STUDIA MATHEMATICA 220 (3) (2014)

Products of Toeplitz operators and Hankel operators

by

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Abstract. We first determine when the sum of products of Hankel and Toeplitz operators is equal to zero; then we characterize when the product of a Toeplitz operator and a Hankel operator is a compact perturbation of a Hankel operator or a Toeplitz operator and when it is a finite rank perturbation of a Toeplitz operator.

1. Introduction. Let D be the open unit disk in the complex plane and ∂D the unit circle. Let $d\sigma$ be the normalized Lebesgue measure on ∂D. Let $L^2 = L^2(\partial D, d\sigma)$ denote the space of Lebesgue square integrable functions on the unit circle. The Hardy space H^2 is the closed subspace of L^2 consisting of analytic functions. Let P be the orthogonal projection from L^2 onto H^2. For $f \in L^\infty$, the *Toeplitz operator* T_f and the *Hankel operator* H_f with symbol f are defined respectively by

$$T_f h = P(fh) \quad \text{and} \quad H_f h = P(U(fh))$$

for h in H^2. Here U is the unitary operator on L^2 defined by $Uh(w) = \bar{w}\tilde{h}(w)$, where $\tilde{h}(w) = h(\bar{w})$. Clearly, $H_f^* = H_{f^*}$, where $f^*(w) = \bar{f}(\bar{w})$. The operator U maps H^2 onto $[H^2]^\perp$ and has the following useful property: $UP = (I-P)U$. As is well known, Hankel and Toeplitz operators are closely related by the equations

$$T_{fg} - T_f T_g = H_{\tilde{f}} H_g, \tag{1.1}$$

$$H_{fg} = H_f T_g + T_{\tilde{f}} H_g. \tag{1.2}$$

The second equality implies that if $g \in H^\infty$, then

$$T_{\tilde{g}} H_f = H_{fg} = H_f T_g.$$

We refer to [5], [7], [6] for the above facts. Some more relationships be-

2010 *Mathematics Subject Classification*: Primary 47B35; Secondary 47B38.
Key words and phrases: finite rank, compact perturbation, products, Toeplitz operator, Hankel operator.

DOI: 10.4064/sm220-3-5

tween these two classes of operators have been studied in several papers. For the problem of commutation, Martínez-Avendaño [11] showed that H_f commutes with T_g if and only if either $f \in H^\infty$, or there exists a constant λ such that $g + \lambda f$ is in H^∞, and both $g + \tilde{g}$ and $g\tilde{g}$ are constants. A natural question is: When does the commutator $[H_f, T_g] = H_f T_g - T_g H_f$ have finite rank? Ding [5] answered this question and another one: When is the product $H_f T_g$ a finite rank perturbation of a Hankel operator H_h? As to the compactness of $[H_f, T_g]$, Guo and Zheng [7] gave a necessary and sufficient condition.

Inspired by these results, we investigate some more relationships between these two classes of operators. We study sums of products of Hankel and Toeplitz operators and operators of the form $H_f T_g + T_h H_k$, and determine when such operators are zero. The classical result of Martínez-Avendaño [11] is recovered as a corollary of our results. Then we characterize when $H_f T_g - H_h$ and $H_f T_g - T_h$ are compact and when $T_f H_g - T_h$ is of finite rank.

In Section 2, we consider a class of operators of the form

$$\sum_{j=1}^{n} H_j T_j,$$

where each H_j is a Hankel operator and each T_j is a Toeplitz operator, and determine when an operator of this type is zero (Theorem 2.5). We also characterize when an operator of the form $H_f T_g + T_h H_k$ is zero (Theorem 2.8). In Section 3, we characterize when $H_f T_g$ is a compact perturbation of a Hankel or a Toeplitz operator (Theorem 3.6 and Corollary 3.7). In Section 4, we characterize when $T_f H_g$ is a finite rank perturbation of a Toeplitz operator (Corollary 4.5).

2. Sums of products of Hankel and Toeplitz operators. In this section, we consider operators that are sums of products of Toeplitz operators and Hankel operators and determine when such an operator is equal to zero.

Given nonzero functions $f, g, h, k \in H^2$, we write $f \otimes g$ for the rank-one operator on H^2 defined by $f \otimes g(h) = \langle h, g\rangle f$. It is well known that $f \otimes g = h \otimes k$ if and only if there exists a nonzero constant $\alpha \in \mathbb{C}$ such that $f = \alpha h$ and $k = \bar{\alpha} g$. More generally, we have the following lemma which is essentially proved in Proposition 4 of [8]. In the following, for a given positive integer n, we let $\mathbb{M}_n$ be the set of all $n \times n$ matrices and $\mathbb{S}_n$ be the set of all permutations of $\{1, \dots, n\}$. If $A \in \mathbb{M}_n$, we let A^* be the conjugate transpose of A.

LEMMA 2.1. *Let $f_j, g_j \in H^2$ for $j = 1, \ldots, n$. Then*

$$\sum_{j=1}^{n} f_j \otimes g_j = 0 \quad \text{on } H^2$$

if and only if there exist $A \in \mathbb{M}_n$ and $\sigma \in \mathbb{S}_n$ such that

$$[A - I] \begin{pmatrix} f_{\sigma(1)} \\ \vdots \\ f_{\sigma(n)} \end{pmatrix} \quad \text{and} \quad A^* \begin{pmatrix} g_{\sigma(1)} \\ \vdots \\ g_{\sigma(n)} \end{pmatrix} = 0.$$

LEMMA 2.2. *$H_{\bar{z}} = 1 \otimes 1$ on H^2.*

Proof. For $f \in H^2$, let $f(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ be the Fourier series of f. Then

$$H_{\bar{z}}(f) = P(U(\bar{z}f)) = P(\tilde{f}) = \tilde{f}(0) = a_0 = \langle f, 1 \rangle = 1 \otimes 1(f). \ \blacksquare$$

LEMMA 2.3. *Let $f, g \in L^\infty$. Then*

$$H_f T_g T_z = T_{\bar{z}} H_f T_g + H_f 1 \otimes H_g^* 1,$$
$$T_f H_g T_z = T_{\bar{z}} T_f H_g - H_{\tilde{f}} 1 \otimes H_g^* 1.$$

Proof. By Lemma 2.2 and formulas (1.1), (1.2), we have

$$\begin{aligned} H_f T_g T_z &= H_f T_{zg} = H_f(T_z T_g + H_{\bar{z}} H_g) = H_f T_z T_g + H_f H_{\bar{z}} H_g \\ &= H_f T_z T_g + H_f [1 \otimes 1] H_g = T_{\bar{z}} H_f T_g + H_f 1 \otimes H_g^* 1, \end{aligned}$$

and

$$\begin{aligned} T_f H_g T_z &= T_f T_{\bar{z}} H_g = (T_{\bar{z}} T_f - H_{\tilde{f}} H_{\bar{z}}) H_g = T_{\bar{z}} T_f H_g - H_{\tilde{f}} H_{\bar{z}} H_g \\ &= T_{\bar{z}} T_f H_g - H_{\tilde{f}} [1 \otimes 1] H_g = T_{\bar{z}} T_f H_g - H_{\tilde{f}} 1 \otimes H_g^* 1. \ \blacksquare \end{aligned}$$

LEMMA 2.4. *For $f \in L^\infty$, the following statements are all equivalent:*

(1) $H_f 1 = 0$.
(2) $H_{\tilde{f}}^* 1 = 0$.
(3) $H_f = 0$.
(4) $f \in H^\infty$.

Proof. Calculate directly using the Fourier series of f. $\blacksquare$

We are now ready to prove the main result of this section. We say a vector is in H_n^∞ if every element of the vector is in H^∞.

THEOREM 2.5. *Let $f_j, g_j \in L^\infty$ for $j = 1, \ldots, n$. Then the operator $T = \sum_{j=1}^{n} H_{f_j} T_{g_j}$ equals 0 on H^2 if and only if there exist $A \in \mathbb{M}_n$ and $\sigma \in \mathbb{S}_n$ such that the following three conditions hold:*

(1) $[A-I]F_\sigma^T \in H_n^\infty$.
(2) $\bar{A}^* G_\sigma^T \in H_n^\infty$.
(3) $G_\sigma A F_\sigma^T \in H^\infty$.

Here $F_\sigma = (f_{\sigma(1)}, \ldots, f_{\sigma(n)})$ *and* $G_\sigma = (g_{\sigma(1)}, \ldots, g_{\sigma(n)})$.

Proof. First assume $T = 0$. By Lemma 2.3, we have

$$\sum_{j=1}^{n} H_{f_j} 1 \otimes H_{g_j}^* 1 = 0.$$

Then by Lemma 2.1, there exist $A = (a_{ij})_{n\times n} \in \mathbb{M}_n$ and $\sigma \in \mathbb{S}_n$ such that

$$[A-I](H_{f_{\sigma(1)}} 1, \ldots, H_{f_{\sigma(n)}} 1)^T = 0, \tag{2.1}$$

$$A^*(H_{g_{\sigma(1)}}^* 1, \ldots, H_{g_{\sigma(n)}}^* 1)^T = 0. \tag{2.2}$$

It follows from (2.1) that

$$H_{\sum_{j=1}^n a_{ij} f_{\sigma(j)}} 1 = \sum_{j=1}^{n} a_{ij} H_{f_{\sigma(j)}} 1 = H_{f_{\sigma(i)}} 1,$$

so

$$H_{\sum_{j=1}^n a_{ij} f_{\sigma(j)} - f_{\sigma(i)}} 1 = 0$$

for each $i = 1, \ldots, n$. By Lemma 2.4, we have

$$\sum_{j=1}^{n} a_{ij} f_{\sigma(j)} - f_{\sigma(i)} \in H^\infty$$

for each i. This shows that $[A-I]F_\sigma^T \in H_n^\infty$, where $F_\sigma = (f_{\sigma(1)}, \ldots, f_{\sigma(n)})$. This implies (1).

Next, using (2.2), we have

$$H^*_{\sum_{i=1}^n a_{ij} g_{\sigma(i)}} 1 = \sum_{i=1}^{n} \overline{a_{ij}} H_{g_{\sigma(i)}}^* 1 = 0$$

for each j and hence

$$\sum_{i=1}^{n} a_{ij} g_{\sigma(i)} \in H^\infty$$

for each j by Lemma 2.4. So (2) holds.

To prove (3), let

$$(h_1, \ldots, h_n)^T = [A-I]F_\sigma^T \quad \text{and} \quad (k_1, \ldots, k_n)^T = \bar{A}^* G_\sigma^T.$$

Then

$$\begin{aligned}\sum_{i=1}^{n} H_{h_i} T_{g_{\sigma(i)}} &= \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} a_{ij} H_{f_{\sigma(j)}} - H_{f_{\sigma(i)}}\Big] T_{g_{\sigma(i)}} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} H_{f_{\sigma(j)}} T_{g_{\sigma(i)}} - \sum_{i=1}^{n} H_{f_{\sigma(i)}} T_{g_{\sigma(i)}} \\ &= \sum_{j=1}^{n} H_{f_{\sigma(j)}} T_{\sum_{i=1}^{n} a_{ij} g_{\sigma(i)}} - \sum_{i=1}^{n} H_{f_i} T_{g_i} \\ &= \sum_{j=1}^{n} H_{f_{\sigma(j)}} T_{k_j} - \sum_{i=1}^{n} H_{f_i} T_{g_i}.\end{aligned}$$

Since $h_j \in H^\infty$ and $k_j \in H^\infty$, we have $H_{h_j} = 0$ by Lemma 2.4 and $H_{f_{\sigma(j)}} T_{k_j} = H_{f_{\sigma(j)} k_j}$ by (1.2) for each j. By the assumption

$$\sum_{j=1}^{n} H_{f_j} T_{g_j} = 0,$$

we have

$$0 = \sum_{j=1}^{n} H_{f_{\sigma(j)}} T_{k_j} = \sum_{j=1}^{n} H_{f_{\sigma(j)} k_j} = H_{\sum_{j=1}^{n} f_{\sigma(j)} k_j},$$

so that $\sum_{j=1}^{n} f_{\sigma(j)} k_j \in H^\infty$ by Lemma 2.4. On the other hand, since $k_i = \sum_{j=1}^{n} a_{ji} g_{\sigma(j)}$ for each i, we have

$$\sum_{i=1}^{n} f_{\sigma(i)} k_i = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{\sigma(i)} a_{ji} g_{\sigma(j)} = \sum_{j=1}^{n} g_{\sigma(j)} \sum_{i=1}^{n} a_{ji} f_{\sigma(i)} = G_\sigma A F_\sigma^T,$$

from which (3) follows.

Now suppose (1)–(3) hold. Let

$$(h_1, \dots, h_n)^T = [A - I] F_\sigma^T \quad \text{and} \quad (k_1, \dots, k_n)^T = \bar{A}^* G_\sigma^T.$$

Then $h_j, k_j \in H^\infty$ for each j. Hence $H_{h_j} = 0$ and $H_{f_{\sigma(j)}} T_{k_j} = H_{f_{\sigma(j)} k_j}$ for each j. Using a similar argument to the above, we have

$$\sum_{i=1}^{n} H_{f_i} T_{g_i} = \sum_{j=1}^{n} H_{f_{\sigma(j)} k_j} = H_{\sum_{j=1}^{n} f_{\sigma(j)} k_j} = H_{G_\sigma A F_\sigma^T} = 0$$

by (3). Thus we have $T = 0$. ∎

If we further specialize to the case $n = 2$ in Theorem 2.5, we obtain a more concrete description in the next corollary.

Corollary 2.6. *Let $f, g, h, k \in L^\infty$. Then $H_f T_g = H_h T_k$ on H^2 if and only if one of the following statements holds:*

(1) $f, h \in H^\infty$.
(2) $g, h, fg \in H^\infty$.
(3) $f, k, hk \in H^\infty$.
(4) $g, k, fg - hk \in H^\infty$.
(5) $f + \alpha h, k + \alpha g, h(k + \alpha g) \in H^\infty$ *for some nonzero constant* $\alpha \in \mathbb{C}$.

Proof. First suppose $H_f T_g = H_h T_k$. By Theorem 2.5(with σ being the identity permutation without loss of generality), we have

$$\begin{aligned} (a-1)f - bh &\in H^\infty, \\ cf - (d-1)h &\in H^\infty, \\ ck + ag &\in H^\infty, \\ dk + bg &\in H^\infty \end{aligned} \tag{2.3}$$

for some constants a, b, c, d. If $f \in H^\infty$ and $b \neq 0$, then the first line above shows $h \in H^\infty$ and (1) holds. If $f \in H^\infty, b = 0$ and $d \neq 0$, then the fourth line above shows $k \in H^\infty$. By Lemma 2.4 and (1.2), $hk \in H^\infty$. Thus (3) holds. If $f \in H^\infty$ and $b = d = 0$, then the second line above shows $h \in H^\infty$, so (1) holds. Therefore, if $f \in H^\infty$, then (1) or (3) holds. Similarly, if $g \in H^\infty$, then (2) or (4) holds. Also, if $h \in H^\infty$, then (1) or (2) holds. Finally, if $k \in H^\infty$, then (3) or (4) holds.

Now assume f, g, h, k are not in H^∞. If $a-1 = b = c = d-1 = 0$, then the third line and fourth line in (2.3) tell us that $g, k \in H^\infty$, which contradicts our assumption. Thus one of $a-1, b, c, d-1$ is nonzero. On the other hand, using the first two conditions in (2.3), we see that $a - 1 \neq 0$ if and only if $b \neq 0$, and $c \neq 0$ if and only if $d - 1 \neq 0$. Thus we have $f + \beta h \in H^\infty$, where $\beta = -b/(a-1)$ or $\beta = -(d-1)/c$. Also, if $a = b = c = d = 0$, then the first two lines in (2.3) show that $f, h \in H^\infty$, which is a contradiction as well. So one of a, b, c, d is nonzero. By the same argument as above we have $k + \gamma g \in H^\infty$, where $\gamma = a/c$ or $\gamma = b/d$. By (2.3), we have

$$\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} \begin{pmatrix} f \\ -h \end{pmatrix} \in H_2^\infty, \qquad \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} k \\ g \end{pmatrix} \in H_2^\infty.$$

If one of the two 2×2 matrices is invertible, then $f, h \in H^\infty$, or $g, k \in H^\infty$, which is a contradiction. Thus the two matrices are not invertible so that their determinants are both zero, which implies $(a-1)(d-1) = bc = ad$ and hence $a + d = 1$. Using this fact, we see that $\beta = \gamma$ for any $\beta \in \{-b/(a-1), -(d-1)/c\}$ and $\gamma \in \{a/c, b/d\}$. Since $f + \beta h \in H^\infty$ and $k + \gamma g \in H^\infty$, we have $H_{f+\beta h} = 0$ and $H_h T_{k+\gamma g} = H_{h(k+\gamma g)}$. It follows that

$$\begin{aligned} H_f T_g &= (H_{f+\beta h} - \beta H_h) T_g = -\beta H_h T_g, \\ H_h T_k &= H_h (T_{k+\gamma g} - \gamma T_g) = H_{h(k+\gamma g)} - \gamma H_h T_g. \end{aligned} \tag{2.4}$$

Since $H_fT_g = H_hT_k$ by assumption and $\beta = \gamma$, we have $H_{h(k+\gamma g)} = 0$ and so $h(k+\gamma g) \in H^\infty$. So (5) follows with $\alpha = \beta = \gamma$.

Conversely, suppose one of the conditions (1)–(5) holds. If one of (1)–(4) holds, we have $H_fT_g = H_hT_k$ by Lemma 2.4 and (1.2). If (5) holds, (2.4) with $\alpha = \beta = \gamma$ shows that $H_fT_g = H_hT_k$. ∎

Taking $h = k = 0$ in Corollary 2.6, we obtain the following result which shows that the product of a Hankel and a Toeplitz operator can be zero only in trivial cases.

Corollary 2.7. *Let $f, g \in L^\infty$. Then $H_fT_g = 0$ on H^2 if and only if one of the following conditions holds:*

(1) $f \in H^\infty$.
(2) $g, fg \in H^\infty$.

Next we consider operators of the form $H_fT_g + T_hH_k$ and characterize when such an operator is zero.

Theorem 2.8. *Let $f, g, h, k \in L^\infty$. Then $H_fT_g + T_hH_k = 0$ on H^2 if and only if one of the following statements holds:*

(1) $f, \tilde{h}, \tilde{h}k \in H^\infty$.
(2) $f, k \in H^\infty$.
(3) $g, k, fg \in H^\infty$.
(4) $g, \tilde{h}, fg + \tilde{h}k \in H^\infty$.
(5) $f - \alpha\tilde{h}, k - \alpha g, \tilde{h}g \in H^\infty$ *for some nonzero constant* α.

Proof. First assume $H_fT_g + T_hH_k = 0$. By Lemma 2.3, we have

$$H_f1 \otimes H_g^*1 = H_{\tilde{h}}1 \otimes H_k^*1. \tag{2.5}$$

If $H_f1 = 0$, then $H_{\tilde{h}}1 = 0$ or $H_k^*1 = 0$, we have either $f, \tilde{h} \in H^\infty$ or $f, k \in H^\infty$. If $f, \tilde{h} \in H^\infty$, then $0 = T_hH_k = H_{\tilde{h}k}$ by our assumption, hence $f, \tilde{h}, \tilde{h}k \in H^\infty$. So (1) or (2) holds. By similar arguments, we see that $H_g^*1 = 0$ implies (3) or (4); $H_{\tilde{h}}1 = 0$ implies (1) or (3); $H_k^*1 = 0$ implies (2) or (4).

If none of $H_f1, H_g^*1, H_{\tilde{h}}1, H_k^*1$ is zero, then $f, g, \tilde{h}, k$ are not in H^∞. By (2.5), we have $H_f1 = \alpha H_{\tilde{h}}1$ and $H_k^*1 = \bar{\alpha}H_g^*1$ for some nonzero constant α. It follows from Lemma 2.4 that $f - \alpha\tilde{h}, k - \alpha g \in H^\infty$. Hence

$$\begin{aligned} H_fT_g &= [H_{f-\alpha\tilde{h}} + \alpha H_{\tilde{h}}]T_g = \alpha H_{\tilde{h}}T_g, \\ T_hH_k &= T_h[H_{k-\alpha g} + \alpha H_g] = \alpha T_hH_g. \end{aligned} \tag{2.6}$$

Since $\alpha \neq 0$ and $H_fT_g + T_hH_k = 0$ by assumption, we have $0 = H_{\tilde{h}}T_g + T_hH_g = H_{\tilde{h}g}$. So $\tilde{h}g \in H^\infty$ and (5) follows.

Conversely, if one of (1)–(4) holds, we have $H_fT_g + T_hH_k = 0$ by Lemma 2.4 and (1.2). If we assume (5), then it follows from (2.6) that $H_fT_g+T_hH_k = \alpha[H_{\tilde{h}}T_g + T_hH_g] = \alpha H_{\tilde{h}g} = 0$. ∎

Taking $h = -g, k = f$ in Theorem 2.8, we obtain the following corollary which coincides with the classical result of Martínez-Avendaño dealing with the commutation problem.

COROLLARY 2.9. *Let $f, g \in L^\infty$. Then $H_fT_g = T_gH_f$ on H^2 if and only if one of the following statements holds:*

(1) $f \in H^\infty$.
(2) $g, \tilde{g} \in H^\infty$.
(3) $f + \alpha g, g + \tilde{g}, g\tilde{g} \in H^\infty$ *for some nonzero constant* α.

3. Compact perturbation. In this section, we investigate when is the product of Hankel operator and Toeplitz operator a compact perturbation of a Hankel or Toeplitz operator. First we introduce some notations. For each z in the unit disk D, the normalized reproducing kernel at z is

$$k_z(w) = \frac{\sqrt{1-|z|^2}}{1-\bar{z}w},$$

it is well known that $k_z \to 0$ weakly as $|z| \to 1^-$. The Möbius transform is denoted by

$$\phi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

To prove our main theorems we will need results about Douglas algebras. A *Douglas algebra* is a closed subalgebra of L^∞ which contains H^∞. The Gelfand space (space of nonzero multiplicative linear functionals) of the Douglas algebra B will be denoted by $M(B)$. If B is a Douglas algebra, then $M(B)$ can be identified with the set of nonzero linear functionals in $M(H^\infty)$ whose representing measures (on $M(L^\infty)$) are multiplicative on B, and we identify the function f with its Gelfand transform on $M(B)$. In particular, $M(H^\infty + C) = M(H^\infty) - D$, and a function $f \in H^\infty$ may be thought of as a continuous function on $M(H^\infty + C)$. A subset of $M(L^\infty)$ is called a *support set* if it is the support of the representing measure for a functional in $M(H^\infty + C)$. For more details, we refer the readers to [9], [2], [12], [14], [3], and [4].

For a function on the unit disk D and $m \in M(H^\infty + C)$, we use the notation $z \to m$ to mean that z converges to m in the maximal ideal space of H^∞, and we write $\lim_{z\to m} F(z) = 0$ if for every net $\{z_\alpha\} \subset D$ converging to m, $\lim_{z_\alpha\to m} F(z_\alpha) = 0$.

The following three lemmas are proved in [7].

LEMMA 3.1. *If $T : H^2 \to H^2$ is a compact operator, then*

$$\lim_{|z|\to 1^-} \|T - T_{\tilde{\phi}_z} T T_{\bar{\phi}_z}\| = 0.$$

LEMMA 3.2. *Suppose that $f, g \in L^\infty$. If $\lim_{z\to m} \|H_g k_z\|_2 = 0$, then $\lim_{z\to m} \|H_g T_f k_z\|_2 = 0$. If $\lim_{z\to m} \|H_g^* k_{\bar{z}}\|_2 = 0$, then $\lim_{z\to m} \|H_g^* T_f k_{\bar{z}}\|_2 = 0$.*

LEMMA 3.3. *A finite sum T of finite products of Toeplitz operators is a compact perturbation of a Toeplitz operator if and only if*

$$\lim_{|z|\to 1^-} \|T - T_{\phi_z}^* T T_{\phi_z}\| = 0.$$

The following lemma of [10, Lemma 2.5] will be used later.

LEMMA 3.4. *Let $f \in L^\infty$ and $m \in M(H^\infty + C)$, and let S be the support set for m. Then $f|_S \in H^\infty|_S$ if and only if $\underline{\lim}_{z\to m} \|H_f k_z\|_2 = 0$.*

A symbol mapping was defined on the Toeplitz algebra in [9]. It was extended in [1] to a contractive *-homomorphism $\sigma : T^+ \to L^\infty$ on the Hankel algebra T^+ which is generated by all Toeplitz operators and all Hankel operators. Moreover, it was shown in [1] that σ is a contractive *-homomorphism, and compact operators and finite products of Toeplitz and Hankel operators with at least one Hankel factor are both contained in $\ker \sigma$.

PROPOSITION 3.5. *For $f, g, h \in L^\infty$, let T denote $H_f T_g - H_h$, then T is compact if and only if*

$$\lim_{|z|\to 1^-} \|T^* T - T_{\phi_z}^* T^* T T_{\phi_z}\| = 0.$$

Proof. The necessity is obvious according to Lemma 3.3; we only prove the sufficiency. We first show that T^*T is a finite sum of finite products of Toeplitz operators:

$$\begin{aligned} T^*T &= (H_f T_g - H_h)^*(H_f T_g - H_h) \\ &= T_g^* H_f^* H_f T_g - T_g^* H_f^* H_h - H_h^* H_f T_g + H_h^* H_h \\ &= T_g^* (H_f^* H_f) T_g - T_g^* (H_f^* H_h) - (H_h^* H_f) T_g + (H_h^* H_h). \end{aligned}$$

Since the product of two Hankel operators is a semicommutator of two Toeplitz operators, T^*T is indeed a finite sum of finite products of Toeplitz operators.

By Lemma 3.3, the assumption tells us that T^*T is a compact perturbation of a Toeplitz operator T_φ, where $\varphi \in L^\infty$. Denote the compact perturbation operator by $K = T^*T - T_\varphi$. Note $T = H_f T_g - H_h$ is in the Hankel algebra T^+ and σ is a *-homomorphism, $\sigma(T) = \sigma(H_f T_g) - \sigma(H_h) = 0$,

$\sigma(T^*T) = \sigma(T)^*\sigma(T) = 0$. So $\varphi = \sigma(T_\varphi) = \sigma(T^*T) - \sigma(K) = 0$ since compact operators are contained in $\ker\sigma$ and this implies that $T^*T = K$ is a compact operator, and of course so is T. ∎

Now we are ready to prove our main result in this section:

THEOREM 3.6. *For $f, g, h \in L^\infty$, $H_f T_g$ is a compact perturbation of H_h if and only if for each support set S, one of the following conditions holds:*

(1) *$f|_S, h|_S$ are in $H^\infty|_S$.*
(2) *$g|_S, (fg-h)|_S$ are in $H^\infty|_S$.*

Proof. First we prove the necessity part. Suppose that $H_f T_g - H_h$ is compact and denoted by T. Then

$$\begin{aligned} T_{\tilde\phi_z} T T_{\bar\phi_z} &= T_{\tilde\phi_z}(H_f T_g - H_h)T_{\bar\phi_z} = T_{\tilde\phi_z} H_f T_g T_{\bar\phi_z} - T_{\tilde\phi_z} H_h T_{\bar\phi_z} \\ &= H_f T_{\phi_z} T_g T_{\bar\phi_z} - H_h T_{\phi_z} T_{\bar\phi_z} \\ &= H_f T_g T_{\phi_z} T_{\bar\phi_z} - H_f H_{\tilde\phi_z} H_g T_{\bar\phi_z} - H_h T_{\phi_z} T_{\bar\phi_z} \\ &= (H_f T_g - H_h) T_{\phi_z} T_{\bar\phi_z} - H_f H_{\tilde\phi_z} H_g T_{\bar\phi_z} \\ &= (H_f T_g - H_h)(1 - H_{\tilde\phi_z} H_{\bar\phi_z}) - H_f H_{\tilde\phi_z} H_g T_{\bar\phi_z} \\ &= (H_f T_g - H_h) - [(H_f T_g - H_h)k_z] \otimes k_z + [H_f k_z] \otimes [T_{\phi_z} H_g^* k_{\bar z}] \\ &= T - [T k_z] \otimes k_z + [H_f k_z] \otimes [T_{\phi_z} H_g^* k_{\bar z}]. \end{aligned}$$

The fourth and sixth equality follow from (1.1) and (1.2), and the seventh equality follows from the equation $H_{\bar\phi_z} = -k_{\bar z} \otimes k_z$ (see [7, Lemma 5]). Noting that k_z converges weakly to zero as $|z| \to 1^-$, we have

$$\lim_{|z|\to 1^-} \|[H_f k_z] \otimes [T_{\phi_z} H_g^* k_{\bar z}]\| = 0$$

by Lemma 3.1. Since

$$\|[H_f k_z] \otimes [H_g^* k_{\bar z}]\| = \|[H_f k_z] \otimes [T_{\phi_z} H_g^* k_{\bar z}] T_{\phi_z}\| \le \|[H_f k_z] \otimes [T_{\phi_z} H_g^* k_{\bar z}]\| \, \|T_{\phi_z}\|,$$

we conclude that

$$\lim_{|z|\to 1^-} \|[H_f k_z] \otimes [H_g^* k_{\bar z}]\| = 0.$$

Let m be in $M(H^\infty + C)$, and let S be the support set of m. By Carleson's Corona Theorem [4], there is a net z converging to m.

Suppose that $\underline{\lim}_{z\to m} \|H_f k_z\|_2 = 0$; note that this is equivalent to

$$\lim_{z\to m} \|H_f k_z\|_2 = 0$$

according to [10, Lemma 2.6], and by Lemma 3.4 we infer that $f|_S$ is in $H^\infty|_S$. Since T is compact,

$$\lim_{z\to m} \|T k_z\|_2 = \lim_{z\to m} \|H_f T_g k_z - H_h k_z\|_2 = 0$$

gives $\lim_{z\to m}\|H_h k_z\|_2 = 0$ since $\lim_{z\to m}\|H_f T_g k_z\|_2 = \lim_{z\to m}\|H_f k_z\|_2 = 0$ by Lemma 3.2. Similarly, $h|_S$ is in $H^\infty|_S$. So condition (1) holds.

Next suppose that there is a constant c such that $\underline{\lim}_{z\to m}\|H_f k_z\|_2 \geq c > 0$. Then $\lim_{z\to m}\|H_g k_z\|_2 = 0$, which follows from the identity $\|H_g^* k_{\bar z}\|_2 = \|H_g k_z\|_2$ (see [7, Lemma 11]). By Lemma 3.4 again, $g|_S$ is in $H^\infty|_S$. Formula (1.2) tells us that

$$H_{fg-h} = H_{fg} - H_h = H_f T_g - H_h + T_{\tilde f} H_g.$$

So

$$\|H_{fg-h} k_z\|_2 \leq \|H_f T_g k_z\|_2 + \|T_{\tilde f} H_g k_z\|_2 \to 0 \quad \text{as } z \to m.$$

Hence $[fg-h]|_S$ is in $H^\infty|_S$ and condition (2) holds. This completes the proof of the necessity part.

Next we prove the sufficiency part. By Proposition 3.5, we need only show

$$\lim_{|z|\to 1^-}\|T^*T - T_{\phi_z}^* T^* T T_{\phi_z}\| = 0.$$

By the Carleson Corona Theorem, the above is equivalent to the condition that for each $m \in M(H^\infty + C)$,

$$\lim_{z\to m}\|T^*T - T_{\phi_z}^* T^* T T_{\phi_z}\| = 0. \tag{3.1}$$

Let $m \in M(H^\infty + C)$, and S be the support set of m. By Carleson's Corona Theorem, there is a net z converging to m.

Suppose that condition (1) holds, i.e., $f|_S, h|_S$ are in $H^\infty|_S$. Lemma 3.4 tells us that

$$\lim_{z\to m}\|H_f^* k_{\bar z}\|_2 = \lim_{z\to m}\|H_f k_z\|_2 = 0, \tag{3.2}$$

$$\lim_{z\to m}\|H_h^* k_{\bar z}\|_2 = \lim_{z\to m}\|H_h k_z\|_2 = 0. \tag{3.3}$$

By Proposition 3.5,

$$\begin{aligned} TT_{\phi_z} &= H_f T_g T_{\phi_z} - H_h T_{\phi_z} = H_f T_{\phi_z} T_g + H_f H_{\tilde\phi_z} H_g - T_{\tilde\phi_z} H_h \\ &= T_{\tilde\phi_z}(H_f T_g - H_h) - [H_f k_z] \otimes [H_g^* k_{\bar z}] = T_{\tilde\phi_z} T - [H_f k_z] \otimes [H_g^* k_{\bar z}]. \end{aligned}$$

The second equality follows from (1.1) and (1.2), the third equality follows from the identity $H_{\tilde\phi_z} = -k_z \otimes k_{\bar z}$. Let $F_z = -[H_f k_z] \otimes [H_g^* k_{\bar z}]$. Then $TT_{\phi_z} = T_{\tilde\phi_z} T - F_z$, and by (3.2), $\lim_{z\to m}\|F_z\| = 0$. So we get

$$\begin{aligned} T_{\phi_z}^* T^* T T_{\phi_z} &= (TT_{\phi_z})^*(TT_{\phi_z}) \\ &= T^* T_{\tilde\phi_z}^* T_{\tilde\phi_z} T + T^* T_{\tilde\phi_z}^* F_z + F_z^* T_{\tilde\phi_z} T + F_z^* F_z \\ &= T^*T - [T^* k_{\bar z}] \otimes [T^* k_{\bar z}] + T^* T_{\tilde\phi_z}^* F_z + F_z^* T_{\tilde\phi_z} T + F_z^* F_z. \end{aligned}$$

The last equality comes from $T^*_{\tilde{\phi}_z} T_{\tilde{\phi}_z} = 1 - k_{\bar{z}} \otimes k_{\bar{z}}$. Combining (3.2) with (3.3) gives

$$T^* k_{\bar{z}} = (H_f T_g - H_h)^* k_{\bar{z}} = T_g^* H_f^* k_{\bar{z}} - H_h^* k_{\bar{z}} \to 0$$

as $z \to m$. Since $\|T\| < \infty$ and $\lim_{z\to m} \|F_z\| = 0$,

$$\lim_{z\to m} \|T^* T^*_{\tilde{\phi}_z} F_z + F_z^* T_{\tilde{\phi}_z} T + F_z^* F_z\| = 0.$$

Clearly this implies (3.1).

Suppose that condition (2) holds. Lemma 3.4 tells us that

$$\lim_{z\to m} \|H^*_{fg-h} k_{\bar{z}}\|_2 = \lim_{z\to m} \|H_{fg-h} k_z\|_2 = 0, \tag{3.4}$$

$$\lim_{z\to m} \|H_g^* k_{\bar{z}}\|_2 = \lim_{z\to m} \|H_g k_z\|_2 = 0. \tag{3.5}$$

Note that

$$T = H_f T_g - H_h = H_{fg} - T_{\tilde{f}} H_g - H_h = H_{fg-h} - T_{\tilde{f}} H_g.$$

Then we have $\lim_{z\to m} \|F_z\| = 0$ by (3.5) and

$$\lim_{z\to m} \|T^* k_{\bar{z}}\|_2 = \lim_{z\to m} \|H^*_{fg-h} k_{\bar{z}} - H_g^* T_{\tilde{f}}^* k_{\bar{z}}\|_2 = 0$$

by (3.4) and Lemma 3.2. This implies (3.1). ∎

Corollary 3.7. *For $f, g, h \in L^\infty, H_f T_g$ is a compact perturbation of T_h if and only if $h = 0$ and for each support set S, one of the following conditions holds:*

(1) *$f|_S$ is in $H^\infty|_S$.*
(2) *$g|_S, [fg]|_S$ are in $H^\infty|_S$.*

Proof. Assume $H_f T_g = T_h + K$, where K is a compact operator. Then

$$\begin{aligned}(H_f 1 \otimes 1 T_g) T_z &= [H_f(1 - T_z T_{\bar{z}}) T_g] T_z = H_f T_g T_z - T_{\bar{z}} H_f T_g \\ &= (T_h + K) T_z - T_{\bar{z}}(T_h + K) = T_{h(z-\bar{z})} + K T_z - T_{\bar{z}} K.\end{aligned}$$

Noting that the leftmost term is a finite rank operator, we infer that $T_{h(z-\bar{z})}$ is a compact operator, which implies that $h = 0$ by [16, Proposition 10.2]. By Theorem 3.6, (1) or (2) holds, proving the "only if" part. The "if" part is obvious by Theorem 3.6. ∎

4. Finite rank perturbation. We need to introduce some notations. Let T, S be bounded linear operators on Hardy space. We write $T = S \bmod (F)$ to denote that the operator $T - S$ has finite rank. The Kronecker theorem [13] states that for $f \in L^\infty$, H_f is of finite rank if and only if f is the sum of an analytic function h and a rational function r. Thus for a rational function $r \in L^\infty$, H_r and $H_{\tilde{r}}$ are both finite rank operators. In fact, we will often use another form of Kronecker's theorem: If $f \in L^\infty$, then H_f has finite rank if and only if there exists a nonzero analytic polynomial $p(z)$ such that $pf \in H^\infty$.

LEMMA 4.1. *For $f, g \in L^\infty$, $H_f = T_g \bmod (F)$ if and only if $g = 0$ and H_f has finite rank.*

Proof. We only need to prove the "only if" part. Assume that $H_f = T_g + F$, where F is a finite rank operator. Multiplying both sides by T_z on the right, we get $H_{fz} = H_f T_z = T_{gz} + FT_z$; then multiplying both sides by $T_{\bar z}$ on the left, we get $H_{fz} = T_{\bar z} H_f = T_{g\bar z} + T_{\bar z} F$. So $T_{g(z-\bar z)}$ is of finite rank, which implies $g = 0$ by [16, Proposition 10.2], and so H_f has finite rank. ∎

COROLLARY 4.2. *For $f, g \in L^\infty$, $H_f = T_g$ if and only if $g = 0$ and f is analytic.*

LEMMA 4.3. *For f_i, g_i, h in L^∞, $i = 1, \ldots, n$, if $\sum_{i=1}^n T_{g_i} H_{f_i} = T_h$, then $h = 0$ and there are constants A_i, B_i with $\sum_{i=1}^n |A_i| > 0$ and $\sum_{i=1}^n |B_i| > 0$ such that*

$$\sum_{i=1}^n A_i f_i \in H^\infty \quad \textit{or} \quad \sum_{i=1}^n B_i \tilde g_i \in H^\infty.$$

Proof. $\sum_{i=1}^n T_{g_i} H_{f_i} = T_h$ implies that

$$\begin{aligned} T_{\bar z}\Big(\sum_{i=1}^n T_{g_i} 1 \otimes 1 H_{f_i}\Big) &= T_{\bar z}\Big(\sum_{i=1}^n T_{g_i}(1 - T_z T_{\bar z}) H_{f_i}\Big) \\ &= T_{\bar z}\Big(\sum_{i=1}^n T_{g_i} H_{f_i}\Big) - \sum_{i=1}^n T_{\bar z} T_{g_i} T_z T_{\bar z} H_{f_i} \\ &= T_{\bar z} T_h - T_h T_z = T_{h(\bar z - z)}. \end{aligned}$$

The leftmost term is a finite rank operator, so the rightmost term $T_{h(\bar z - z)}$ is a finite rank Toeplitz operator, which implies it is zero, and so $h = 0$. Furthermore, $\sum_{i=1}^n H_{f_i^*} T_{\bar g_i} = 0$, and it follows from [5, Theorem 2.1] that there exist constants A_i, B_i with $\sum_{i=1}^n |A_i| > 0$ and $\sum_{i=1}^n |B_i| > 0$ such that

$$\sum_{i=1}^n A_i f_i \in H^\infty \quad \text{or} \quad \sum_{i=1}^n B_i \tilde g_i \in H^\infty$$

since $g \in H^\infty$ if and only if $g^* \in H^\infty$, and $\bar g^* = \tilde g$. ∎

LEMMA 4.4. *For f_i, g_i, h in L^∞, $i = 1, \ldots, n$, if $\sum_{i=1}^n T_{g_i} H_{f_i} - T_h$ has rank k, then $h = 0$ and there are analytic polynomials $A_i(z), B_i(z)$ with $\max\{\deg A_i(z) : 1 \le i \le n\} = k$ and $\max\{\deg B_i(z) : 1 \le i \le n\} = k$ such that $\sum_{i=1}^n A_i f_i \in H^\infty$ or $\sum_{i=1}^n B_i \tilde g_i \in H^\infty$.*

Proof. Assume that $\sum_{i=1}^{n} T_{g_i} H_{f_i} - T_h = F$, where F is an operator of rank k. We have

$$\begin{aligned} T_{\bar z}\Big(\sum_{i=1}^{n} T_{g_i} 1\otimes 1 H_{f_i}\Big) &= T_{\bar z}\Big(\sum_{i=1}^{n} T_{g_i}(1-T_zT_{\bar z})H_{f_i}\Big)\\ &= T_{\bar z}\Big(\sum_{i=1}^{n} T_{g_i}H_{f_i}\Big) - \sum_{i=1}^{n} T_{\bar z}T_{g_i}T_zT_{\bar z}H_{f_i}\\ &= T_{\bar z}(T_h+F)-(T_h+F)T_z = T_{h(\bar z - z)} + T_{\bar z}F - FT_z. \end{aligned}$$

This implies $T_{h(\bar z - z)}$ is a finite rank Toeplitz operator, so $h=0$. Furthermore, $\sum_{i=1}^{n} H_{f_i^*}T_{\bar g_i} = F^*$, and it follows from [5, Theorem 2.2] that there exist analytic polynomials $A_i(z), B_i(z)$ with $\max\{\deg A_i(z) : 1\le i\le n\} = k$, and $\max\{\deg B_i(z) : 1\le i\le n\} = k$ such that $\sum_{i=1}^{n} A_i f_i \in H^\infty$ or $\sum_{i=1}^{n} B_i\tilde g_i \in H^\infty$. ∎

Corollary 4.5. *For $f,g,h\in L^\infty$, $T_gH_f = T_h \bmod (F)$ if and only if $h=0$ and one of the following conditions holds:*

(1) *H_f has finite rank.*
(2) *$H_{\tilde g}$ and $H_{f\tilde g}$ have finite rank.*

Proof. First we prove the "only if" part. Suppose $T_gH_f = T_h \bmod (F)$. By Lemma 4.4, there are nonzero analytic polynomials $A(z)$ and $B(z)$ such that $A(z)f\in H^\infty$ or $B(z)\tilde g\in H^\infty$. If $A(z)f\in H^\infty$, then H_f has finite rank. If $B(z)\tilde g\in H^\infty$, then $H_{\tilde g}$ has finite rank. Because $T_gH_f = H_{f\tilde g} - H_{\tilde g}T_f = H_{f\tilde g} \bmod (F)$, we have $H_{f\tilde g} = T_h \bmod (F)$, which implies $H_{f\tilde g}$ is a finite rank operator by Lemma 4.1.

The "if" part is easy and follows from the same argument as above. ∎

Corollary 4.6. *For $f,g,h\in L^\infty$, $T_gH_f = T_h$ if and only if $h=0$ and one of the following conditions holds:*

(1) *$f\in H^\infty$.*
(2) *$\tilde g\in H^\infty$ and $f\tilde g\in H^\infty$.*

Proof. It is sufficient to prove the "only if" part since the "if" part is obvious. Suppose $T_gH_f = T_h$. It follows from Lemma 4.3 that $f\in H^\infty$ or $\tilde g\in H^\infty$. If $\tilde g\in H^\infty$, then $T_gH_f = H_{f\tilde g} = 0$, so $f\tilde g\in H^\infty$. ∎

Theorem 4.7. *For $f_1,f_2,g_1,g_2,h\in L^\infty$, we have*

$$T_{g_1}H_{f_1} + T_{g_2}H_{f_2} = T_h \bmod (F)$$

if and only if $h=0$ and one of the following conditions holds:

(1) *H_{f_1}, H_{f_2} have finite rank.*
(2) *$H_{f_1}, H_{\tilde g_2}, H_{f_2\tilde g_2}$ have finite rank.*
(3) *$H_{\tilde g_1}, H_{f_2}, H_{f_1\tilde g_1}$ have finite rank.*
(4) *$H_{\tilde g_1}, H_{\tilde g_2}, H_{f_1\tilde g_1 + f_2\tilde g_2}$ have finite rank.*

(5) *There exist nonzero analytic polynomials* A_1, A_2, B_1, B_2, R *such that* $A_1B_1 + A_2B_2 = 0$ *and that* $A_1f_1 + A_2f_2$, $B_1\tilde{g}_1 + B_2\tilde{g}_2$ *and* $R[A_2f_2(B_1\tilde{g}_1 + B_2\tilde{g}_2)]$ *are analytic.*

Proof. Suppose

$$T_{g_1}H_{f_1} + T_{g_2}H_{f_2} = T_h \bmod (F).$$

By Lemma 4.4, $h = 0$. So we get

$$T_{g_1}H_{f_1} + T_{g_2}H_{f_2} = 0 \bmod (F),$$

which implies that

$$H_{f_1^*}T_{\bar{g}_1} + H_{f_2^*}T_{\bar{g}_2} = 0 \bmod (F).$$

It follows from [5, Theorem 4.2] that the above holds if and only if one of the conditions (1)–(5) holds. ∎

Acknowledgements. The authors would like to thank the referees for their excellent suggestions. This research is supported by NSFC No. 11271059.

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Received June 12, 2013
Revised version November 10, 2013 (7798)