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Weighted Fréchet spaces of holomorphic functions

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Abstract. This article deals with weighted Fréchet spaces of holomorphic functions which are defined as countable intersections of weighted Banach spaces of type H^∞. We characterize when these Fréchet spaces are Schwartz, Montel or reflexive. The quasinormability is also analyzed. In the latter case more restrictive assumptions are needed to obtain a full characterization.

I. Introduction. This article deals with the weighted Fréchet spaces $HW(G)$ and $HW_0(G)$ of holomorphic functions. For an increasing sequence $W = (w_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions (weights) on an open subset G of $\mathbb{C}^N$ we consider the projective limit of the Banach spaces $Hw_n(G) := \{f \in H(G) ; \|f\|_n := \sup_{z\in G} w_n(z)|f(z)| < \infty\}$ resp. $H(w_n)_0(G) := \{f \in H(G) ; w_n f$ vanishes at ∞ on $G\}$, $n \in \mathbb{N}$. Under rather general assumptions we give a characterization of being Schwartz, Montel and reflexive in terms of the sequence of weights (which are considered as growth conditions in the sense of [13]) or in terms of their associated growth conditions. Using the class $\mathcal{W}$ of weights on the unit disk which was introduced by Bierstedt and Bonet [9] we get a necessary and sufficient condition for the quasinormability of $HW(G)$ resp. $HW_0(G)$.

In the case of Köthe echelon spaces these characterizations were obtained by Köthe, Grothendieck, Dieudonné–Gomes, Bierstedt–Meise–Summers, and Valdivia. For spaces of continuous functions the characterizations in terms of the weights were obtained by Bierstedt, Meise and Summers (see [14] and [15]). In this paper these questions are studied in the setting of weighted spaces of holomorphic functions. In contrast to the case of continuous functions the so-called *associated growth conditions* mentioned by Andersen and Duncan in [1] and studied thoroughly by Bierstedt, Bonet and Taskinen in [13] are needed to get the characterizations. While we get results

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for being Schwartz, Montel and reflexive under rather mild assumptions, the characterization of the quasinormability of $HW(G)$ resp. $HW_0(G)$ requires the restriction to the unit disk $\mathbb{D}$ as well as to the class $\mathcal{W}$ of radial weights which was introduced by Bierstedt and Bonet (see [9]). This class provides a suitable frame to decompose holomorphic functions in a certain sense.

This article is organized as follows. Section II gives the necessary notation and definitions. Section III studies weighted Fréchet spaces of holomorphic functions which are reflexive resp. Montel. We show that the algebraic identity $HW(G) = HW_0(G)$ is equivalent to the reflexivity of $HW_0(G)$ if $W = (w_n)_{n\in\mathbb{N}}$ is an increasing sequence of strictly positive continuous functions on an open set $G \subset \mathbb{C}^N$, $N \geq 1$. Moreover we characterize in terms of weights considered as growth conditions and associated growth conditions when $HW(G)$ is a Montel space. Here we use methods of Bierstedt and Bonet (see [11]). We finish this section with some remarks about the Schwartz property of weighted Fréchet spaces of holomorphic functions. A result of Bonet, Friz and Jordá [19] yields a characterization in terms of the associated growth conditions under rather general conditions.

Section IV consists of two parts. The first part presents a necessary condition for the quasinormability of $HW(G)$ resp. $HW_0(G)$. In the second part we prove that this condition is also sufficient when we restrict our attention to the class $\mathcal{W}$ of radial weights on the unit disc. Under some additional assumptions we show that $HW_0(G)$ is quasinormable if and only if it satisfies condition (QNo) of Peris.

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II. Notation. Our notation for locally convex spaces is standard; see for example Jarchow [26], Köthe [27], Meise and Vogt [31] and Pérez Carreras and Bonet [32]. For a locally convex space E, we denote by E^* the space of all linear functionals on E while E' is the topological dual and E'_{b} the strong dual. If E is a locally convex space, $\mathcal{U}_0(E)$ and $\mathcal{B}(E)$ stand for the families of all absolutely convex 0-neighborhoods and absolutely convex bounded sets in E, respectively.

In what follows, G denotes an open subset of $\mathbb{C}^N$, $N \geq 1$. The space $H(G)$ of all holomorphic functions on G will usually be endowed with the topology co of uniform convergence on compact subsets of G. Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous functions on G. For every $n \in \mathbb{N}$ the spaces

$$Hw_n(G) := \{f \in H(G)\ ;\ \|f\|_n := \sup_{z\in G} w_n(z)|f(z)| < \infty\},$$

$$H(w_n)_0(G) := \{f \in H(G)\ ;\ w_n f \text{ vanishes at } \infty \text{ on } G\}$$

endowed with the norm $\|\cdot\|_n$ are Banach spaces. The weighted Fréchet spaces of holomorphic functions are defined by

$$HW(G) := \mathrm{proj}_n Hw_n(G), \qquad HW_0(G) := \mathrm{proj}_n H(w_n)_0(G).$$

For each $n \in \mathbb{N}$, let B_n, resp. $B_{n,0}$, be the closed unit ball of $Hw_n(G)$, resp. $H(w_n)_0(G)$, and $C_n := B_n \cap HW(G)$, resp. $C_{n,0} := B_{n,0} \cap HW_0(G)$. By $\overline{B}_n$, $\overline{B}_{n,0}$, $\overline{C}_n$, $\overline{C}_{n,0}$ we denote the *co*-closures of the corresponding sets. The sequence $\left(\frac{1}{n}C_n\right)_{n\in\mathbb{N}}$, resp. $\left(\frac{1}{n}C_{n,0}\right)_{n\in\mathbb{N}}$, constitutes a 0-neighborhood base of $HW(G)$, resp. $HW_0(G)$. Without loss of generality we may assume that $(C_n)_{n\in\mathbb{N}}$, resp. $(C_{n,0})_{n\in\mathbb{N}}$, is a 0-neighborhood base. Put

$$\overline{W} := \{\overline{w} : G \to]0,\infty[\ ;$$
$$\overline{w} \text{ continuous on } G,\ w_n\overline{w} \text{ is bounded on } G \text{ for all } n \in \mathbb{N}\},$$

and $C_{\overline{w}} := \{f \in HW(G)\ ;\ |f| \le \overline{w} \text{ on } G\}$, $C_{\overline{w},0} := C_{\overline{w}} \cap HW_0(G)$, $\overline{w} \in \overline{W}$. We write $\overline{C}_{\overline{w}}$ and $\overline{C}_{\overline{w},0}$ for the respective *co*-closures. Then $(C_{\overline{w}})_{\overline{w}\in\overline{W}}$, resp. $(C_{\overline{w},0})_{\overline{w}\in\overline{W}}$, is a fundamental system of bounded subsets of $HW(G)$, resp. $HW_0(G)$. Each $C_{\overline{w}}$ is absolutely convex and *co*-compact. (See [7, Section 3.A].)

Let v be a weight on G considered as a growth condition in the sense of [13]. Its *associated growth condition* is defined by

$$\widetilde{v}(z) := \sup\{|g(z)|\ ;\ g \in H(G),\ |g| \le v\}, \qquad z \in G.$$

A weight v on a balanced domain $G \subset \mathbb{C}^N$, $N \ge 1$, is said to be *radial* if $v(z) = v(\lambda z)$ for every $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. On a balanced open subset G of $\mathbb{C}^N$, $N \ge 1$, each $f \in H(G)$ has a Taylor series representation about zero,

$$f(z) = \sum_{k=0}^{\infty} p_k(z), \qquad z \in G,$$

where p_k is a k-homogeneous polynomial ($k = 0, 1, \dots$). The series converges to f uniformly on each compact subset of G. The *Cesàro means* of the partial sums of the Taylor series of f are denoted by $S_n(f)$ ($n = 0, 1, \dots$); that is,

$$[S_n(f)](z) = \frac{1}{n+1}\sum_{l=0}^{n}\Big(\sum_{k=0}^{l} p_k(z)\Big), \qquad z \in G.$$

Each $S_n(f)$ is a polynomial (of degree $\le n$) and $S_n(f) \to f$ uniformly on every compact subset of G ($f \in H(G)$ arbitrary).

Bierstedt, Bonet and Galbis (see [12]) used Cesàro means to show that if $W = (w_n)_{n\in\mathbb{N}}$ is an increasing sequence of non-negative continuous and radial functions on a balanced open set $G \subset \mathbb{C}^N$, $N \ge 1$, such that $HW_0(G)$ contains the polynomials, then $\overline{B}_{n,0} = B_n$ and $\overline{C}_n = B_n$ for every $n \in \mathbb{N}$.

Weighted Fréchet spaces of continuous functions are defined similarly on locally compact spaces X. Bierstedt and Meise (see [14]) and Bastin (see [2], [3]) characterized properties like Schwartz, Montel, quasinormable, distinguished and the density condition for these spaces. They gave conditions in terms of the weights which, in a modified form, also play a role in this article.

We now mention some examples of weighted Fréchet spaces of holomorphic functions which were studied in the literature.

(I) In [22] Epifanov considered a bounded convex domain G in $\mathbb{C}^N$, $N \geq 1$, a decreasing sequence $(\varphi_n)_{n\in\mathbb{N}}$ of bounded non-negative convex functions on G and defined the sequence $W = (w_n)_{n\in\mathbb{N}}$ by

$$w_n(z) = e^{-\varphi_n(z)}, \qquad z \in G,\ n \in \mathbb{N}.$$

He showed that under certain assumptions the strong dual of $HW(G)$ is topologically isomorphic to the inductive limit $\mathcal{V}H(\mathbb{C}^N)$, where $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ is given by

$$v_n(z) = e^{-\sup\{\mathrm{Re}\langle z,t\rangle - \varphi_n(t)\ ;\ t\in G\}}, \qquad z \in \mathbb{C}^N,\ N \geq 1.$$

For further information on weighted inductive limits of holomorphic functions see e.g. [16] and [6].

(II) Let a plurisubharmonic function $p : \mathbb{C}^N \to [0,\infty)$ satisfy the conditions $\log(1+|z|^2) = O(p(z))$, $p(z) = p(|z|)$ and $p(2z) = O(p(z))$. The increasing sequence $W = (w_n)_{n\in\mathbb{N}}$ is defined by

$$w_n(z) := e^{-p(z)/n}, \qquad z \in G,\ n \in \mathbb{N}.$$

The weighted Fréchet space $HW(G)$ was considered by Berenstein, Chang and Li in [4] and denoted by $A_p^0(G)$. Note that $HW(G)$ is a Schwartz space. They studied geometrical conditions on $V := f^{-1}(0)$, where $f \in A_p^0(\mathbb{C}^N)$, in order that V is an interpolating variety for the corresponding ring. In case $G = \mathbb{C}$ and $p(z) = |z|^{\varrho}$, $\varrho > 0$, we obtain the Fréchet space of all entire functions of order ϱ and of type 0.

(III) Weighted Fréchet spaces of holomorphic functions are used by Bonet and Meise to consider (LF)-spaces in connection with ultradistributions of Roumieu type (see [20]). These (LF)-spaces appear as the Fourier–Laplace transform of spaces of ultradistributions of compact support. By [20] the (LF)-spaces are nuclear and reflexive. The "steps" of the inductive limit are given by the following Fréchet spaces. Let Ω be a non-empty open and convex subset of $\mathbb{R}^N$ with $0 \in \Omega$, and $(K_n)_{n\in\mathbb{N}}$ be a fundamental sequence of convex and compact subsets of Ω such that $0 \in \mathring{K}_1$ and $K_n \subset \mathring{K}_{n+1}$ for every $n \in \mathbb{N}$. The increasing sequences $W_n = (w_{n,k})_{k\in\mathbb{N}}$, $n \in \mathbb{N}$, are defined by

$$w_{n,k}(z) := \exp\left(-h_n(\mathrm{Im}(z)) - \frac{1}{k}\,\omega(z)\right), \qquad z \in \mathbb{C}^N,\ k \in \mathbb{N},$$

where h_n is given by $h_n(x) := \sup_{y\in K_n}\langle x, y\rangle$, $x \in \mathbb{R}^N$, $n \in \mathbb{N}$, and ω denotes a non-quasianalytic function. For further information on non-quasianalytic functions we refer to [20].

III. Montel and reflexive spaces. To get the desired characterization when $HW_0(G)$ is reflexive we need the following lemma.

LEMMA 1. *Consider a sequence* $(f_j)_{j\in\mathbb{N}} \subset HW_0(G)$. *The following are equivalent*:

(a) $f_j \to 0$ *with respect to* $\sigma(HW_0(G), HW_0(G)')$.
(b) $(f_j)_{j\in\mathbb{N}}$ *is bounded*, *and* $(f_j)_{j\in\mathbb{N}}$ *converges to* 0 *with respect to co.*

The proof is standard (cf. [8], [17], [34]).

COROLLARY 2. *On each bounded subset* B *of* $HW_0(G)$, *co is stronger than the weak topology.*

The proof follows directly from Lemma 1.

THEOREM 3. *Let* $C_{\overline{w}} = \overline{C}_{\overline{w},0}$ *for every* $\overline{w}$ *in a subset* $\overline{A}$ *of* $\overline{W}$ *such that* $(C_{\overline{w}})_{\overline{w}\in\overline{A}}$ *is a fundamental system of bounded sets in* $HW(G)$. *Then the following assertions are equivalent*:

(a) $HW(G) = HW_0(G)$.
(b) $HW_0(G)$ *is reflexive.*

Proof. (b)⇒(a). We fix $f \in HW(G)$. There is $\overline{w} \in \overline{A}$ with $f \in C_{\overline{w}}$. Since $C_{\overline{w}} = \overline{C}_{\overline{w},0}$, we can find a sequence $(f_j)_{j\in\mathbb{N}} \subset C_{\overline{w},0} \subset HW_0(G)$ with $f_j \to f$ with respect to *co*; thus, $(f_j)_{j\in\mathbb{N}}$ is relatively weakly compact by (b). We can find a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ of $(f_j)_{j\in\mathbb{N}}$ which converges weakly to an element g of $HW_0(G)$, hence pointwise to g. We conclude $g = f \in HW_0(G)$ and finally $HW(G) = HW_0(G)$.

(a)⇒(b). We assume $HW(G) = HW_0(G)$. By definition we have to show that each bounded subset B of $HW_0(G)$ is relatively weakly compact. There is $\overline{w} \in \overline{A}$ such that $B \subset C_{\overline{w},0}$. Since $HW(G) = HW_0(G)$ we have $C_{\overline{w}} = C_{\overline{w},0}$, and $C_{\overline{w},0}$ is *co*-compact. By Corollary 2 and Montel's theorem B is weakly relatively compact. ∎

For a sequence $W = (w_n)_{n\in\mathbb{N}}$ with $w_n = w_{n+1} = w$ for every $n \in \mathbb{N}$ we obtain a Banach space $Hw(G)$. For spaces of this type the following is known:

THEOREM 4 (Bonet and Wolf [21]). *Let* G *be an open subset of* $\mathbb{C}^N$, $N \geq 1$, *and let* v *be a strictly positive and continuous weight on* G. *Then* *the space $Hv_0(G)$ embeds almost isometrically into c_0. In particular, if this space is infinite-dimensional, then $Hv_0(G)$ and $Hv(G)$ are not reflexive.*

There are many examples of sequences $W = (w_n)_{n\in\mathbb{N}}$ such that $C_{\overline{w}} = \overline{C}_{\overline{w},0}$ for every $\overline{w} \in \overline{A}$. See [12] and [24].

Remark 5 (Bierstedt, Bonet and Galbis [12, Proposition 1.2(c)]). Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous and radial functions on a balanced open set $G \subset \mathbb{C}^N$, $N \geq 1$. The $C_{\overline{w}}$ for $\overline{w} \in \overline{W}$ radial form a basis of bounded sets in $HW(G)$ and $C_{\overline{w}} = \overline{C}_{\overline{w}} \cap HW_0(G) = \overline{C}_{\overline{w},0}$ for every $\overline{w} \in \overline{W}$ radial.

Proposition 6 (Holtmanns [24, Proposition 4.2.8]). *Let $G = \{z \in \mathbb{C}\ ;\ \mathrm{Im}(z) > 0\}$ and $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous functions on G such that $\lim_{\mathrm{Im}(z)\to 0} w_n(z) = 0$ and $w_n(z) \leq w_n(z + ip)$ for every $0 < p < 1$ and every $n \in \mathbb{N}$. Then $C_{\overline{w}} = \overline{C}_{\overline{w},0}$ for every $\overline{w} \in \overline{W}$.*

In what follows we first study under which conditions $HW(G)$ is Montel and then the connection between Montel spaces and reflexive spaces. We need the following lemma.

Lemma 7. *The following assertions are equivalent*:

(a) *$HW(G)$ is Montel.*
(b) *The topology τ of $HW(G)$ and co coincide on each bounded subset of $HW(G)$.*

Proof. (a)⇒(b). Let $HW(G)$ be a Montel space. By definition each bounded subset B of $HW(G)$ is τ-relatively compact. By [27, 3.2.(6)], τ coincides with each weaker topology, so in particular with *co*. Hence (b) follows.

(b)⇒(a). We fix a bounded subset A of $HW(G)$. Then there is $\overline{w} \in \overline{W}$ such that $A \subset C_{\overline{w}}$ and $C_{\overline{w}}$ is *co*-compact. Hence A is *co*-relatively compact and by (b) relatively compact in $HW(G)$. By definition this shows that $HW(G)$ is a Montel space. ∎

Theorem 8. *The following assertions are equivalent*:

(a) *$HW(G) = HW_0(G)$, and $HW(G)$ is a Montel space.*
(b) *For every $n \in \mathbb{N}$ and every $\overline{w} \in \overline{W}$, $w_n(\overline{w})^\sim$ vanishes at ∞ on G.*

Proof. (a)⇒(b). We assume that $HW(G)$ is a Montel space and that $HW(G) = HW_0(G)$. We fix $n \in \mathbb{N}$ and put $\mathcal{A} := \{w_n(z)\delta_z\ ;\ z \in G\}$. Since $\mathcal{A}$ is contained in C_n°, $\mathcal{A}$ is equicontinuous in $HW(G)'$. It follows that

$$\sigma(HW(G)', HW(G))|_{\mathcal{A}} = \lambda(HW(G)', HW(G))|_{\mathcal{A}},$$

where $\lambda(HW(G)', HW(G))$ denotes the topology of uniform convergence on compact subsets of $HW(G)$.

For fixed $U \in \mathcal{U}(HW(G)', \sigma(HW(G)', HW(G)))$ there is a finite set $(f_1, \ldots, f_s) \subset HW(G) = HW_0(G)$ such that

$$\mathcal{A} \cap (f_1, \ldots, f_s)^\circ \subset U \cap \mathcal{A}.$$

We write $V := (f_1, \ldots, f_s)^\circ$ and because of $HW(G) = HW_0(G)$ we can find $K \subset\subset G$ with

$$w_n(z)\delta_z \in V \quad \text{for every } z \in G \setminus K.$$

We fix $\overline{w} \in \overline{W}$ and $\varepsilon > 0$. Since $HW(G)$ is Montel, $C_{\overline{w}}$ is compact in $HW(G)$. Hence for the $\lambda(HW(G)', HW(G))$-0-neighborhood $\varepsilon C_{\overline{w}}^\circ \subset HW(G)'$ we can find $K \subset\subset G$ such that $w_n(z)\delta_z \in \varepsilon C_{\overline{w}}^\circ$ for every $z \in G \setminus K$, i.e. there exists $K \subset\subset G$ such that

$$\varepsilon \geq w_n(z) \sup\{|f(z)| \; ; \; f \in C_{\overline{w}}\} = w_n(z)(\overline{w})^\sim(z)$$

for every $z \in G \setminus K$. So (b) follows.

(b)⇒(a). In order to show $HW(G) = HW_0(G)$, we fix $f \in HW(G)$. We can find $\overline{w} \in \overline{W}$ with $|f| \leq (\overline{w})^\sim$ on G. For fixed $n \in \mathbb{N}$ we obtain

$$w_n|f| \leq w_n(\overline{w})^\sim \quad \text{on } G.$$

This together with assertion (b) implies that $w_n|f|$ vanishes at ∞ on G. Hence f belongs to $HW_0(G)$ and $HW(G) = HW_0(G)$ as desired.

It remains to show that $HW(G)$ is Montel. We fix $\overline{w} \in \overline{W}$ and claim that $HW(G)$ and $(H(G), co)$ induce the same topology on $C_{\overline{w}}$. Since $(C_{\overline{w}})_{\overline{w} \in \overline{W}}$ is a fundamental system of bounded subsets of $HW(G)$, Lemma 7 then implies that $HW(G)$ is a Montel space. We fix $n \in \mathbb{N}$ and $\varepsilon > 0$. By assumption there is $K \subset\subset G$ such that

$$w_n(z)(\overline{w})^\sim(z) \leq \varepsilon$$

for every $z \in G \setminus K$. We put

$$U := \{g \in HW(G) \; ; \; w_n(z)|g(z)| \leq \varepsilon \text{ for all } z \in G\}$$

and have to show that there is a 0-neighborhood V in $(H(G), co)$ with

$$V \cap C_{\overline{w}} \subset U \cap C_{\overline{w}}.$$

We put $V := \{g \in H(G) \; ; \; |g(z)| \leq \varepsilon / \max_{y \in K} w_n(y) \text{ for all } z \in K\}$ and fix $g \in V \cap C_{\overline{w}}$. Thus, we obtain $w_n(z)|g(z)| \leq \max_{y \in K} w_n(y) \varepsilon / \max_{y \in K} w_n(y)$ $= \varepsilon$ for $z \in K$. For $z \in G \setminus K$ we distinguish two cases. If $(\overline{w})^\sim(z) > 0$, we get

$$w_n(z)|g(z)| = w_n(z)(\overline{w})^\sim(z) \, \frac{|g(z)|}{(\overline{w})^\sim(z)} \leq \varepsilon.$$

If we have $(\overline{w})^\sim(z) = 0$, this implies $g(z) = 0$ because g belongs to $C_{\overline{w}}$. Then $w_n(z)|g(z)| = 0 < \varepsilon$. Thus g is in $U \cap C_{\overline{w}}$. ∎

Corollary 9. *Let* $C_{\overline{w}} = \overline{C}_{\overline{w},0}$ *for every* $\overline{w} \in \overline{A} \subset \overline{W}$ *such that* $(C_{\overline{w}})_{\overline{w}\in\overline{A}}$ *is a fundamental system of bounded sets in* $HW(G)$. *Then the following assertions are equivalent*:

(a) $HW(G)$ *is Montel.*
(b) *For every* $n \in \mathbb{N}$ *and every* $\overline{w} \in \overline{W}$, $w_n(\overline{w})^\sim$ *vanishes at* ∞ *on* G.

Proposition 10. *Let* G *be an open and connected subset of* $\mathbb{C}$ *such that* $\mathbb{C}^* \setminus G$ *has no one-point component. Then* $HW(G) = HW_0(G)$ *implies condition* (b) *in Theorem* 8.

Proof. Assume that condition (b) is not satisfied. There are $\overline{w} \in \overline{W}$, $n \in \mathbb{N}$ and a sequence $(z_k)_{k\in\mathbb{N}} \subset G$ which converges to z_0 in the boundary of G in $\mathbb{C}^*$ such that for every $k \in \mathbb{N}$ we have

$$w_n(z_k)(\overline{w})^\sim(z_k) \geq \varepsilon. \tag{1}$$

By [13, Proposition 1.2(iv)] for every $k \in \mathbb{N}$ there is $f_k \in H(G)$ such that $|f_k| \leq \overline{w}$ on G and $|f_k(z_k)| = (\overline{w})^\sim(z_k)$. Each f_k is an element of $HW(G)$.

Without loss of generality we may assume that z_0 is contained in a closed, connected set $L \subset \mathbb{C}^* \setminus G$ of more than one point. It is easy to see that $U := \mathbb{C}^* \setminus L$ is conformally equivalent to $\mathbb{D}$.

The sequence $(z_k)_{k\in\mathbb{N}}$ contains an $H^\infty(U)$-interpolating subsequence which will also be denoted by $(z_k)_{k\in\mathbb{N}}$. (See [23, p. 195].)

Then the proof of [35, III.E.4] yields the existence of a sequence of functions $(\varphi_j)_{j\in\mathbb{N}} \subset H^\infty(U)$ such that $\varphi_j(z_k) = \delta_{jk}$ and $\sum_{j=1}^\infty |\varphi_j| \leq M < \infty$. Put $f := \sum_{k=1}^\infty \varphi_k f_k$. We have $|f(z)| \leq \sum_{k=1}^\infty |\varphi_k(z) f_k(z)| \leq M\overline{w}(z)$ for every $z \in G$. Thus the series converges uniformly on compact subsets of G, hence $f \in H(G)$.

It remains to show that f belongs to $HW(G) \setminus HW_0(G)$. First we prove that f is an element of $HW(G)$. For every $z \in G$ and every $m \in \mathbb{N}$, we have

$$w_m(z)|f(z)| \leq w_m(z) \sum_{k=1}^\infty |\varphi_k(z)|\,|f_k(z)| \leq w_m(z) M\overline{w}(z) < \infty.$$

On the other hand, $|f(z_j)| = |\sum_{k=1}^\infty \varphi_k(z_j) f_k(z_j)| = |f_j(z_j)| = (\overline{w})^\sim(z_j)$ for each $j \in \mathbb{N}$. An application of (1) shows $w_n(z_k)|f(z_k)| = w_n(z_k)(\overline{w})^\sim(z_k) \geq \varepsilon$ for every $n \in \mathbb{N}$. So $f \notin H(w_n)_0(G)$. This contradicts $HW(G) = HW_0(G)$. ■

Corollary 11. *Let* $G \subset \mathbb{C}$ *be open and connected such that* $\mathbb{C}^* \setminus G$ *has no one-point component. Assume that* $C_{\overline{w}} = \overline{C}_{\overline{w},0}$ *for every* $\overline{w} \in \overline{A}$. *Then the following assertions are equivalent*:

(a) $HW(G) = HW_0(G)$.
(b) $HW_0(G)$ *is reflexive.*
(c) $HW(G)$ *is reflexive.*
(d) *For every* $n \in \mathbb{N}$ *and every* $\overline{w} \in \overline{W}$, $w_n(\overline{w})^\sim$ *vanishes at* ∞ *on* G.

(e) *$HW(G)$ is Montel.*
(f) *$HW_0(G)$ is Montel.*

Proof. The equivalence of (a) and (b) is given by Theorem 3. By Theorem 8 and Proposition 10 we deduce that (a) is equivalent to (d). Since $HW_0(G)$ is a closed topological subspace of $HW(G)$, (c) implies (b) by [26, Proposition 11.4.5] and (e) yields (f) by [26, Proposition 11.5.4]. By Theorem 8, (e) follows from (d). Since $HW(G)$ and $HW_0(G)$ are Fréchet spaces we obtain (e)⇒(c) and (f)⇒(b). ∎

REMARK 12. The following condition (M') was introduced by Bierstedt, Meise and Summers in [16] for Köthe matrices and studied for sequences of weights by Bierstedt and Meise in [14].

An increasing sequence $W = (w_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions on an open set $G \subset \mathbb{C}^N$, $N \geq 1$, is said to satisfy *condition* (M') if for any $n \in \mathbb{N}$ and any subset Y of G which is not relatively compact there is $m = m(n, Y) > n$ such that $\inf_{y\in Y} w_n(y)/w_m(y) = 0$.

They showed that the algebraic equality $CW(G) = CW_0(G)$ is equivalent to (M') (see [14, Proposition 5.5]). It is an easy consequence that (M') implies $HW(G) = HW_0(G)$. A simple computation shows that (M') is satisfied if and only if the following holds: For every $n \in \mathbb{N}$ and every $\overline{w} \in \overline{W}$, $w_n\overline{w}$ vanishes at ∞ on G.

This remark shows that (b) in Theorem 8 differs from (M') exactly by the use of an associated growth condition. We will show that this is really a difference by constructing an example in which $HW(G) = HW_0(G)$, but $CW(G) \neq CW_0(G)$. The idea for the construction of this example is taken from [13]. First we need some auxiliary results.

The following condition (ND') was introduced by Bierstedt and Meise in [14]. It is well known that a sequence W of weights which has (ND') does not satisfy (M').

REMARK 13. Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous functions on an open set $G \subset \mathbb{C}^N$, $N \geq 1$. Then W is said to satisfy *condition* (ND') if there are $n \in \mathbb{N}$ and a decreasing sequence $(J_k)_{k\in\mathbb{N}}$ such that for every $k \geq n$ we have

(i) $\inf_{z\in J_k} w_n(z)/w_k(z) > 0$,
(ii) there is $l(k) > k$ with $\inf_{z\in J_k} w_n(z)/w_{l(k)}(z) = 0$.

Another condition which is needed is the following condition (S') for an increasing sequence $W = (w_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions on an open set $G \subset \mathbb{C}^N$, $N \geq 1$, which was studied by Bierstedt, Meise and Summers in [15] in the setting of Köthe sequence spaces:

(S') For every $n \in \mathbb{N}$ there is $m > n$ such that w_n/w_m vanishes at ∞ on G.

It is well known that (S') implies (M'). But the converse is not true (see [15]).

LEMMA 14. *Let G_1 be an open subset of $\mathbb{C}$, and put $G := G_1 \times \mathbb{C}$. Let $T = (t_n)_{n\in\mathbb{N}}$ and $U = (u_n)_{n\in\mathbb{N}}$ be increasing sequences of weights on G_1 and $\mathbb{C}$, respectively. Let $0 < p < 1$, and assume that for each $n \in \mathbb{N}$ and each $z \in \mathbb{C}$, $1 \le u_n(z) \le (1+|z|)^p$. Choose $S = (s_n)_{n\in\mathbb{N}}$ with $s_n(z) := u_n(z)/(1+|z|)^p$, $z \in \mathbb{C}$, and put $W = (w_n)_{n\in\mathbb{N}}$, $w_n(z_1,z_2) = t_n(z_1)s_n(z_2)$ for $(z_1,z_2) \in G$, $n \in \mathbb{N}$. Then:*

(a) *Every $f \in HW(G)$ is constant in the second coordinate z_2, i.e. we have $f(z_1,z_2) = f(z_1,z_2')$ for $z_2, z_2' \in \mathbb{C}$, $z_1 \in G_1$.*
(b) *$HW(G)$ and $HT(G_1)$ are canonically isomorphic.*

Proof. (a) We fix $f \in HW(G)$. By definition there is $M > 0$ such that $w_n(z_1,z_2)|f(z_1,z_2)| = t_n(z_1)s_n(z_2)|f(z_1,z_2)| \le M$ for every $(z_1,z_2) \in G$. By assumption we have $1/(1+|z_2|)^p \le s_n(z_2) \le 1$ for every $z_2 \in \mathbb{C}$, hence we obtain

$$|f(z_1,z_2)| \le \frac{M}{t_n(z_1)}\,\frac{1}{s_n(z_2)} \le \frac{M}{t_n(z_1)}\,(1+|z_2|)^p \quad \text{on } G.$$

By Liouville's theorem the claim follows (see [13, Lemma 2.2].

(b) Consider the maps

$$\begin{aligned} \psi &: HW(G) \to HT(G_1), \qquad f \mapsto (z_1 \mapsto f(z_1,0)) \\ \psi^{-1} &: HT(G_1) \to HW(G), \qquad g \mapsto ((z_1,z_2) \mapsto g(z_1)), \end{aligned}$$

which obviously are topological isomorphisms. ■

EXAMPLE 15. In Lemma 14 consider $G_1 = \mathbb{C}$ and choose an increasing sequence $T = (t_n)_{n\in\mathbb{N}}$ of weights t_n on $\mathbb{C}$ such that $t_n(0) = 1$ for every $n \in \mathbb{N}$ and such that T has (S'). Hence $HT(\mathbb{C}) = HT_0(\mathbb{C})$. For every increasing sequence $S = (s_n)_{n\in\mathbb{N}}$ on $\mathbb{C}$ with $\lim_{|z|\to\infty} s_n(z) = 0$ for every $n \in \mathbb{N}$ and such that S satisfies the assumptions of Lemma 14 we define $W = (w_n)_{n\in\mathbb{N}}$ on $\mathbb{C}^2 = \mathbb{C}\times\mathbb{C}$ as in Lemma 14. We claim $HW(\mathbb{C}^2) = HW_0(\mathbb{C}^2)$. To show this we have to distinguish two cases.

CASE 1: We suppose $|z_1| \to \infty$ and fix $f \in HW(\mathbb{C}^2)$ as well as $n \in \mathbb{N}$. By Lemma 14 we have $HW(\mathbb{C}^2) \cong HT(\mathbb{C})$. Since $\lim_{|z_2|\to\infty} s_n(z_2) = 0$ there is $M > 0$ with $|s_n(z)| \le M$ for every $z \in \mathbb{C}$. Fix $\varepsilon > 0$ and put $\varepsilon' := \varepsilon/M$. Since $HT(\mathbb{C}) = HT_0(\mathbb{C})$, we can find $K_1 \subset\subset \mathbb{C}$ such that, by the proof of Lemma 14,

$$w_n(z_1,z_2)|f(z_1,z_2)| = t_n(z_1)s_n(z_2)|f(z_1,0)| \le Mt_n(z_1)|f(z_1,0)| \le \varepsilon' M = \varepsilon$$

for every $z_1 \in \mathbb{C}\setminus K_1$ and every $z_2 \in \mathbb{C}$. Now, take $K = K_1 \times \{0\}$ to get $w_n(z_1,z_2)|f(z_1,z_2)| \le \varepsilon$ for every $z = (z_1,z_2) \in \mathbb{C}^2\setminus K$. Hence $f \in HW_0(\mathbb{C}^2)$.

CASE 2: We assume $|z_2| \to \infty$ and fix $f \in HW(\mathbb{C}^2)$ as well as $n \in \mathbb{N}$. By Lemma 14 we have $HW(\mathbb{C}^2) \cong HT(\mathbb{C}) = HT_0(\mathbb{C})$. Hence there is $m > 0$ with

$$w_n(z_1, z_2)|f(z_1, z_2)| = t_n(z_1)|f(z_1, 0)|s_n(z_2) \leq m s_n(z_2)$$

for every $z = (z_1, z_2) \in \mathbb{C}^2$. Now we fix $\varepsilon > 0$ and put $\varepsilon' := \varepsilon/m$. Since $\lim_{|z_2|\to\infty} s_n(z_2) = 0$ we can find $K_2 \subset\subset \mathbb{C}$ such that

$$w_n(z_1, z_2)|f(z_1, z_2)| \leq m s_n(z_2) \leq m\varepsilon \quad \text{for all } z_1 \in \mathbb{C} \text{ and } z_2 \in \mathbb{C} \setminus K_2.$$

Take $K = \{0\} \times K_2$ to get $w_n(z_1, z_2)|f(z_1, z_2)| \leq \varepsilon$ for every $z = (z_1, z_2) \in \mathbb{C}^2 \setminus K$. Thus, $f \in HW_0(\mathbb{C}^2)$.

Next we construct a sequence $S = (s_n)_{n\in\mathbb{N}}$ as above such that in addition $W = (w_n)_{n\in\mathbb{N}}$ has (ND'), hence not (M'). Therefore $CW(G) \neq CW_0(G)$ (see [14, Proposition 5.5]).

For $n \in \mathbb{N}$, $n \geq 3$, and arbitrary $k \in \mathbb{N}_0$ let first $u_n : \mathbb{R}_+ \to \mathbb{R}_+$ be given on $[2k, 2k + 2]$ by

$$u_n(r) := \begin{cases} 1 & \text{for } r \in [2k, 2k + 2^{-n}] \text{ and } r \in [2k + 1, 2k + 2], \\ (1 + r)^{(n-1)/2n} & \text{for } r \in [2k + 3 \cdot 2^{-(n+1)}, 2k + 1 - 2^{-(n+1)}], \end{cases}$$

with u_n affine on $[2k + 2^{-n}, 2k + 3 \cdot 2^{-(n+1)}]$ and $[2k + 1 - 2^{-(n+1)}, 2k + 1]$. Distinguishing several cases it is elementary to verify that $(u_n)_{n\geq 3}$ is an increasing sequence. Now extend u_n radially, $u_n(z) = u_n(|z|)$ for $z \in \mathbb{C}$, $n \in \mathbb{N}$, to obtain an increasing sequence $(u_n)_{n\in\mathbb{N}}$ on $\mathbb{C}$ for which the assumption of Lemma 14 holds with $p = 1/2$.

At this point it suffices to show that W satisfies (ND'). For arbitrary $n \in \mathbb{N}$, take $J_n = \{(0, 2k + 2^{-m}) \; ; \; k, m \geq n\} \subset \mathbb{C}^2$. Given $n \geq n_0 := 3$, for any $(0, 2k + 2^{-m}) \in J_n$ we have

$$w_{n_0}(0, 2k + 2^{-m}) = t_{n_0}(0)s_{n_0}(2k + 2^{-m}) = \frac{1}{(1 + 2k + 2^{-m})^{1/2}},$$
$$w_n(0, 2k + 2^{-m}) = t_n(0)s_n(2k + 2^{-m}) = \frac{1}{(1 + 2k + 2^{-m})^{1/2}}$$

since $m \geq n$; hence $\inf_{J_n} w_{n_0}/w_n = 1 > 0$. But for $l_n := n + 1$, $k \geq n$,

$$w_{n+1}(0, 2k + 2^{-n}) = t_{n+1}(0)s_{n+1}(2k + 2^{-n}) = \frac{(1 + 2k + 2^{-n})^{n/(2n+2)}}{(1 + 2k + 2^{-n})^{1/2}},$$
$$\frac{w_{n_0}(0, 2k + 2^{-n})}{w_{l_n}(0, 2k + 2^{-n})} = (1 + 2k + 2^{-n})^{-n/(2n+2)}.$$

We obtain $\inf_{J_n} w_{n_0}/w_{l_n} \leq (1 + 2k + 2^{-n})^{-n/(2n+2)} \to 0$ $(k \to \infty)$, which proves that W satisfies (ND').

Now we are interested in the question when $HW(G)$ is Schwartz.

Proposition 16. *Let $\overline{C}_n = B_n$ as well as $B_n = \overline{B}_{n,0}$ for every $n \in \mathbb{N}$. The following are equivalent*:

(a) *$HW(G)$ is a Schwartz space.*
(b) *For every $n \in \mathbb{N}$ there is $m > n$ such that $Hw_m(G) \hookrightarrow Hw_n(G)$ is compact.*
(c) *For every $n \in \mathbb{N}$ there is $m > n$ such that $H(w_m)_0(G) \hookrightarrow H(w_n)_0(G)$ is compact.*
(d) *For every $n \in \mathbb{N}$ there is $m > n$ such that for every $\varepsilon > 0$ there is $K \subset\subset G$ with*

$$w_n(z) < \varepsilon \widetilde{w}_m(z) \quad \text{for every } z \in G \setminus K.$$

Proof. The equivalence of (b), (c) and (d) follows from [19, Theorem 8].

(b)⇒(a). This is [25, Proposition 3.15.9].

(a)⇒(b). If $HW(G)$ is a Schwartz space, for every $n \in \mathbb{N}$ there is $m > n$ such that for every $\varepsilon > 0$ we can find a finite set $F \subset HW(G)$ with $C_m \subset F + \varepsilon C_n$. This implies

$$C_m \subset F + \varepsilon C_n \subset F + \varepsilon B_n.$$

Since εB_n is *co*-compact and F is *co*-closed, from [25, Proposition 2.10.5] it follows that $F + \varepsilon B_n$ is *co*-closed. We get $B_m = \overline{C}_m \subset F + \varepsilon B_n$ and hence (b) follows. ∎

IV. The quasinormability of $HW(G)$ resp. $HW_0(G)$. First we give a necessary condition for the quasinormability of $HW(G)$ resp. $HW_0(G)$.

Proposition 17. *Let $\overline{C}_n = B_n$ for every $n \in \mathbb{N}$ and $HW(G)$ be quasinormable. Then the following holds*: *For every $n \in \mathbb{N}$ there is $m > n$ such that for every $\alpha > 0$ we can find $\overline{w} \in \overline{W}$ with*

$$\left(\frac{1}{w_m}\right)^{\sim} \leq \overline{w} + \frac{\alpha}{w_n} \quad \text{on } G. \tag{2}$$

Proof. By definition, for every $n \in \mathbb{N}$ there is $m > n$ such that for every $\alpha > 0$ there exists $\overline{w} \in \overline{W}$ with $C_m \subset C_{\overline{w}} + \alpha C_n \subset C_{\overline{w}} + \alpha B_n$. Since $C_{\overline{w}}$ is *co*-compact and αB_n is *co*-closed, $C_{\overline{w}} + \alpha B_n$ is *co*-closed and we obtain

$$\overline{C}_m = B_m \subset C_{\overline{w}} + \alpha B_n. \tag{3}$$

It is enough to show that (3) implies (2). For this we fix $f \in H(G)$ with $|f| \leq 1/w_m$ on G. Then f is contained in B_m and hence in $C_{\overline{w}} + \alpha B_n$ by (3). Thus $f = g_1 + g_2$ with $g_1 \in C_{\overline{w}}$ and $g_2 \in \alpha B_n$. We get $|f| \leq |g_1| + |g_2| \leq \overline{w} + \alpha/w_n$ on G. If we take the supremum over all f, we obtain (2). ∎

Proposition 18. *Inequality* (2) *implies the following condition*: *For every $n \in \mathbb{N}$ there is $m > n$ such that for every $k > n$ and every $\mu > 0$ there* *is* $\xi > 0$ *with*

$$\left(\frac{1}{w_m}\right)^{\sim} \leq \frac{\xi}{w_k} + \frac{\mu}{w_n} \quad \text{on } G. \tag{4}$$

Proof. First, we fix $n \in \mathbb{N}$, select $m > n$ as in (2) and fix $k > n$ as well as $\mu > 0$. Applying (2) we obtain $(1/w_m)^{\sim} \leq \overline{w} + \mu/w_n$ on G. By definition we can find $\xi > 0$ such that $\overline{w} \leq \xi/w_k$ on G. This implies $(1/w_m)^{\sim} \leq \overline{w} + \mu/w_n \leq \xi/w_k + \mu/w_n$ on G. ∎

PROPOSITION 19. *Let* $C_{\overline{w}} = \overline{C}_{\overline{w},0}$ *for every* $\overline{w} \in \overline{A} \subset \overline{W}$ *such that* $(C_{\overline{w}})_{\overline{w} \in \overline{A}}$ *is a fundamental system of bounded sets in* $HW(G)$. *The following are equivalent*:

(a) $HW(G)$ *is quasinormable.*
(b) $HW_0(G)$ *is quasinormable.*

Proof. By [7, Proposition 10], $HW(G)$ and $(HW_0(G)'_{\mathrm{b}})'_{\mathrm{b}}$ are topologically equal. A Fréchet space E is quasinormable if and only if the strong bidual E'' is quasinormable (see for example [10, Theorem 3]). ∎

The following construction was introduced by Bierstedt and Bonet in [9]. Let $\mathcal{W}$ be a class of strictly positive continuous radial weights v on the unit disc $\mathbb{D}$ which satisfy $\lim_{r\to 1-} v(r) = 0$ and for which the restriction of v to $[0, 1)$ is non-increasing. We suppose that the class $\mathcal{W}$ is stable under finite minima and under multiplication by positive scalars.

Next, we assume that there is a sequence $R_n : H(\mathbb{D}) \to H(\mathbb{D})$, $n \in \mathbb{N}$, of linear operators which are continuous for the compact open topology and such that the range of each R_n is a finite-dimensional subspace of the polynomials. It is also assumed that $R_n R_m = R_{\min(n,m)}$ for any n, m with $n \neq m$ and that for each polynomial p there is n such that $R_n p = p$, from which it follows that $R_m p = p$ for each $m \geq n$. Moreover, we suppose that there is $c > 0$ such that $\sup_{|z|=r} |R_n p(z)| \leq c \sup_{|z|=r} |p(z)|$ for all n, all $r \in (0, 1)$ and any polynomial p.

Finally, setting $R_0 := 0$, and putting $r_n := 1 - 2^{-n}$, $n \in \mathbb{N} \cup \{0\}$, we assume that the following conditions are satisfied by the class $\mathcal{W}$:

(P1) There is $C \geq 1$ such that, for each $v \in \mathcal{W}$ and for each polynomial p,

$$\frac{1}{C} \sup_n \Big(\sup_{|z|=r_n} |(R_{n+2} - R_{n-1})p(z)| \Big) v(r_n) \leq \|p\|_v \leq C \sup_n \Big(\sup_{|z|=r_n} |(R_{n+1} - R_n)p(z)| \Big) v(r_n).$$

(P2) For each $v \in \mathcal{W}$ there is $D(v) \geq 1$ such that for each sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials of which only finitely many are non-zero,

$$\sup_{z\in\mathbb{D}}\Big|\sum_{n=1}^{\infty}(R_{n+1}-R_n)p_n(z)\Big|v(z)\le D(v)\sup_k(\sup_{|z|=r_k}|p_k(z)|)v(r_k).$$

The *main example* for $\mathcal{W}$ will be the set of all strictly positive continuous radial weights v on $\mathbb{D}$ which satisfy $\lim_{r\to 1-}v(r)=0$, are non-increasing on $[0,1)$, and for which there are $\varepsilon_0>0$ and $k(0)\in\mathbb{N}$ such that:

(L1) $\inf_k v(r_{k+1})/v(r_k)\ge\varepsilon_0$,
(L2) $\limsup_{k\to\infty}v(r_{k+k(0)})/v(r_k)<1-\varepsilon_0$.

In this case, R_n is the *convolution with the de la Vallée Poussin kernel*, i.e. for a holomorphic function f on $\mathbb{D}$, $f(z)=\sum_{k=0}^{\infty}a_kz^k$, we have

$$R_nf(z):=\sum_{k=0}^{2^n}a_kz^k+\sum_{k=2^n+1}^{2^{n+1}}\frac{2^{n+1}-k}{2^n}a_kz^k.$$

Conditions (L1) and (L2) are a uniform version of the conditions introduced by W. Lusky in [28], [29], and they also appear in the sequence space representations for weighted (LB)-spaces given by Mattila, Saksman and Taskinen [30]. Bierstedt and Bonet showed that (L1) and (L2) imply (P1) and (P2) (see [9]).

The following lemma is well known. Nevertheless, we give the proof.

Lemma 20. *Let E be a locally convex space and F a dense subspace of E. Assume that F is quasinormable. Then E also is quasinormable.*

Proof. We fix a 0-neighborhood base $\mathcal{U}$ in F. Since F is dense in E, a 0-neighborhood base of E is given by $\overline{\mathcal{U}}:=\{\overline{U}\ ;\ U\in\mathcal{U}\}$. By [26, Proposition 10.7.1] the quasinormability of F implies that for every $U\in\mathcal{U}$ there is $V\subset U$, $V\in\mathcal{U}$, such that for every $\lambda>0$ there is a bounded set $B\subset F$ with $V\subset B+\lambda U$. We obtain $\overline{V}\subset\overline{B+\lambda U}\subset\overline{B}+2\lambda\overline{U}$. Since $\overline{B}$ is bounded, E is quasinormable. ■

The converse of this lemma is not true. In [18] Bonet and Dierolf constructed examples of reflexive and quasinormable Fréchet spaces such that there is a non-distinguished dense subspace.

Theorem 21. *Let $W=(w_n)_{n\in\mathbb{N}}\subset\mathcal{W}$ be an increasing sequence of weights on $\mathbb{D}$ such that $HW_0(\mathbb{D})$ contains the polynomials. The following are equivalent*:

(a) *$HW_0(\mathbb{D})$ is quasinormable.*
(b) *$HW(\mathbb{D})$ is quasinormable.*
(c) *For every $n\in\mathbb{N}$ there is $m>n$ such that for every $k>n$ and every $\varepsilon>0$ there is $\lambda>0$ with*

$$C_{m,0}\subset\lambda C_{k,0}+\varepsilon C_{n,0}.$$

(d) *For every $n \in \mathbb{N}$ there is $m > n$ such that for every $k > n$ and every $\mu > 0$ there is $\xi > 0$ with*

$$\left(\frac{1}{w_m}\right)^{\sim} \le \frac{\xi}{w_k} + \frac{\mu}{w_n} \quad \text{on } \mathbb{D}.$$

(e) *For every $n \in \mathbb{N}$ there is $m > n$ such that for every $\alpha > 0$ there is $\overline{w} \in \overline{W}$ with*

$$\left(\frac{1}{w_m}\right)^{\sim} \le \overline{w} + \frac{\alpha}{w_n} \quad \text{on } \mathbb{D}.$$

Proof. (a)⇔(c). This is [31, Definition before Lemma 26.13].

(a)⇔(b). By Remark 5 we have $C_{\overline{w}} = \overline{C}_{\overline{w},0}$ for every $\overline{w} \in \overline{W}$ radial. Thus, we can apply Proposition 19 to obtain the desired equivalence.

(a)⇒(e). We have $B_n = \overline{B}_{n,0}$ and $B_n = \overline{C}_n$ for every $n \in \mathbb{N}$ (see [12] as mentioned above). Hence Proposition 17 implies (d).

(e)⇒(d) See the proof of Proposition 18.

(d)⇒(c). By [12] the polynomials are dense in $HW_0(\mathbb{D})$. Hence it is enough to consider only polynomials. We fix $n \in \mathbb{N}$, choose $m > n$ as in (d) and fix $k > n$ and $\varepsilon > 0$. Now, we put $\mu := \varepsilon/(2c^2 + D_2)2C$, where c and C are the constants from (P1), and the condition before (P1), and $D_2 = D(w_n)$ in (P2). For μ, (d) yields ξ. We fix $p \in \mathcal{P} \cap C_{m,0}$. Hence $|p| \le 1/w_m$, or equivalently, $|p| \le (1/w_m)^{\sim}$ on $\mathbb{D}$. Condition (d) implies

$$\left(\frac{1}{w_m}\right)^{\sim} \le \frac{\xi}{w_k} + \frac{\mu}{w_n} \le \max\left(\frac{2\xi}{w_k}, \frac{2\mu}{w_n}\right) \quad \text{on } \mathbb{D}.$$

Now, we put $u := \min(w_k/2\xi, w_n/2\mu)$. Then u belongs to the class $\mathcal{W}$, and we have

$$|p| \le \left(\frac{1}{w_m}\right)^{\sim} \le \max\left(\frac{2\xi}{w_k}, \frac{2\mu}{w_n}\right) = \frac{1}{u}.$$

Hence $u|p| \le 1$ on $\mathbb{D}$. Put $\kappa_1 := 1/2\xi$, $\kappa_2 := 1/2\mu$, $u_1 := w_k$, $u_2 := w_n$, i.e. $u = \min(\kappa_1 u_1, \kappa_2 u_2)$.

We have $p = \sum_{n=0}^{\infty}(R_{n+1} - R_n)p = R_1 p + \sum_{n=1}^{\infty}(R_{n+1} - R_n)p$, and the sum is finite. We first treat the term $R_1 p$.

By the condition before (P1) and the estimate on $u|p|$, we get

$$u(r_1) \sup_{|z|=r_1} |R_1 p(z)| \le c u(r_1) \sup_{|z|=r_1} |p(z)| \le c.$$

We select $i \in \{1, 2\}$ with $u(r_1) = \kappa_i u_i(r_1)$. From the second inequality in (P1), applied to the polynomial $R_1 p$ and u, and once more the condition before (P1), we conclude

$$\begin{aligned}\sup_{z\in\mathbb{D}} u_i(z)|R_1p(z)| &\le C\sup_n u_i(r_n)(\sup_{|z|=r_n}|(R_{n+1}-R_n)R_1p(z)|)\\ &= Cu_i(r_1)\sup_{|z|=r_1}|(R_2-R_1)R_1p(z)|\\ &= C(\kappa_i)^{-1}u(r_1)\sup_{|z|=r_1}|(R_2-R_1)R_1p(z)|\\ &\le 2cC(\kappa_i)^{-1}u(r_1)\sup_{|z|=r_1}|R_1p(z)|\le 2c^2C(\kappa_i)^{-1}.\end{aligned}$$

This implies $R_1p\in 4Cc^2\xi C_{k,0}$ or $R_1p\in 4Cc^2\mu C_{n,0}$.

To treat the other term $p-R_1p=\sum_{n=1}^{\infty}(R_{n+1}-R_n)p$, we first apply the first inequality in (P1) for u and the estimate for $u|p|$ to get

$$u(r_n)(\sup_{|z|=r_n}|(R_{n+2}-R_{n-1})p(z)|)\le C. \tag{5}$$

We can write $\mathbb{N}$ as a disjoint union $J_1\cup J_2$ such that

$$u(r_j)=\begin{cases}\kappa_1u_1(r_j) & \text{for } j\in J_1,\\ \kappa_2u_2(r_j) & \text{for } j\in J_2.\end{cases}$$

Now, for $i=1,2$ put $g_i=\sum_{n\in J_i}(R_{n+1}-R_n)p$, which is a polynomial; clearly $p-R_1p=g_1+g_2$. We fix $i\in\{1,2\}$ and let $p_n^i:=(R_{n+2}-R_{n-1})p$ for $n\in J_i$ and $p_n^i:=0$ otherwise. The properties of the sequence $(R_n)_{n\in\mathbb{N}}$ imply

$$g_i=\sum_{n\in J_i}(R_{n+1}-R_n)(R_{n+2}-R_{n-1})p=\sum_{n=1}^{\infty}(R_{n+1}-R_n)p_n^i,$$

and all the sums are finite; hence

$$\sup_{z\in\mathbb{D}}u_i(z)|g_i(z)|=\sup_{z\in\mathbb{D}}u_i(z)\Big|\sum_{n=1}^{\infty}(R_{n+1}-R_n)p_n^i\Big|.$$

Since only a finite number of the p_n^i are non-zero and all the weights belong to the class $\mathcal{W}$, we can apply (P2) and the estimate (5) to conclude

$$\begin{aligned}\sup_{z\in\mathbb{D}}u_i(z)|g_i(z)| &\le D_i\sup_n(\sup_{|z|=r_n}|p_n^i(z)|)u_i(r_n)\le D_i\sup_{n\in J_i}(\sup_{|z|=r_n}|p_n^i(z)|)u_i(r_n)\\ &= D_i\sup_{n\in J_i}(\sup_{|z|=r_n}|(R_{n+2}-R_{n-1})p(z)|)u_i(r_n)\\ &\le D_i(\kappa_i)^{-1}\sup_{n\in J_i}(\sup_{|z|=r_n}|(R_{n+2}-R_{n-1})p(z)|)u(r_n)\\ &\le D_i(\kappa_i)^{-1}C.\end{aligned}$$

This yields $g_1\in 2\xi D_1CC_{k,0}$ and $g_2\in 2\mu D_1CC_{n,0}$. Thus

$$p=R_1p+g_1+g_2\in(2c^2+D_1)2\xi CC_{k,0}+\varepsilon C_{n,0}.$$

Put $\lambda:=(2c^2+D_1)2\xi C$ to obtain the claim. ∎

The following condition was introduced by Peris in [33].

DEFINITION 22. A locally convex space E is said to satisfy *condition* (QNo) if for every $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that for all $\varepsilon > 0$ we can find $P \in L(E, E)$ with

(i) $P(V) \in \mathcal{B}(E)$,
(ii) $(\mathrm{id} - P)(V) \in \varepsilon U$.

If a Fréchet space E is (QNo) then it is quasinormable (see [33]).

LEMMA 23 (Peris [33]). *Let E be a locally complete locally convex space and F a dense subspace with (QNo). Then E also enjoys (QNo).*

PROPOSITION 24. *Let $W = (w_n)_{n\in\mathbb{N}} \subset \mathcal{W}$ be an increasing sequence of weights on $\mathbb{D}$ such that the polynomials $\mathcal{P}$ are dense in $HW_0(\mathbb{D})$ and $\overline{V} := \{1/\overline{w}\ ;\ \overline{w} \in \overline{W}\}$ is a subset of $\mathcal{W}$. The following are equivalent*:

(a) *$HW_0(\mathbb{D})$ is quasinormable.*
(b) *$HW_0(\mathbb{D})$ is (QNo).*

Proof. (b)⇒(a). This follows from the definition (see [33, remarks after Definition 3.2]).

(a)⇒(b). This proof is analogous to that of Theorem 21. Using the same notations we write $T_i p := \sum_{n=1}^{\infty}(R_{n+1} - R_n)p_n^i$ for $i \in \{1, 2\}$ and find that $T_1 p \in 2D_1CC_{\overline{w},0}$ and $T_2 p \in 2\lambda D_2CC_{n,0}$. Thus

$$p = R_1p + T_1p + T_2p \in (2c^2 + D_1)2CC_{\overline{w},0} + 2D_2C\lambda C_{n,0}$$

or

$$p = R_1p + T_1p + T_2p \in 2D_1CC_{\overline{w},0} + (2c^2 + D_2)2\lambda CC_{n,0}.$$

We distinguish the following cases:

CASE 1: We assume $u(r_1) = \kappa_1 u_1(r_1)$. We define $T : \mathcal{P} \to \mathcal{P}$ by $T(p) = R_1p + T_1p$. Then

$$T(C_{m,0} \cap \mathcal{P}) \subset (2c^2 + D_1)2C(C_{\overline{w},0} \cap \mathcal{P}) \in \mathcal{B}(\mathcal{P}),$$
$$(I - T)(C_{m,0} \cap \mathcal{P}) \subset 2D_2C\lambda(C_{n,0} \cap \mathcal{P}) \subset \varepsilon(C_{n,0} \cap \mathcal{P}).$$

CASE 2: We suppose $u(r_1) = \kappa_2 u_2(r_1)$. We define $T : \mathcal{P} \to \mathcal{P}$ by $T(p) = T_1p$ and get

$$T(C_{m,0} \cap \mathcal{P}) \subset 2D_1C(C_{\overline{w},0} \cap \mathcal{P}) \in \mathcal{B}(\mathcal{P}),$$
$$(I - T)(C_{m,0} \cap \mathcal{P}) \subset (2c^2 + D_2)2C\lambda(C_{n,0} \cap \mathcal{P}) \subset \varepsilon(C_{n,0} \cap \mathcal{P}).$$

Thus, $\mathcal{P}$ and $HW_0(\mathbb{D})$ have (QNo). ∎

We finish this article with an example of a sequence $W = (w_n)_{n\in\mathbb{N}}$ such that $\overline{V}$ belongs to the class $\mathcal{W}$.

Proposition 25. *Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous functions on $\mathbb{D}$. Assume that there are $\varepsilon_0 > 0$ and $k_0 \in \mathbb{N}$ such that the following conditions are satisfied:*

(L1) $\inf_k w_n(r_{k+1})/w_n(r_k) \geq \varepsilon_0$ *for every* $n \in \mathbb{N}$.

(L2) *There is* $k_1 \in \mathbb{N}$ *with*

$$w_n(r_{k+k_0}) < (1-\varepsilon_0)w_n(r_k)$$

for every $k \geq k_1$ *and* $n \in \mathbb{N}$.

Then $\overline{V} \subset \mathcal{W}$.

Proof. We fix $\bar{v} \in \overline{W}$. There is $\overline{w} \in \overline{W}$ with $\bar{v} \leq \overline{w}$ such that there is a sequence $(\beta_n)_{n\in\mathbb{N}}$ of positive numbers such that for every $r > 0$ there is $k(r) \in \mathbb{N}$ with

$$(6)\qquad (1/\overline{w})(z) = \frac{1}{\min_{1\leq n\leq k(r)} \beta_n/w_n(z)} = \max_{1\leq n\leq k(r)} \frac{1}{\beta_n} w_n(z).$$

We have to show that there are $\varepsilon_2 > 0$ and $k_2 \in \mathbb{N}$ such that

(L1) $\inf_k \frac{(1/\overline{w})(r_{k+1})}{(1/\overline{w})(r_k)} \geq \varepsilon_2$.

(L2) $\limsup_{k\to\infty} \frac{(1/\overline{w})(r_{k+k_2})}{(1/\overline{w})(r_k)} < 1-\varepsilon_2$.

We choose $\varepsilon_2 = \varepsilon_0$ and $k_2 = k_0$ and prove (L1). For a fixed $k \in \mathbb{N}$ it remains to show

$$\varepsilon_2(1/\overline{w})(r_k) \leq (1/\overline{w})(r_{k+1}).$$

We select $0 < r_{k+1} < s < 1$ and by (6) get

$$(1/\overline{w})(z) = \max_{1\leq n\leq k(s)} \frac{1}{\beta_n} w_k(z)$$

for $z \in \mathbb{D}$ with $|z| \leq s$. We distinguish the following cases:

(i) If $(1/\overline{w})(r_k) = (1/\beta_j)w_j(r_k)$ and $(1/\overline{w})(r_{k+1}) = (1/\beta_j)w_j(r_{k+1})$, then

$$\varepsilon_2(1/\overline{w})(r_k) = \frac{\varepsilon_2}{\beta_j} w_j(r_k) \leq \frac{1}{\beta_j} w_j(r_{k+1}) = (1/\overline{w})(r_{k+1}).$$

(ii) If $(1/\overline{w})(r_k) = (1/\beta_j)w_j(r_k)$ and $(1/\overline{w})(r_{k+1}) = (1/\beta_l)w_l(r_{k+1})$, then

$$\varepsilon_2(1/\overline{w})(r_k) = \frac{\varepsilon_2}{\beta_j} w_j(r_k) \leq \frac{1}{\beta_j} w_j(r_{k+1}) \leq \frac{1}{\beta_l} w_l(r_{k+1}) = (1/\overline{w})(r_{k+1}).$$

Thus $\inf_k \frac{(1/\overline{w})(r_k)}{(1/\overline{w})(r_{k+k_1})} \geq \varepsilon_2$.

It remains to show (L2). As before we choose $\varepsilon_2 = \varepsilon_0$ and $k_2 = k_0$. We have to prove that there is $N_0 \in \mathbb{N}$ such that for every $k \geq N_0$,

$$(1/\overline{w})(r_{k+k_2}) < (1-\varepsilon_2)(1/\overline{w})(r_k).$$

There is $k_1 \in \mathbb{N}$ such that for every $k \geq k_1$ and for every $n \in \mathbb{N}$ we have

$$w_n(r_{k+k_2}) < (1-\varepsilon_2)w_n(r_k).$$

We fix $k \geq k_1$ and select $0 < k + k_2 < s < 1$. Hence, by (6),

$$(1/\overline{w})(z) = \max_{1\leq n\leq k(s)} \frac{1}{\beta_n} w_n(z)$$

for every $z \in \mathbb{D}$ with $|z| \leq s$. We put $N_0 := k_1$ and distinguish the following cases:

(i) If $(1/\overline{w})(r_{k+k_2}) = (1/\beta_j)w_j(r_{k+k_2})$ and $(1/\overline{w})(r_k) = w_j(r_k)$, then

$$(1/\overline{w})(r_{k+k_2}) = \frac{1}{\beta_j} w_j(r_{k+k_2}) \leq (1-\varepsilon_2)\frac{1}{\beta_j} w_j(r_k) = (1-\varepsilon_2)(1/\overline{w})(r_k).$$

(ii) If $(1/\overline{w})(r_{k+k_2}) = (1/\beta_j)w_j(r_k)$ and $(1/\overline{w})(r_k) = (1/\beta_l)w_l(r_k)$, then

$$\begin{aligned}(1/\overline{w})(r_{k+k_2}) &= \frac{1}{\beta_j} w_j(r_{k+k_2}) \leq (1-\varepsilon_2)\frac{1}{\beta_j} w_j(r_k)\\ &\leq (1-\varepsilon_2)\frac{1}{\beta_l} w_l(r_k) = (1-\varepsilon_2)(1/\overline{w})(r_k).\end{aligned}$$

The claim follows. ∎

The following is a special case of Bierstedt [5, Satz 3.5(1)].

THEOREM 26 (Bierstedt [5, Satz 3.5(1)]). *Let $U_1 = (u_{1,k})_{k\in\mathbb{N}}$, $U_2 = (u_{2,k})_{k\in\mathbb{N}}, \dots, U_m = (u_{m,k})_{k\in\mathbb{N}}$ be increasing sequences of strictly positive continuous and radial functions on $\mathbb{D}$. If*

$$W := \bigotimes_{i=1}^m U_i = \Big\{\bigotimes_{i=1}^m u_{i,k}\ ;\ k \in \mathbb{N}\Big\},$$

then

$$HW_0(\mathbb{D}^m) = \Big(\widetilde{\bigotimes}_\varepsilon\Big)_{i=1}^m (HU_i)_0(\mathbb{D}).$$

Since the tensor product of two Fréchet spaces with (QNo) also satisfies (QNo) (see Peris [33]) we get the following consequence.

COROLLARY 27. *Let $U_1 = (u_{1,k})_{k\in\mathbb{N}}$, $U_2 = (u_{2,k})_{k\in\mathbb{N}}, \dots, U_m = (u_{m,k})_{k\in\mathbb{N}}$ be increasing sequences of strictly positive continuous and radial functions on $\mathbb{D}$ such that each $H(U_i)_0(\mathbb{D})$ contains the polynomials. Then $HW_0(\mathbb{D}^m) = (\widetilde{\bigotimes}_\varepsilon)_{i=1}^m H(U_i)_0(\mathbb{D})$ satisfies (QNo) if and only if each $H(U_i)_0(\mathbb{D})$ has (QNo).*

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