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Positive bases in ordered subspaces with the Riesz decomposition property

by

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Abstract. In this article we suppose that E is an ordered Banach space whose positive cone is defined by a countable family $\mathcal{F} = \{f_i \mid i \in \mathbb{N}\}$ of positive continuous linear functionals on E, i.e. $E_+ = \{x \in E \mid f_i(x) \geq 0 \text{ for each } i\}$, and we study the existence of positive (Schauder) bases in ordered subspaces X of E with the Riesz decomposition property. We consider the elements x of E as sequences $x = (f_i(x))$ and we develop a process of successive decompositions of a quasi-interior point of X_+ which at each step gives elements with smaller support. As a result we obtain elements of X_+ with minimal support and we prove that they define a positive basis of X which is also unconditional. In the first section we study ordered normed spaces with the Riesz decomposition property.

1. Introduction and notations. The most typical examples of ordered Banach spaces E with a rich class of ordered subspaces are the universal spaces $C[0,1]$ and ℓ_∞. As shown in [8, Theorem 4.1] ([1]) each separable ordered Banach space with closed and normal positive cone is order-isomorphic to an ordered subspace of $C[0,1]$, therefore the study of positive bases in separable ordered Banach spaces is equivalent to the study of such bases in closed ordered subspaces of $C[0,1]$.

In this article we study the general problem of existence of positive bases in ordered subspaces X of E, as formulated in the abstract, by developing a method of decomposition of a quasi-interior point of X. To develop this method we study the subspaces X of E with the *maximum support property*. In such X the quasi-interior points of X and of its closed principal solid subspaces are characterized as the positive vectors of those subspaces with maximum support. We show that in such subspaces the extremal points of X_+ are the nonzero elements of X_+ with minimal support; this property

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([1]) This result is shown by a slight modification of the classical proof of the universality of $C[0,1]$.

turns out to be important for the study of positive bases. Also, this class of subspaces is large. Indeed, as shown in [7, Lemma 5.1], each Banach lattice with a positive basis is order isomorphic to a closed, ordered subspace of ℓ_∞ with the maximum support property with respect to the family $\mathcal{F}$ of the Dirac measures δ_i supported at the natural numbers i, and a similar result is also true for the space $C[0,1]$ (see [8, Theorem 5.1]). Therefore the class of ordered subspaces of ℓ_∞ or $C[0,1]$ with the maximum support property is large and contains, in the sense of order isomorphism, the class of Banach lattices with a positive basis.

To develop our method of decompositions we also study the ordered subspaces X of E with the following property which we call the *ws-property*: for any $x \in X_+$ and any $f_i \in \mathcal{F}$ the set $K = \{y \in X_+ \mid y \leq x \text{ and } f_i(y) = 0\}$ has at least one maximal element. According to the terminology of vector optimization, X has the ws-property if and only if the set K has Pareto efficient points with respect to X_+. If E is a Banach lattice with order continuous norm or if E is a dual space, we show, in Corollaries 20 and 21, that the ordered subspaces of E have the ws-property. In the main result of this article, Theorem 32, we prove that the maximum support property and the ws-property are sufficient conditions for the existence of positive bases in the ordered subspaces of E with the Riesz decomposition property. As an application we show (Theorem 36) that the maximum support property and the ws-property are necessary and sufficient for a positive biorthogonal system of an ordered Banach space E with the Riesz decomposition property to define a positive basis of E.

This article is a generalization of [7] where the same problem is studied in lattice-subspaces of E. In the first section of this paper we study ordered normed spaces with the Riesz decomposition property and we prove some results necessary for our method of decompositions. Specifically we study quasi-interior points and we generalize the results existing for normed lattices to ordered normed spaces with the Riesz decomposition property (Theorems 4 and 6).

Finally, note that each Banach space with an unconditional basis, ordered by the positive cone of the basis, is a Banach lattice with respect to an equivalent norm. The problem of existence of unconditional basic sequences in Banach spaces, known as the unconditional basic sequence problem, was solved in the negative in 1993 by W. T. Gowers and B. Maurey [3]. Our results give necessary conditions for the existence of unconditional basic sequences in ordered Banach spaces.

Let Y be a (partially) ordered normed space with positive cone Y_+. If $Y = Y_+ - Y_+$ then the cone Y_+ is *generating* or *reproducing*, and if there exists a real number $a > 0$ so that $x, y \in Y_+$ with $x \leq y$ implies that $\|x\| \leq a\|y\|$, the cone Y_+ is *normal*. Recall that a convex set P in a linear space is a *cone* if $\lambda x \in P$ for any real $\lambda \geq 0$ and any $x \in P$, and $P \cap (-P) = \{0\}$. For $x, y \in Y$ with $x \leq y$, the set $[x, y] = \{z \in Y \mid x \leq z \leq y\}$ is the *order interval* xy. A point $x \in Y_+$, $x \neq 0$, is an *extremal point* of Y_+ if for any $y \in Y$ with $0 < y < x$ there exists $\lambda \in \mathbb{R}_+$ such that $y = \lambda x$.

The space Y has the *Riesz decomposition property* (RDP) if for any $x, y_1, y_2 \in Y_+$ with $x \leq y_1 + y_2$ there exist $x_1, x_2 \in Y_+$ such that $x = x_1 + x_2$ and $0 \leq x_1 \leq y_1$, $0 \leq x_2 \leq y_2$. A subspace Z of Y is *solid* if for any $x, y \in Z$ with $x \leq y$, the order interval $[x, y]$ is contained in Z. We say that the cone Y_+ *gives an open decomposition* of Y or that Y_+ is *nonflat* if $U_+ - U_+$ is a neighborhood of zero, where $U_+ = U \cap Y_+$ is the positive part of the closed unit ball U of Y, or equivalently, if any $x \in Y$ has a representation $x = x_1 - x_2$ with $x_1, x_2 \in Y_+$ and $\|x_1\|, \|x_2\| \leq M\|x\|$, where M is a constant real number.

A linear functional f on Y is *positive* if $f(x) \geq 0$ for each $x \in Y_+$, and *strictly positive* if $f(x) > 0$ for each $x \in Y_+$, $x \neq 0$. Denote by Y^* the set of continuous linear functionals of Y and by Y_+^* the set of positive ones.

Suppose that Y is an ordered Banach space. A sequence $\{e_n\}$ in Y is a (Schauder) *basis* of Y if each $x \in Y$ has a unique expansion $x = \sum_{n=1}^\infty \lambda_n e_n$ with $\lambda_n \in \mathbb{R}$ for each n. If moreover $Y_+ = \{x = \sum_{n=1}^\infty \lambda_n e_n \mid \lambda_n \geq 0$ for each $n\}$, then $\{e_n\}$ is a *positive basis* of Y. A positive basis is unique in the sense that if $\{b_n\}$ is another positive basis of Y, then each element of $\{b_n\}$ is a positive multiple of an element of $\{e_n\}$. If $\{e_n\}$ is a positive basis of Y then, by [9, Theorem 16.3] and [4, Theorems 3.5.2 and 4.1.5], the following statements are equivalent:

(i) the basis $\{e_n\}$ is unconditional,
(ii) the cone Y_+ is generating and normal,
(iii) Y is a Banach lattice with respect to an equivalent norm.

A linear operator T from Y onto an ordered normed space Z is an *order-isomorphism* of Y onto Z if T is one-to-one, T and T^{-1} are continuous and for each $x \in Y$ we have: $x \in Y_+$ if and only if $T(x) \in Z_+$. For undefined notions and terminology regarding ordered spaces we refer to [4], [5], [1], [6] and [10]. For Schauder bases we refer to [9].

2. Quasi-interior points in spaces with the Riesz decomposition property. In this section we denote by Y an ordered normed space with the Riesz decomposition property whose positive cone Y_+ is closed, normal and gives an open decomposition of Y. Then, by the Riesz–Kantorovich theorem, the set Y^{b} of order bounded linear functionals on Y is an order complete linear lattice. For any $x \in Y_+$,

$$I_x = \bigcup_{n \in \mathbb{N}} [-nx, nx]$$

is the *solid subspace of* Y *generated by* x, and the closure of I_x is the *closed solid subspace of* Y *generated by* x. We prove below that the closure of I_x is again solid. Recall the following properties of an ordered Banach space W which we use in this article: (i) If W_+ is closed and generating, then W_+ gives an open decomposition of W (Krein–Šmulian) and any order bounded linear functional on W is continuous and (ii) the cone W_+ is normal if and only if $W^* = W_+^* - W_+^*$ (M. Krein); see for example [4, Theorems 3.5.2, 3.5.6 and 3.4.8]. We start with the following obvious result.

Proposition 1. *Any solid subspace of* Y *has the Riesz decomposition property.*

Proposition 2. *Suppose that* $x \in Y_+$, $x \neq 0$ *and* I *is the closure of* I_x. *Then*:

(i) *for any* $y \in I_+$, *there exists an increasing sequence* $\{y_n\}$ *in* I_x *which converges to* y, *with* $0 \leq y_n \leq y$ *for each* n,
(ii) I *is a solid subspace of* Y,
(iii) *the positive cone* I_x^+ *on* I_x *is generating*,
(iv) *if we suppose moreover that* Y *is a Banach space then each positive*, *continuous*, *linear functional on* I *has a positive*, *continuous*, *linear extension onto* Y.

Proof. Let $y \in I_+$, $y \neq 0$. First we shall show that there exists a sequence $\{y_n'\}$ in $I_x \cap [0, y]$ convergent to y. Since $y \in I$, we have $y = \lim_{n\to\infty} t_n$ where $t_n \in [-\kappa_n x, \kappa_n x]$ and $\{\kappa_n\}$ is an increasing sequence of natural numbers. Hence $t_n - y \to 0$, therefore by [4, Theorem 3.3.5], there exist sequences $\{w_n\}$, $\{v_n\}$ in Y_+ with $t_n - y = w_n - v_n$ and $w_n, v_n \to 0$. Then $t_n + v_n - y = w_n \geq 0$, and therefore

$$y \leq t_n + v_n \leq \kappa_n x + v_n. \tag{1}$$

By the RDP we know that $y = y_n' + y_n''$ where $0 \leq y_n' \leq \kappa_n x$ and $0 \leq y_n'' \leq v_n$. Since the cone Y_+ is normal and the sequence v_n converges to zero, the sequence y_n'' also converges to zero, hence $y_n' \to y$, as desired. So for any positive real number ε, we have $\|y - y_n'\| < \varepsilon/2$ for a proper n. We put $r_1 = y_n'$. Similarly there exists $r_2 \in I_x \cap [0, y - r_1]$ with $\|y - r_1 - r_2\| < \varepsilon/2^2$ and continuing this process we find a sequence $\{r_n\}$ in I_x with $r_n \in [0, y - \sum_{i=1}^{n-1} r_i]$ and $\|y - \sum_{i=1}^{n} r_i\| < \varepsilon/2^n$ for each n. Then $y_n = \sum_{i=1}^{n} r_i$ is an increasing sequence in $[0, y]$ which converges to y, proving (i).

For the proof of (ii) it is enough to show that $[0, y] \subseteq I_+$ for any $y \in I_+$. So let $y \in I_+$ and $z \in [0, y]$. As in the proof of (i) we find again that y satisfies (1) and by the RDP we have $z = z_n' + z_n''$ where $0 \leq z_n' \leq \kappa_n x$, $0 \leq z_n'' \leq v_n$ and as before $z_n'' \to 0$. Hence $z_n' \to z$, therefore $z \in I$ and statement (ii) follows.

Statement (iii) is obvious because for any $y \in [-nx, nx]$ we have $0 \leq y + nx \leq 2nx$, therefore $y + nx = a + b$ where $a, b \in Y_+$ with $a \leq nx$, $b \leq nx$, and hence $y = a - (nx - b)$.

Finally, suppose that f is a positive, continuous linear functional on I. For any $y \in Y_+$ we put $L_y = \{z \in I_x^+ \mid z \leq y\}$. Then L_y is bounded because the cone Y_+ is normal. For any $y \in Y_+$ we put $g(y) = \sup\{f(z) \mid z \in L_y\}$. By the RDP and by the fact that I_x is solid we have $L_y + L_w = L_{y+w}$. Therefore g is positively homogeneous and additive on Y_+. Hence g has a linear and positive extension onto Y which we denote again by g, i.e. $g(x) = g(x_1) - g(x_2)$ for any $x = x_1 - x_2 \in Y$ with $x_1, x_2 \in Y_+$. By [4, Corollary 3.5.6], g is continuous. By the definition of g and by the fact that I_x is solid, we have $g(y) = f(y)$ for any $y \in I_x^+$, therefore g coincides with f on I_x because $I_x = I_x^+ - I_x^+$. Since I_x is dense in I we conclude that g is also equal to f on I, therefore g is an extension of f from I to Y. ■

DEFINITION 3. An element $u \in Z_+$ of an ordered topological linear space Z is a *quasi-interior point* of Z_+ (or a quasi-interior positive element of Z) if the solid subspace $\bigcup_{n\in\mathbb{N}}[-nu, nu]$ of Z generated by u is dense in Z.

The above definition extends the notion of the quasi-interior point (see [1, p. 259]) from normed lattices to ordered topological linear spaces. It is clear that if u is a quasi-interior point of Z_+ then $f(u) > 0$ for any positive, continuous, and nonzero linear functional f on Z. In [5, p. 24], the points u of an ordered Banach space Z with $f(u) > 0$ for any positive, continuous, nonzero linear functional f on Z are called quasi-interior points of Z_+. In Theorem 6 we show that in ordered Banach spaces with the RDP, these two definitions are equivalent. By Proposition 2 we get the following result:

THEOREM 4. *An element $u \in Y_+$ is a quasi-interior point of Y_+ if and only if for each $x \in Y_+$ there exists an increasing sequence $\{x_n\}$ in I_u which converges to x with $0 \leq x_n \leq x$ for each n.*

PROPOSITION 5. *If u is a quasi-interior point of Y_+, then $[0, x] \cap [0, u] \neq \{0\}$ for each $x \in Y_+$, $x \neq 0$.*

Proof. By the above theorem there exists an increasing sequence $\{x_n\}$ in I_u with $0 < x_n \leq x$ which converges to x, therefore the proposition is true. ■

THEOREM 6. *If moreover Y is a Banach space and $u \in Y_+$, then the following statements are equivalent*:

(i) *u is a quasi-interior point of Y_+,*
(ii) *$f(u) > 0$ for each $f \in Y_+^*$, $f \neq 0$.*

Proof. (i)⇒(ii) is obvious because $f(u) = 0$ implies that $f = 0$ on Y. For the converse suppose that (ii) holds and that the closure I of I_u is a proper subspace of Y. So there exists $g \in Y^*$, $g \neq 0$, which is identically zero on I. Then $|g| \in Y^*$ because Y is a Banach space and $|g|$ is positive. It is known that $|g|(y) = \sup g([-y, y])$ for any $y \in Y_+$. Since $g \neq 0$ and the positive cone of Y is generating we see that $g(y) \neq 0$ for at least one $y \in Y_+$, which implies that $|g| \neq 0$. Therefore $|g|(u) > 0$. Since $|g|(u) = \sup g([-u, u])$ it follows that g is not identically zero on the interval $[-u, u]$, a contradiction because g is identically zero on I and $[-u, u] \subseteq I$. Therefore u is a quasi-interior point of Y_+ and (ii)⇒(i) is proved. ∎

PROPOSITION 7. *Let Z be an ordered normed space and suppose that its positive cone Z_+ is complete. Then the following statements are equivalent*:

(i) *every $y \in Z_+$, $y \neq 0$, is a quasi-interior point of Z_+*,
(ii) $\dim Z = 1$.

Proof. Suppose that (i) is true. First we show that the boundary ϑZ_+ of Z_+ is equal to $\{0\}$. By the Bishop–Phelps theorem (see for example [4, Theorem 3.8.14]) the support points of Z_+ are dense in ϑZ_+. Suppose that r is a support point of Z_+ which is supported by the functional $x^* \in Z^*$, $x^* \neq 0$, i.e. $x^*(r) = \min\{x^*(t) \mid t \in Z_+\}$. Then $x^*(r) \leq 0$ because $0 \in Z_+$. If we suppose that x^* is not positive, there exists $a \in Z_+$ with $x^*(a) < 0$. Then x^*, restricted to the halfline defined by a, takes any negative real value, therefore $x^*(r) = -\infty$, a contradiction. Hence x^* is positive. If we suppose that $r \neq 0$, then r is a quasi-interior point of Z_+, therefore $x^*(r) > 0$, a contradiction, because we have found before that $x^*(r) \leq 0$, hence $r = 0$ and $\vartheta Z_+ = \{0\}$.

We now show that $Z = Z_+ \cup (-Z_+)$. So suppose that $w \in Z \setminus Z_+$ and $y \in Z_+$, $y \neq 0$. Suppose also that z is a point of the line segment $[y, w]$ with $z \in \vartheta Z_+$. Then $z = 0$, therefore $w \in -Z_+$, hence $Z = Z_+ \cup (-Z_+)$. Suppose now that w is a fixed point of $Z \setminus Z_+$. As shown before, for any point $y \in Z_+$, $y \neq 0$, the line segment $[y, w]$ contains 0, therefore y belongs to the line defined by w and 0, hence Z_+ is a halfline and $\dim Z = 1$. So (i) implies (ii). The converse is clear. ∎

DEFINITION 8. Let Z be an ordered space and $x, y \in Z_+$ with $x, y \neq 0$. If $[0, x] \cap [0, y] = \{0\}$, we say that x, y are *disjoint in* Z_+ and write $\inf_{Z_+}\{x, y\} = 0$.

The next result will be used later for the study of positive bases. Statement (i) is an easy consequence of the Riesz decomposition property.

PROPOSITION 9. *Let Z be an ordered normed space with the Riesz decomposition property. Then the following statements are true*:

(i) *If the vectors* $y_1, \ldots, y_n$ *are pairwise disjoint in* Z_+ *and* $x \in Z_+$ *with* $x \le y_1 + \cdots + y_n$, *then:*

(a) x *has a unique decomposition* $x = x_1 + \cdots + x_n$ *with* $0 \le x_i \le y_i$ *for each* $i = 1, \ldots, n$,

(b) *if* $x \ge y_i$ *for each* $i = 1, \ldots, n$, *then* $x = y_1 + \cdots + y_n$,

(c) *if* Φ_1, Φ_2 *are subsets of* $\{1, \ldots, n\}$, $y_{\Phi_1} = \sum_{i \in \Phi_1} \lambda_i y_i$, $y_{\Phi_2} = \sum_{i \in \Phi_2} \mu_i y_i$, *where* λ_i, μ_i *are positive real numbers and* $h \le y_{\Phi_1}$, $h \le y_{\Phi_2}$ *then* h *has a unique decomposition* $h = \sum_{i \in \Phi_1 \cap \Phi_2} h_i$ *where* $0 \le h_i \le \min\{\lambda_i, \mu_i\} y_i$ *for each* $i \in \Phi_1 \cap \Phi_2$. *If* $\Phi_1 \cap \Phi_2 = \emptyset$ *then* y_{Φ_1}, y_{Φ_2} *are disjoint in* Z_+.

(ii) *If the positive cone* Z_+ *of* Z *is normal and the vectors* y_i, $i \in \mathbb{N}$, *are pairwise disjoint in* Z_+, *and the sum* $\sum_{i=1}^{\infty} y_i$ *exists, then*

(a) $\inf_{Z_+}\{\sum_{i=1}^{n} y_i, \sum_{i=n+1}^{\infty} y_i\} = 0$ *for each* n,

(b) *each element* x *of* Z_+ *with* $0 \le x \le \sum_{i=1}^{\infty} y_i$ *has a unique expansion* $x = \sum_{i=1}^{\infty} x_i$ *with* $0 \le x_i \le y_i$ *for each* i.

Proof. The proof of (i) is the following: By the RDP we have $x = x_1 + \cdots + x_n$ with $0 \le x_i \le y_i$ for each i. Suppose that also $x = x_1' + \cdots + x_n'$ with $0 \le x_i' \le y_i$ for each i. Then $0 \le x_j' \le x_1 + \cdots + x_n$, therefore $x_j' = x_1'' + \cdots + x_n''$ with $0 \le x_i'' \le x_i \le y_i$ for each i, and hence $x_i'' = 0$ for each $i \ne j$ because y_i and y_j are disjoint. So $x_j' \le x_j$ and similarly $x_j \le x_j'$, therefore $x_j = x_j'$ for each j, and the expansion of x is unique.

If $y_j \le x$ for each j, then $y_j = y_{j1} + y_{j2} + \cdots + y_{jn}$ with $0 \le y_{ji} \le x_i \le y_i$ for each i. But $0 \le y_{ji} \le y_j$, hence $y_{ji} = 0$ for each $i \ne j$. So $y_j = y_{jj} \le x_j \le y_j$, therefore $y_j = x_j$ for each j and (b) is proved.

To prove (c) we remark that $0 \le h \le y_{\Phi_1}$ implies that $h = \sum_{i \in \Phi_1} h_i$ with $0 \le h_i \le \lambda_i y_i$ for each $i \in \Phi_1$. Since $h \le y_{\Phi_2}$ we have $h_i = \sum_{j \in \Phi_2} h_i^j$ with $0 \le h_i^j \le \mu_j y_j$ for any $j \in \Phi_2$. Since the vectors y_i are disjoint we infer that $h_i^j = 0$ for each $j \ne i$, therefore $h_i = h_i^i \le \min\{\lambda_i, \mu_i\} y_i$ and (c) is proved.

To prove statement (a) of (ii) we suppose that $0 \le h \le \sum_{i=1}^{n} y_i, \sum_{i=n+1}^{\infty} y_i$. Then $h = \sum_{i=1}^{n} h_i$ with $0 \le h_i \le y_i$ for each $i = 1, \ldots, n$. Also $h_i \le y_{n+1} + \sum_{i=n+2}^{\infty} y_i$, therefore $h_i = h_{n+1} + h_{n+1}'$ where $0 \le h_{n+1} \le y_{n+1}$ and $0 \le h_{n+1}' \le \sum_{i=n+2}^{\infty} y_i$. Since y_i and y_{n+1} are disjoint we deduce that $h_{n+1} = 0$, therefore $0 \le h_i = h_{n+1}' \le \sum_{i=n+2}^{\infty} y_i$ and by induction $0 \le h_i \le \sum_{i=n+m}^{\infty} y_i$ for each $m \in \mathbb{N}$. Since the cone is normal and the sequence $\sum_{i=n+m}^{\infty} y_i$ converges to zero, we have $h_i = 0$ for each $i = 1, \ldots, n$. Therefore $h = 0$ and (a) is proved.

To prove (b) suppose that $0 \le x \le \sum_{i=1}^{n} y_i + \sum_{i=n+1}^{\infty} y_i$. Then x has a unique decomposition $x = \sum_{i=1}^{n} x_i + x_n'$ with $0 \le x_i \le y_i$ for each $i = 1, \ldots, n$ and $0 \le x_n' \le \sum_{i=n+1}^{\infty} y_i$. If we suppose that $m > n$ and $x = \sum_{i=1}^{m} v_i + v_m'$ with $0 \le v_i \le y_i$ for $i = 1, \ldots, m$ and $0 \le v_m' \le \sum_{i=m+1}^{\infty} y_i$, then $x =$

$\sum_{i=1}^{n} v_i+(\sum_{i=n+1}^{m} v_i+v'_m)$, therefore $x_i = v_i$ for each $i = 1,\dots,n$. Hence the vectors x_i, $i \in \mathbb{N}$, are uniquely determined and the expansion $x = \sum_{i=1}^{\infty} x_i$ with $0 \le x_i \le y_i$ for each i is unique. ∎

For a further study of the Riesz decomposition property on the space of operators between Banach lattices we refer to [2] and the references therein.

3. Ordered subspaces. In this section we denote by E an infinite-dimensional ordered Banach space whose positive cone E_+ is defined by a countable family $\mathcal{F} = \{f_i \mid i \in \mathbb{N}\}$ of positive, continuous linear functionals on E, i.e. $E_+ = \{x \in E \mid f_i(x) \ge 0 \text{ for each } i\}$. Also we denote by X an *ordered subspace* of E, i.e. X is a subspace of E ordered by the induced ordering. It is clear that E_+ is closed and that $X_+ = X \cap E_+$ is the positive cone of X. For any $x, y \in X$, denote by $\sup_X\{x, y\}$ the supremum and by $\inf_X\{x, y\}$ the infimum of $\{x, y\}$ in X whenever they exist. If $\sup_X\{x, y\}$ and $\inf_X\{x, y\}$ exist for any $x, y \in X$, we say that X is a *lattice-subspace* of E. According to our notations, for any $x, y \in X$ with $x \le y$ the set $[x, y]_X = \{z \in X \mid x \le z \le y\}$ is the *order interval xy* in X; if $x, y \in X_+$ with $[0, x]_X \cap [0, y]_X = \{0\}$, we say that x, y are *disjoint in* X_+ and we write $\inf_{X_+}\{x, y\} = 0$. Also for any $x \in X_+$, $x \ne 0$, we denote by $I_x(X) = \bigcup_{n=1}^{\infty}[-nx, nx]_X$ the *solid subspace of* X *generated by* x. The closure $\overline{I_x(X)}$ of $I_x(X)$ in X is the *closed solid subspace of* X *generated by* x. If $\overline{I_x(X)} = X$, then x is a quasi-interior point of X_+.

3.1. *The minimal and maximum support properties.* The minimal and maximum support properties have been introduced in [7]. For any point $x \in E$ we denote by $x(i)$ the real number $f_i(x)$ and by $\operatorname{supp}(x) = \{i \in \mathbb{N} \mid x(i) \ne 0\}$ the *support* of x (with respect to $\mathcal{F}$). The set $\operatorname{supp}(X_+) = \bigcup_{x \in X_+} \operatorname{supp}(x)$ is the *support* of X_+ (with respect to $\mathcal{F}$). An element $x \ne 0$ of X_+ *has minimal support* in X_+ (with respect to $\mathcal{F}$) if for any $y \in X_+$, $\operatorname{supp}(y) \subsetneq \operatorname{supp}(x)$ implies $y = 0$.

Definition 10. The ordered subspace X of E has the *minimal support property* (with respect to $\mathcal{F}$) if for each $x \in X_+ \setminus \{0\}$ we have: x is an extremal point of X_+ if and only if x has minimal support in X_+.

Proposition 11. *Suppose I is the closed solid subspace of X generated by a nonzero, positive element x of X_+. Then* $\operatorname{supp}(u) = \operatorname{supp}(I_+)$ *for any quasi-interior point u of I_+.* (*The converse is not always true.*)

Proof. It is clear that $\operatorname{supp}(u) \subseteq \operatorname{supp}(I_+)$. If $f_i(u) = 0$ for some $i \in \operatorname{supp}(I_+)$, then f_i is identically zero on $I_u(X)$ and therefore also on I, a contradiction because we have supposed that $i \in \operatorname{supp}(I_+)$. Hence $f_i(u) > 0$ and $\operatorname{supp}(u) = \operatorname{supp}(I_+)$. By Example 15(ii) below, the converse is not always true. ∎

Definition 12. The ordered subspace X of E has the *maximum support property* (with respect to $\mathcal{F}$) if each subspace F of X which is either X or a closed solid subspace of X generated by a nonzero element of X_+ has the property: an element $x \in F_+$ is a quasi-interior point of F_+ if and only if $\operatorname{supp}(x) = \operatorname{supp}(F_+)$.

Proposition 13. *If X_+ is closed and X has the maximum support property*, *then X_+ has quasi-interior points.*

Proof. For each $i \in \operatorname{supp}(X_+)$ there exists $x_i \in X_+$ with $f_i(x_i) > 0$. So

$$u = \sum_{i \in \operatorname{supp}(X_+)} \frac{x_i}{2^i \|x_i\|}$$

is a quasi-interior point of X_+ because X has the maximum support property and $\operatorname{supp}(u) = \operatorname{supp}(X_+)$. ∎

The proof of the next proposition is the same as that of Proposition 3.4 of [7]. The extra assumption here that X_+ is closed is made in order to be able to use Proposition 7.

Proposition 14. *If X_+ is closed and X has the maximum support property*, *then X has the minimal support property.*

Example 15. (i) The sequence spaces c_0 and ℓ_p for $1 \le p < \infty$ have the maximum support property with respect to the family $\mathcal{F} = \{\delta_i\}$ of Dirac measures $\delta_i(x) = x(i)$ supported at the natural numbers i. The space ℓ_∞ of bounded real sequences does not have the maximum support property with respect to $\mathcal{F}$. Indeed, the vector x with $x(i) = 1/i$ for any i has maximum support and the closed solid subspace generated by x is c_0. On the other hand, ℓ_∞ has the minimal support property because the extremal points of ℓ_∞^+, being positive multiples of the vectors e_i, have minimal support.

(ii) The family $\{\delta_{r_i} \mid i \in \mathbb{N}\}$ of Dirac measures supported at the rational numbers r_i in $[0,1]$ and also the family $\mathcal{G} = \{\mu_i \mid i \in \mathbb{N}\}$ of Lebesgue measures μ_i restricted to I_i, where $\{I_i\}$ is a sequence of subintervals of $[0,1]$ so that each interval (a,b) of $[0,1]$ contains at least one I_i, define the positive cone of the space $E = C[0,1]$ of continuous, real-valued functions defined on $[0,1]$. The space E does not have the maximum support property with respect to these families. Indeed, if $x \in E_+$ with $x(t_0) = 0$ for some irrational number t_0 and $x(t) > 0$ for each $t \neq t_0$, then $\operatorname{supp}(x) = \mathbb{N}$ but x is not a quasi-interior point of E_+.

Theorem 16 ([8, Proposition 2.5]). *If X is closed and X has a positive basis $\{b_n\}$*, *the following statements are equivalent*:

(i) *X has the maximum support property with respect to $\mathcal{F}$*,

(ii) *there exists a sequence* $\{i_n\}$ *in* $\mathbb{N}$ *such that* $f_{i_n}(b_n) > 0$ *and* $f_{i_n}(b_m) = 0$ *for any* $m \neq n$, *i.e. the coefficient functionals of the basis* $\{b_n\}$ *can be extended onto* E *to positive multiples of elements of* $\mathcal{F}$.

Here is an example of an ordered subspace with a positive basis, but without the maximum support property.

EXAMPLE 17. Let $\{b_n\}$ be a sequence in ℓ_∞ so that $b_1(4n) = 1/2^n$, $b_1(4n+1) = 1/3^n$ and $b_1(i) = 0$ in the other cases, $b_2(4n) = 1/3^n$, $b_2(4n+1) = 1/2^n$ and $b_2(i) = 0$ in the other cases, and $b_n = e_{4n+2}$ for $n \geq 3$. Then $\{b_n\}$ is a positive basis of the closed subspace X of ℓ_∞ generated by it. The subspace X does not have the maximum support property with respect to the family $\mathcal{F}$ of Dirac measures δ_i supported at the natural numbers i. Indeed, $\mathrm{supp}(b_1) = \mathrm{supp}(b_2)$, therefore $\delta_i(b_1) > 0$ if and only if $\delta_i(b_2) > 0$, and by Theorem 16, X does not have the maximum support property.

3.2. *The ws-property.* The notion of the s-property (supremum property) has been introduced in [7]. We define here a weaker property, which we call the ws-property (weak s-property), as follows:

DEFINITION 18. An ordered subspace X of E has the *ws-property* (with respect to $\mathcal{F}$) if for each $x \in X_+$, $x \neq 0$, and for each $i \in \mathrm{supp}(X_+)$ the set $\{y \in [0,x]_X \mid y(i) = 0\}$ has at least one maximal element.

If in the above definition the set $\{y \in [0,x]_X \mid y(i) = 0\}$ has a maximum element, then X has the s-property. If X has the ws-property, each solid subspace Z of X has this property. In the theory of vector optimization the maximal elements of a subset K of a normed space Z with respect to an ordering cone P of Z are the *Pareto efficient points* of K. In our case, the ws-property ensures the existence of Pareto efficient points with respect to X_+. We start with the following easy result.

THEOREM 19. *Suppose that* τ *is a linear topology on* E *and*

(i) X_+ *is* τ*-closed*,
(ii) *each increasing net in* X_+, *order bounded in* X, *has a* τ*-convergent subnet*,
(iii) *for each* i *the positive part* $K_i^+ = \{y \in X_+ \mid f_i(y) = 0\}$ *of the kernel of* f_i *in* X *is* τ*-closed.*

Then X *has the ws-property.*

Proof. Suppose that $x \in X_+$ and that A is a totally ordered subset of the τ-closed set $[0,x]_X \cap K_i^+$. For each finite subset Φ of A denote by x_Φ the maximum of Φ. Then $\{x_\Phi\}$, being an increasing, order bounded net in $[0,x]_X \cap K_i^+$, is convergent to $x_0 \in [0,x]_X \cap K_i^+$ which is an upper bound of A, and by Zorn's lemma the set $[0,x]_X \cap K_i^+$ has maximal elements. ∎

COROLLARY 20. *If E is a Banach lattice with order continuous norm and X_+ is closed, then X has the ws-property.*

Proof. Each order interval in E is weakly compact. Since X_+ is weakly closed, each order interval in X is weakly compact, hence X has the ws-property. ∎

COROLLARY 21. *If E is a dual space, the functionals f_i are weak-star continuous and X_+ is weak-star closed and normal, then X has the ws-property.*

Proof. For each $x \in X_+$ the order interval $[0, x]_X$ is weak-star closed and bounded because X_+ is normal, therefore $[0, x]_X$ is weak-star compact. Hence X has the ws-property. ∎

COROLLARY 22. *If X is closed with a positive basis, then X has the ws-property.*

Proof. By [11, Theorem 5], each order interval of X is compact. ∎

EXAMPLE 23. (i) The spaces c_0 and ℓ_p with $1 \le p < \infty$ and also the spaces $L_p^+(\mu)$, $1 \le p < \infty$, being Banach lattices with order continuous norm, have the ws-property with respect to any countable family which defines their positive cone. Also all their closed ordered subspaces have the ws-property.

(ii) By Corollary 21, ℓ_∞ and its weak-star closed ordered subspaces have the ws-property with respect to the family of Dirac measures supported at natural numbers.

(iii) $C[0, 1]$ does not have the ws-property with respect to the family of Dirac measures supported at rational numbers in $[0, 1]$. It is easy to show that the set $\{y \in C[0, 1] \mid 0 \le y \le x \text{ and } y(1/2) = 0\}$, where $x \in C_+[0, 1]$ with $x(1/2) > 0$, does not have maximal elements.

If P, Q, R are subcones of X_+ with $R = P + Q$ and $P \cap Q = \{0\}$, we say that R is the *direct sum* of P, Q and write $P \oplus Q = R$.

PROPOSITION 24. *Suppose that X is closed, X_+ is generating and normal, and X has the Riesz decomposition property and the ws-property with respect to $\mathcal{F}$. Let $x \in X_+$, $x \neq 0$, $i \in \operatorname{supp}(X_+)$ and denote by z_i a maximal element of $\{y \in [0, x]_X \mid y(i) = 0\}$. Then $z_i' = x - z_i$ is a minimal element of $\{y \in [0, x]_X \mid y(i) = x(i)\}$. If I, J, W are the closed solid subspaces of X generated respectively by x, z_i, z_i', then:*

(i) $\inf_{X_+}\{z_i, z_i'\} = 0$.

(ii) *The functional f_i is identically zero on J. If $f_i(x) > 0$ then f_i is strictly positive on W. If $f_i(x) = 0$, then $z_i = x$, and if f_i is* *strictly positive on I, then $z_i' = x$. If f_i is not identically zero and non-strictly positive on I then $0 < z_i < x$ and $0 < z_i' < x$.*

(iii) *If $f_i(x) > 0$, then $I_{z_i}^+(X) \oplus I_{z_i'}^+(X) = I_x^+(X)$ and $J_+ \oplus W_+ = I_+$.*

Proof. Suppose that $z \in A = \{y \in [0,x]_X \mid y(i) = x(i)\}$ with $z_i' > z$. Then $x - z > z_i$ and $f_i(x-z) = 0$, which contradicts the definition of z_i. Therefore z_i' is a minimal element of A.

(i) Let $h \in X$ with $0 < h \leq z_i, z_i'$. Then $0 \leq h(i) \leq z_i(i) = 0$, hence $h(i) = 0$. So $h + z_i \leq x$ and $(h + z_i)(i) = 0$, a contradiction. It follows that $\inf_{X_+}\{z_i, z_i'\} = 0$.

(ii) Since $z_i(i) = 0$, f_i is identically zero on I_{z_i} and therefore also on J. Suppose that $f_i(x) > 0$. Then $z_i < x$, hence $z_i' > 0$ and $W_+ \neq \{0\}$. Suppose that $w \in W_+$, $w > 0$ and $w(i) = 0$. Then by Theorem 4, w is the limit of an increasing sequence of elements of $I_{z_i'}^+(X)$, therefore $y(i) = 0$ for at least one $y \in X$ with $0 < y \leq z_i'$. Then $y + z_i \leq x$ and $(y + z_i)(i) = 0$, a contradiction, therefore f_i is strictly positive on W. If we suppose that $f_i(x) = 0$, then by the definition of z_i we have $z_i = x$ and if f_i is strictly positive on I then $z_i = 0$, therefore $z_i' = x$. Suppose now that f_i is nonzero and also non-strictly positive on I. Then $x(i) > 0$ and also $v(i) = 0$ for at least one nonzero point v of I_+. Since v is the limit of an increasing sequence of elements of $I_x^+(X)$, we have $y(i) = 0$ for at least one nonzero $y \in [0,x]_X$. This implies that $z_i > 0$ because if $z_i = 0$ then $z_i < y$, which contradicts the definition of z_i. Also $z_i < x$ because $x(i) > 0$. So $0 < z_i < x$ and $0 < z_i' < x$.

(iii) Let $f_i(x) > 0$. Suppose that $h \in J_+ \cap W_+$. Then $h \in J_+$ and therefore $h(i) = 0$. Since f_i is strictly positive on W we have $h = 0$, therefore $J_+ \cap W_+ = \{0\}$. Suppose that $y \in [0,x]_X$. Then $y \leq z_i + z_i'$ and by the RDP we have $y = y_1 + y_2$ with $y_1 \in [0, z_i]_X$ and $y_2 \in [0, z_i']_X$. By the above remarks the first assertion of (iii) is proved.

Suppose now that $y \in I_+$. By Theorem 4, y is the limit of an increasing sequence y_n in $I_x^+(X)$ with $y_n \leq y$ for each n. Hence $y_{n+1} - y_n \in I_x^+(X)$, therefore $y_{n+1} - y_n \leq k_n x = k_n(z_i + z_i')$, and by the RDP we have $y_{n+1} - y_n = a_{n+1} + b_{n+1}$ with $a_{n+1} \in I_{z_i}^+(X)$ and $b_{n+1} \in I_{z_i'}^+(X)$. If $y_1 = a_1 + b_1$ with $a_1 \in I_{z_i}^+(X)$ and $b_1 \in I_{z_i'}^+(X)$, then

$$y_n = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n).$$

If $s_n = a_1 + \cdots + a_n$ and $r_n = b_1 + \cdots + b_n$, then $s_{n+1} - s_n = a_{n+1} \leq y_{n+1} - y_n$, therefore the sequence $\{s_n\}$ is convergent, because $\{y_n\}$ is convergent and the cone X_+ is normal. Similarly, $\{r_n\}$ is convergent and therefore $y = y' + y''$ with $y' \in J_+$ and $y'' \in W_+$. Hence $I_+ = J_+ \oplus W_+$. ■

Definition 25. Let X be a closed ordered subspace of E as in the previous proposition, and suppose that x is a nonzero element of X_+ and $f_i \in \mathcal{F}$. If f_i is not identically zero and non-strictly positive on $I_x(X)$ and

$x = x_1 + x_2$ where x_1 is a maximal element of the set $\{y \in [0,x]_X \mid y(i) = 0\}$, then we say that $x = x_1 + x_2$ is a *decomposition of x with respect to f_i* (or *with respect to i*) and also that x is *decomposable into x_1, x_2 with respect to f_i*. If f_i is identically zero on $I_x(X)$ or if f_i is strictly positive on $I_x(X)$, we say that x is *indecomposable with respect to f_i* (or with respect to i).

3.3. *Existence of positive bases.* In what follows we will denote by X a closed, ordered subspace of E so that:

(i) X has the Riesz decomposition property,
(ii) the positive cone X_+ of X is closed, normal and generating,
(iii) X has the maximum support property and the ws-property with respect to $\mathcal{F}$.

As noted at the beginning of the previous section, (i) and (ii) imply that X_+ gives an open decomposition of X and that X^* is an order complete linear lattice. We will also denote by M the following subset of $\mathbb{N}$:

$$M = \{i \in \operatorname{supp}(X_+) \mid f_i \text{ is non-strictly positive on } X\}.$$

Therefore for each $x \in X_+$, $x \neq 0$, we have $x(i) > 0$ for each $i \in \operatorname{supp}(X_+) \setminus M$. Also $M \neq \emptyset$ because $M = \emptyset$ implies $\operatorname{supp}(x) = \operatorname{supp}(X_+)$ for each $x \in X_+$, $x \neq 0$, therefore $\dim X = 1$ by Proposition 7. In order to prove the existence of extremal points of X_+ we develop a process of successive decompositions of a quasi-interior point of X_+. So suppose that u is a quasi-interior point of X_+ (such a point exists by Proposition 13); we decompose u as follows:

STEP 1. We put $i_1 = \min M$ and we decompose u into x_1, x_2 with respect to i_1. Then $u = x_1 + x_2$ and $\inf_{X_+}\{x_1, x_2\} = 0$. Also f_{i_1} is identically zero on I_1 and strictly positive on I_2 where I_1, I_2 are the closed solid subspaces of X generated by x_1, x_2 respectively. The set $m_1 = \{x_1, x_2\}$ is the *front* and the natural number i_1 is the *index* of the first decomposition.

STEP $\nu + 1$. Suppose that we have accomplished the νth step and that m_ν is the front and i_ν the index of the νth decomposition. Then at least one of the elements of m_ν is decomposable with respect to an $i \in M$. Indeed, if no element x of m_ν is decomposable with respect to any $i \in M$ then for any $i \in M$, f_i is strictly positive or identically zero on the closed solid subspace I of X generated by x and it is easy to show that $\operatorname{supp}(y) = \operatorname{supp}(I_+)$ for any $y \in I_+$, $y \neq 0$, so y is a quasi-interior point of I. Hence $\dim I = 1$ and X is finite-dimensional because m_ν is finite. Put $i_{\nu+1} = \min\{i \in M \mid$ at least one element of m_ν is decomposable with respect to $i\}$. Then $i_{\nu+1} > i_\nu$ and we decompose with respect to $i_{\nu+1}$ the elements of m_ν which allow such a decomposition. We denote by $m_{\nu+1}$ the set which contains the elements of m_ν which are indecomposable with respect to $i_{\nu+1}$ and also the elements that arise from the decomposition of the elements of m_ν with respect to $i_{\nu+1}$. The set $m_{\nu+1}$ is the *front* and $i_{\nu+1}$ is the *index* of the $(\nu+1)$th decomposition. The set

$$\delta(u)=\bigcup_{\nu=0}^{\infty} m_\nu,$$

where $m_0=\{u\}$, is the *tree of decompositions* of u.

PROPOSITION 26. *In the above process of decompositions of u we have*:

(i) *the sequence $\{i_\nu\}$ of indices of decompositions is strictly increasing*,
(ii) *for each $i\in M$ with $i\le i_\nu$ and for each $x\in m_\nu$, x is indecomposable with respect to i, so f_i is strictly positive or identically zero on $I=\overline{I_x(X)}$*,
(iii) *the elements of m_ν are nonzero with sum u. Also* $\inf_{X_+}\{x,y\}=0$ *for any $x,y\in m_\nu$ with $x\neq y$*,
(iv) $\inf_{X_+}\{x,u-x\}=0$ *for each $x\in\delta(u)$.*

Proof. Statements (i)–(iii) are obvious. To prove (iv) we suppose that $x\in m_\nu$ for some ν and that $m_\nu=\{x,y_1,\ldots,y_k\}$. Since the elements of m_ν are pairwise disjoint in X_+ with sum u we have $u-x=\sum_{i=1}^k y_i$ and (iv) is true by Proposition 9. ∎

For any $x\in m_\nu$ with $\nu\ge 1$ it is easy to show that there exists a unique vector $y\in m_{\nu-1}$ with $y\ge x$. Also for any $x\in m_\nu$ there exists at least one $y\in m_{\nu+1}$ with $x\ge y$. If $x,y\in\delta(u)$ with $x\in m_\nu$, $y\in m_{\nu+\mu}$ and $y\le x$, we say that x is the *presuccessor* of y in m_ν, or that y is a *successor* of x in $m_{\nu+\mu}$. If moreover $y\in m_{\nu+1}$ we say that x is the *first presuccessor* of y or that y is a *first successor* of x.

PROPOSITION 27. *The following are true*:

(i) *for any $x\in m_\nu$ the sum of the successors of x in $m_{\nu+\mu}$ is equal to x*,
(ii) *if y is a successor of x with $x>y$ and I is the closed solid subspace of X generated by x, then* $\inf_{X_+}\{y,x-y\}=0$ *and y is not a quasi-interior point of I_+*,
(iii) *for each $x\in\delta(u)$ and each $i\in M\cap\operatorname{supp}(x)$, there exists a successor y of x such that the functional f_i is strictly positive on the closed solid subspace I of X generated by y.*

Proof. (i) Any element of $\delta(u)$ is the sum of its first successors, therefore the proposition is true for $\mu=1$ and continuing, we deduce it for any μ.

(ii) Since $x-y\le u-y$ and $\inf_{X_+}\{y,u-y\}=0$ we have $\inf_{X_+}\{y,x-y\}=0$, therefore y is not a quasi-interior point of I_+ by Proposition 5.

(iii) Let $x\in m_\kappa$. Since the sequence $\{i_\nu\}$ is strictly increasing, there exists $\nu\in\mathbb{N}$ with $\nu>\kappa$ and $i\le i_\nu$. Then f_i is strictly positive or identically zero on any closed solid subspace of X generated by an element of m_ν. But $x = \sum_{j=1}^{r} x_j$ where $x_1, \ldots, x_r$ are the successors of x in m_ν and $f_i(x) > 0$ because $i \in \operatorname{supp}(x)$, therefore f_i is strictly positive on at least one of the closed solid subspaces of X generated by $x_1, \ldots, x_r$, which proves (iii). ∎

If $x \in \delta(u)$ and $x \in m_\nu$ for each $\nu \geq \nu_0$, then we will say that *the process of decomposition stops* at x. In other words, the process of decomposition stops at x if there exists $\nu_0 \in \mathbb{N}$ so that $x \in m_{\nu_0}$ and for each $i \in M$ with $i > i_{\nu_0}$, the functional f_i is strictly positive or identically zero on the closed solid subspace I of X generated by x. Then for each $i \in M$ with $i \leq i_{\nu_0}$, f_i is strictly positive or identically zero on I (Proposition 26), therefore $\operatorname{supp}(z) = \operatorname{supp}(I_+)$ for any $z \in I_+$, $z \neq 0$, hence any nonzero vector of I_+ is a quasi-interior point of I, which implies that $\dim I = 1$. So x is an extremal point of X_+ and we have proved the following:

Proposition 28. *If the process of decompositions of u stops at an element $x_0 \in \delta(u)$ then x_0 is an extremal point of X_+.*

A sequence $\{x_\nu\}$ in $\delta(u)$ is a *branch* of $\delta(u)$ if $x_\nu > x_{\nu+1}$ for each $\nu \in \mathbb{N}$.

Proposition 29. *Each branch of $\delta(u)$ converges to zero.*

Proof. It is enough to show that any branch $\{x_\nu\}$ of $\delta(u)$ with $x_0 = u$ converges to zero. Let $z_\nu = x_{\nu-1} - x_\nu$ for all $\nu \geq 1$. Then for all ν, μ, we have

$$u = z_1 + \cdots + z_\nu + x_\nu \quad \text{and} \quad x_\nu = z_{\nu+1} + \cdots + z_{\nu+\mu} + x_{\nu+\mu}. \tag{2}$$

The vectors $z_1, \ldots, z_\nu, x_\nu$ are pairwise disjoint in X_+. Indeed, we have $\inf_{X_+}\{x_\nu, u - x_\nu\} = 0$, hence $\inf_{X_+}\{x_\nu, \sum_{i=1}^{\nu} z_i\} = 0$, therefore $\inf_{X_+}\{x_\nu, z_i\} = 0$ for each $i \leq \nu$, because $z_i \leq \sum_{j=1}^{\nu} z_j$. Suppose that $j > i$. Then $z_j \leq x_i$ and $\inf_{X_+}\{z_i, x_i\} = 0$, therefore $\inf_{X_+}\{z_j, z_i\} = 0$. Hence $\inf_{X_+}\{z_j, z_i\} = 0$ for any $i \neq j$.

Let $u_0 = \sum_{\nu=1}^{\infty} z_\nu / 2^\nu$. We shall show that $\operatorname{supp}(u_0) = \operatorname{supp}(X_+)$. For each $i \in \operatorname{supp}(X_+) \setminus M$ we have $x(i) > 0$ for each $x \in X_+$, $x \neq 0$, hence $i \in \operatorname{supp}(u_0)$. Suppose that $i \in M$ and that x_ν is decomposed at the κ_νth decomposition. Since $\{i_{\kappa_\nu}\}$ is strictly increasing, there exists $\mu \in \mathbb{N}$ with $i < i_{\kappa_\mu}$. By Proposition 26(ii), f_i is strictly positive or identically zero on $I = \overline{I_{x_\mu}(X)}$. We shall show that in both cases $i \in \operatorname{supp}(u_0)$. If f_i is strictly positive on I we have $z_{\mu+1}(i) > 0$ because $0 < z_{\mu+1} < x_\mu$, and therefore $i \in \operatorname{supp}(u_0)$. If f_i is identically zero on I then $x_\mu(i) = 0$, therefore

$$f_i(z_1 + \cdots + z_\mu) = f_i(z_1 + \cdots + z_\mu + x_\mu) = f_i(u) > 0,$$

hence $f_i(z_j) > 0$ for at least one j, so $i \in \operatorname{supp}(u_0)$. Therefore $\operatorname{supp}(X_+) = \operatorname{supp}(u_0)$ and u_0 is a quasi-interior point of X.

By Theorem 4, there is an increasing sequence $\phi_n \in [0, u]_X \cap [0, r_n u_0]_X$, where $\{r_n\}$ is a strictly increasing sequence of natural numbers with

$\lim_{n\to\infty}\phi_n = u$. Let

$$h_\nu = \sum_{\mu=1}^{\infty} r_\nu \,\frac{z_{r_\nu+\mu}}{2^{r_\nu+\mu}}.$$

Since $0 \le \phi_n \le r_n u_0$ we have $0 \le \phi_n \le r_n z_1 + \cdots + r_n z_{r_n} + h_n$ and by Proposition 9, ϕ_n has a unique decomposition $\phi_n = \phi_n^1 + \cdots + \phi_n^{r_n} + H_n$ with $0 \le \phi_n^i \le r_n z_i$ for each i and $0 \le H_n \le h_n$. The last inequality implies that $\lim_{n\to\infty} H_n = 0$, because $\lim_{n\to\infty} h_n = 0$ and the cone X_+ is normal. Also we have $0 \le \phi_n^i \le u, r_n z_i$ for $i = 1, \ldots, r_n$, therefore $\phi_n^i = a_1 + \cdots + a_{r_n} + b_n$ with $0 \le a_j \le z_j$ for each j and $0 \le b_n \le x_{r_n}$. Since the vectors $z_1, \ldots, z_{r_n}, x_{r_n}$ are pairwise disjoint in X_+ we have $\phi_n^i = a_i$, therefore $0 \le \phi_n^i \le z_i$ for each $i = 1, \ldots, r_n$. Since $H_n \le u$, we have $H_n = \gamma_1 + \cdots + \gamma_{r_n} + c_n$ with $0 \le \gamma_j \le z_j$ for each $j = 1, \ldots, r_n$ and $0 \le c_n \le x_{r_n}$. Since $H_n \le h_n$ we also have $\gamma_j \le h_n$ for each j. Since the vectors z_j, $j = 1, \ldots, r_n$, and h_n are pairwise disjoint in X_+ we have $\gamma_j = 0$ for each $j = 1, \ldots, r_n$, hence $H_n = c_n$, therefore $H_n \le x_{r_n}$. So $\lim_{n\to\infty}(u - (\phi_n^1 + \cdots + \phi_n^{r_n} + H_n)) = 0$, therefore

$$\lim_{n\to\infty} [(z_1 - \phi_n^1) + \cdots + (z_{r_n} - \phi_n^{r_n}) + (x_{r_n} - H_n)] = 0.$$

Since the members in the above limit are positive and the cone of X_+ is normal we infer that $\lim_{n\to\infty}(x_{r_n} - H_n) = 0$. We have shown above that $\lim H_n = 0$, therefore $\lim x_{r_n} = 0$. Since the sequence $\{x_n\}$ is decreasing it converges to zero and the proposition is proved. ∎

Proposition 30. *For each $x \in \delta(u)$ at least one successor of x is an extremal point of X_+.*

Proof. Let $x \in \delta(u)$. If at least one successor x' of x does not belong to a branch of $\delta(u)$, then the process of decomposition stops after a finite number of steps at any successor of x', therefore any successor of x' is an extremal point of X_+ dominated by x and the assertion is proved. So suppose that any successor of x belongs to a branch of $\delta(u)$. Also we may suppose that $x < u$ because in the case where $x = u$, it is enough to show the assertion for one of its successors.

Let I be the closed solid subspace of X generated by x and set

$$L = \{i \in \operatorname{supp}(x) \mid f_i \text{ is not strictly positive on } I_x(X)\}.$$

Then $L \subseteq M$. Also $\operatorname{supp}(x) = \operatorname{supp}(I_+)$. If L is finite, then after a finite number of steps the decomposition stops at any successor of x and the assertion holds. So suppose that L is infinite. Let $j_1 = \min L$. Then by Proposition 27(iii), there exists $x_1 \in \delta(u)$ such that $x_1 \le x$ and f_{j_1} is strictly positive on $\overline{I_{x_1}(X)}$. Since x_1 is an element of a branch of $\delta(u)$ dominated by x, and any such branch of $\delta(u)$ converges to zero, we may suppose that there exists $y_1 \in \delta(u)$ such that $y_1 < x_1 \le x$ and $\|y_1\| \le 2^{-1}\varepsilon$, where ε

is a constant real number with $0 < \varepsilon < \|x\|$. Note also that f_{j_1} is strictly positive on $\overline{I_{y_1}(X)}$ because it is strictly positive on $\overline{I_{x_1}(X)}$ and $0 < y_1 < x_1$. By Proposition 27 we know that $\inf_{X_+}\{y_1, x - y_1\} = 0$, hence y_1 is not a quasi-interior point of I. Therefore $\mathrm{supp}(y_1) \neq \mathrm{supp}(I_+)$, hence there exists $i \in \mathrm{supp}(I_+)$ with $i \notin \mathrm{supp}(y_1)$, so there exists $i \in L$ with $y_1(i) = 0$. We put $j_2 = \min\{i \in L \mid y_1(i) = 0\}$. Then $j_1 < j_2$ and as before we can find a vector $y_2 \in \delta(u)$ so that $y_2 < x$, $\|y_2\| \leq 2^{-2}\varepsilon$ and f_{j_2} is strictly positive on $\overline{I_{y_2}(X)}$. Then $\inf_{X_+}\{y_1, y_2\} = 0$, because for any $h \in X$ with $0 \leq h \leq y_1, y_2$ we have $0 \leq h(j_2) \leq y_1(j_2) = 0$, therefore $h = 0$ because f_{j_2} is strictly positive on $\overline{I_{y_2}(X)}$. In view of the method of selecting y_2 (as a sufficiently small member of a branch which converges to zero) we may also suppose that $y_1 \in m_{\nu_1}$ and $y_2 \in m_{\nu_2}$ with $\nu_1 < \nu_2$. We may moreover suppose that ν_2 is sufficiently large so that m_{ν_2}, besides the successors of x and the element y_2, contains at least one extra element so that

$$m_{\nu_2} = \{y_2, a_1, \ldots, a_k, b_1, \ldots, b_r, c_1, \ldots, c_l\},$$

where $a_1, \ldots, a_k$ are the successors of y_1 and $y_2, a_1, \ldots, a_k, b_1, \ldots, b_r$ are the successors of x. We put $s_1 = y_1$ and $s_2 = y_1 + y_2$. Then $s_1 < x$ and $s_2 < x$. The first inequality is obvious and the second holds because x is the sum of its successors in m_{ν_2}. Also $s_1(j_1) > 0$ and by the definition of j_2, we have $s_2(i) > 0$ for each $i \in L$ with $i \leq j_2$. By Proposition 9, $\inf_{X_+}\{s_i, x - s_i\} = 0$ for each $i = 1, 2$, because the successors of x in m_{ν_2} are pairwise disjoint. Since $\inf_{X_+}\{s_2, x - s_2\} = 0$ we deduce that s_2 is not a quasi-interior point of I_+, hence there exists $i \in L$ with $s_2(i) = 0$. Let $j_3 = \min\{i \in L \mid s_2(i) = 0\}$. Then $j_2 < j_3$ and as before we can find $y_3 \in m_{\nu_3}$ such that $\nu_2 < \nu_3$, $\|y_3\| \leq 2^{-3}\varepsilon$, f_{j_3} is strictly positive on $\overline{I_{y_3}(X)}$ and the set of successors of x in m_{ν_3} contains the successors of y_1, the successors of y_2, the element y_3 and at least one extra element. As before we can show that $\inf_{X_+}\{y_1, y_3\} = \inf_{X_+}\{y_2, y_3\} = 0$. We put $s_3 = s_2 + y_3$.

Continuing this process we obtain a sequence $\{j_\nu\}$ in L and sequences $\{y_\nu\}$, $\{s_\nu\}$ in X_+ such that $s_1 = y_1$, $s_\nu = s_{\nu-1} + y_\nu$ for each $\nu = 2, 3, \ldots$, with the following properties:

(i) $0 < s_\nu < s_{\nu+1} < x$,

(ii) $\|s_{\nu+1} - s_\nu\| = \|y_{\nu+1}\| \leq 2^{-\nu-1}\varepsilon$ and $y_\nu \in m_{k_\nu}$ with $k_\nu < k_{\nu+1}$ for each ν,

(iii) $\inf_{X_+}\{s_\nu, x - s_\nu\} = 0$ for each ν,

(iv) $\{j_\nu\}$ is a strictly increasing sequence in L and for each $i \in L$ with $i < j_{\nu+1}$ we have $s_\nu(i) > 0$.

By (ii), $\{s_\nu\}$ is a Cauchy sequence; set $s = \lim_{\nu\to\infty} s_\nu$. Then $0 \leq s_\nu \leq s \leq x$ for each ν. Since $\|s_\nu\| \leq \sum_{i=1}^{\nu} \|y_i\| \leq \varepsilon < \|x\|$, we have $s < x$. Also by (iv) and the fact that $\{s_\nu\}$ is increasing we see that $s(i) > 0$ for each $i \in L$, therefore $\operatorname{supp}(s) = \operatorname{supp}(I_+)$. Hence s is a quasi-interior point of I_+.

We will show that $\inf_{X_+}\{s, x-s\} = 0$. To this end we suppose that $0 \le h \le s, x-s$. Since $s = s_\nu + (s - s_\nu)$ we have $h = h_\nu + h'_\nu$ with $0 \le h_\nu \le s_\nu$ and $0 \le h'_\nu \le (s - s_\nu)$. Since the cone is normal and $\lim(s - s_\nu) = 0$ we have $\lim h'_\nu = 0$, therefore $h = \lim h_\nu$. Since $0 \le h_\nu \le s_\nu$ and $h_\nu \le x - s \le x - s_\nu$ we infer that $h_\nu = 0$ for each ν, by (iii). Therefore $h = 0$, hence $\inf_{X_+}\{s, x-s\} = 0$. Since I is solid we also obtain $\inf_{I_+}\{s, x-s\} = 0$, which contradicts the fact that s is a quasi-interior point of I_+ (Proposition 5). Hence at least one successor x' of x does not belong to a branch of $\delta(u)$, therefore at least one successor x_0 of x is an extremal point of X_+ and the proposition is proved. ∎

Proposition 31. *Any extremal point x_0 of X_+ is a positive multiple of a unique element of $\delta(u)$.*

Proof. By Proposition 5, since x_0 is an extremal point of X_+, there exists a real number $r > 0$ with $rx_0 \le u$. Hence $r \le a\|u\|/\|x_0\|$, where a is the constant of the normal cone X_+. Therefore $\sup\{r \in \mathbb{R}_+ \mid rx_0 \le u\} = \lambda > 0$. Let $z_0 = \lambda x_0$. Then $0 < z_0 \le u$. Since $u = \sum_{z \in m_\nu} z$ and the elements of m_ν are pairwise disjoint, there exists a unique $y_\nu \in m_\nu$ so that $z_0 \le y_\nu$. Then $\inf_{X_+}\{z_0, x\} = 0$ for each $x \in m_\nu$, $x \ne y_\nu$. Also $y_\nu \ge y_{\nu+1} \ge z_0$ for each ν. Since each branch of $\delta(u)$ converges to zero, the process of decomposition stops at a point y_μ which is an extremal point of X_+ with $z_0 \le y_\mu$. Hence $y_\mu = \lambda' x_0$. Also $y_\mu \le u$ and by the definition of λ we have $\lambda' \le \lambda$, therefore $y_\mu \le z_0$, which implies that $y_\mu = z_0$ and $z_0 \in \delta(u)$. If $z'_0 = kx_0 \in \delta(u)$, then $kx_0 \le u$, therefore $k \le \lambda$ and $z'_0 \le z_0$. Hence z'_0 is a successor of z_0. If $z'_0 < z_0$ we get a contradiction because z_0, being an extremal point of X_+, is indecomposable. Therefore $z'_0 = z_0$ and the proposition is proved. ∎

In our main result below we prove that X has a positive basis. This basis is also unconditional because X_+ is generating and normal. For convenience we repeat the standing assumptions on E and X.

Theorem 32. *Let E be an ordered Banach space and suppose that E_+ is defined by the family $\mathcal{F} = \{f_i \mid i \in \mathbb{N}\} \subset E^*_+$. Let X be a closed ordered subspace of E with the Riesz decomposition property and suppose that X_+ is normal and generating. If X has the maximum support property and the ws-property with respect to $\mathcal{F}$, then X has a positive basis.*

Proof. Let B be the set of extremal points of X_+ with norm 1. By Proposition 30, $B \ne \emptyset$, and by the previous proposition the map $T : B \to \delta(u)$ so that $T(x) = \lambda x \in \delta(u)$ is one-to-one. Since $\delta(u)$ is countable, so is B, say $B = \{u_i : i \in \mathbb{N}\}$ and $b_i = \lambda_i u_i \in \delta(u)$. Let $u_0 = \sum_{i=1}^{\infty} b_i/2^i$. For each

$i \in M$ there exists $z \in \delta(u)$ so that f_i is strictly positive on $I = \overline{I_z(X)}$. By Proposition 30, $z \geq b_j$ for at least one j, therefore $b_j(i) > 0$. Hence $\mathrm{supp}(u_0) = \mathrm{supp}(X_+)$ and u_0 is a quasi-interior point of X_+.

Let $x \in X_+$. Then there exists an increasing sequence $x_n \in [0, x] \cap [0, k_n u_0]$ where the sequence k_n is strictly increasing and $\lim_{n\to\infty} x_n = x$. Since $0 \leq x_n \leq k_n u_0$, each x_n has a unique expansion $x_n = \sum_{i=1}^{\infty} \sigma_{ni} u_i$ with $\sigma_{ni} \in \mathbb{R}_+$, by Proposition 9. The sequence $\{\sigma_{ni} \mid n \in \mathbb{N}\}$ is increasing. Indeed, for $m > n$ we take again the expansion $x_m - x_n = \sum_{i=1}^{\infty} a_i u_i$ and we have $\sigma_{mi} = \sigma_{ni} + a_i \geq \sigma_{ni}$. Set $\sigma_i = \lim_{n\to\infty} \sigma_{ni}$. Then $0 \leq \sigma_i u_i \leq x$, because $0 \leq \sigma_{ni} u_i \leq x$ for each i. For each $m \in \mathbb{N}$ we have $\sum_{i=1}^{m} \sigma_{ni} u_i \leq x_n \leq x$ and by taking limits as $n \to \infty$ we see that $\sum_{i=1}^{m} \sigma_i u_i \leq x$. Since $\{x_n\}$ converges to x there exists a strictly increasing sequence m_n of natural numbers so that the sequence $y_n = \sum_{i=1}^{m_n} \sigma_{ni} u_i$ converges to x. Then $\sum_{i=1}^{m_n} \sigma_{ni} u_i \leq \sum_{i=1}^{m_n} \sigma_i u_i \leq x$, which yields $x = \sum_{i=1}^{\infty} \sigma_i u_i$. Let $\overline{u}_j = \sum_{i\neq j} b_i/2^i$. Then $\overline{u}_j$ is not a quasi-interior point of X_+, because $\inf_{X_+}\{b_j, \overline{u}_j\} = 0$ by Proposition 9. Therefore $\mathrm{supp}(\overline{u}_j)$ is a proper subset of $\mathrm{supp}(X_+)$, hence there exists $k_j \in M$ with $f_{k_j}(\overline{u}_j) = 0$. Hence $f_{k_j}(u_i) = 0$ for each $i \neq j$. Also $f_{k_j}(u_j) > 0$ because $f_{k_j}(u_0) > 0$. Let $g_j = f_{k_j}/f_{k_j}(u_j)$. Then for each $x \in X_+$ we have $g_j(x) = \sigma_j$, therefore $x = \sum_{i=1}^{\infty} g_i(x) u_i$. Since the cone X_+ is generating we conclude that $x = \sum_{i=1}^{\infty} g_i(x) u_i$ for each $x \in X$ and this expansion is unique. Therefore $\{u_n\}$ is a positive basis of X. ∎

By the previous result and Corollaries 20 and 21 we have:

COROLLARY 33. *Let E be a Banach lattice with order continuous norm and suppose that E_+ is defined by a countable family $\mathcal{F} \subset E_+^*$. Let X be a closed ordered subspace of E with the Riesz decomposition property and generating positive cone X_+. If X has the maximum support property with respect to $\mathcal{F}$, then X has a positive basis.*

COROLLARY 34. *Let E be an ordered Banach space whose positive cone is defined by a family $\mathcal{F} = \{f_i \mid i \in \mathbb{N}\} \subset E_+^*$. Suppose also that E is a dual space and that the functionals f_i are weak-star continuous. If X is a closed ordered subspace of E with the Riesz decomposition property, and X_+ is weak-star closed, normal and generating, and X has the maximum support property with respect to $\mathcal{F}$, then X has a positive basis.*

REMARK 35. In the special case where $E = \ell_\infty$ and X is a weak-star closed ordered subspace of ℓ_∞ with the RDP and generating positive cone X_+ we have: If X has the maximum support property with respect to the family of Dirac measures supported at natural numbers, then X has a positive basis.

4. Biorthogonal systems. The results of the previous section can be applied to the problem: *under what conditions does a biorthogonal system define a positive basis?* So in this section we suppose that E is an ordered Banach space with a *positive biorthogonal system* $\{(e_i, f_i) \mid i \in \mathbb{N}\}$, i.e. $e_i \in E$ and $f_i \in E_+^*$ for each i, $f_i(e_i) = 1$, $f_i(e_j) = 0$ for all $j \neq i$, and the family $\mathcal{F} = \{f_i \mid i \in \mathbb{N}\}$ defines the positive cone of E. In the next results the positive basis of E is also unconditional.

Theorem 36. *Let E be an ordered Banach space with a positive biorthogonal system $\{(e_i, f_i) \mid i \in \mathbb{N}\}$. If E_+ is normal and generating and E has the Riesz decomposition property, then the following statements are equivalent:*

(i) *The sequence $\{e_i\}$ of the biorthogonal system is a positive basis of E,*
(ii) *E has the maximum support property and the ws-property with respect to the family $\mathcal{F} = \{f_i \mid i \in \mathbb{N}\}$.*

Proof. Suppose that $\{e_i\}$ is a positive basis of E. Since $\{(e_i, f_i)\}$ is a positive biorthogonal system of E we know that $f_i(e_i) = 1$ and $f_i(e_j) = 0$ for all $j \neq i$; therefore, by Theorem 16, E has the maximum support property with respect to $\mathcal{F}$. Since $\{e_i\}$ is a positive basis of E, by Corollary 22, E has the ws-property, so (i) implies (ii). Suppose now that (ii) holds. Then E has a positive basis $\{b_n\}$. Since E has the maximum support property with respect to $\mathcal{F}$, E has the minimal support property, therefore an element x_0 of E_+ is an extremal point of E_+ if and only if x_0 has minimal support in E_+. Therefore the extremal points of E_+ are the positive multiples of the elements e_n ($\mathrm{supp}(e_n) = \{n\}$). Since the elements of the positive basis define the extremal rays of E_+ it follows that the basis $\{b_n\}$ coincides, up to a scalar multiple and proper enumeration, with the sequence $\{e_n\}$. ■

Corollary 37. *Let E be an ordered Banach space with a positive biorthogonal system $\{(e_i, f_i) \mid i \in \mathbb{N}\}$ and suppose that E has the Riesz decomposition property and either*

(a) *E is a Banach lattice with order continuous norm, or*
(b) *E is a dual space, the positive cone E_+ of E is weak-star closed, normal and generating, and the functionals f_i are weak-star continuous.*

Then the following statements are equivalent:

(i) *the sequence $\{e_i\}$ of the biorthogonal system is a positive basis of E,*
(ii) *E has the maximum support property with respect to the family $\{f_i \mid i \in \mathbb{N}\}$.*

Remark 38. According to Corollary 37, the sequence $\{e_i\}$ of the usual biorthogonal system $\{e_i, \delta_i\}$ of ℓ_∞ is not a positive basis of ℓ_∞ because it does not have the maximum support property with respect to the family $\{\delta_i\}$ (Example 15).

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, New York, 1985.

[2] N. Danet, *The Riesz decomposition property for the space of regular operators*, Proc. Amer. Math. Soc. 129 (2000), 539–542.

[3] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.

[4] G. J. O. Jameson, *Ordered Linear Spaces*, Lecture Notes in Math. 141, Springer, Heidelberg, 1970.

[5] M. A. Krasnosel'skij, Je. A. Lifshits, and A. V. Sobolev, *Positive Linear Systems—The Method of Positive Operators*, Sigma Ser. Appl. Math. 5, Heldermann, 1989.

[6] P. Meyer-Nierberg, *Banach Lattices*, Springer, 1991.

[7] I. A. Polyrakis, *Schauder bases in locally solid lattice Banach spaces*, Math. Proc. Cambridge Philos. Soc. 101 (1987), 91–105.

[8] —, *Lattice-subspaces of* $C[0,1]$ *and positive bases*, J. Math. Anal. Appl. 184 (1994), 1–15.

[9] I. Singer, *Bases in Banach Spaces—I*, Springer, Heidelberg, 1970.

[10] B. Z. Vulikh, *Introduction to the Theory of Cones in Normed Spaces*, Kalinin, 1977 (in Russian).

[11] A. W. Wickstead, *Compact subsets of partially ordered Banach spaces*, Math. Ann. 212 (1975), 271–284.

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