinite sets L_1, L_2 as in the definition of W in Section 3. Let W_4 be the smallest subset of $c_{00}(\mathbb{N})$ such that:

(1) $(\pm e_n)_n \in W_4$, where $(e_n)_n$ is the canonical basis of $c_{00}(\mathbb{N})$,
(2) for any $f \in W_4$ and $g \in c_{00}(\mathbb{N})$ with $|f| = |g|$ also $g \in W_4$,
(3) W_4 is closed under projections on intervals of $\mathbb{N}$,
(4) W_4 is closed under the $(\mathcal{A}_{n_{2j}}, m_{2j}^{-1})$-operations on any block sequences,
(5) W_4 is closed under the $(\mathcal{A}_{n_{2j+1}}, m_{2j+1}^{-1})$-operations on $(2j+1)$-special sequences,
(6) W_4 is closed under the G-operation, defined as follows. For any set $F = \{n_1 < \cdots < n_{2q}\} \subset \mathbb{N}$ which is Schreier (i.e. $2q \leq n_1$) we set
$$S_F f = \chi_{\bigcup_{p=1}^q [n_{2p-1}, n_{2p})} f.$$
The G-operation associates with any $f \in c_{00}$ the vector $g = \frac{1}{2} S_F f$, for any F as above.

In order to complete the definition we define special sequences. A sequence $f_1 < \cdots < f_{n_{2j+1}}$ in W_4 is a $(2j+1)$-*special sequence* if:

(1) for every $i = 1, \ldots, n_{2j+1}$, $w(f_i) = m_{2j_i}$ where $j_1 \in L_1$, $j_i \in L_2$ for any $i > 1$ and $n_{2j+1} < m_{2j_1} < \cdots < m_{2j_{n_{2j+1}}}$,
(2) $m_{2j_{i+1}} > (\max \operatorname{supp} f_i) m_{2j_i}$ for any $1 \leq i < n_{2j+1}$,
(3) for $1 < i \leq n_{2j+1}$ the sequence $(|f_1|, \ldots, |f_{i-1}|)$ is uniquely determined by $w(f_i)$.

Note that the norming set K of the mixed Tsirelson space $T[(\mathcal{A}_{n_j}, 1/m_j)_j]$ is closed under projections on subsets of $\mathbb{N}$ and $m_1 = 2$. It follows that $W_4 \subset K$. This observation together with unconditionality of the basis in $\mathcal{X}_{(4)}$ allows repeating in $\mathcal{X}_{(4)}$ the argument of [M] that ℓ_∞ is finitely disjointly representable in every infinite-dimensional subspace of $T[(\mathcal{A}_{n_j}, 1/m_j)_j]$. This reasoning only uses the estimation of the action of any functional $f \in K$ on a linear combination of some block sequence by the action of another functional $g \in K$ on an analogous combination of the basis (e_n) by means of modifying the tree-analysis of f into the tree-analysis of g. As $W_4 \subset K$ we can adapt the reasoning of [M] to $\mathcal{X}_{(4)}$, obtaining the following theorem, which answers question (2) of [FR1].

THEOREM 5.1. *The space $\mathcal{X}_{(4)}$ is locally minimal, i.e. $\mathcal{X}_{(4)}$ is finitely represented in any of its infinite-dimensional subspaces.*

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