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Thickness conditions and Littlewood–Paley sets

by

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Abstract. We consider sets in the real line that have Littlewood–Paley properties LP(p) or LP and study the following question: How thick can these sets be?

1. Introduction. Let E be a closed Lebesgue measure zero set in the real line $\mathbb{R}$ and let I_k, $k = 1, 2, \ldots$, be the intervals complementary to E, i.e., the connected components of the complement $\mathbb{R} \setminus E$. Let S_k be the operator defined by

$$\widehat{S_k f} = 1_{I_k} \cdot \widehat{f}, \qquad f \in L^2 \cap L^p(\mathbb{R}),$$

where 1_{I_k} is the characteristic function of I_k, and $\widehat{\ }$ stands for the Fourier transform. Consider the corresponding quadratic Littlewood–Paley function:

$$S(f) = \Big(\sum_k |S_k f|^2 \Big)^{1/2}.$$

Following [12] we say that E has *property* LP(p) ($1 < p < \infty$) if for all $f \in L^p(\mathbb{R})$ we have

$$c_1 \|f\|_{L^p(\mathbb{R})} \le \|S(f)\|_{L^p(\mathbb{R})} \le c_2 \|f\|_{L^p(\mathbb{R})},$$

where c_1, c_2 are positive constants independent of f. When a set has property LP(p) for all p, $1 < p < \infty$, we say that it *has property* LP.

The role of such sets in harmonic analysis and particularly in multiplier theory is well-known. We recall that if G is a locally compact Abelian group and Γ is the group dual to G, then a function $m \in L^\infty(\Gamma)$ is called an *L^p-Fourier multiplier*, $1 \le p \le \infty$, if the operator Q given by

$$\widehat{Qf} = m \cdot \widehat{f}, \qquad f \in L^p \cap L^2(G),$$

is bounded from $L^p(G)$ to itself (here $\widehat{\ }$ is the Fourier transform on G). The space of all such multipliers is denoted by $M_p(\Gamma)$. Provided with the norm

$$\|m\|_{M_p(\Gamma)} = \|Q\|_{L^p(G) \to L^p(G)},$$

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the space $M_p(\Gamma)$ is a Banach algebra (with the usual multiplication of functions). For basic facts on multipliers in the cases when $\Gamma = \mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$, where $\mathbb{Z}$ is the group of integers and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the circle, see [1], [13, Chap. IV], [7].

A classical example of an infinite set that has property LP is the set $E = \{\pm 2^k : k \in \mathbb{Z}\} \cup \{0\}$ (see, e.g., [13, Chap. IV, Sec. 5]). From the arithmetic and combinatorial point of view, sets that have property LP(p) or LP were studied extensively: see, e.g., [1]–[3], [12]. With the exception of [12] these works deal with countable sets, particularly, with subsets of $\mathbb{Z}$. At the same time there exist uncountable sets that have property LP. This was first established by Hare and Klemes [3]; see also [8] and [9, Sec. 4].

In this paper we study the following question: How thick can a set $E \subseteq \mathbb{R}$ that has property LP(p) $(p \neq 2)$ or property LP be? In Theorems 1 and 2 we show that such a set cannot be metrically very thick, namely it is porous and the measure of the δ-neighbourhood of any portion of it tends to zero quite rapidly (as $\delta \to +0$). As a consequence we obtain (see Corollary) an estimate for the Hausdorff dimension of these sets. An immediate consequence of our estimate is that if a set has property LP, then its Hausdorff dimension is zero. In Theorem 3 we show that there exist sets which are thin in several senses simultaneously but have property LP(p) for no $p \neq 2$. In Theorem 4 we show that a set can be quite thick but at the same time have property LP. In part our arguments are close to those used by other authors to study subsets of $\mathbb{Z}$, but the mere fact of existence of uncountable (i.e. thick in the sense of cardinality) sets that have property LP brings some specific details to the subject.

It is well-known that a set has property LP(p) if and only if it has property LP(q), where $1/p + 1/q = 1$ (see, e.g., [12]). Thus, it suffices to consider the case when $1 < p < 2$.

We use the following notation. For a set $F \subseteq \mathbb{R}$ we denote its open δ-neighbourhood $(\delta > 0)$ by $(F)_\delta$. If F is measurable, then $|F|$ means its Lebesgue measure. A *portion* of a set $F \subseteq \mathbb{R}$ is a set of the form $F \cap I$, where I is a bounded interval. By $\dim F$ we denote the Hausdorff dimension of F. For basic properties of the Hausdorff dimension we refer the reader to [11]. For a set $F \subseteq \mathbb{R}$ and a point $t \in \mathbb{R}$ we put $F + t = \{x + t : x \in F\}$. By $\operatorname{card} A$ we denote the number of elements of a finite set A. By an *arithmetic progression of length* N we mean a set of the form $\{a + kd : k = 1, \dots, N\}$, where $a, d \in \mathbb{R}$ and $d \neq 0$. We use $c, c(p), c(p, E), \dots$ to denote various positive constants which may depend only on p and the set E.

2. Results. We recall that a set $F \subseteq \mathbb{R}$ is said to be *porous* if there exists a constant $c > 0$ such that every bounded interval $I \subseteq \mathbb{R}$ contains a subinterval J with $|J| \geq c|I|$ and $J \cap F = \emptyset$.

THEOREM 1. *Let $E \subseteq \mathbb{R}$ be a closed set of measure zero. Suppose that E has property* LP(p) *for some p, $p \neq 2$. Then E is porous.*

Earlier Hare and Klemes showed that if a set in $\mathbb{Z}$ has property LP then it is porous [2, Theorem 3.7].

To prove Theorem 1 we need certain lemmas.

LEMMA 1. *Let $1 < p < \infty$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a nonconstant affine mapping. Suppose that a function $m \in M_p(\mathbb{R})$ is continuous at each point of the set $\varphi(\mathbb{Z}^n)$. Then the restriction $m \circ \varphi_{|\mathbb{Z}^n}$ of the superposition $m \circ \varphi$ to $\mathbb{Z}^n$ belongs to $M_p(\mathbb{Z}^n)$, and $\|m \circ \varphi_{|\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)} \leq c\|m\|_{M_p(\mathbb{R})}$, where $c = c(p) > 0$ is independent of φ, m and the dimension n.*

Proof. The proof is a trivial combination of two well-known assertions on multipliers. The first one is the theorem on superpositions with affine mappings [4, Chap. I, Sec. 1.3], which implies that for every $m \in M_p(\mathbb{R})$ we have $m \circ \varphi \in M_p(\mathbb{R}^n)$ and $\|m \circ \varphi\|_{M_p(\mathbb{R}^n)} = \|m\|_{M_p(\mathbb{R})}$. The second one is the de Leeuw theorem [10] (see also [5]) on restrictions to $\mathbb{Z}^n$, according to which if a function $g \in M_p(\mathbb{R}^n)$ is continuous at all points of $\mathbb{Z}^n$, then $g_{|\mathbb{Z}^n} \in M_p(\mathbb{Z}^n)$ and $\|g_{|\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)} \leq c(p)\|g\|_{M_p(\mathbb{R}^n)}$. ■

LEMMA 2. *Let $E \subseteq \mathbb{R}$ be a nowhere dense set and let $F \subseteq \mathbb{R}$ be a finite or countable set. Then for each $\delta > 0$ there exists $\xi \in \mathbb{R}$ such that $|\xi| < \delta$ and $(F + \xi) \cap E = \emptyset$.*

Proof. The set

$$\bigcup_{t \in F} (E - t),$$

being a union of at most countable family of nowhere dense sets, cannot contain the whole interval $(-\delta, \delta)$, hence there exists $\xi \in (-\delta, \delta)$ that does not belong to the union. ■

We say that a (finite or countable) set $F \subseteq \mathbb{R}$ *splits* a closed set $E \subseteq \mathbb{R}$ if $F \subseteq \mathbb{R} \setminus E$ and no two distinct points of F are contained in the same interval complementary to E.

LEMMA 3. *Let $1 < p < 2$. Let $E \subseteq \mathbb{R}$ have property* LP(p). *Suppose that F is a subset of an arithmetic progression of length N, and F splits E. Then* $\operatorname{card} F \leq c(p, E)N^{2/q}$, *where $1/p + 1/q = 1$.*

Proof. This lemma can be deduced from Theorems 1.2 and 1.3 of [12]. We give an independent simple proof based on a quite standard argument. Consider an arithmetic progression $\{a + kd : k = 1, \ldots, N\}$. We can assume that $d > 0$. Suppose that a set $F = \{a + k_j d : j = 1, \ldots, \nu\}$, where $1 \leq k_j \leq N$, splits E. For $j = 1, \ldots, \nu$ let Δ_j be the interval of length δ centered at $a + k_j d$, where $\delta > 0$ is so small that $\delta < d$ and $\Delta_j \cap E = \emptyset$, $j = 1, \ldots, \nu$.

We put

$$m_\theta = \sum_{j=1}^{\nu} r_j(\theta) \cdot 1_{\Delta_j},$$

where $r_j(\theta) = \operatorname{sign} \sin 2^j \pi\theta$, $\theta \in [0,1]$, $j = 1,2,\dots$, are the Rademacher functions.

It is well-known that if a set E has property $\mathrm{LP}(p)$, then it has the Marcinkiewicz property $\mathrm{Mar}(p)$, namely ([1]), for each function $m \in L^\infty(\mathbb{R})$ whose variations $\mathrm{Var}_{I_k} m$ on the intervals I_k complementary to E are uniformly bounded, we have $m \in M_p(\mathbb{R})$ and

$$\|m\|_{M_p(\mathbb{R})} \le c(p,E)\Big(\|m\|_{L^\infty(\mathbb{R})} + \sup_k \mathrm{Var}_{I_k} m\Big). \tag{1}$$

Thus we have $\|m_\theta\|_{M_p(\mathbb{R})} \le c$, where $c > 0$ is independent of N and θ. Consider the affine mapping $\varphi(x) = a + dx$, $x \in \mathbb{R}$. Using Lemma 1 for $n = 1$, we see that

$$\|m_\theta \circ \varphi_{|\mathbb{Z}}\|_{M_p(\mathbb{Z})} \le c(p)\|m_\theta\|_{M_p(\mathbb{R})} \le c_1(p).$$

Thus

$$\Big\|\sum_k m_\theta(a+kd) c_k e^{ikx}\Big\|_{L^p(\mathbb{T})} \le c_1(p)\Big\|\sum_k c_k e^{ikx}\Big\|_{L^p(\mathbb{T})}$$

for every trigonometric polynomial $\sum_k c_k e^{ikx}$. In particular,

$$\Big\|\sum_{k=1}^{N} m_\theta(a+kd) e^{ikx}\Big\|_{L^p(\mathbb{T})} \le c_1(p)\Big\|\sum_{k=1}^{N} e^{ikx}\Big\|_{L^p(\mathbb{T})}.$$

Hence,

$$\Big\|\sum_{j=1}^{\nu} r_j(\theta) e^{ik_j x}\Big\|_{L^p(\mathbb{T})} \le c_1(p)\Big\|\sum_{k=1}^{N} e^{ikx}\Big\|_{L^p(\mathbb{T})}. \tag{2}$$

It is easy to verify that

$$\Big\|\sum_{k=1}^{N} e^{ikx}\Big\|_{L^p(\mathbb{T})} \le c(p) N^{1/q},$$

so (2) yields

$$\int_{\mathbb{T}} \Big|\sum_{j=1}^{\nu} r_j(\theta) e^{ik_j x}\Big|^p dx \le c_2(p) N^{p/q}.$$

([1]) Actually $\mathrm{LP}(p)$ and $\mathrm{Mar}(p)$ are equivalent: see, e.g., [12, Theorem 1.1].

By integrating this inequality with respect to $\theta \in [0,1]$ and using the Khintchine inequality

$$\Big(\int_0^1 \Big|\sum_j c_j r_j(\theta)\Big|^p d\theta\Big)^{1/p} \ge c\Big(\sum_j |c_j|^2\Big)^{1/2}, \quad 1 \le p < 2,$$

(see, e.g., [14, Chap. V, Sec. 8]), we obtain $\nu^{p/2} \le c_3(p) N^{p/q}$. ∎

Proof of Theorem 1. We can assume that $1 < p < 2$. For a bounded interval $I \subseteq \mathbb{R}$ let

$$d(I) = \sup\{|J| : J \text{ is an interval}, \ J \subseteq I, \ J \cap E = \emptyset\}.$$

Suppose that E is not porous. Then, for each positive integer N we can find a (bounded) interval I such that $0 < d(I) < |I|/3N$. Let $d = 2d(I)$. Consider an arithmetic progression $t_k = a + kd$, $k = 1, \dots, N$, that lies in the interior of I. Using Lemma 2, we can find ξ such that $t_k + \xi \notin E$, $k = 1, \dots, N$, and ξ is so small that $\{t_k + \xi : k = 1, \dots, N\} \subseteq I$. Note that since $d = 2d(I)$, no two distinct points of the progression $\{t_k + \xi : k = 1, \dots, N\}$ lie in the same interval complementary to E. Thus this progression splits E. By Lemma 3 this is impossible if N is sufficiently large. ∎

THEOREM 2. *Let $1 < p < 2$. Let $E \subseteq \mathbb{R}$ be a closed set of measure zero. Suppose that E has property* LP(p). *Then each portion $E \cap I$ of E satisfies*

$$|(E \cap I)_\delta| \le c|I|^{2/q}\delta^{1-2/q},$$

where $1/p + 1/q = 1$ and the constant $c = c(p, E) > 0$ is independent of I and δ.

Theorem 2 immediately implies an estimate for the Hausdorff dimension of sets that have property LP(p):

COROLLARY. *If $1 < p < 2$ and a set $E \subseteq \mathbb{R}$ has property* LP(p), *then* $\dim E \le 2/q$, *where $1/p + 1/q = 1$. Thus, if E has property* LP, *then* $\dim E = 0$.

Proof of Theorem 2. Consider an arbitrary portion $E \cap I$ of E. Let J be the interval concentric with I and of twice its length. Denote the left endpoint of J by a. Fix a positive integer N and consider the progression $a + kd$, $k = 1, \dots, N$, where $d = |J|/N$. By Lemma 2 one can find ξ such that no element of $\{a + kd + \xi : k = 1, \dots, N\}$ is in E and $I \subseteq J + \xi = (a + \xi, a + Nd + \xi)$.

We define intervals J_k by

$$J_k = (a + (k-1)d + \xi, \ a + kd + \xi), \quad k = 1, \dots, N.$$

Consider the intervals J_{k_j} such that $J_{k_j} \cap E \ne \emptyset$. Obviously their right endpoints split E, so, by Lemma 3, their number is at most $c(p)N^{2/q}$. Thus $E \cap I$ is covered by at most $c(p)N^{2/q}$ intervals of length $d = 2|I|/N$ each.

Let $\delta > 0$. We can assume that $\delta < |I|$ (otherwise the assertion of the theorem is trivial). Choosing a positive integer N so that

$$\frac{2|I|}{N} \le \frac{\delta}{3} < \frac{4|I|}{N},$$

we see that $E \cap I$ can be covered by at most $c(p)(12|I|/\delta)^{2/q}$ intervals of length $\delta/3$ each. It remains to replace each of these intervals with the corresponding concentric interval of nine times its length. This proves the theorem. The corollary follows. ∎

We note now that a set can be quite thin and at the same time have property LP(p) for no $p \neq 2$. Consider the set

$$F = \Big\{ \sum_{k=1}^{\infty} \varepsilon_k l_k : \varepsilon_k = 0 \text{ or } 1 \Big\}, \tag{3}$$

where l_k, $k = 1, 2, \ldots$, are positive numbers with $l_{k+1} < l_k/2$. It was shown by Sjögren and Sjölin [12] that such sets have property LP(p) for no p, $p \neq 2$. (In particular, the Cantor triadic set does not have property LP(p) for $p \neq 2$.) Taking a rapidly decreasing sequence $\{l_k\}$ one can obtain a set F of the form (3) that is porous and has the property that the measure of its δ-neighbourhood rapidly tends to zero. Still, in a sense, any set of the form (3) is thick: it is uncountable and all its points are its accumulation points. Theorem 3 below shows that a set can be thin in several senses simultaneously, and at the same time have property LP(p) for no p, $p \neq 2$.

THEOREM 3. *Let ψ be a positive function on an interval $(0, \delta_0)$, $\delta_0 > 0$, with $\lim_{\delta \to +0} \psi(\delta)/\delta = +\infty$. There exists a strictly increasing bounded sequence $a_1 < a_2 < \cdots$ such that the set $E = \{a_k\}_{k=1}^{\infty} \cup \{\lim_{k\to\infty} a_k\}$ satisfies the following conditions:* 1) *E is porous;* 2) *$|(E)_\delta| \le \psi(\delta)$ for all sufficiently small $\delta > 0$;* 3) *E has property* LP(p) *for no p, $p \neq 2$.*

Proof. Given (real) numbers a and $l_1, \ldots, l_n$ consider the set of all points $a + \sum_{j=1}^{n} \varepsilon_j l_j$, where $\varepsilon_j = 0$ or 1. Assume that the cardinality of this set is 2^n. Following [6] we call such a set an *n-chain* (2).

We shall need the following refinement of the Sjögren and Sjölin result on the sets (3). This refinement also provides a partial extension of Proposition 3.4 of [2], that treats subsets of integers, to the general case of closed measure zero sets in the line.

LEMMA 4. *Let $E \subseteq \mathbb{R}$ be a closed set of measure zero. Suppose that E contains n-chains with arbitrarily large n. Then E has property* LP(p) *for no $p \neq 2$.*

(2) An n-chain is a particular case of what is called a parallelepiped of dimension n, that is, of a set of cardinality 2^n, obtained as the Minkowski sum of n two-element sets.

Proof. Suppose that, contrary to the assertion, E has property LP(p) for some p, $p \neq 2$. We can assume that $1 < p < 2$.

Let n be such that E contains an n-chain

$$a + \sum_{j=1}^{n} \varepsilon_j l_j, \quad (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n. \tag{4}$$

Consider the set

$$B = \Big\{a + \sum_{j=1}^{n} k_j l_j : (k_1, \dots, k_n) \in \mathbb{Z}^n\Big\}.$$

By Lemma 2 there exists an arbitrarily small ξ such that

$$(B + \xi) \cap E = \emptyset. \tag{5}$$

Clearly, if ξ is small enough, then no two distinct points of the chain obtained by the same shift ξ of the chain (4) can lie in the same interval complementary to E. Thus, there exists ξ such that (5) holds and the n-chain

$$a + \xi + \sum_{j=1}^{n} \varepsilon_j l_j, \quad (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n,$$

splits E.

For each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n$ let I_ε denote the interval complementary to E that contains the point $a + \xi + \sum_{j=1}^{n} \varepsilon_j l_j$. For an arbitrary choice of signs $\pm$ consider the function

$$m = \sum_{\varepsilon \in \{0,1\}^n} \pm 1_{I_\varepsilon}.$$

We have (see (1))

$$\|m\|_{M_p(\mathbb{R})} \le c, \tag{6}$$

where $c > 0$ is independent of n and the choice of signs.

Consider the following affine mapping φ:

$$\varphi(x) = a + \xi + \sum_{j=1}^{n} x_j l_j, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Note that condition (5) implies that the function m is continuous at each point of $\varphi(\mathbb{Z}^n)$. Using Lemma 1, we obtain (see (6)) $m \circ \varphi_{|\mathbb{Z}^n} \in M_p(\mathbb{Z}^n)$ and

$$\|m \circ \varphi_{|\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)} \le c,$$

where the constant $c > 0$ is independent of n and the choice of signs.

Therefore, for every trigonometric polynomial $\sum_{k \in \mathbb{Z}^n} c_k e^{i(k,t)}$ on the torus $\mathbb{T}^n$,

$$\Big\| \sum_{k \in \mathbb{Z}^n} m \circ \varphi(k) c_k e^{i(k,t)} \Big\|_{L^p(\mathbb{T}^n)} \le c \Big\| \sum_{k \in \mathbb{Z}^n} c_k e^{i(k,t)} \Big\|_{L^p(\mathbb{T}^n)}.$$

(We use (k,t) to denote the usual inner product of vectors $k \in \mathbb{Z}^n$ and $t \in \mathbb{T}^n$.) In particular, taking $c_k = 1$ for $k \in \{0,1\}^n$ and $c_k = 0$ for $k \notin \{0,1\}^n$, we obtain

$$\Big\| \sum_{\varepsilon\in\{0,1\}^n} m\Big(a+\xi+\sum_{j=1}^n \varepsilon_j l_j\Big) e^{i(\varepsilon,t)}\Big\|_{L^p(\mathbb{T}^n)} \le c\Big\| \sum_{\varepsilon\in\{0,1\}^n} e^{i(\varepsilon,t)}\Big\|_{L^p(\mathbb{T}^n)}.$$

That is,

$$\Big\| \sum_{\varepsilon\in\{0,1\}^n} \pm e^{i(\varepsilon,t)}\Big\|_{L^p(\mathbb{T}^n)} \le c\Big\| \sum_{\varepsilon\in\{0,1\}^n} e^{i(\varepsilon,t)}\Big\|_{L^p(\mathbb{T}^n)}.$$

Raising this inequality to the power p and averaging with respect to the signs $\pm$ (i.e., using the Khintchine inequality), we obtain

$$\Big\| \sum_{\varepsilon\in\{0,1\}^n} e^{i(\varepsilon,t)}\Big\|_{L^2(\mathbb{T}^n)} \le c\Big\| \sum_{\varepsilon\in\{0,1\}^n} e^{i(\varepsilon,t)}\Big\|_{L^p(\mathbb{T}^n)}. \tag{7}$$

Note that

$$\sum_{\varepsilon\in\{0,1\}^n} e^{i(\varepsilon,t)} = \prod_{j=1}^n (1+e^{it_j}), \quad t=(t_1,\dots,t_n)\in\mathbb{T}^n,$$

so (7) yields

$$\|1+e^{it}\|^n_{L^2(\mathbb{T})} \le c\|1+e^{it}\|^n_{L^p(\mathbb{T})}. \tag{8}$$

Since n can be arbitrarily large, relation (8) implies

$$\|1+e^{it}\|_{L^2(\mathbb{T})} \le \|1+e^{it}\|_{L^p(\mathbb{T})},$$

which, as one can easily verify, is impossible for $1<p<2$. ∎

LEMMA 5. *Let l_k, $k=1,2,\dots$, be positive numbers satisfying $l_{k+1} < l_k/2$. Then the set F defined by* (3) *contains a strictly increasing sequence $S=\{a_k\}_{k=1}^\infty$ that contains an n-chain for every n.*

Proof. For $n=1,2,\dots$ let

$$\alpha_n = \sum_{k=1}^{n^2} l_k, \quad \beta_n = \sum_{k=1}^{n^2+n} l_k.$$

Clearly $\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots$, so the closed intervals $[\alpha_n,\beta_n]$, $n=1,2,\dots$, are pairwise disjoint.

Define sets $F_n\subseteq F$, $n=1,2,\dots$, as follows:

$$F_n = \Big\{ l_1+l_2+\cdots+l_{n^2} + \sum_{k=n^2+1}^{n^2+n} \varepsilon_k l_k : \varepsilon_k = 0 \text{ or } 1\Big\}.$$

Note that $F_n\subseteq[\alpha_n,\beta_n]$ for all $n=1,2,\dots$.

It remains to put $S=\bigcup_{n=1}^\infty F_n$. ∎

We shall now complete the proof of the theorem. Replacing, if needed, the function $\psi(\delta)$ with

$$\widetilde{\psi}(\delta) = \delta \inf_{0<t\le\delta} \frac{\psi(t)}{t},$$

we can assume that $\psi(\delta)/\delta \nearrow +\infty$ as $\delta \searrow 0$.

Take a strictly increasing sequence of positive integers n_k, $k = 1, 2, \ldots$, so that

$$6 \cdot 2^k \le \psi(3^{-n_k})/3^{-n_k}, \qquad k = 1, 2, \ldots. \tag{9}$$

Consider the set

$$F = \Big\{ \sum_{k=1}^{\infty} \varepsilon_k 3^{-n_k} : \varepsilon_k = 0 \text{ or } 1 \Big\}.$$

It is clear that F is porous (as a subset of the Cantor triadic set).

Assuming that $\delta > 0$ is sufficiently small, we can find k such that

$$3^{-n_{k+1}} \le \delta < 3^{-n_k}. \tag{10}$$

Note that F can be covered by 2^{k+1} closed intervals of length $3^{-n_{k+1}}$ each. Consider the δ-neighbourhood of each of these intervals. We infer that (see (10))

$$|(F)_\delta| \le 2^{k+1} 3\delta.$$

Hence, taking (9), (10) into account, we obtain

$$|(F)_\delta| \le \frac{\psi(3^{-n_k})}{3^{-n_k}} \delta \le \psi(\delta).$$

Using Lemma 5 we can find a strictly increasing sequence $S = \{a_k\}_{k=1}^{\infty}$ contained in F such that for every n the sequence S contains an n-chain. Let $E = S \cup \{a\}$, where $a = \lim_{k\to\infty} a_k$. It remains to apply Lemma 4. ∎

Our next goal is to construct a set that has property LP(p) or property LP and at the same time is thick. Theorem 2 implies that if $1 < p < 2$ and a bounded set E has property LP(p), then $|(E)_\delta| = O(\delta^{1-2/q})$ as $\delta \to +0$. Hence, if a bounded set E has property LP, then $|(E)_\delta| = O(\delta^{1-\varepsilon})$ for all $\varepsilon > 0$. The author does not know if these estimates are sharp. A partial solution to this problem is given by Theorem 4 below. It is a simple consequence of the Hare and Klemes theorem [3, Theorem A], which provides a sufficient condition for a set to have property LP(p). Stated for sets in $\mathbb{Z}$, this theorem, as noted at the end of [3], easily transfers to sets in $\mathbb{R}$ and allows one to construct perfect sets that have this property.

We shall use the version of the Hare and Klemes theorem stated in [9, Sec. 4]. According to this version, for each p, $1 < p < \infty$, there is a constant τ_p ($0 < \tau_p < 1$) with the following property. Let E be a closed set of measure zero in the interval $[0, 1]$. Suppose that, under an appropriate numbering, the intervals I_k, $k = 1, 2, \ldots$, complementary to E in $[0, 1]$ (i.e., the connected components of $[0,1]\setminus E$) satisfy

(11) $$\delta_{k+1}/\delta_k \le \tau_p, \quad k=1,2,\ldots,$$

where $\delta_k=|I_k|$. Then E has property LP(p). This in turn implies that if

(12) $$\lim_{k\to\infty} \delta_{k+1}/\delta_k = 0,$$

then E has property LP.

Theorem 4.

(a) *Let* $1<p<\infty$. *There exists a perfect set* $E\subseteq[0,1]$ *which has property* LP(p) *and at the same time satisfies* $|(E)_\delta|\ge c\delta\log(1/\delta)$ *for all sufficiently small* $\delta>0$.

(b) *Let* $\gamma(\delta)$ *be a positive nondecreasing function on* $(0,\infty)$ *with* $\lim_{\delta\to+0}\gamma(\delta)=0$. *There exists a perfect set* $E\subseteq[0,1]$ *which has property* LP *and at the same time satisfies* $|(E)_\delta|\ge c\,\gamma(\delta)\delta\log(1/\delta)$.

Proof. Let δ_k, $k=1,2,\ldots$, be a sequence of positive numbers with

(13) $$\sum_k \delta_k = 1.$$

Let $E\subseteq[0,1]$ be a closed set. Assume that, under an appropriate numbering, the intervals I_k, $k=1,2,\ldots$, complementary to E in $[0,1]$ satisfy $|I_k|=\delta_k$, $k=1,2,\ldots$. In this case we say that E is *generated* by the sequence $\{\delta_k\}$. (Certainly $|E|=0$.) Note that for each sequence $\{\delta_k\}$ of positive numbers with (13) there exists a perfect set $E\subseteq[0,1]$ generated by $\{\delta_k\}$.

It is easy to see that if E is generated by a positive sequence $\{\delta_k\}$ satisfying (13), then for all $\delta>0$ we have

(14) $$|(E)_\delta| \ge 2\delta\,\mathrm{card}\{k:\delta_k>2\delta\}.$$

Indeed, if $I_k=(a_k,b_k)$ is an arbitrary interval complementary to E in $[0,1]$ such that $|I_k|>2\delta$, then the δ-neighbourhood of E contains the intervals $(a_k,a_k+\delta)$ and $(b_k-\delta,b_k)$.

We now prove part (a) of the theorem. Fix p, $1<p<\infty$. Let

$$\delta_k = ae^{-kb}, \quad k=1,2,\ldots,$$

where the positive constants a and b are chosen so that conditions (11), (13) hold. Consider a perfect set $E\subseteq[0,1]$ generated by $\{\delta_k\}$. Using (14), we see that

$$|(E)_\delta| \ge 2\delta\left(\frac{1}{b}\log\frac{a}{2\delta}-1\right),$$

which proves (a).

Now we prove (b). Without loss of generality we can assume that $\gamma(1/e) = 1/4$. Let

$$b(x)=\frac{1}{\gamma(e^{-x})}, \quad x>0.$$

The function b is nondecreasing, $b(x)\to\infty$ as $x\to\infty$, and $b(1)=4$.

Define

$$\delta_k = ae^{-kb(k)}, \quad k = 1, 2, \ldots,$$

where $a > 0$ is chosen so that (13) holds. Note that

$$\delta_{k+1}/\delta_k = e^{-((k+1)b(k+1)-kb(k))} \le e^{-b(k)} \to 0 \quad \text{as } k \to \infty,$$

and thus (12) holds.

Consider a perfect set $E \subseteq [0,1]$ generated by the sequence $\{\delta_k\}$.

Let $\delta > 0$ be sufficiently small. Choose a positive integer $k = k(\delta)$ so that

$$\delta_{k+1} \le 2\delta < \delta_k. \tag{15}$$

We have

$$\operatorname{card}\{k : \delta_k > 2\delta\} \ge k(\delta).$$

So (see (14))

$$|(E)_\delta| \ge 2\delta k(\delta). \tag{16}$$

Note that (15) implies

$$kb(k) < \log \frac{a}{2\delta} \le (k+1)b(k+1).$$

Hence, for all sufficiently small $\delta > 0$ we have

$$\frac{1}{2} kb(k) < \log \frac{1}{\delta} \le 2(k+1)b(k+1). \tag{17}$$

The left-hand inequality in (17) yields (recall that $b(1) = 4$)

$$2k = \frac{1}{2} kb(1) \le \frac{1}{2} kb(k) < \log \frac{1}{\delta},$$

whence

$$b(2k) \le b\left(\log \frac{1}{\delta}\right) = \frac{1}{\gamma(\delta)}.$$

Combining this inequality and the right-hand inequality in (17), we see that

$$\log \frac{1}{\delta} \le 2(k+1)b(k+1) \le 4kb(2k) \le 4k \frac{1}{\gamma(\delta)}.$$

So,

$$\frac{1}{4} \gamma(\delta) \log \frac{1}{\delta} \le k = k(\delta).$$

Thus (see (16)),

$$|(E)_\delta| \ge \frac{1}{2} \gamma(\delta) \delta \log \frac{1}{\delta}. \ ■$$

Remark. As far as the author knows, the problem of the existence of a set that has property LP(p) for some p, $p \ne 2$, but does not have property LP is open.

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