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Isometries between groups of invertible elements in C^*-algebras

by

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Abstract. We describe all surjective isometries between open subgroups of the groups of invertible elements in unital C^*-algebras. As a consequence the two C^*-algebras are Jordan $*$-isomorphic if and only if the groups of invertible elements in those C^*-algebras are isometric as metric spaces.

1. Introduction. This paper arises from a desire to study isometries between groups of invertible elements in unital Banach algebras (cf. [4–7]). As is proved in [5] a surjective isometry between open subgroups of the groups of all invertible elements in unital semisimple Banach algebras extends to a surjective real-linear isometry between the underlying Banach algebras. In particular, if the given Banach algebras are commutative, then the extended map is a real isomorphism followed by multiplication by some element. We might conjecture that the Jordan structure is essentially preserved in general. In this paper we give a complete description for the case of unital C^*-algebras.

Throughout the paper the group of all invertible elements in a unital C^*-algebra A is denoted by A^{-1}, and the identity in A is denoted by I_A.

2. The results. The description of real-linear isometries between C^*-algebras, given below, might be already known (cf. [2, Corollary 3.3]). Unfortunately we could not find it in the literature and we present it with a sketch of proof for the convenience of the readers.

Proposition 2.1. *Let A and B be unital C^*-algebras. Suppose that ϕ is a surjective isometry from A onto B with $\phi(0) = 0$. Then ϕ is real-linear and there exist a central projection P in B and a complex-linear Jordan $*$-isomorphism J from A onto B such that*

$$\phi(a) = \phi(I_A)PJ(a) + \phi(I_A)(I_B - P)J(a)^*, \qquad a \in A. \tag{2.1}$$

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Conversely every operator of this form is a real-linear isometry from A onto B.

Proof. First applying the celebrated theorem of Mazur and Ulam [8] (cf. [9]) we see at once that ϕ is a real-linear map. A C^*-algebra is a real C^*-algebra in the sense of [1], hence by [1, Theorem 6.4] the equality

$$\phi(ab^*c + cb^*a) = \phi(a)\phi(b)^*\phi(c) + \phi(c)\phi(b)^*\phi(a) \tag{2.2}$$

holds for every triple $a, b, c \in A$. Choose $b \in A$ so that $\phi(b^*)$ is invertible. Letting $a = c = I_A$ in (2.2) we find that $\phi(I_A)$ is invertible. Furthermore substituting $b = I_A$ we see that $\phi(I_A)$ is unitary. Denoting $\phi_0(\cdot) = (\phi(I_A))^{-1}\phi(\cdot)$, an elementary calculation shows that ϕ_0 is a surjective real-linear isometry from A onto B such that $\phi_0(I_A) = I_B$, and

$$\phi_0(ab^*c + cb^*a) = \phi_0(a)\phi_0(b)^*\phi_0(c) + \phi_0(c)\phi_0(b)^*\phi_0(a) \tag{2.3}$$

for every triple $a, b, c \in A$. By (2.3) we obtain the equalities $\phi_0(b^*) = \phi_0(b)^*$ for every $b \in A$ (ϕ_0 *is $*$-preserving*) and $\phi_0(a^2) = (\phi_0(a))^2$ for every $a \in A$ (ϕ_0 *is square-preserving*).

Put

$$P = \frac{-i\phi_0(iI_A) + I_B}{2}.$$

Define a real-linear operator $J : A \to B$ by

$$J(a) = P\phi_0(a) + (I_B - P)\phi_0(a)^* \tag{2.4}$$

for every $a \in A$. As ϕ_0 is $*$-preserving and square-preserving, the equalities $(-i\phi_0(iI_A))^2 = I_B$ and $(-i\phi_0(iI_A))^* = -i\phi_0(iI_A)$ follow. We also see that $\phi_0(ia) = \phi_0(a)\phi_0(iI_A) = \phi_0(iI_A)\phi_0(a)$ for every $a \in A$. Applying these equalities we observe that these P and J are just as desired. Note that J is surjective. To prove this let $b \in B$. Then there exists $a \in A$ with $\phi_0(a) = Pb + (I_B - P)b^*$ as ϕ_0 is surjective. By a simple calculation $J(a) = b$ as desired.

Conversely, suppose that U is unitary in B, P is a central projection in B, and J is a complex-linear Jordan $*$-isomorphism from A onto B such that

$$\phi(a) = UPJ(a) + U(I_B - P)J(a)^*$$

for every $a \in A$. A calculation shows that ϕ is surjective. We now prove that ϕ is an isometry between A and B. Since P is a central projection we see that

$$\|Pb + (I_B - P)c\| = \max\{\|Pb\|, \|(I_B - P)c\|\} \tag{2.5}$$

for all b and c in B. Applying (2.5) for $b = J(a)$ and $c = J(a)^*$ or $J(a)$ we can easily check that

$$\begin{aligned}\|PJ(a)+(I_B-P)J(a)^*\| &= \max\{\|PJ(a)\|,\|(I_B-P)J(a)^*\|\}\\ &= \max\{\|PJ(a)\|,\|(I_B-P)J(a)\|\}=\|J(a)\|\end{aligned}$$

for every $a\in A$ as I_B-P is central. Since J is an isometry (cf. [3, Theorem 6.2.5]) and U is unitary, the real-linear map ϕ is an isometry. ∎

THEOREM 2.2. *Let A and B be unital C^*-algebras and $\mathcal{A}$ and $\mathcal{B}$ open subgroups of A^{-1} and B^{-1} respectively. Suppose that T is a bijection from $\mathcal{A}$ onto $\mathcal{B}$. Then T is an isometry if and only if $T(I_A)$ is unitary in B and there are a central projection P in B, and a complex-linear Jordan $*$-isomorphism J from A onto B such that*

$$T(a)=T(I_A)PJ(a)+T(I_A)(I_B-P)J(a)^*,\qquad a\in\mathcal{A}. \tag{2.6}$$

Furthermore the operator $T(I_A)PJ(\cdot)+T(I_A)(I_B-P)J(\cdot)^*$ defines a surjective real-linear isometry from A onto B.

Proof. Suppose that $T:\mathcal{A}\to\mathcal{B}$ is a surjective isometry. By [5, Theorem 3.2], T extends to a real-linear isometry $\tilde{T}$ from A onto B since the radical $\mathrm{rad}(B)$ of B is $\{0\}$ as B is a C^*-algebra. Applying Proposition 2.1 we see that (2.6) holds.

Conversely suppose that T is given by (2.6). Then by Proposition 2.1 we see that $T(I_A)PJ(\cdot)+T(I_A)(I_B-P)J(\cdot)^*$ defines a surjective real-linear isometry from A onto B. ∎

For a complex Hilbert space H the algebra of all bounded linear operators on H is denoted by $B(H)$. We describe the structure of all surjective isometries between open subgroups of $B(H)^{-1}$. We need the following notation. Beside the adjoint operation on the algebra $B(H)$ we shall also need the operation of transposition. It is defined by choosing a complete orthonormal system in H and for any operator a considering the operator a^T whose matrix in the given basis is the transpose of the corresponding matrix of a. It can be seen that the map $a\mapsto a^T$ is a well-defined linear $*$-antiautomorphism of $B(H)$. Then the conjugate $\overline{a}$ of $a\in B(H)$ is defined by the formula $\overline{a}=(a^*)^T$. Our result reads as follows.

COROLLARY 2.3. *Let H_1 and H_2 be complex Hilbert spaces. Suppose that T is a surjective isometry from $\mathcal{A}$ onto $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are open subgroups of $B(H_1)^{-1}$ and $B(H_2)^{-1}$ respectively. Then $T(I_{B(H_1)})$ is a unitary operator and there is a unitary operator w (complex-linear isometry) from H_1 onto H_2 such that T is of one of the following forms:*

(1) $T(a)=T(I_{B(H_1)})waw^*$ *for all $a\in\mathcal{A}$,*
(2) $T(a)=T(I_{B(H_1)})wa^*w^*$ *for all $a\in\mathcal{A}$,*
(3) $T(a)=T(I_{B(H_1)})wa^Tw^*$ *for all $a\in\mathcal{A}$,*
(4) $T(a)=T(I_{B(H_1)})w\overline{a}w^*$ *for all $a\in\mathcal{A}$.*

Proof. The center of $B(H_2)$ consists of the scalar operators, hence P is a trivial projection (the zero operator or $I_{B(H_2)}$). Then Theorem 2.2 asserts that there is a Jordan $*$-isomorphism J such that $T(a) = T(I_{B(H_1)})J(a)$ for all $a \in \mathcal{A}$ or $T(a) = T(I_{B(H_1)})J(a)^*$ for all $a \in \mathcal{A}$. A Jordan $*$-isomorphism from $B(H_1)$ onto $B(H_2)$ is an algebra isomorphism or an algebra antiisomorphism, hence there is a unitary operator w from H_1 onto H_2 such that $J(b) = wbw^*$ for every $b \in B(H_2)$ or $J(b) = wb^T w^*$ for every $b \in B(H_2)$. Thus we have the conclusion. ∎

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